Punctured torus and Lagrangian triangle groups in $PU(2, 1)$.

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Abstract

We embed the Teichmüller space of the once punctured torus $T_{(1,1)}$ into the set of conjugacy classes of groups generated by three anti-holomorphic involutions I_1, I_2 and I_3 (Lagrangian triangle groups), acting on the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$. We deform this embedding, and obtain a three dimensional family E of discrete, faithful and type preserving representations of the fundamental group of the once punctured torus.

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1 Introduction

Triangle groups are among the most studied objects in two-dimensional complex hyperbolic geometry. They are generated by three involutions, and may thus be seen as representations of $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ into the isometry group of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ (see [Sch02] for a survey). One of the main problems is to find conditions for such a representation to be discrete and faithful. A classical approach to this problem is to begin with a representation ρ_0 whose image stabilizes a two-dimensional totally geodesic subspace, and to study the possible deformations of this representation. If ρ_0 is flexible, and if ρ_t is a deformation of ρ_0, a natural problem is to determine the maximal τ such that ρ_t remains discrete and faithful for $t \in [0, \tau]$. The usual obstruction for ρ_t to remain discrete and/or faithful is when a loxodromic element turns elliptic during the deformation. This is the complex hyperbolic version of a classical phenomenon for Kleinian groups (see [GP92], [FK00]). Our main result addresses this problem of maximal deformation in the case of an embedding of the whole Teichmüller space instead of a single deformation.

In this work, we are interested in triangle groups generated by three anti-holomorphic involutions, each of which fixes pointwise a Lagrangian plane. We refer to these groups as *Lagrangian triangle groups*. Examples of Lagrangian triangle groups are studied for instance in [FK00].
Throughout this paper, we will use the following notation:

- Γ_1 is the group having presentation $\langle i_1, i_2, i_3 \,|\, i_k^2 = 1\rangle$.
- Γ_2 is the group having presentation $\langle a, b, c \,|\, [a, b]\, c = 1\rangle$. It is the fundamental group of the punctured torus. Γ_2 is embedded (with index two) in Γ_1 by $a \to i_1 i_2$ and $b \to i_3 i_2$.
- $T_{(1,1)}$ is the Teichmüller space of the once punctured torus (see section 2) .

- $\widehat{PU(2,1)}$ (resp. $\widehat{\mathrm{PSL}(2,\mathbb{R})}$) is the full group of isometries of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ (resp. the complex hyperbolic line $\mathbb{H}^1_{\mathbb{C}}$), including holomorphic and anti-holomorphic isometries (see section 3).

In the case of the complex hyperbolic line $\mathbb{H}^1_{\mathbb{C}}$, triangle groups have been used to study the representations of the free group on two generators $F_2 = \langle a, b\rangle$ into PSL(2,$\mathbb{R}$)(see [Mat82], [Gil95]). Among these representations are the punctured torus groups, that is, the discrete, faithful and type preserving representations of the fundamental group of the once punctured torus into PSL(2,$\mathbb{R}$). If ρ is a punctured torus group, it is possible to decompose the generators of its image under the form :

$$\rho(a) = I_1 \circ I_2 \text{ and } \rho(b) = I_3 \circ I_2, \tag{1}$$

where the I_k's are half-turns. The commutator $[\rho(a), \rho(b)] = (I_1I_2I_3)^2$ generates the cyclic subgroup of the punctured torus fundamental group corresponding to a loop around the cusp.

We wish to generalize this approach to the case of two dimensional complex hyperbolic geometry, using anti-holomorphic involutions instead of half-turns. We will call a discrete, faithful and type preserving representation of Γ_2 in $PU(2,1)$ an $\mathbb{H}^2_{\mathbb{C}}$ *punctured torus group.* The purpose of this work is the following:

I. Describe the set $\mathfrak{R}$ of $\widehat{PU(2,1)}$-conjugacy classes of Lagrangian triangle groups $\langle I_1, I_2, I_3\rangle$ such that the cyclic product $\gamma = (I_1I_2I_3)^2$ is parabolic.

II. In $\mathfrak{R}$, identify a three dimensional family of groups containing an $\mathbb{H}^2_{\mathbb{C}}$ punctured torus group with index 2. This family is obtained by deforming a natural embedding of $T_{(1,1)}$ into $\mathfrak{R}$.

All conjugacy classes of $\mathbb{H}^2_{\mathbb{C}}$ punctured torus groups are in $\mathcal{M} = Hom(F_2, PU(2,1))/PU(2,1)$, which has dimension 8. More precisely, they are in the open subset $\mathcal{M}^{lox}$ of $\mathcal{M}$ where the generators of F_2 are represented by loxodromic elements. The subset of $\mathcal{M}^{lox}$ formed by those classes of representations $[\rho]$ such that the pair $(\rho(a), \rho(b))$ admits the same decomposition as in (1) where the half-turns are replaced by Lagrangian involutions form a closed subset of dimension 7 (see [Wil05]). If we add the condition that the commutator be parabolic, the dimension drops to 6. The main result of this work is the following theorem:

Theorem 1. *There exists a three dimensional subset* $\mathfrak{F}$ *of* $\mathfrak{R}$ *homeomorphic to* $\mathcal{T} \times [0, \frac{\pi}{2}[$ *having the following properties:*

1. $\mathcal{T}$ *is an embedding of* $T_{(1,1)}$ *into* $\mathfrak{R}$.
2. *If* $\rho \in E = \mathcal{T} \times [0, \frac{\pi}{4}]$, $\rho(\Gamma_1)$ *is discrete and faithful, and contains an index two subgroup which is an* $\mathbb{H}^2_{\mathbb{C}}$ *punctured torus group.*
3. E *is maximal in the following sense: for any* $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$, *there is a point* $m \in \mathcal{T}$ *such that any group represented by* (m, α) *contains an elliptic element.*

α has a geometric meaning, as explained in section 6 .

We start with a description of the Teichmüller space of the once punctured torus. This space has been studied intensively as the simplest non-trivial Teichmüller space of a non-compact Riemann surface of finite volume. Our description is based on the normalization of the parabolic cycle instead of the fixed points of the generators. The coordinates on $\mathfrak{R}$, introduced in section 5, will follow along the same lines.

After a quick review of the basic properties of the complex hyperbolic plane in section 3, we study the Lagrangian planes (also called $\mathbb{R}$-planes) in section 4. We define the angle between two Lagrangian subspaces of $\mathbb{H}^2_{\mathbb{C}}$ in section 4.3. The parameter α of theorem 1 is the measure of the angle between two Lagrangian planes. In 4.4, we describe a special kind of $\mathbb{R}$-sphere (i.e. a sphere foliated by Lagrangian planes). These $\mathbb{R}$-spheres are invariant under inversion in their leaves (see section 4.4, and [Sch05]).

In section 5, we deal with I. If $\rho \in \mathfrak{R}$, the fixed point of $\rho(\gamma)$ gives rise to a cycle C_ρ:

$$p_2 \xrightarrow{\rho(i_1)} p_3 \xrightarrow{\rho(i_2)} p_1 \xrightarrow{\rho(i_3)} p_2.$$

$\mathfrak{R}$ contains those classes of Lagrangian triangle groups such that p_1, p_2 and p_3 are mutually distinct. We normalize this cycle using Cartan's angular invariant. From the ideal triangle Δ having these vertices one naturally obtains three $\mathbb{R}$-planes, each of which corresponds to an order two symmetry of Δ. We will refer to this triple as the "base configuration", and denote it $(P_1(\mathbb{A}), P_2(\mathbb{A}), P_3(\mathbb{A}))$, where $\mathbb{A}$ is the Cartan invariant. All the configurations we are interested in are related to this base configuration by three loxodromic isometries $h_{23}^{z_1}$, $h_{13}^{z_2}$ and $h_{12}^{z_3}$, where $h_{ij}^{z_k}$ is the loxodromic isometry fixing p_i and p_j with multiplier $z_k \in \mathbb{C}$ (see (5) in section 3.4). Our coordinates on $\mathfrak{R}$ will be the three complex multipliers (z_1, z_2, z_3) of the loxodromic isometries, and $\mathbb{A}$, the angular invariant of the cycle.

In section 6, in which we focus on II, we prove Theorem 1. To that end, we make use of the $\mathbb{R}$-balls described in section 4.4. We describe a one parameter family of domains F^α $(0 < \alpha < \pi/4)$, bounded by three $\mathbb{R}$-balls, and having the property that for any $m \in \mathcal{T}$, F^α is a fundamental domain for the group $(m, \alpha) \in \mathcal{T} \times [0, \pi/4]$. Each F^α is used to show discreteness and faithfulness of a two-parameter family of groups. The main technical point is to show that the $\mathbb{R}$-balls bounding F^α are disjoint as long as $\alpha \in [0, \pi/4]$.

To put our work in perspective, note that a complete classification of the punctured torus groups of PSL(2,$\mathbb{C}$) has been established by Minsky in [Min99]. It is still out of reach in the case of PU(2,1).

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2 Punctured torus and triangle groups in PSL(2,$\mathbb{R}$)

2.1 The Teichmüller space of the once punctured torus.

We start with a classical proposition describing the subgroups of PSL(2,$\mathbb{R}$) uniformizing a punctured torus.

Proposition 1. *Let A and B be two elements of PSL(2,$\mathbb{R}$), and call G the group generated by A and B. Assume that the following conditions hold:*

1. *A and B are hyperbolic, and their axes meet in precisely one point inside inside $\mathbb{H}^1_{\mathbb{C}}$*
2. *the commutator $[A, B]$ is parabolic*

Then G is Fuchsian and the Riemann surface $\mathbb{H}^1_{\mathbb{C}}/G$ is a once punctured torus. Conversely, any once punctured torus is uniformized by a group having these properties.

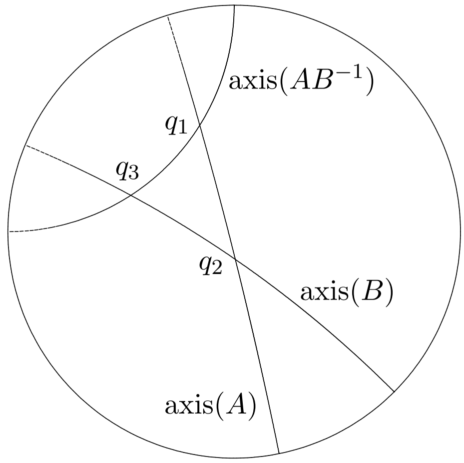


Figure 1: Decomposition of A and B.

For a complete proof of this proposition, see [Kee71].

Definition 1. A *punctured torus group* is a representation $\rho : F_2 \longrightarrow$ PSL(2,$\mathbb{R}$) such that $\rho(a)$ and $\rho(b)$ satisfy conditions 1 and 2 of proposition 1.

Recall that the Teichmüller space of the once punctured torus may be seen as the set

$$\{\rho : \Gamma_2 \longrightarrow \text{PSL(2,}\mathbb{R})\} / \widehat{\text{PSL(2,}\mathbb{R})},$$

where ρ is a discrete, faithful and type-preserving representation of Γ_2 into PSL(2,$\mathbb{R}$) and $\widehat{\text{PSL(2,}\mathbb{R})}$ acts by conjugation. Note that in this case, type preserving means that the only non-hyperbolic elements of $\rho(\Gamma_2)$ are parabolic and are conjugate to the powers of $\rho([a, b])$. Proposition 1 shows that the Teichmüller space of the once punctured torus is the set of $\widehat{\text{PSL(2,}\mathbb{R})}$-conjugacy classes of punctured torus groups. Call A, B and C the images of a, b and c by ρ, and choose lifts $\tilde{A}$, $\tilde{B}$ of A and B to SL(2,$\mathbb{R}$) such that $x = Tr(\tilde{A}) > 2$, $y = Tr(\tilde{B}) > 2$ and $z = Tr(\tilde{A}\tilde{B}) > 2$. Then, the Teichmüller space of the once punctured torus is parametrized by

$$x^2 + y^2 + z^2 = xyz \quad x > 2,\, y > 2,\, z > 2. \tag{2}$$

See [Kee71] for details. This relation was already known in [FK26]. See [Wol83] for a description of the associated moduli space, and a description of its Kähler structure.
The decomposition of the generators as products of involutions is a standard tool in the study of the two-generator subgroups of PSL(2,$\mathbb{R}$)(see [Gil95]). If G is a punctured torus group, it is possible to find a group dectri$G^\star$ generated by three half-turns such that G is of index two in $G^\star$, which is easier to analyze. This decomposition is provided by the following classical lemma. (See figure 1).

Lemma 1. *Let A and B be two elements of PSL(2,$\mathbb{R}$) satisfying condition (1) of proposition 1. There exists a unique triple of half-turns (E_1, E_2, E_3) such that $A = E_1 \circ E_2$ and $B = E_3 \circ E_2$.*

Note that $[A, B] = (E_1 E_2 E_3)^2$.

2.2 Classical triangle groups

Recall that $\widehat{\text{PSL(2,}\mathbb{R})}$ is the group generated by PSL(2,$\mathbb{R}$) and the reflections in geodesics. Recall that Γ_1 is the group having presentation $\langle i_1, i_2, i_3 \,|\, i_k^2 = 1\rangle$. Γ_2 is embedded as an index two subgroup of Γ_1.

Definition 2. A triangle group is a representation $\rho : \Gamma_1 \longrightarrow \mathrm{PSL}(2,\mathbb{R})$.

In this section, we only consider triangle groups with holomorphic generators, that is, generated by three half-turns. Such a triangle group is determined by the fixed point of each of the $\rho(i_k)$'s. A systematic analysis of the discreteness of groups generated by three half-turns in $\mathbb{H}^1_{\mathbb{C}}$ may be found in [Bea83] or [Gil95].

Definition 3. Define

$$\mathcal{T} = \left\{ \rho \text{ triangle group} \,\middle|\, \begin{array}{l} \text{the } \rho(i_k)\text{'s are distinct half-turns} \\ \rho(\gamma) \text{ is parabolic.} \end{array} \right\} / \widehat{\mathrm{PSL}(2,\mathbb{R})}$$

We now describe a special family of triangle groups that yields coordinates on $\mathcal{T}$. Pick the following three points in the upper-half plane:

$$p_1 = 1,\ p_3 = -1 \text{ and } p_2 = \infty.$$

Call γ_{ij} the geodesic joining p_i to p_j $(i \neq j)$ and Δ the ideal triangle $p_1p_2p_3$. Orient the boundary of Δ as follows: γ_{12} toward p_2, γ_{32} toward p_3, and γ_{13} toward p_1. We shall use the following notations:

- For distinct i, j, k let s_k be the orthogonal projection of p_k onto γ_{ij} ($s_2 = i$, $s_1 = -1 + 2i$ and $s_3 = 1 + 2i$).
- For $r > 0$ and $r \neq 1$, let h^r_{ij} be the hyperbolic element having fixed points p_i and p_j and multiplier r. Assume moreover that $r > 1$ corresponds to the case where h^r_{ij} translates in the positive direction along γ_{ij}. If $r = 1$, define $h^1_{ij} = Id$.
- Define $q^r_k = h^r_{ij}(s_k)$ for distinct i, j, k and $r > 0$, and E^r_k the half-turn fixing q^r_k.

The three points s_1, s_2 and s_3 will play the role of a base configuration. These objects are depicted on figure 2 in the unit disk model of $\mathbb{H}^1_{\mathbb{C}}$.

Definition 4. To any triple (r_1, r_2, r_3) of positive numbers, associate the triangle group $T(r_1, r_2, r_3)$ defined by $\rho(i_k) = E^{r_k}_k$ $(k = 1, 2, 3)$.

The three half-turns $E^{r_1}_1$, $E^{r_2}_2$ and $E^{r_3}_2$ are distinct. The following lemma gives a necessary and sufficient condition for $T(r_1, r_2, r_3)$ to be a representative of a point of $\mathcal{T}$.

Lemma 2. *Given a triple (r_1, r_2, r_3) of positive numbers, the isometry $(E^{r_1}_1 E^{r_2}_2 E^{r_3}_3)^2$ is parabolic if and only if $r_1r_2r_3 = 1$.*

Proof. For each $m = u + iv$ ($u \in \mathbb{R}$ and $v > 0$) in the upper half-plane we write E_m for the half-turn fixing m. It admits as a lift to $\mathrm{SL}(2, \mathbb{R})$ the matrix

$$d_{u,v} = \begin{bmatrix} -u/v & \left(u^2 + v^2\right)/v \\ -1/v & u/v \end{bmatrix}.$$

In turn, we obtain matrices for the lifts of the half-turns E^r_k :

$$q^{r1}_1 = -1 + \frac{2i}{r_1^2},\ q^{r3}_3 = 1 + 2ir_3^2,\ \text{and } q^{r2}_2 = \frac{-1 + r_2^4}{1 + r_2^4} + i\frac{2r_2^2}{r_2^4 + 1}.$$

One verifies directly that $(E^{r_1}_1 \circ E^{r_2}_2 \circ E^{r_3}_3)^2$ has matrix form

$$\begin{bmatrix} (r_1r_2r_3)^{-4} & \tau \\ 0 & (r_1r_2r_3)^4 \end{bmatrix} \qquad \text{with } \tau = -\left(2 + (r_1r_2r_3)^4 + (r_1r_2r_3)^{-4} + 2r_2^4r_3^4 + 2r_3^4 + \frac{2}{r_1^4} + \frac{2}{r_1^4r_2^4}\right).$$

Since τ is never zero, $(E^{r_1}_1 \circ E^{r_2}_2 \circ E^{r_3}_3)^2$ is parabolic precisely when the two diagonal entries of the above matrix are equal to 1. The result follows. □

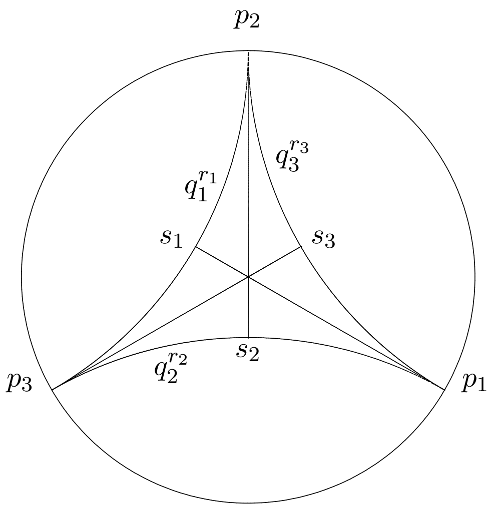


Figure 2: Δ, and $T(r_1, r_2, r_3)$ for $r_1 < 1$, $r_2 < 1$ and $r_3 > 1$.

Remark 1. It would have been simpler to compute $E_1^{r_1} E_2^{r_2} E_3^{r_3}$ instead of its square. However, in the case of $PU(2, 1)$ the half-turns E_k will be replaced by anti-holomorphic involutions I_k, and the product $I_1 I_2 I_3$ will be anti-holomorphic, so that its square is more convenient.

Proposition 2. *Any point of $\mathcal{T}$ is represented by a unique triple $(E_1^{r_1}, E_2^{r_2}, E_3^{r_3})$ with r_1, r_2, $r_3 > 0$ and $r_1 r_2 r_3 = 1$.*

Proof. Let E_1, E_2 and E_3 be three distinct half-turns. $(E_1 E_2 E_3)^2$ is parabolic if and only if $E_1 E_2 E_3$ is. Hence, if $\langle E_1, E_2, E_3\rangle$ is a representative of a point of $\mathcal{T}$, pick m_2 the fixed point of $E_1 E_2 E_3$. m_2 gives rise to a cycle of length 3 :

$$m_2 \xrightarrow{E_3} m_1 \xrightarrow{E_2} m_3 \xrightarrow{E_1} m_2.$$

This cycle is non-degenerate: if for instance, we had $m_1 = m_2$, then E_1, E_2 and E_3 would stabilize the geodesic $m_1 m_3$, and the group generated by $E_1 E_2$ and $E_3 E_2$ would be Abelian, so we would have $(E_1 E_2 E_3)^2 = 1$. Now, conjugating the E_k's by the unique element g of $\widehat{\mathrm{PSL(2,\mathbb{R})}}$ such that $g(m_i) = p_i$ clearly doesn't change the point of $\mathcal{T}$. This shows the result. $\square$

Lemma 1 shows any punctured torus group is contained with index two a triangle group, wich by the above proposition is conjugate to a unique $T(r_1, r_2, r_3)$ satisfying $r_1 r_2 r_3 = 1$. Conversely, if ρ is a point of $\mathcal{T}$, the subgroup generated by $E_1^{r_1} \circ E_2^{r_2}$ and $E_3^{r_3} \circ E_2^{r_2}$ is a punctured torus when $r_1 r_2 r_3 = 1$, as showed by the classical Poincaré polygon theorem in PSL(2,$\mathbb{R}$). As a consequence, given a punctured torus group G, there exists unique $r_1 > 0$ and $r_3 > 0$ such that G is conjugate to the index two subgroup of $\langle E_1^{r_1}, E_2^{(r_1 r_3)^{-1}}, E_3^{r_3}\rangle$ generated by $E_{r_1}^1 \circ E_2^{(r_1 r_3)^{-1}}$ and $E_3^{r_3} \circ E_2^{(r_1 r_3)^{-1}}$. Hence, (r_1, r_3) is a set of coordinates on the Teichmüller space of the once punctured torus.

The (x, y, z)-coordinates of section 2.1 (relation (2)) describe a punctured torus using the length of the geodesics representing generators of the fundamental group. This is done through the relation: $\cosh^2(l/2) = Tr(g)^2/4$, where l is the translation length, and g a lift to SL(2,$\mathbb{R}$) of the associated isometry. The symmetric punctured torus is the one with coordinates $x = y = z = 3$. It is of index 2 in the element of $\mathcal{T}$ having coordinates $(1, 1, 1)$.

3 The complex hyperbolic plane and its isometries

It is convenient to switch between two sets of coordinates for $\mathbb{H}^2_{\mathbb{C}}$, analogous to the Poincaré disk and the upper half-plane for $\mathbb{H}^1_{\mathbb{C}}$. We describe first a set of coordinates for those two models. For more

details, see [Gol99].
We denote by $\mathbf{P}$ the projectivization map $\mathbb{C}^3 \setminus \{0\} \longrightarrow \mathbb{C}P^2$.

3.1 The ball model.

Define V the set of vectors of $\mathbb{C}^3$ having negative norm with respect to the Hermitian form $(X, Y) = \bar{X}^T JY$, where $\cdot^T$ is the transposition and

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In this model,

$$\mathbf{P}(V) = \mathbb{H}^2_{\mathbb{C}} = \left\{(w_1, w_2) \in \mathbb{C}^2 \middle| |w_1|^2 + |w_2|^2 < 1\right\}.$$

3.2 The Siegel model.

It is obtained in the same way as the previous model, this time using the form given by

$$J_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

In this model,

$$\mathbb{H}^2_{\mathbb{C}} = \left\{(z_1, z_2) \in \mathbb{C}^2 \middle| 2\mathrm{Re}(z_1) < -|z_2|^2\right\}.$$

We will use horospherical coordinates (z, t, u), definied by:

$$z_2 = z\sqrt{2} \in \mathbb{C},\ t = \mathrm{Im}(z_1) \in \mathbb{R},\ 2u = -|z_2|^2 - 2\mathrm{Re}(z_1) \in \mathbb{R}_+.$$

In this model, a copy of $\mathbb{H}^2_{\mathbb{R}}$ corresponds to the set of points having horospherical coordinates $(x, 0, u)$ with $x \in \mathbb{R}$ and $u \in \mathbb{R}_+$. It is an example of an $\mathbb{R}$-plane (see section 4). A lift to $\mathbb{C}^3$ of a point of $\mathbb{H}^2_{\mathbb{C}}$ is given in horospherical coordinates by

$$(z, t, u) \longrightarrow \begin{bmatrix} -|z|^2 - u + it \\ \sqrt{2}z \\ 1 \end{bmatrix} \tag{3}$$

The boundary of $\mathbb{H}^2_{\mathbb{C}}$ is the set $\{u = 0\}$. It is equipped with a Heisenberg group structure, with product

$$[z, t].[z', t'] = [z + z', t + t' + 2\mathrm{Im}(z\bar{z'})].$$

Note that the Heisenberg translations extend to isometries of $\mathbb{H}^2_{\mathbb{C}}$ (see section 3.4).

3.3 The Cayley transform.

The Cayley transform exchanges biholomorphically the above two models. It is the collineation c associated to the linear automorphism of $\mathbb{C}^3$ with matrix:

$$\tilde{c} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$\tilde{c}$ conjugates J to J_0, and satisfies $\tilde{c}^2 = Id$. In coordinates:

$$c : (w_1, w_2) \longrightarrow (z_1, z_2) = \left(\frac{w_1 + 1}{w_1 - 1}, \sqrt{2}\frac{w_2}{w_1 - 1} \right)$$

We denote by π the restriction of c to the boundary of the ball which is the stereographic projection from S^3 onto the Heisenberg group :

$$\pi(w_1, w_2) = \left[\frac{w_2}{w_1 - 1}; \frac{-2\mathrm{Im}(w_1)}{|w_1 - 1|^2} \right] \qquad \text{and} \qquad \pi^{-1}([z, t]) = \left(\frac{-|z|^2 + it + 1}{-|z|^2 + it - 1}, \frac{2z}{-|z|^2 + it - 1} \right).$$

3.4 Automorphisms of $\mathbb{H}^2_{\mathbb{C}}$.

Definition 5. Let f be the polynomial

$$f(z) = |z|^4 - 8\mathrm{Re}(z^3) + 18|z|^2 - 27.$$

f provides a trace criterion for matrices of $SU(2,1)$ representing automorphisms of $\mathbb{H}^2_{\mathbb{C}}$:

Lemma 3. *Let M be in $SU(2,1)$, let τ be its trace, and A the isometry associated to M. Then,*

- *If $f(\tau) < 0$, A is regular elliptic.*
- *If $f(\tau) > 0$, A is loxodromic.*
- *If $f(\tau) = 0$, then A is either parabolic or special elliptic.*

By special elliptic, we mean an elliptic element whose lifts have repeated eigenvalues. See chapter 6 of [Gol99] for detailed statements and proofs.

Remark 2. If x, $y \in \mathbb{R}$,

$$f(x + iy) = y^4 + y^2 \left(x + 6 - 3\sqrt{3}\right) \left(x + 6 + 3\sqrt{2}\right) + (x + 1)(x - 3)^3.$$

Thus, as a consequence of Lemma 3, we see that if $\mathrm{Re}(Tr(M)) > 3$, A is loxodromic.

The following special types of isometries will be useful later. They take a particularly simple form in Heisenberg coordinates.

- The Heisenberg (left) translation by $[z, t]$ admits the lift to $SU(2,1)$:

$$\begin{bmatrix} 1 & -\sqrt{2}\bar{z} & -|z|^2 + it \\ 0 & 1 & \sqrt{2}z \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

  It is a parabolic element fixing ∞. Heisenberg translations and their conjugates are known as "pure-parabolic" isometries.

- The Heisenberg dilation by $re^{i\theta}$: $[z, t] \longmapsto [re^{i\theta}z, r^2 t]$ $(r > 0)$ admits the lift to $U(2,1)$ given by

$$\begin{bmatrix} r & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1/r \end{bmatrix}. \tag{5}$$

  It is a loxodromic element fixing $[0, 0]$ and ∞ if $r \neq 1$, and a complex reflection if $r = 1$. Any loxodromic element h of PU(2,1) is conjugate in PU(2,1) to a unique Heisenberg dilation by $re^{i\theta}$ with $r > 1$ and $\theta \in [0, \pi]$. We will refer to the number $re^{i\theta}$ as the *complex multiplier* of h.

4 $\mathbb{R}$-planes.

4.1 Definition.

We call $\mathbb{R}$-planes the totally real totally geodesic subspaces of $\mathbb{H}^2_{\mathbb{C}}$. $\mathbb{R}$-planes are Lagrangian submanifolds of $\mathbb{H}^2_{\mathbb{C}}$, and we might sometimes refer to them as Lagrangian planes (or simply Lagrangians). Every Lagrangian P is the fixed point set of a unique anti-holomorphic involution of $\mathbb{H}^2_{\mathbb{C}}$, called inversion in P. The intersection of a Lagrangian plane with $\partial\mathbb{H}^2_{\mathbb{C}}$, called an $\mathbb{R}$-circle, is homeomorphic to a circle (see [Gol99]). Each $\mathbb{R}$-circle bounds one and only one $\mathbb{R}$-plane, and we shall call inversion in an $\mathbb{R}$-circle the action of the inversion in the corresponding $\mathbb{R}$-plane induced on the boundary.

Definition 6. The $\mathbb{R}$-plane $\mathbb{H}^2_{\mathbb{R}}$ is the set of points with real coordinates in the ball model of $\mathbb{H}^2_{\mathbb{C}}$. We call P_0 the $\mathbb{R}$-plane $P_0 = \left\{(ix_1, ix_2) \in \mathbb{H}^2_{\mathbb{C}}, x_i \in \mathbb{R}\right\} = i\mathbb{H}^2_{\mathbb{R}}$. Let R_0 be the $\mathbb{R}$-circle associated to P_0.

All $\mathbb{R}$-planes are images of $\mathbb{H}^2_{\mathbb{R}}$ under $PU(2,1)$. For the next two definitions, we will only make use of the Siegel model of $\mathbb{H}^2_{\mathbb{C}}$.

Definition 7. Let R be an $\mathbb{R}$-circle, and I_R the associated inversion. the point$I_R(\infty)$ is called the center of R.

Definition 8. Let R be a finite $\mathbb{R}$-circle (that is, not containing ∞). There exists a unique parabolic element T fixing ∞, and a unique Heisenberg dilation,

$$d : [z,t] \longrightarrow [re^{i\theta}z, r^2 t]$$

such that $T(R) = d(R_0)$. The radius of R is defined to be $r^2 e^{2i\theta}$ (see [Gol99]).

Remark that via stereographic projection, $\partial\mathbb{H}^2_{\mathbb{R}}$ is mapped to the x-axis of the Heisenberg group, and that R_0 has center $[0,0]$ and radius 1. For this reason R_0 is sometimes called the *standard* $\mathbb{R}$*-circle*.

4.2 Inversion in an $\mathbb{R}$-plane.

We first describe the action of the inversion in the standard $\mathbb{R}$-circle R_0.

Definition 9. Let P be an $\mathbb{R}$-plane, and I_P the associated inversion. We will say that $M \in$ U(2,1) is a *matrix for* I_P if for any $z \in \mathbb{H}^2_{\mathbb{C}}$ and any lift $\tilde{z}$ of z,

$$\mathbf{P}\left(M.\overline{\tilde{z}}\right) = I_P\left(z\right). \tag{6}$$

(Recall that $\mathbf{P}$ is the projection $\mathbb{C}^3 \setminus \{0\} \to \mathbb{C}P^2$).

Remark 3. Given any $h \in \widehat{\mathrm{PU}(2,1)}$, by "a matrix for h", we mean either any lift of h to U(2,1) (if h is holomorphic), or any matrix that satisfies relation (6) (if h is antiholomorphic).

In the Siegel model, the inversion in the standard $\mathbb{R}$-circle R_0 has matrix J_0, and its action in vectorial homogeneous coordinates is:

$$\begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} \longmapsto J_0 \begin{bmatrix} \bar{z_1} \\ \bar{z_2} \\ 1 \end{bmatrix}.$$

Note that this gives J_0 a double interpretation: it is both the matrix of the bilinear form defining $\mathbb{H}^2_{\mathbb{C}}$ and a matrix for the inversion in P_0.

If h is an isometry with matrix $M \in PU(2,1)$, then $I_{R_0} \circ h$ has matrix $J_0\overline{M}$. This is used to show the following lemma together with the matrices for Heisenberg translations (4) given in section 3.4.

Lemma 4. *Let R be the $\mathbb{R}$-circle with center $[z,t]$ and radius $r^2e^{2i\theta}$. The inversion I_R in R has matrix*

$$J_R = \begin{bmatrix} a & r^2ac-b & r^2a^2+b^2e^{-2i\theta}+r^2 \\ c & r^2c^2+e^{2i\theta} & r^2ac-b \\ \frac{1}{r^2} & c & a \end{bmatrix}$$

where $a = \frac{-|z|^2+it}{r^2}$, $b = \bar{z}e^{2i\theta}\sqrt{2}$ and $c = \frac{z\sqrt{2}}{r^2}$.

Since $r^2 = \dfrac{|b|}{|c|}$ and $e^{2i\theta} = \dfrac{b|b|}{\bar{c}|c|}$, J_R actually depends only on a, b and c. Note that $det\,(J_R) = -e^{2i\theta}$, thus $J_R \in$ U(2,1), and, in order to work with traces, we will normalize J_R to SU(2,1) by multiplying it by $-e^{-\frac{2i\theta}{3}}$. The matrix relation corresponding to the fact that I_R is a anti-holomorphic involution is $J_R\overline{J_R} = Id$.

We will need the following lemma from [FZ99]:

Lemma 5. *Let P_1 and P_2 be two $\mathbb{R}$-planes. Then,*

1. *$I_{P_1} \circ I_{P_2}$ is parabolic if and only if P_1 and P_2 intersect in one boundary point.*
2. *$I_{P_1} \circ I_{P_2}$ is loxodromic if and only if P_1 and P_2 are disjoint.*
3. *$I_{P_1} \circ I_{P_2}$ is regular elliptic if and only if P_1 and P_2 intersect in precisely one point inside $\mathbb{H}^2_{\mathbb{C}}$.*

Remark 4.

1. Note that if two Lagrangian inversions have matrices M_1 and M_2, then their product has matrix $M_1\overline{M_2}$.
2. In order to show that two $\mathbb{R}$-planes are disjoint, we thus have to verify that the product of the two inversions is loxodromic.

4.3 Angle between two intersecting $\mathbb{R}$-planes.

4.3.1 Definitions.

Definition 10. Two pairs (L_1, L_2) and (L'_1, L'_2) of intersecting $\mathbb{R}$-planes are said to have the same angle if and only if there exits an element g of $PU(2,1)$ such that

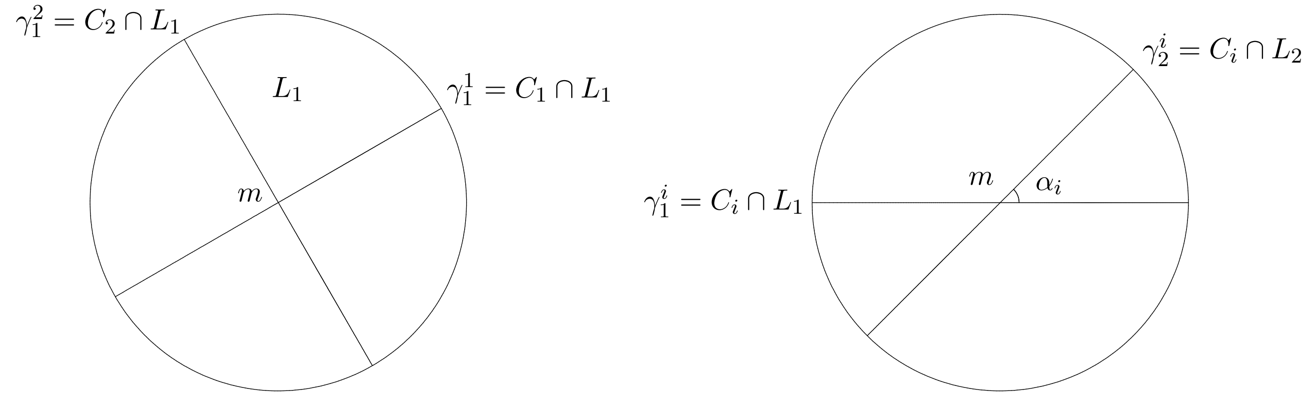
$$L'_i = g(L_i),\ \ i = 1, 2.$$

To measure the angle between two $\mathbb{R}$-planes, we use the following simple lemma:

Lemma 6. *Consider two $\mathbb{R}$-planes L_1 and L_2, intersecting at one point p inside $\mathbb{H}^2_{\mathbb{C}}$. There exists an element $g \in PU(2,1)$ such that $g\,(P_1) = \mathbb{H}^2_{\mathbb{R}} = \{(x,y)\,, x, y \in \mathbb{R}\}$, and $g\,(P_2) = \left\{\left(e^{i\alpha_1}x, e^{i\alpha_2}y\right), x, y \in \mathbb{R}\right\}$, with $0 \le \alpha_1 \le \alpha_2 < \pi$.*

Definition 11. Given a pair (L_1, L_2) of intersecting $\mathbb{R}$-planes, the angle between L_1 and L_2 is denoted by $(\widehat{L_1, L_2})$. Define the measure of $(\widehat{L_1, L_2})$ to be the pair (α_1, α_2) provided by lemma 6.

Remark 5. According to Lemma 6, the elliptic element $f = I_{L_2} \circ I_{L_1}$ has two stable complex lines, C_1 and C_2, and f acts on C_1 (resp. C_2) as a rotation through α_1 (resp. α_2). Hence, we will refer to α_1 (resp. α_2) as the angle between L_1 and L_2 "read in C_1" (resp. "read in C_2"). This terminology is justified by the fact that both I_{P_1} and I_{P_2} stabilize C_1 and C_2, and thus, that both L_1 and L_2 meet C_i along geodesics γ_1^i and γ_2^i. The angle between γ_1^i and γ_2^i has measure α_i. See also [FZ99].



Intersection of C_1 and C_2 with L_1. Intersection of L_1 and L_2 with C_i

Figure 3: Angle between L_1 and L_2 and stable complex lines of $I_1 \circ I_2$.

front view side view

Figure 4: Torus of $\mathbb{R}$-planes having angle $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ with $\mathbb{H}^2_{\mathbb{R}}$ through the origin.

Lemma 6 together with the discussion in remark 5 shows that there is a circle of $\mathbb{R}$-planes through a point $m \in L_1$ having a given angle with L_1. When $\alpha_1 = \alpha_2$, the circle collapses to a point, since in that case the product of the inversions commutes with all the elements of the stabilizer of m.

Example 1. Assume $L_1 = \mathbb{H}^2_{\mathbb{R}}$ and $m = (0,0)$. The set of $\mathbb{R}$-circles corresponding to $\mathbb{R}$-planes having angles $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ is depicted figure 4. It is a torus foliated by linked $\mathbb{R}$-circles (see lemma 8).

Example 2. The standard $\mathbb{R}$-circle R_0 corresponds to the $\mathbb{R}$-plane $i\mathbb{H}^2_{\mathbb{R}}$ through $(0,0)$, using ball-model coordinates. It has angle $(\pi/2, \pi/2)$ with $\mathbb{H}^2_{\mathbb{R}}$.

Example 3. Consider an $\mathbb{R}$-plane P intersecting $\mathbb{H}^2_{\mathbb{R}}$. ∂P is centered at the point p having Heisenberg coordinates $[x, 0]$ with $x \in \mathbb{R}$ if and only if $I_P(\infty) = p$. In this case, I_P stabilizes the complex line C spanned by ∞ and p, and its angle with $\mathbb{H}^2_{\mathbb{R}}$ read in C is $\pi/2$.

4.4 Intersection of $\mathbb{R}$-planes.

Lemma 7. *Let P and P' be two $\mathbb{R}$-planes. Call I_P and $I_{P'}$ the respective inversions. If $P \cap P' = \emptyset$, then $P \cap I_{P'}(P) = \emptyset$*

Proof. Assume $X \in P \cap I_{P'}(P) : X = I_{P'}(Y)$, with $Y \in P$. If $X = Y$, then $X \in P'$, which contradicts the assumption. If not, the geodesic γ spanned by X and Y is stable under $I_{P'}$, thus contains a fixed point p for $I_{P'}$. $X, Y \in P$, so γ is drawn in P, because P is totally geodesic. Hence $p \in P \cap P'$. This is a contradiction. □

Lemma 8 compares the different $\mathbb{R}$-planes having the same angle with a given $\mathbb{R}$-plane at a given point.

Lemma 8. *Consider three $\mathbb{R}$-planes P, P_1 and P_2, all containing a point m, and so that*

$$\widehat{(P, P_1)} = \widehat{(P, P_2)} = (\alpha, \beta) \ \text{ with } \alpha \neq \beta.$$

Then $P_1 \bigcap P_2 = \{m\}$ if $P_1 \neq P_2$.

The proof follows from the normalization in lemma 6.

Lemma 9. *Consider two intersecting $\mathbb{R}$-planes P and Q, having angle (α, β). Then I_P stabilize Q if and only if we are in one of the following cases :*

1. *$\alpha = \beta = 0$. In this case $P = Q$.*
2. *$\alpha = 0$ and $\beta = \frac{\pi}{2}$. In this case $I_P|_Q$ is the inversion in the geodesic $P \cap Q$.*
3. *$\alpha = \beta = \frac{\pi}{2}$. In this case $I_P|_Q$ is a half turn fixing the point $P \cap Q$.*

Proof. We use ball coordinates. We may normalize so that $Q = \mathbb{H}^2_{\mathbb{R}}$, and $P \bigcap Q \ni (0, 0)$. Then P is parametrized by

$$P = \left\{ \left(e^{i\alpha} x_1, e^{i\beta} x_2 \right), x_1^2 + x_2^2 < 1 \right\},$$

and I_P is

$$(w_1, w_2) \longrightarrow \left(\bar{w_1} e^{2i\alpha}, \bar{w_2} e^{2i\beta} \right).$$

The result follows. □

Lemma 10. *Consider three $\mathbb{R}$-planes P_i, $i = 1, 2, 3$, so that the following holds :*

1. *$P_i \cap P_1 = \{m_i\}$ for $i = 2, 3$, and $m_2 \neq m_3$.*
2. *$I_2 \circ I_1$ and $I_3 \circ I_1$ both stabilize the complex line C containing m_2 and m_3.*
3. *$\widehat{(P_2, P_1)} = \left(\frac{\pi}{2}, \beta\right) = \widehat{(P_3, P_1)}$, and the $\frac{\pi}{2}$ angle is read in C.*

Then P_2 and P_3 are disjoint.

Proof. If $\beta = \pi/2$, the result is clear because P_2 and P_3 are distinct fibers of the orthogonal projection onto P_1. If $\beta \neq \pi/2$, call R the complex reflection having mirror C and angle $\pi/2 - \beta$. P_2 and P_3 have angle $(\pi/2, \pi/2)$ with $P_1' = R(P_1)$. The result follows. □

Definition 12. An $\mathbb{R}$-*ball* is a 3-dimensional ball foliated by $\mathbb{R}$-planes.

See also [Sch01].

Remark 6. Lemma 10 is the main tool to build a special type of $\mathbb{R}$-balls, used in section 6 to describe fundamental domains for the groups we are interested in. This is done in the following way:
Let γ be a geodesic, and C the associated complex line. Let m_s $(s > 0)$ a parametrization of γ, and P some $\mathbb{R}$-plane containing γ. For any s call Q_s the $\mathbb{R}$-plane through m_s having angle $(\pi/2, \beta)$ with P, and such that $I_{Q_s} \circ I_P$ stabilizes C. Then $S = \bigcup_{s>0} Q_s$ is an $\mathbb{R}$-ball.

Definition 13. We call the $\mathbb{R}$-ball constructed in remark 6 the $\mathbb{R}$-*ball over* γ *with angle* β *with respect to* P, and we denote it by $S^{\beta}_{\gamma,P}$.

The next lemma is one of the main tools in the proof of the theorem (see section 6).

Lemma 11. *Let P be a Lagrangian, $\gamma \subset P$ a geodesic. For any β, the $\mathbb{R}$-ball $S^{\beta}_{\gamma,P}$ is invariant under inversion in any of its leaves.*

Proof. The proof of Lemma 10 shows that any $S^{\beta}_{\gamma,P}$ is the inverse image of γ under the orthogonal projection onto a Lagrangian meeting P along γ. The result follows. □

Remark 7. $\mathbb{R}$-balls with constant angle are very similar to bisectors. The geodesic γ is the analogue of the real spine, and P the analogue of the complex spine. It could be called a "Lagrangian spine". Contrary to the case of bisectors, γ does not determine uniquely P. Note that $S^{\beta}_{\gamma,P}$ contain only one complex line, which is the one spanned by γ. The boundaries $\partial S^{\beta}_{\gamma,P}$ are so-called $\mathbb{R}$-spheres, analogues of spinal spheres for bisectors. Some examples are depicted on figures 5, 6 and 7.

5 Lagrangian triangle groups.

5.1 Introduction.

We now wish to generalize the approach of section 2 to the case of $\mathbb{H}^2_{\mathbb{C}}$. A priori, the simplest way to do so would be to study subgroups of $PU(2,1)$ generated by three holomorphic involutions, but this would impose a restriction on the conjugacy class of the generators:

Lemma 12. *If E_1, $E_2 \in PU(2,1)$ are two holomorphic involutions, then any lift of $E_1 \circ E_2$ to $SU(2,1)$ has real trace.*

On the other hand, if I_1 and I_2 are Lagrangian inversions, $I_1 \circ I_2$ may be in any conjugacy class of $PU(2,1)$. We will thus define an analogue of $\mathcal{T}$, (the set of classical triangle groups described in section 2) in the case of PU(2,1), using Lagrangian inversions.

5.2 Description of $\mathfrak{R}$.

Recall that $\Gamma_1 = \langle i_1, i_2, i_3 \,|\, i_1^2 = i_2^2 = i_3^2 = 1 \rangle$, $\gamma = (i_1 i_2 i_3)^2$ and Γ_2 is the fundamental group of the once punctured torus, and F_2 is the free group on two generators: $\langle a, b \rangle$.

Definition 14.
1. A Lagrangian triangle group is a representation $\rho : \Gamma_1 \longrightarrow \widehat{PU(2,1)}$ such that $\rho(i_k)$ is a Lagrangian inversion for $k = 1, 2, 3$.
2. An $\mathbb{H}^2_{\mathbb{C}}$-*punctured torus group* is a discrete, faithful and type-preserving representation of Γ_2 into PU(2,1).

Remark 8. A Lagrangian triangle group is fully defined by a triple of $\mathbb{R}$-planes: given such a triple, $\tau = (P_1, P_2, P_3)$, ρ is the unique representation such that $\rho(i_k) = I_k$, the inversion in P_k. Thus, we will often refer to "the Lagrangian triangle group $\langle I_1, I_2, I_3 \rangle$", where the I_k's are Lagrangian inversions.

We will be specially interested in the following set :

Definition 15. Let $\mathfrak{R}$ be the set

$$\mathfrak{R} = \left\{ \text{Lagrangian triangle group } \rho \;\middle|\; \begin{array}{c} \text{the } \rho(i_k)\text{'s are distinct} \\ \rho(\gamma) \text{ is parabolic} \\ \rho \text{ verifies condition (C)} \end{array} \right\} \Big/ \widehat{PU(2,1)}.$$

(C) is a condition of non-degeneracy which is stated in remark 10 and definition 16 below.

There is a natural map from the set of Lagrangian triangle groups into $Hom(F_2, \mathrm{PU(2,1)})$ given by:

$$H : \rho \mapsto \rho_h = \left\{ \begin{array}{l} a \mapsto \rho(i_1 i_2) \\ b \mapsto \rho(i_3 i_2) \end{array} \right\}.$$

$\rho_h(F_2)$ is the index 2 subgroup of $\rho(\Gamma_1)$ containing the holomorphic elements. We will call it the *holomorphic subgroup* of $\rho(\Gamma_1)$.

Lemma 13. *Let ρ be a Lagrangian triangle group. For any choice of matrices for the $\rho(i_k)$'s, the associated matrix for $\rho(\gamma)$ is in $SU(2,1)$ and has real trace.*

Proof. Let $J_k \in \mathrm{U(2,1)}$ be a matrix for $I_k = \rho(i_k)$. The action of I_k may be written in coordinates by $I_k(z) = \mathbf{P}\left(J_k \bar{\tilde{z}}\right)$. Thus, $\rho(\gamma)$ has matrix $M = J_1 \bar{J}_2 J_3 \bar{J}_1 J_2 \bar{J}_3$ (see remark 4) . Clearly, $det(M) = 1$, and $Tr(M) = \overline{Tr(M)}$. □

Proposition 3. *Consider a Lagrangian triangle group ρ, with $\rho \in \mathfrak{R}$. Then, $\rho(\gamma)$ is pure parabolic (that is, conjugate to a Heisenberg translation).*

Proof. According to Lemma 13, any lift of $\rho(\gamma)$ to $SU(2,1)$ has real trace. Since it is parabolic, $\rho(\gamma)$ is either pure parabolic ($\mathrm{Tr}\rho(\gamma) = 3$) or screw parabolic with rotation of angle π ($\mathrm{Tr}\rho(\gamma) = -1$). Now, $\rho(\gamma) = h \circ h$, where h is the anti-holomorphic isometry having matrix form $N = I_1 \bar{I}_2 I_3$. h has at least one fixed point in the closure of $\mathbb{H}^2_{\mathbb{C}}$ (by Brouwer's theorem), and any point fixed by h is fixed by $\rho(\gamma)$. Hence, h has exactly one fixed point on the boundary of $\mathbb{H}^2_{\mathbb{C}}$, which we may assume to be ∞ (using the Siegel model). Normalized in this way, the matrix N is upper triangular. $N\bar{N}$ is a matrix for $\rho(\gamma)$. It is clearly upper triangular with positive real diagonal entries. □

Remark 9. As a consequence, an $\mathbb{H}^2_{\mathbb{C}}$ punctured torus group generated by A and B such that $[A, B]$ is not pure parabolic can never be decomposed using Lagrangian inversions in the form $A = I_1 \circ I_2$ and $B = I_3 \circ I_2$. See [FP03] for an example of non Lagrangian decomposable punctured torus group (contained with index 6 in a representation of the modular group). See [Wil05] for a necessary and sufficient condition for decomposability.

Remark 10.

1. Let ρ be a Lagrangian triangle group such that $\rho(i_1 i_2 i_3)$ has a fixed point in $\partial\mathbb{H}^2_{\mathbb{C}}$. Calling this fixed point m_2, we obtain an ordered triple (m_2, m_1, m_3) of points C_ρ contained in $\partial\mathbb{H}^2_{\mathbb{C}}$, satisfying :

$$m_2 \xrightarrow{\rho(i_3)} m_1 \xrightarrow{\rho(i_2)} m_3 \xrightarrow{\rho(i_1)} m_2. \quad (7)$$

   The fixed point argument in the proof of proposition 3 shows that this is the case for any $\rho \in \mathfrak{R}$. This will be an important point to set coordinates on $\mathfrak{R}$.

2. We are only interested in the case where $\sharp(C_\rho) = 3$ i.e. where C_ρ is non-degenerate. Note that when it is degenerate, it is easily shown that either $\rho(\Gamma_1)$ is contained in a maximal parabolic subgroup of PU(2,1), or contains a complex reflection.

As a consequence of part *2.* of remark 10, we set the following definition :

Definition 16. Let ρ be a Lagrangian triangle group such that $\rho(\gamma)$ is parabolic. We say that ρ verifies condition (C) if $\sharp(C_\rho) = 3$.

If two Lagrangian triangle groups ρ_1 and ρ_2 are conjugate in $PU(2,1)$, say $\rho_2 = g\rho_1 g^{-1}$, then $g(C_{\rho_1}) = C_{\rho_2}$. Thus, in order to normalize the elements of $\mathfrak{R}$, we need some information about the triples of points of $\partial\mathbb{H}^2_{\mathbb{C}}$. Given a point q of $\partial\mathbb{H}^2_{\mathbb{C}}$ denote by $\tilde{q}$ the lift of q to $\mathbb{C}^3$ provided by (3) (see section 3.2). Recall the

Definition 17. Given three points x_1, x_2 and x_3 in $\partial\mathbb{H}^2_{\mathbb{C}}$, the Cartan invariant of the x_k's is

$$\mathbb{A}(x_1, x_2, x_3) = -\arg\left(\langle\tilde{x_1}, \tilde{x_2}\rangle\langle\tilde{x_2}, \tilde{x_3}\rangle\langle\tilde{x_3}, \tilde{x_1}\rangle\right)$$

Recall that $\mathbb{A} = 0$ (resp. $\pm\frac{\pi}{2}$) if and only if the three points lie in an $\mathbb{R}$-plane (resp. a complex line).

Proposition 4. *Let (x_1, x_2, x_3) and (y_1, y_2, y_3) be two triples of points of $\partial\mathbb{H}^2_{\mathbb{C}}$. There exists $g \in PU(2,1)$ such that $g(x_i) = y_i$ if and only if $\mathbb{A}(x_1, x_2, x_3) = \mathbb{A}(y_1, y_2, y_3)$. This g is unique unless the three points lie in a complex line.*

See [Gol99] (theorems 7.1.1 and 7.1.2) for a proof of this proposition and a geometric interpretation of the Cartan invariant.

Lemma 14. *Consider a triple of pairwise distinct points of $\partial\mathbb{H}^2_{\mathbb{C}}$, (m_1, m_2, m_3), not in a common complex geodesic. Then :*

1. *There exists a unique Lagrangian plane L_1, with inversion I_{L_1}, such that*

$$I_{L_1}(m_2) = m_3,\ I_{L_1}(m_3) = m_2 \text{ and } I_{L_1}(m_1) = m_1$$

   (see [Gol99] lemma 7.1.7).

2. *Given any Lagrangian plane l_1 such that the inversion in l_1 exchanges m_2 and m_3, there exists an isometry h_1, which is either loxodromic or a complex reflection, fixing m_2 and m_3 and satisfying $h_1(L_1) = l_1$. Moreover, h_1 is unique up to an order 2 reflection in the complex geodesic generated by m_2 and m_3.*

Proof. The proof of *1.* is in [Gol99]. Let us prove *2.* Call h the isometry $I_{l_1} \circ I_{L_1}$. h fixes m_2 and m_3, thus is either loxodromic or a complex reflection. Write $re^{i\alpha}$ for its complex multiplier (note that h is a complex reflection if and only if $r = 1$). There are two isometries having the required property: h_1 (resp. h_1'), fixing m_2 and m_3 and having multiplier $\sqrt{r}e^{i\alpha}$ (resp. $\sqrt{r}e^{i(\alpha+\pi)}$). The result follows. □

The following corollary is a consequence of the first part of Lemma 14

Corollary 1. *Given a triple $(m_1, m_2, m_3) \in \left(\partial\mathbb{H}^2_{\mathbb{C}}\right)^3$, there exists an elliptic element E of order three such that $E(m_1) = m_2$ and $E(m_2) = m_3$.*

Proof. Apply Lemma 14 to obtain two Lagrangian inversions I_1 (resp. I_2) fixing m_1 (resp. m_2) and exchanging m_2 and m_3 (resp. m_1 and m_3). Then $E = I_1 \circ I_2$ satisfies the above property. See also [Gol99]. □

5.3 Coordinates on $\mathfrak{R}$

In this section, we transpose the results of section 2.2 to the setting of $\mathbb{H}^2_{\mathbb{C}}$. We first describe a family of normalized Lagrangian triangle groups having a cycle of length 3. We then provide necessary and sufficient conditions for an element of this family to be in $\mathfrak{R}$, and deduce a natural set of coordinates on $\mathfrak{R}$. In section 2.2, the three points s_1, s_2 and s_3 played the role of a base configuration, they are replaced here by the three $\mathbb{R}$-planes provided by Lemma 14.

From now on, we will call p_1, p_2 and p_3 the boundary points having Heisenberg coordinates:

$$p_1 = [0,0], \quad p_2 = \infty \text{ and } p_3(\mathbb{A}) = [1, \tan \mathbb{A}].$$

These three points have lifts to $\mathbb{C}^{2,1}$:

$$\tilde{p_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \tilde{p_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \tilde{p_3(\mathbb{A})} = \begin{bmatrix} -1 + i \tan \mathbb{A} \\ \sqrt{2} \\ 1 \end{bmatrix}$$

and verify $\mathbb{A}\left(p_1, p_2, p_3(\mathbb{A})\right) = \mathbb{A}$.
To simplify notation, we will replace $p_3(\mathbb{A})$ by p_3 when this causes no ambiguity.
Applying Lemma 14, we obtain three Lagrangian inversions $I_1(\mathbb{A})$, $I_2(\mathbb{A})$, and $I_3(\mathbb{A})$ such that $I_k(\mathbb{A})$ fixes p_k and exchanges the two other points. Call $P_1(\mathbb{A})$, $P_2(\mathbb{A})$ and $P_3(\mathbb{A})$ the associated $\mathbb{R}$-planes. This is the base configuration.

These three inversions have respective matrices :

$$J_1(\mathbb{A}) = \begin{bmatrix} -e^{-i\mathbb{A}} & 0 & 0 \\ \sqrt{2}\cos \mathbb{A} & e^{i\mathbb{A}} & 0 \\ \cos \mathbb{A} & \sqrt{2}\cos \mathbb{A} & -e^{-i\mathbb{A}} \end{bmatrix} \quad J_2(\mathbb{A}) = \begin{bmatrix} 1 & \sqrt{2} & -1 + i\tan \mathbb{A} \\ 0 & -1 & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$J_3(\mathbb{A}) = \begin{bmatrix} 0 & 0 & 1/\cos \mathbb{A} \\ 0 & -e^{i\mathbb{A}} & 0 \\ \cos \mathbb{A} & 0 & 0 \end{bmatrix}$$

We call Δ the ideal triangle $p_1 p_2 p_3$, and γ_{ij} the geodesic connecting p_i to p_j, with the orientation described in section 2: γ_{12} toward p_2, γ_{32} toward p_3, and γ_{13} toward p_1. We shall also use the following notation :

- If $|z| \neq 1$, $h_{ij}^{z,\mathbb{A}}$ is the loxodromic element fixing p_i and p_j, having multiplier z and such that $h_{ij}^{z,\mathbb{A}}$ translates along γ_{ij} in the positive direction when $|z| > 1$. If $|z| = 1$, $h_{ij}^{z,\mathbb{A}}$ is the complex reflection fixing p_i and p_j having complex multiplier z.

- Call $P_k^{z,\mathbb{A}}$ the $\mathbb{R}$-plane $h_{ij}^{z,\mathbb{A}}(P_k)$, for distinct i, j, k, and $I_k^{z,\mathbb{A}}$ the inversion associated to $P_k^{z,\mathbb{A}}$.

Writing $z = re^{i\theta}$ and $w = e^{i\mathbb{A}} \cos \mathbb{A}$, the translations $h_{ij}^{z,\mathbb{A}}$ admit the following lifts to $U(2,1)$:

$$h_{32}^{z,\mathbb{A}} \sim \begin{bmatrix} r^{-1} & \sqrt{2}r^{-1}(1-z) & 2e^{i\theta} - (r\bar{w})^{-1} - rw^{-1} \\ 0 & e^{i\theta} & \sqrt{2}r\left(1-\bar{z}^{-1}\right) \\ 0 & 0 & r \end{bmatrix}$$

$$h_{31}^{z,\mathbb{A}} \sim \begin{bmatrix} r^{-1} & 0 & 0 \\ -wr^{-1}(1-z)\sqrt{2} & e^{i\theta} & 0 \\ 2e^{i\theta}\cos^2 \mathbb{A} - r\bar{w} - r^{-1}w & -r\bar{w}\left(1-\bar{z}^{-1}\right) & r \end{bmatrix}$$

$$h_{12}^{z,\mathbb{A}} \sim \begin{bmatrix} r & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1/r \end{bmatrix}$$

Finally, matrices for the inversions $I_k^{z,\mathbb{A}}$ are obtained by applying the relation:

$$J_i^{z,\mathbb{A}} = h_{jk}^{z,\mathbb{A}} J_i(\mathbb{A}) \overline{h_{jk}^{z,\mathbb{A}}}^{-1} \tag{8}$$

Definition 18. For any $(z_1, z_2, z_3, \mathbb{A}) \in \mathbb{C}^3 \times]-\frac{\pi}{2}, \frac{\pi}{2}[$, call $\mathcal{R}(z_1, z_2, z_3, \mathbb{A})$ the Lagrangian triangle group defined by

$$\rho(i_k) = I_k^{z_k,\mathbb{A}} \text{ for } k = 1, 2, 3.$$

We now compute $\rho(\gamma)$, in order to obtain conditions for a point of $\mathcal{R}(z_1, z_2, z_3, \mathbb{A})$ to be in $\mathfrak{R}$. Writing $z_k = r_k e^{i\theta_k}$ for $k = 1, 2, 3$, we obtain for $\rho(\gamma)$ the matrix

$$\begin{bmatrix} (r_1r_2r_3)^{-4} & -\sqrt{2}\omega_1 & \omega_3 \\ 0 & 1 & \sqrt{2}\bar{\omega}_2 \\ 0 & 0 & (r_1r_2r_3)^4 \end{bmatrix}$$

with the notations

$$\begin{aligned}
\omega_1 &= (z_1\bar{z}_2z_3)^{-2}\left(1 - \bar{z_1}^{-2} + (\bar{z}_1z_2)^{-2}\right) - \left(1 - z_1^{-2} + (z_1\bar{z}_2)^{-2}\right) \\
\omega_2 &= (r_1r_2r_3)^4\,\omega_1 \\
\omega_3 &= -(r_1r_2r_3)^4\,|\omega_1|^2 + i\,(t + \mathrm{Im}(z)) \\
\text{with} & \\
t &= \tan\mathbb{A}\left(\left(-1 + \frac{1}{r_1^4} + \frac{1}{r_1^4r_2^4}\right) - r_3^4\left(-1 + r_2^4 - r_1^4r_2^4\right)\right) \\
\text{and} & \\
z &= +2\,(z_1\bar{z}_2z_3)^2\left(1 - \bar{z_3}^2 + (z_2\bar{z}_3)^2\right) + 2\,(z_1\bar{z}_2)^{-2}\left(-1 + z_3^2 - \bar{z_1}^{-2}\right) \\
&\quad +2\,(\bar{z}_2z_3)^2\left(\bar{z_3}^2 - 2 + z_1^{-2}\right) + 4z_3^2 - 4z_3^2z_1^{-2} + 2z_1^{-2}.
\end{aligned}$$

Hence,

$$\mathrm{Tr}(\rho(\gamma)) = (r_1r_2r_3)^{-4} + 1 + (r_1r_2r_3)^4.$$

Remark 11. 1. $\mathrm{Tr}(\rho(\gamma))$ depends neither on the θ_i's nor on $\mathbb{A}$. When $r_1r_2r_3 \neq 1$, $\rho(\gamma)$ is loxodromic, and its trace fully determines its conjugacy class.

2. When $r_1r_2r_3 = 1$, the expressions simplify: t vanishes, ω_1 and ω_2 satisfy:

$$\omega_2 = \omega_1 = -\bar{z_3}^2 - \overline{z_2z_3}^2 - \frac{1}{z_1^2z_2^2} - 1 + \frac{1}{z_1^2} + (z_1z_2z_3)^{-2}.$$

Thus, when $r_1r_2r_3 = 1$, $\rho(\gamma)$ does not depends on $\mathbb{A}$ anymore.

As a consequence:

Proposition 5. *The $\widehat{PU(2,1)}$ conjugacy class of $\mathcal{R}(z_1, z_2, z_3, \mathbb{A})$ is in $\mathfrak{R}$ if and only if:*

$$|z_1z_2z_3| = 1 \text{ and } \overline{\left(\frac{z_3}{z_1}\right)}\left(\bar{z_2}^{-1} + \bar{z}_2\right) + z_1\bar{z}_2z_3 \notin \mathbb{R}.$$

Proof. Let ρ be the representation of Γ_1 associated to $\mathcal{R}(z_1, z_2, z_3, \mathbb{A})$. For simplicity, denote by I_k the inversion $\rho(i_k)$. By construction, the cycle of $\rho(\Gamma_1)$ is non-degenerate and ρ is in $\mathfrak{R}$ if and only if $\rho(\gamma)$ is parabolic. Thus, the condition $|z_1 z_2 z_3| = 1$ is necessary. We still have to ensure that $\rho(\gamma)$ is not the identity. Call M_k a matrix form for I_k, and $N = M_1 \overline{M_2} M_3 \in U(2,1)$. Then $N\bar{N}$ is a matrix for $\rho(\gamma)$. As a consequence, $\rho(\gamma) = Id$ if and only if $N^{-1} = \bar{N}$, that is, if $I_1 \circ I_2 \circ I_3$ is a Lagrangian inversion. Using the matrices above and the relation $|z_1 z_2 z_3| = 1$, one verifies

$$N^{-1} - \bar{N} = \begin{bmatrix} 0 & c & 0 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{bmatrix}$$

The parameter c is given by

$$c = 2i\sqrt{2}e^{i(\theta_1 - \theta_2 + \theta_3)} \left(r_3^2 \sin(\theta_1 - \theta_2 - \theta_3) - \sin(\theta_1 - \theta_2 + \theta_3) + r_1^{-2} \sin(-\theta_1 - \theta_2 + \theta_3) \right)$$

where we have written $z_k = r_k e^{i\theta_k}$. The result follows, using the relation $r_1 r_2 r_3 = 1$. □

Proposition 6. *Let $[\varphi]$ be a point of $\mathfrak{R}$ such that $\varphi(\Gamma_1)$ does not stabilize any complex line. Then, $[\varphi]$ is represented by a unique $\rho : \Gamma \longrightarrow \widehat{PU(2,1)}$, defined by*

$$\rho(i_k) = I_k^{z_k, \mathbb{A}}, \quad k = 1, 2, 3,$$

and satisfying

$$|z_1 z_2 z_3| = 1 \text{ and } \overline{\left(\frac{z_3}{z_1}\right)} \left(\bar{z}_2^{-1} + \bar{z}_2\right) + z_1 \bar{z}_2 z_3 \notin \mathbb{R}.$$

Here, denoting $z_k = r_k e^{i\theta_k}$, $r_k > 0$, $\theta_k \in [0, \pi[$, and $\mathbb{A} \in [0, \frac{\pi}{2}[$

Proof. Consider a point of $\mathfrak{R}$, and choose a representative ρ of this point. As in section 5.2, consider the cycle (m_1, m_2, m_3). There exists a unique $\beta \in [0, \pi/2]$ and a unique $g \in \widehat{PU(2,1)}$ such that

$$g(m_1) = p_1,\ g(m_2) = p_2,\ g(m_3) = p_3(\mathbb{A}) \text{ and } |\mathbb{A}(m_1, m_2, m_3)| = \beta.$$

Conjugating ρ by g, and applying Lemma 14 and Proposition 5, we obtain the proposition. □

Remark 12. $(z_1, z_2, z_3, \mathbb{A})$ is actually a set of coordinates on the set of conjugacy classes of Lagrangian triangle groups such that $\rho(i_1 i_2 i_3)$ has at least one fixed point on $\partial\mathbb{H}^2_{\mathbb{C}}$.

6 Proof of the theorem.

We first consider representations ρ such that $\rho(\Gamma_1)$ stabilizes an $\mathbb{R}$-plane, which we normalize to be $\mathbb{H}^2_{\mathbb{R}}$. In this case, the cycle $\mathcal{C}_\rho$ is contained in $\partial\mathbb{H}^2_{\mathbb{R}}$. The $\mathbb{R}$-planes fixed by the three Lagrangian inversions generating $\rho(\Gamma_1)$ are orthogonal to $\mathbb{H}^2_{\mathbb{R}}$, and the corresponding Lagrangian triangle groups are embeddings of the classical triangle groups described in section 2. This step is described in section 6.1.

In section 6.2, we describe a one parameter deformation of all the embedded groups. We next decribe fundamental domains for these deformed configurations having $\mathbb{R}$-balls with constant angle for their faces. The main point is to show that these hypersurfaces (called the S_i^α's) are disjoint. This is done in Lemma 15. Last, in section 6.3, we prove the theorem. The main part is to show that the deformed representations are type-preserving. This is done in section 6.3.2.

To simplify the exposition of the proof, we make the following change of notation: from now on $I_k^{r,\alpha}$ will be the inversion $I_k^{re^{i\alpha},0}$. Denote also by $J_k^{r,\alpha}$ the associated matrix form, and by $P_k^{r,\alpha}$ the associated $\mathbb{R}$-plane. We will denote by B^c the closure in $\mathbb{H}^2_{\mathbb{C}} \cup \partial\mathbb{H}^2_{\mathbb{C}}$ of a set B.

6.1 Step 1: Embedding of the classical triangle groups into $\mathfrak{R}$.

The third part of Lemma 9 provides a way to embed any triangle group of PSL(2,$\mathbb{R}$) into $\widehat{\text{PU}(2,1)}$. This is done in the next proposition.

Proposition 7. *Let $T = \langle E_1, E_2, E_3 \rangle$ be a triangle group of $PSL(2,\mathbb{R})$. There exists a representation φ_0 of T into $\widehat{PU(2,1)}$ having the following properties*

1. *$\varphi_0(T)$ is a Lagrangian triangle group. It stabilizes $\mathbb{H}^2_{\mathbb{R}} \subset \mathbb{H}^2_{\mathbb{C}}$, and $\varphi_0(E_k)_{|\mathbb{H}^2_{\mathbb{R}}}$ is a half-turn, for $k = 1, 2, 3$.*

2. *φ_0 is discrete, faithful, and type-preserving.*

Proof. For $i = 1, 2, 3$, call q_i the fixed point of E_i. Let h be an conformal embedding of $\mathbb{H}^1_{\mathbb{C}}$ into $\mathbb{H}^2_{\mathbb{C}}$ with image $\mathbb{H}^2_{\mathbb{R}}$. Call Π the orthogonal projection $\mathbb{H}^2_{\mathbb{C}} \longrightarrow \mathbb{H}^2_{\mathbb{R}}$, and define for $k = 1, 2, 3$, $P_k = \Pi^{-1}(h(q_k))$. P_k is an $\mathbb{R}$-plane having angle $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $\mathbb{H}^2_{\mathbb{R}}$. Let I_k be the Lagrangian inversion in P_k. Define φ_0 by $\varphi_0(E_k) = I_k$ for $k = 1, 2, 3$. Then :

1. According to Lemma 9, the first part of the proposition is true.

2. Call d_1 and d_2 the distance functions on $\mathbb{H}^1_{\mathbb{C}}$ and $\mathbb{H}^2_{\mathbb{C}}$. Since h is conformal, it is clear that for any $g \in T$ and $m \in \mathbb{H}^1_{\mathbb{C}}$,
$$h(g.m) = \varphi_0(g).h(m). \tag{9}$$
Hence , if $\varphi_0(g) = Id$, $d_2(\varphi_0(g) \cdot h(m), h(m)) = 0 = d_1(g \cdot m, m)$, thus φ_0 is faithful. The same kind of argument shows discreteness and preservation of types. This shows the second part.

□

Corollary 2. *$\mathcal{T}$ is naturally embedded in $\mathfrak{R}$.*

Proof. The normalization from sections 2 and 5, together with the previous proposition shows that the mapping
$$\begin{array}{rcl} \Psi : \mathcal{T} & \longrightarrow & \mathfrak{R} \\ T(r_1, r_2, r_3) & \longmapsto & \mathcal{R}\left(r_1 e^{i\frac{\pi}{2}}, r_2 e^{i\frac{\pi}{2}}, r_3 e^{i\frac{\pi}{2}}, 0\right) \end{array}$$
is an embedding. From now on, we will thus identify $\mathcal{T}$ with $\Psi(\mathcal{T}) \subset \mathfrak{R}$. □

Since the Lagrangian inversions preserve orthogonality, the 3 balls $S^0_i = \Pi^{-1}(\gamma_{jk})$ (i, j, k, distinct) are stable under $I_i^{r_i, \frac{\pi}{2}}$. As a consequence, F^0, the inverse image of Δ by the orthogonal projection, Π, is a fundamental domain for the groups $\mathcal{R}\left(r_1 e^{i\frac{\pi}{2}}, r_2 e^{i\frac{\pi}{2}}, r_3 e^{i\frac{\pi}{2}}, 0\right)$. Let us summarize the properties of the S^0_i's:

1. For distinct i, j, k, S^0_i is $S^{\frac{\pi}{2}}_{\gamma_{jk}, \mathbb{H}^2_{\mathbb{R}}}$, the $\mathbb{R}$-ball over γ_{jk} with angle $\pi/2$ with respect to $\mathbb{H}^2_{\mathbb{R}}$ (see definition 13 of section 4.3).

2. $I_i^{r_i, \frac{\pi}{2}}$ exchanges the two components of $\mathbb{H}^2_{\mathbb{C}} \setminus S^0_i$

3. $(S^0_i)^c \cap (S^0_j)^c = \{p_k\}$ with i, j, k mutually distinct.

Definition 19. An $\mathbb{R}$-ball $S \subset \mathbb{H}^2_{\mathbb{C}}$ is a 3-dimensional ball foliated by $\mathbb{R}$-planes.

6.2 Step 2: Deformation of the embedded groups.

All the $\mathbb{R}$-planes we have used in step 1 were orthogonal to the $\mathbb{R}$-plane $\mathbb{H}^2_{\mathbb{R}}$. The idea of the deformation of the embedding of $\mathcal{T}$ into $\mathfrak{R}$, is to move all the angles from $(\pi/2, \pi/2)$ to $(\pi/2, \pi/2+\alpha)$. This induces a deformation of the balls S_i^0 into S_i^α, and we will check that if $\alpha \in [0, \pi/4]$, the deformed spheres remains disjoint.

Definition of the deformed $\mathbb{R}$-balls . For disjoint i, j, k , define $S_i^\alpha = S^{\frac{\pi}{2}+\alpha}_{\gamma_{jk}, \mathbb{H}^2_{\mathbb{R}}}$, the $\mathbb{R}$-ball over γ_{jk} with angle $\frac{\pi}{2}+\alpha$ with respect to $\mathbb{H}^2_{\mathbb{R}}$. See definition 13 of section 4.3

Note that $S_i^\alpha = \bigcup_{r_i>0} P_i^{r_i, \frac{\pi}{2}+\alpha}$. Recall that from lemma 11 (section 4.3), we know that S_i^α is invariant under inversion in any of its leaves. The following lemma is the essential tools to prove the theorem.

Lemma 15. *For $i = 1, 2, 3$, and $\alpha \in [\frac{-\pi}{4}, \frac{\pi}{4}]$, $(S_i^\alpha)^c \bigcap (S_j^\alpha)^c = \{p_k\}$*

Proof. Because of the symmetry of order 3 described in Corollary 1 , it is sufficient to show that the leaves of S_1^α and S_3^α are disjoint for these values of α. According to lemma 5, this is equivalent to show that as long as $\alpha \in [\frac{-\pi}{4}, \frac{\pi}{4}]$,

$$I_1^{r_1, \frac{\pi}{2}+\alpha} \circ I_3^{r_3, \frac{\pi}{2}+\alpha} \text{ is loxodromic } \forall\, (r_1, r_3) \in]0, +\infty[^2$$

Using the matrices provided in section 5.3, one checks that $I_1^{r_1, \frac{\pi}{2}+\alpha}$ and $I_3^{r_3, \frac{\pi}{2}+\alpha}$ have matrices

$$M_1^{r_1}(\alpha) = \begin{bmatrix} -r_1^2 & -\sqrt{2}\left(r_1^2 + e^{2i\alpha}\right) & r_1^2 + r_1^{-2} + 2e^{2i\alpha} \\ \sqrt{2}r_1^2 & 2r_1^2 + e^{2i\alpha} & -\sqrt{2}\left(r_1^2 + e^{2i\alpha}\right) \\ r_1^2 & \sqrt{2}r_1^2 & -r^2 \end{bmatrix}$$

and

$$M_3^{r_3}(\alpha) = \begin{bmatrix} 0 & 0 & r_3^2 \\ 0 & e^{2i\alpha} & 0 \\ r_3^{-2} & 0 & 0 \end{bmatrix}.$$

The matrix $M = M_1^{r_1}(\alpha)\overline{M_3^{r_3}(\alpha)} \in \mathrm{SU}(2,1)$ is a matrix for $I_1^{r_1, \frac{\pi}{2}+\alpha} \circ I_3^{r_3, \frac{\pi}{2}+\alpha}$ (see remark 4). To show that the isometry associated to M is loxodromic, we compute its trace. A direct calculation yields

$$\mathrm{Re}(\mathrm{Tr}(M_1^{r_1}(\alpha)\overline{M_3^{r_3}(\alpha)})) = r_1^2 r_3^2 + 1 + \frac{1}{r_1^2 r_3^2} + 2\cos 2\alpha \left(r_1^2 + \frac{1}{r_3^2}\right) + \frac{r_1^2}{r_3^2}$$

and

$$\mathrm{Im}(\mathrm{Tr}(M_1^{r_1}(\alpha)\overline{M_3^{r_3}(\alpha)})) = 2\sin 2\alpha \left(r_1^2 - \frac{1}{r_3^2}\right).$$

As a consequence, as long as $\cos 2\alpha$ remains positive,

$$\mathrm{Re}(\mathrm{Tr}(M_1^{r_1}(\alpha)\overline{M_3^{r_3}(\alpha)})) > \frac{1}{r_1^2 r_3^2} + r_1^2 r_3^2 + 1 \geq 3,$$

and the isometry associated to M is loxodromic (see Lemma 2). This completes the proof of proposition 15. □

Since S_i^α contains γ_{jk}, for distinct i, j, k, S_j^α and S_k^α are in the same connected component of $\mathbb{H}^2_{\mathbb{C}} \setminus S_i^\alpha$.

Definition 20. For $i = 1, 2, 3$, let B_i^α be the connected component of $\mathbb{H}^2_{\mathbb{C}} \setminus S_i^\alpha$ not containing S_j^α and S_k^α for distinct i, j, k.

The previous lemma shows that, for i, j, k distinct $(B_i^\alpha)^c \cap (B_j^\alpha)^c = \{p_k\}$, ($B^c$ denotes the closure of the set B). We go now to the proof of the theorem.

6.3 Proof of the theorem

Let $\mathfrak{F}$ be the subset of $\mathfrak{R}$ defined by

$$\mathfrak{F} = \left\{ \mathcal{R}\left(r_1 e^{i(\frac{\pi}{2}+\alpha)}, r_2 e^{i(\frac{\pi}{2}+\alpha)}, r_3 e^{i(\frac{\pi}{2}+\alpha)}, 0\right) \mid (r_1, r_2, r_3, \alpha) \in]0, \infty[^3 \times [0, \frac{\pi}{2}], r_1 r_2 r_3 = 1 \right\}.$$

It is represented by the groups $G(r_1, r_3, \alpha) = \langle I_1^{r_1,\alpha}, I_2^{(r_1 r_3)^{-1},\alpha}, I_3^{r_3,\alpha} \rangle$ described above. Let E be the subset of $\mathfrak{F}$ where $0 \leq \alpha \leq \frac{\pi}{4}$.

6.3.1 Part 1 of the theorem

$\mathfrak{F}$ is homeomorphic to $\mathcal{T} \times [0, \pi/2]$, and it follows from sections 2 and 6.1 that $\mathcal{T} = \mathcal{T} \times \{0\}$ is an embedding of the Teichmüller space $T_{(1,1)}$ in $\mathfrak{R}$.

6.3.2 Part 2 of the theorem

The two lemmas 11 and 15 describe three balls in $\mathbb{H}^2_{\mathbb{C}}$, B_1^α, B_2^α and B_3^α, bounded by S_1^α, S_2^α and S_3^α satisfying the following properties:

(i) S_k^α is invariant by $I_k^{r_k,\alpha}$ for $r_k > 0$.

(ii) The two connected components of $\mathbb{H}^2_{\mathbb{C}} \setminus S_k^\alpha$ are exchanged by $I_k^{r_k,\alpha}$.

(iii) For $\alpha \in [0, \frac{\pi}{4}]$, $B_k^\alpha \cap B_j^\alpha = \emptyset$ and $(B_k^\alpha)^c \cap (B_j^\alpha)^c = \{p_i\}$.

1. **Discreteness and faithfulness .** Using the above balls, the standard proof for Schottky groups works without changes (see [Rat94] for instance).

2. **Type preserving property.** Consider $w = w_1 \cdots w_{2n} \neq Id$, a holomorphic word of $\rho(\Gamma_1)$, and conjugate it, so that $w_{2n} \neq w_1$. For any l, we will denote by D_l the ball $B_{k_l}^\alpha$ invariant by w_l. The properties (i), (ii), (iii) above show that D_1 and D_{2n} are stable under w. Hence w has at least one fixed point in both D_1^c and D_{2n}^c. But w has at most two fixed points or else, it would be a complex reflection, and this would contradict either discreteness or faithfulness. Hence, there are only two possibilities :

   (a) w has two distinct fixed points $q_1 \in D_1^c$ and $q_{2n} \in D_{2n}^c$, and it is loxodromic.

   (b) w fixes one of the p_k's.

   If (b) happens and, for instance, w fixes p_2, a standard argument shows that w is a (possibly negative) power of γ.
   This shows that the only non-loxodromic elements of the holomorphic subgroup of $\rho(\Gamma_1)$ are parabolic, and are conjugate to powers of the cusp element. Thus the holomorphic subgroup of $\rho(\Gamma_1)$ is an $\mathbb{H}^2_{\mathbb{C}}$ punctured torus group.

6.3.3 Part 3 of the theorem

Assume that $r_3 = r_1^{-1}$. Then, as in the proof of Lemma 15, it is seen that

$$\mathrm{Re}\left(\mathrm{Tr}\left(I_1^{r_1,\alpha} \circ I_3^{r_1^{-1},\alpha}\right)\right) = 3 + r_1^2\left(r_1^2 + \cos 2\alpha\right) \text{ and } \mathrm{Im}\left(\mathrm{Tr}\left(I_1^{r_1,\alpha} \circ I_3^{r_1^{-1},\alpha}\right)\right) = 0.$$

Hence, if $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$, and $r_1^2 + \cos 2\alpha < 0$, $I_1^{r_1,\alpha} \circ I_3^{r_1^{-1},\alpha}$ is elliptic. To be more precise, if we set $r_3 = \frac{t}{r_1}$, there is a neighborhood $U(\alpha, r_1)$ of 1 such that:

$$t \in U(\alpha, r_1) \Longleftrightarrow I_1^{r_1,\alpha} \circ I_3^{\frac{t}{r_1},\alpha} \text{ is elliptic}$$

7 Observations.

Remark 13. Let ρ be the representation of Γ_1 associated to a group $G(r_1, r_3, \alpha)$, with $r_1, r_3 > 0$ and $\frac{\pi}{4} \geq \alpha > 0$. Lemma 9 shows that $G(r_1, r_3, \alpha)$ do not stabilize $\mathbb{H}^2_{\mathbb{R}}$. It is easily seen that any $\mathbb{R}$-plane stabilized by $G(r_1, r_3, \alpha)$, must contain the triple C_ρ. Hence, if $\frac{\pi}{4} \geq \alpha > 0$, $G(r_1, r_3, \alpha)$ does not stabilize any $\mathbb{R}$-plane.

Remark 14. We have constructed our deformation in such a way that the element $\gamma = (I_1 I_2 I_3)^2$ remains purely parabolic everywhere. $\rho(\gamma)$ has matrix form:

$$\begin{bmatrix} 1 & -\sqrt{2}\bar{\omega} & -|\omega|^2 + i\tau \\ 0 & 1 & \sqrt{2}\omega \\ 0 & 0 & 1 \end{bmatrix}.$$

In the case of $G(r_1, r_3, \alpha)$, ω and τ become :

$$\omega = 2e^{i\alpha} \cos\alpha \left(1 + r_3^2 + r_1^{-2}\right)$$

$$\tau = -2\sin 2\alpha \left(\left(r_3^2 + 1\right)^2 - \frac{1}{r_1^4}\right).$$

There are two PU(2,1)-conjugacy classes of Heisenberg translations : vertical translations form a conjugacy class, and non-vertical translations another. In our case $-\pi/4 \leq \alpha \leq \pi/4$, thus $\omega \neq 0$. Hence, all the groups we have described are 2-generator subgroups with fixed conjugacy class of the commutator.

Remark 15. The proof of the third part of the theorem showed that when $r_1 r_3 = 1$, the length 2 word $I_1 I_3$ of $G(r_1, r_3, \alpha)$ remains loxodromic when $r_1^2 + \cos 2\alpha$ is negative. However, when this last condition is not satisfied, another word can become elliptic, but it seems hard to determine which one. As an example, consider the case where $r_1 = r_3^{-1} = 2$, keeping the condition $r_1 r_2 r_3 = 1$. A computation shows that all the length two words are loxodromic for any $\alpha \in [0, \frac{\pi}{2}[$. An experimental study shows that the length 8 word $I_1 I_3 I_1 I_2 I_3 I_2 I_3 I_2$, which has trace

$$3 + 1154\cos^4\alpha - 429\cos^2\alpha - 1150\, i \sin\alpha \cos^3\alpha$$

is elliptic on the segment $\alpha_0 < \alpha < \frac{\pi}{2}$, with $0.468\pi < \alpha_0 < 0.469\pi$. It is the first word (that is, the shortest) to become elliptic for these values of r_1, r_2 and r_3. For a given value of α, it seems difficult to determine which word will be the first to become elliptic (in the spirit of the Schwartz conjectures, see [Sch02]) .

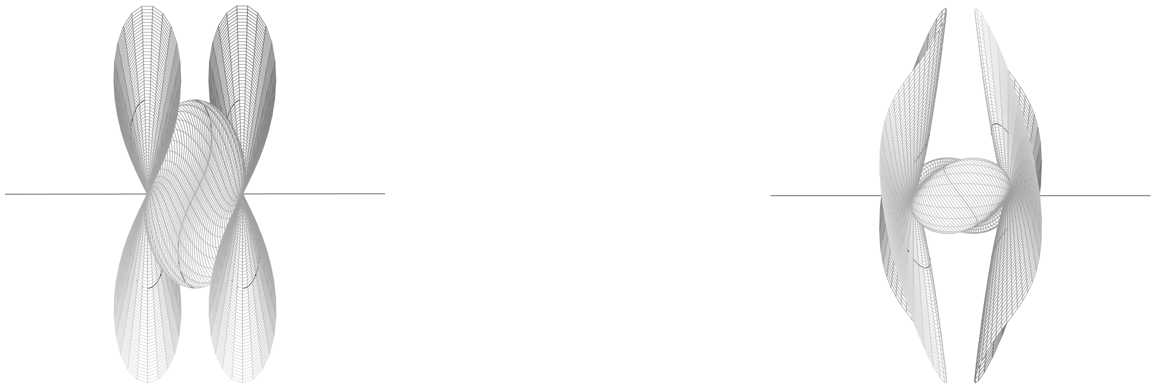


Figure 5: Top and side view of the fundamental domain for the embedding of the classical case.

Figure 6: Top and side view of the limit fundamental domain for $\alpha = \frac{\pi}{4}$.

Figure 7: Top and side view of the $\mathbb{R}$-sphere S_2 alone, for $\alpha = \frac{\pi}{10}$.

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