Bending Fuchsian representations of fundamental groups of cusped surfaces in PU(2,1).

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Abstract

We describe a new family of representations of $\pi_1(\Sigma)$ in PU(2,1), where Σ is a hyperbolic Riemann surface with at least one deleted point. This family is obtained by a bending process associated to an ideal triangulation of Σ. We give an explicit description of this family by describing a coordinate system on it which is in the spirit of Fock and Goncharov coordinates on the Teichmüller space. We identify within this family new examples of discrete, faithful and type-preserving representations of $\pi_1(\Sigma)$. In turn, we obtain a 1-parameter family of embeddings of the Teichmüller space of Σ in the PU(2,1)-representation variety of $\pi_1(\Sigma)$. These results generalise to arbitrary Σ the result we had obtained in [32] for the 1-punctured torus.

Key words: Complex hyperbolic geometry, representations of surface groups, Teichmüller space, deformation.

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1 Introduction

Let Σ be an oriented surface with negative Euler characteristic. Describing the representation variety of the fundamental group $\pi_1(\Sigma)$ in a given Lie group G has been a major problem during the last two decades. The central object in this field is the character variety

$$\mathrm{Rep}_{\pi_1(\Sigma),G} = \mathrm{Hom}(\pi_1(\Sigma), G)/G. \tag{1}$$

This problem finds its source in the study of the Teichmüller space of Σ, which classifies hyperbolic metrics or complex structures on Σ. Riemann's uniformization theorem implies that the Teichmüller space of Σ may be seen as the subset of $\mathrm{Rep}_{\pi_1(\Sigma),\mathrm{PSL}(2,\mathbb{R})}$ consisting of conjugacy classes of discrete, faithful and type-preserving representations. In [14], Goldman has classified the connected components of $\mathrm{Rep}_{\pi_1(\Sigma),\mathrm{PSL}(2,\mathbb{R})}$ using the Euler number of a representation in the case where Σ is closed without boundary. It turns out that the extremal values of the Euler number correspond to two connected components of $\mathrm{Rep}_{\pi_1(\Sigma),\mathrm{PSL}(2,\mathbb{R})}$ which are copies of the Teichmüller space of Σ. These two connected components are actually conjugate one to the other by an antiholomorphic isometry of the upper half-plane. The question of classifying the connected components of $\mathrm{Rep}_{\pi_1(\Sigma),G}$ and understanding the situation of discrete and faithful representations within it has since then been addressed for many Lie groups (see for instance [4, 5, 11, 20, 23]), leading to what is sometimes called *higher Teichmüller theory.* The frame of this work is the case where G =PU(2,1), which is the group of holomorphic isometries of its symmetric space, the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$. In this setting, only a few examples are known (see [1, 12, 16, 18, 19]), and determining whether or not a finitely generated subgroup of PU(2,1) is discrete remains a difficult task.

In the case where Σ is closed without boundary, the connected components of the variety $\mathrm{Rep}_{\pi_1(\Sigma),\mathrm{PU}(2,1)}$ are classified by the Toledo invariant of a representation ρ (see [5, 30]), which we will denote by $\mathbf{tol}(\rho)$. This invariant is defined as the integral over Σ of the pull-back of the Kähler form on $\mathbf{H}^2_{\mathbb{C}}$ by a ρ-equivariant embedding of $\tilde{\Sigma}$ into $\mathbf{H}^2_{\mathbb{C}}$. However, contrary to the case of PSL(2,$\mathbb{R}$), discrete and faithful representations are not contained in specific components of $\mathrm{Rep}_{\Sigma,\mathrm{PU}(2,1)}$, as shown for instance by [16] or [1, 2, 12]. In the case where Σ has punctures, then the Toledo invariant is defined for type-preserving representation (that is, mapping classes of loops around punctures to parabolics), but it does not classify the topological components (see [19]) of $\mathrm{Rep}_{\pi_1(\Sigma),\mathrm{PU}(2,1)}$. The latter case is the one we are going to be concerned with.

A classical way to produce non-trivial examples of representations of surface groups in PU(2,1) is to start with a representation ρ_0 preserving a totally geodesic subspace V of $\mathbf{H}^2_{\mathbb{C}}$ and to deform it. There are a priori two ways to do so in our case since there are two kinds of maximal totally geodesic subspaces in $\mathbf{H}^2_{\mathbb{C}}$, namely *complex lines* and totally geodesic Lagrangian planes, or *real planes.* Complex lines are embeddings of $\mathbf{H}^1_{\mathbb{C}}$ with sectional curvature -1, and real planes are embeddings of $\mathbf{H}^2_{\mathbb{R}}$ with sectional curvature $-1/4$. Their respective stabilisers are P(U(1,1)×U(1)) and PO(2,1) (see for instance [15]). A discrete and faithful representation preserving a complex line (resp. a real plane) is called $\mathbb{C}$-*Fuchsian* (resp. $\mathbb{R}$-*Fuchsian*).

The behaviours of $\mathbb{C}$ and $\mathbb{R}$-Fuchsian representations under deformation are different. The main reason for that is that a $\mathbb{C}$-Fuchsian representation is reducible, whereas an $\mathbb{R}$-Fuchsian representation is not. Indeed, in the case where Σ is closed without boundary, Goldman's rigidity theorem (see [13]) asserts that any deformation of a $\mathbb{C}$-Fuchsian representation is still $\mathbb{C}$-Fuchsian. In contrast, it is proved in [26] that any $\mathbb{R}$-Fuchsian representation is contained in an open set of maximal dimension in $\mathrm{Rep}_{\Sigma,\mathrm{PU}(2,1)}$ containing only discrete and faithful representations, but non necessarily $\mathbb{R}$-Fuchsian. In the case of a surface with punctures, Goldman's rigidity theorem does not hold. Several counter examples can be found in the literature, for instance in [8, 17, 19].

1. We first describe a family of deformations obtained by a bending process. These representations arise as holonomies of equivariant mappings from the Farey set of a

surface Σ to the boundary of $\mathbf{H}^2_{\mathbb{C}}$. We see the Farey set as the set of vertices of a lift to $\tilde{\Sigma}$ of an ideal triangulation T of Σ and consider applications mapping the triangles of $\tilde{T}$ to ideal triangles contained in a real plane. Roughly speaking, we are bending along the edges of an ideal triangulation of Σ, and the case were the representation preserves a real plane corresponds to vanishing bending angles. This is Theorem 2.

2. We identify within this family of examples a subfamily of discrete, faithful, and type-preserving representations. The proof of discreteness is done by showing the the action of $\rho(\pi_1)$ on $\mathbf{H}^2_{\mathbb{C}}$ is discontinuous. We obtain in turn a 1-parameter family of embeddings of the Teichmüller space of Σ into $\mathrm{Rep}_{\pi_1(\Sigma),\mathrm{PU}(2,1)}$containing only classes of discrete, faithful representations with unipotent boundary holonomy. This result is Theorem 3

In the context of $\mathbf{H}^2_{\mathbb{C}}$, deformations by bending were first described by Apanasov in [3]. More recently, in [29], Platis has described the complex hyperbolic version of Thurston's *quakebending deformations* for deformations of $\mathbb{R}$-Fuchsian representations of groups in the case of closed surfaces without boundary. He shows that if ρ_0 is an $\mathbb{R}$-Fuchsian representation of $\pi_1(\Sigma)$, Λ is a finite geodesic lamination with a complex transverse measure μ, then there exists $\epsilon > 0$ such that any quakebend deformation $\rho_{t\mu}$ of ρ_0 is complex hyperbolic quasi-Fuchsian for all $t < \epsilon$. The proof of discreteness in [29] rests on the main result in [26] where the proof of discreteness is done by building a fundamental domain.

In order to sum up our work, let us introduce some notation.

Throughout this work, we will denote by Σ a oriented surface of genus g with $n > 0$ deleted points, which we denote by $x_1, \cdots x_n$. We will assume that Σ has negative Euler characteristic, that is, $2 - 2g - n < 0$. We will denote by $\pi_1(\Sigma)$ or π_1 the fundamental group of Σ. It admits the following presentation

$$\pi_1(\Sigma) \sim \langle a_1, b_2, \cdots a_g, b_g, c_1 \cdots c_n | \prod [a_i, b_i] \prod c_j = 1 \rangle,$$

where the c_j's are homotopy classes of loops enclosing the punctures of Σ. The universal cover of Σ is an open disc $\tilde{\Sigma}$, with a π_1-invariant family of points on its boundary corresponding to the deleted points of Σ. This set of boundary points is called the *Farey set* of Σ, and denoted by $\mathcal{F}_\infty$. We will denote by $\mathcal{DF}(\Sigma)$ the set of PSL(2,$\mathbb{R}$)-classes of discrete and faithful representations of $\pi_1(\Sigma)$ in PSL(2,$\mathbb{R}$), and by $\mathcal{T}(\Sigma)$ the Teichmüller space of Σ, which can be seen as the subset of $\mathcal{DF}(\Sigma)$ consisting of those classes of type-preserving representations. If T is an ideal triangulation of Σ, a decoration of T is a mapping $\mathtt{d} : e(T) \longrightarrow \mathbb{R}$, where $e(T)$ is the set of unoriented edges of T. Such a decoration is called *positive* whenever $\mathtt{d}(e) > 0$ for all edge e.

An $\mathbf{H}^1_{\mathbb{C}}$-realization of $\mathcal{F}_\infty$ is a pair (ϕ, ρ), where ϕ is an application $\mathcal{F}_\infty \longrightarrow \partial\mathbf{H}^1_{\mathbb{C}}$, ρ is a discrete and faithful representation $\pi_1(\Sigma) \longrightarrow$PSL(2,$\mathbb{R}$), and ϕ is $(\pi_1(\Sigma), \rho)$-equivariant, that is, $\phi(\gamma \cdot m) = \rho(\gamma)\phi(m)$ for all $\gamma \in \pi_1(\Sigma)$ and $m \in \mathcal{F}_\infty$. The group PSL(2,$\mathbb{R}$) acts on the set of $\mathbf{H}^1_{\mathbb{C}}$-realizations of $\mathcal{F}_\infty$ by

$$g \cdot (\phi, \rho) = (g \circ \phi, g\rho g^{-1}), \tag{2}$$

and we will denote by $\mathcal{DF}^+$ the set PSL(2,$\mathbb{R}$)-classes of $\mathbf{H}^1_{\mathbb{C}}$ realizations of $\mathcal{F}_\infty$. The set $\mathcal{DF}^+$ is a $2^n : 1$ cover of of $\mathcal{DF}$, branched over the set of classes of representations ρ such that $\rho(c_i)$ is parabolic for at least one index i. The starting point of our work is the following classical theorem.

Theorem 1. *Let T be an ideal triangulation of Σ. There is a canonical bijection between $\mathcal{DF}^+$ and the set of positive decorations of T.*

The proof of this result goes as follows. To any positively decorated triangulation, it is possible to associate a unique class of $\mathbf{H}^1_{\mathbb{C}}$-realization (ϕ, ρ). This is done by interpreting the positive numbers attached to the edges of T as cross-ratios and use it to develop Σ in $\mathbf{H}^1_{\mathbb{C}}$. The mapping ϕ appears as the developing map, and ρ as the associated holonomy representation of $\pi_1(\Sigma)$. Conversely, it is possible to reconstruct the decoration $\mathtt{d}$ from the data (ϕ, ρ). We give a detailed point for us is the fact that the image by a representation ρ associated to a positively decorated triangulation of a class of loop γ is given explicitly as a product of elementary isometries which play the role of transition maps between the charts of $\tilde{\Sigma}$. These elementary isometries are the following. First, an elliptic element of order three cyclically permuting the three point ∞, -1 and 0 in the upper half-plane model of $\mathbf{H}^1_{\mathbb{C}}$. Second, a one parameter family of involutions $(I_x)_{x\in\mathbb{R}_{>0}}$ characterised by the conditions

$$I_x(\infty) = 0, I_x(-1) = x \text{ and } I_x^2 = \text{Id}.$$

As shown in [24], it is possible to describe a similar system of explicit coordinates on an open subset of $\text{Rep}_{\pi_1(\Sigma),\text{PU}(2,1)}$which contains all the classes of discrete and faithful representations. However, identifying those classes of representations that are indeed discrete remains out of reach, due to the complexity of the coordinates. Therefore, we will restrict ourselves to a family of representations, obtained by making an additional geometric assumption. More precisely, our first goal is to classify what we call *T-bent realizations* of $\mathcal{F}_\infty$ in $\mathbf{H}^2_{\mathbb{C}}$, where T is an ideal triangulation of Σ, that is pairs (ϕ, ρ), where

- ρ is a representation $\pi_1(\Sigma) \longrightarrow \text{Isom}(\mathbf{H}^2_{\mathbb{C}})$,
- $\phi : \mathcal{F}_\infty \longrightarrow \partial\mathbf{H}^2_{\mathbb{C}}$ is a (π_1, ρ)-equivariant mapping,
- for any face Δ of $\hat{T}$ with vertices $a, b, c \in \mathcal{F}_\infty$, the three points $\phi(a)$, $\phi(b)$ and $\phi(c)$ form a real ideal triangle of $\mathbf{H}^2_{\mathbb{C}}$, that is, they belong to the boundary of a real plane of $\mathbf{H}^2_{\mathbb{C}}$.

In the context of PSL(2,$\mathbb{R}$), the parameter x decorating the edges of an ideal triangulation is a cross-ratio: the unique invariant of a pair of ideal triangles of $\mathbf{H}^1_{\mathbb{C}}$ sharing an edge. To parametrise the isometry classes of T-bent realizations of $\mathcal{F}_\infty$ associated to T, we will decorate the edges of T using the invariant of an ordered pair of real ideal triangles (τ_1, τ_2) in $\mathbf{H}^2_{\mathbb{C}}$, which is a complex number $z \in \mathbb{C} \setminus \{-1, 0\}$ denoted by $\mathtt{Z}(\tau_1, \tau_2)$. The parameter z is similar to the Koranyi-Reimann cross-ratio on the Heisenberg group (see [15, 21] and remark 9 in section 3.2), and is actually the same parameter used by Falbel in [7] to glue ideal tetrahedra in $\mathbf{H}^2_{\mathbb{C}}$. Note that Falbel needs two such parameters to describe the isometry class of an ideal tetrahedra. We only need one such parameter since we only consider pairs of real ideal triangles, which correspond in his terminology to the particular case of symmetric tetrahedra (see section 4.3 of [7]).

The modulus of z is similar to the cross-ratio in $\mathbf{H}^1_{\mathbb{C}}$, and its argument is the *bending parameter*, which might be seen as the measure of an angle between real planes. In particular, if z is real, the two adjacent real ideal triangles are contained in a common real plane. We will therefore call a *bending decoration* of T any application $\mathtt{D} : e(T) \longrightarrow \mathbb{C} \setminus \{-1, 0\}$ (the two cases where $z = 0$ or $z = -1$ correspond to degenerate triangles).

As in the case of PSL(2,$\mathbb{R}$), it is possible to associate T-bent realizations to bending decorations. We obtain again an explicit expression for the images of classes of loops by ρ as products of elementary isometries. This time, one of the elementary isometries is again an elliptic element of order 3, cyclically permuting the three points ∞, $[-1, 0]$ and $[0, 0]$ of $\partial\mathbf{H}^2_{\mathbb{C}}$ seen as the one point compactification of the 3-dimensional Heisenberg group $\mathbb{C} \times \mathbb{R}$. The other kind of elementary isometries is a family of involutions $(\sigma_z)_{z \in \mathbb{C} \setminus \{-1,0\}}$, where σ_z is characterised by the conditions

$$\sigma_z(\infty) = [0, 0], \sigma_z([-1, 0]) = [z, 0] \text{ and } \sigma_z^2 = \mathrm{Id}.$$

It turns out that σ_z is antiholomorphic. This is related to the fact that $\mathtt{Z}$ classifies ordered pairs of triangles (in particular, $\mathtt{Z}(\tau_1, \tau_2) = \overline{\mathtt{Z}(\tau_2, \tau_1)}$, see section 3.2). Therefore a product of elementary isometries is not always holomorphic. This is why the representation ρ is taken in Isom($\mathbf{H}^2_{\mathbb{C}}$) rather than in PU(2,1), which is the index two subgroup of Isom($\mathbf{H}^2_{\mathbb{C}}$) containing holomorphic isometries. However, for some special triangulations, the image of the representation is actually contained in PU(2,1). Namely, we show in section 4.4, that the representation ρ associated to a T-bent realization of $\mathcal{F}_\infty$ is holomorphic if and only if T is *bipartite*, that is if its dual graph is bipartite . Now, any cusped surface Σ admits a bipartite ideal triangulation (this is Proposition 11). This bending process produces thus representations of $\pi_1(\Sigma)$ in PU(2,1) for any genus and number of punctures of Σ.
Let $\mathcal{BD}_T$ be the set of bending decorations of an ideal triangulation T and $\mathcal{BR}^*_T$ the quotient of $\mathcal{BD}_T$ by the action of complex conjugation. The first result of our work is the following.

The bending theorem

Theorem 2. *There is a bijection between $\mathcal{BD}^*_T$ and $\mathcal{BR}_T$.*

We will see that to any bending decoration is naturally associated a unique pair (r_1, r_2) of PU(2,1)-classes of bent realizations of $\mathcal{F}_\infty$ which represent the same Isom($\mathbf{H}^2_{\mathbb{C}}$)-class of realization. The complex conjugation of bending representations corresponds to the permutation $(r_1, r_2) \longrightarrow (r_2, r_1)$.

After having classified T-bent realizations, we focus on a special kind: those T-bent realizations corresponding to *regular* bending decorations. A bending decoration is said to be *regular* if it has constant argument, that is, if it is of the form $\mathtt{D} = \mathtt{d}e^{i\theta}$, where $\mathtt{d}$ is a positive decoration of T, and $\theta \in]-\pi,\pi]$ is a fixed real number. When $\theta = 0$, we obtain T-bent realizations where all the images of the points of $\mathcal{F}_\infty$ are contained in a real plane. The corresponding representations are $\mathbb{R}$-Fuchsian. For any positive decoration $\mathtt{d}$ of T and any $\theta \in]-\pi,\pi]$, $\mathtt{d}e^{i\theta}$ is the regular bending decoration of T associating to any edge $e \in e(T)$ the complex number $\mathtt{d}(e)e^{i\theta}$.

We now state the main result of our work.

The discreteness theorem

Theorem 3. *Let T be a bipartite ideal triangulation of Σ, and $\theta \in]-\pi,\pi[$ be a real number. For any positive decoration $\mathtt{d}$ of T, let $\rho_{\mathtt{d}}$ be a representative of the unique PSL(2,$\mathbb{R}$)-class of representation $\pi_1(\Sigma) \longrightarrow$PSL(2,$\mathbb{R}$) associated with $\mathtt{d}$, and $\rho^\theta_{\mathtt{d}}$ be a representative of the unique Isom($\mathbf{H}^2_{\mathbb{C}}$)-class of representation $\pi_1(\Sigma) \longrightarrow$PU(2,1) associated to $\mathtt{d}^\theta$. Then*

1. *For any index i, $\rho^\theta_{\mathtt{d}}(c_i)$ is loxodromic (resp. parabolic) if and only if $\rho_{\mathtt{d}}(c_i)$ is hyperbolic (resp. parabolic).*
2. *The representation $\rho^\theta_{\mathtt{d}}$ do not preserve any totally geodesic subspace of $\mathbf{H}^2_{\mathbb{C}}$ unless $\theta = 0$ in which case it is $\mathbb{R}$-Fuchsian.*
3. *As long as $\theta \in [\pi/2, \pi/2]$, the representation $\rho^\theta_{\mathtt{d}}$ is discrete and faithful.*

The two Theorems 2 and 3 are generalisations to the case of any punctured surface of results we had obtained in [32] in the case of the punctured torus.
In particular, when we restrict to those positive decorations $\mathtt{d}$ such that the associated representations $\rho_{\mathtt{d}}$ are type-preserving, we obtain a 1-parameter family parametrised by $\theta \in [-\pi/2, \pi/2]$ of embeddings of the Teichmüller space of Σ in $\mathrm{Rep}_{\pi_1(\Sigma),\mathrm{PU}(2,1)}\pi_1(\Sigma)$ of which images contain only classes of discrete, faithful and type-preserving representations. Note that the family of embedding which we obtain depends on the initial choice of the triangulation.
The proof of Theorem 3 goes as follows. The bent realization associated to the bending decoration $\mathtt{d}^\theta$ provides a family F of real ideal triangles in $\mathbf{H}^2_{\mathbb{C}}$. If Δ and Δ' are two adjacent triangles within this family, we define a canonical real hypersurface of $\mathbf{H}^2_{\mathbb{C}}$ called the *splitting surface* of τ and τ' and denoted by $\mathrm{Spl}(\tau,\tau')$. The main technical point is to show that if τ is any triangle in F, and $\tau_i, i = 1,2,3$ are its neighbours, the three associated splitting surfaces are mutually disjoint in $\mathbf{H}^2_{\mathbb{C}}$. More precisely, denoting by p_i the unique vertex of τ which is also a vertex of τ_{i+1} and τ_{i+2}, then the intersection of the closures of $\mathrm{Spl}(\tau,\tau_{i+1})$ and $\mathrm{Spl}(\tau,\tau_{i+2})$ is $\{p_i\}$. This technical result is Theorem 5. This allows us to define a prism $\mathfrak{p}_\tau$ associated to τ. If τ and τ' are two adjacent real ideal triangles, the two (closed) prisms $\mathfrak{p}_\tau$ and $\mathfrak{p}_{\tau'}$ intersect along the splitting surface $\mathrm{Spl}(\tau,\tau')$. This is sufficient in order to prove that the action of $\rho(\pi_1)$ is discontinuous on the union of all the $\mathfrak{p}_\tau$'s, and therefore that $\rho(\pi_1)$ is discrete and faithful. Splitting surfaces are examples of what we call *spinal $\mathbb{R}$-surfaces*, which are the inverse images of geodesics by the orthogonal projection on real planes (see section 5.2). This terminology refers to spinal surfaces, defined by

Mostow in [25], which are the inverse images of geodesic by the orthogonal projection on a complex line (spinal surfaces are often called *bisectors*, see [15]). Spinal $\mathbb{R}$-surfaces appeared first in [32] under the name of $\mathbb{R}$-balls. They were generalised by Parker and Platis in [26] under the name of *packs*. In their terminology, spinal $\mathbb{R}$-surfaces are *flat packs*. In particular, the characterisation of spinal $\mathbb{R}$-surfaces given in the Lemma 5 is similar to their definition of packs.

To put our results in perspective, let us sum up a few facts about $\text{Rep}_{\pi_1(\Sigma),\text{PU}(2,1)}$.

- If Σ has genus g and n punctures, the representation variety $\text{Rep}_{\pi_1(\Sigma),\text{PU}(2,1)}$ has real dimension $16g-16+8n$, and its subset containing the classes of type preserving representations has real dimension $16g-16+7n$. Any ideal triangulation of Σ has $6g-6+3n$ edges. Therefore the dimension of $\mathcal{DF}^+$ is $6g-6+3n$, and $\mathcal{DF}^+$ may be seen as $\mathbb{R}_{>0}^{6g-6+3n}$. The subset corresponding to type preserving representation, which is in fact the Teichmüller space of Σ is a real subvariety of dimension $6g-6+2n$,
- If T is an ideal triangulation of Σ, $\mathcal{BR}_T$ and $\mathcal{BR}_T^*$ have real dimension $12g-12+6n$, and may be seen respectively as $(\mathbb{C}\setminus\{-1,0\})^{6g-6+3n}$ and its quotient by the complex conjugation. The real dimension of the family of the classes of discrete and faithful representations obtained by examining regular bending decorations of T is $6g-6+3n+1$ and falls to $6g-6+2n+1$ if we add the condition of type-preservation.
- The Toledo invariant **tol** (see [19, 30]) of a representation is defined for representations of fundamental groups of compact surfaces, and for type preserving representations of surfaces with deleted points. All type-preserving representations we obtain here have vanishing Toledo invariant. This is a direct consequence of the fact that the representations are constructed from families of real ideal triangles (see section 6).

Our work is organised as follows. We provide in section 2 the necessary background about the complex hyperbolic plane and its isometries. The invariant of a pair of real ideal triangles is described in section 3. In we expose Fock and Goncharov's version of Thurston's coordinates on $\mathcal{DF}^+$, and give a complete proof of Theorem 1. Although this material is classical, we thought it was worth including a detailed exposition of it. Section 4 is devoted to the classification of T-bent realizations of $\mathcal{F}_\infty$. The characterisation of bent realization giving representations on PU(2,1) in terms of bipartite triangulations is given in 4.4, and we study the holonomy of loops around deleted points in 4.5. We turn then to the proof of Theorem 5. We define spinal $\mathbb{R}$-surfaces and splitting surfaces in 5.2, and prove the discreteness part of the theorem in 5.3. Section 6 is devoted to some remarks and comments. In particular we draw the connection between our work and the previously known families of examples studied in [8, 19, 32].

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2 The complex hyperbolic 2-space

We refer the reader to [6, 15] for more precise information and more references about the material exposed in this section.

2.1 $\mathbf{H}^2_{\mathbb{C}}$ and its isometries

Let $\mathbb{C}^{2,1}$ denote the vector space $\mathbb{C}^3$ equipped with the Hermitian form of signature (2,1) given by the matrix

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{3}$$

The hermitian product of two vectors X and Y is given by $\langle X, Y \rangle = X^T J \bar{Y}$, where X^T denotes the transposed of X. We denote by V^- (resp. V^0) the negative (resp. null) cone associated to the hermitian form, and by P the projectivisation $P : \mathbb{C}^{2,1} \longrightarrow \mathbb{C}P^2$.

In the rest of the paper, whenever m is a point in $\mathbb{C}P^2$, we will denote by $\mathbf{m}$ a lift of it to $\mathbb{C}^{2,1}$.

Definition 1. The complex hyperbolic 2-space $\mathbf{H}^2_{\mathbb{C}}$ is the projectivisation of V^- equipped with the distance function d given by

$$\cosh^2 d\left(\frac{m,n}{2}\right) = \frac{\langle \mathbf{m}, \mathbf{n} \rangle \langle \mathbf{n}, \mathbf{m} \rangle}{\langle \mathbf{m}, \mathbf{m} \rangle \langle \mathbf{n}, \mathbf{n} \rangle} \quad \forall (m,n) \in P(V)^2 \tag{4}$$

Proposition 1. *The isometry group of $\mathbf{H}^2_{\mathbb{C}}$ is generated by PU(2,1), the projective unitary group associated to J and the complex conjugation.*

The group PU(2,1) is the group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$, and is the identity component of Isom($\mathbf{H}^2_{\mathbb{C}}$). The other component contains the antiholomorphic isometries, all of which may be written in the form $\phi \circ \sigma$, where ϕ is a holomorphic isometry and σ is the complex conjugation.

Horospherical coordinates The complex hyperbolic 2-space is biholomorphic to the unit ball of $\mathbb{C}^2$, and its boundary is diffeomorphic to the 3-sphere S^3. The projective model of $\mathbf{H}^2_{\mathbb{C}}$ associated to the matrix J given by (3) is often referred to as the *Siegel model* of $\mathbf{H}^2_{\mathbb{C}}$. In this model, any point m of $\mathbf{H}^2_{\mathbb{C}}$ admits a unique lift to $\mathbb{C}^3$ given by

$$\mathbf{m} = \begin{bmatrix} -|z|^2 - u + it \\ z\sqrt{2} \\ 1 \end{bmatrix}, \text{ with } z \in \mathbb{C},\ t \in \mathbb{R} \text{ and } u > 0. \tag{5}$$

The boundary of $\mathbf{H}^2_{\mathbb{C}}$ corresponds to those vectors for which u vanishes together with the point of $\mathbb{C}\mathbf{P^2}$ corresponding to the vector $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. It might thus be seen as the one point compactification of $\mathbb{R}^3$.

The triple (z,t,u) given by (5) is called the *horospherical coordinates* of m (the hypersurfaces $\{u=u_0\}$ are the horospheres centred at the point ∞ of $\partial\mathbf{H}^2_{\mathbb{C}}$, which corresponds to the vector $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$).

The boundary of $\mathbf{H}^2_{\mathbb{C}}$ has naturally the structure of the 3-dimensional Heisenberg group, seen as the maximal unipotent subgroup of PU(2,1) fixing ∞. We will call *Heisenberg coordinates* of the boundary point with horospherical coordinates $(z,t,0)$ the pair $[z,t]$. In these coordinates the group structure is given by

$$[z,t]\cdot[w,s]=[z+w,s+t+2\mathrm{Im}\,(z\bar{w})]$$

The ball model of $\mathbf{H}^2_{\mathbb{C}}$ The same construction can be done using a different hermitian form of signature $(2,1)$ on $\mathbb{C}^3$. Using the special form associated to the matrix $J_0=\mathrm{diag}(1,1,-1)$, we would obtain the so-called *ball model* of the complex hyperbolic 2-space, which lead to a description of $\mathbf{H}^2_{\mathbb{C}}$ as the unit ball $\mathbb{C}^2$.

2.2 Totally geodesic subspaces

The maximal totally geodesic subspaces of $\mathbf{H}^2_{\mathbb{C}}$ have real dimension 2. There are two types of such subspaces: the *complex lines*, and the *real planes*.

The complex lines. These subspaces are the images under projectivisation of those complex planes of $\mathbb{C}^3$ intersecting the negative cone V^-. The standard example is the subset C_0 of $\mathbf{H}^2_{\mathbb{C}}$ containing points of horospherical coordinates $(0,t,u)$ with $t\in\mathbb{R}$ and $u>0$. This is an embedded copy of the usual Poincaré upper half-plane. We will refer to this particular complex line as $\mathbf{H}^1_{\mathbb{C}}\subset\mathbf{H}^2_{\mathbb{C}}$. All the other complex lines are the images of $\mathbf{H}^1_{\mathbb{C}}$ by an element of PU(2,1). Note that any complex line C is fixed pointwise by a unique holomorphic involutive isometry, called the *complex symmetry about C*.

The real planes. These subspaces are the images of the Lagrangian vector subspaces of $\mathbb{C}^{2,1}$ under projectivisation. The standard example is the subset containing points of horospherical coordinates $(x,0,u)$ with $x\in\mathbb{R}$ and $u>0$. The image of the mapping

$$x+iu\longmapsto(x,0,u) \tag{6}$$

is again an embedded copy of the usual Poincaré upper half-plane and we will refer to this particular real plane as $\mathbf{H}^2_{\mathbb{R}}\subset\mathbf{H}^2_{\mathbb{C}}$. All other real planes are images of the standard one by an element of PU(2,1). There is also a unique involution fixing pointwise a real plane R which is called the *real symmetry about R*. It is antiholomorphic, and, in the case of $\mathbf{H}^2_{\mathbb{R}}$, is complex conjugation. If σ is a real symmetry, we will call the real plane which is its fixed point set its *mirror*.

Remark 1. In the ball model, the standard complex line is the first axis of coordinates $\{(z,0),|z|<1\}$. The standard real plane $\mathbf{H}^2_{\mathbb{R}}$ is again the set of points with real coordinates $\{(x_1,x_2),x_1^2+x_2^2<1\}$.

Computing with real symmetries The following proposition is of great use to work with real symmetries.

Proposition 2. *Let Q be an $\mathbb{R}$-plane, and σ_Q be the symmetry about Q. There exists a matrix $M_Q \in \mathrm{SU}(2,1)$ such that*

$$M_Q\overline{M_Q} = 1 \text{ and } \sigma_Q(m) = \mathbf{P}(M_Q.\bar{\mathbf{m}}) \text{ for any } m \in \mathbf{H}^2_{\mathbb{C}} \text{ with lift } \mathbf{m}, \tag{7}$$

where $\mathbf{P}$:$\mathbb{C}^3 \longrightarrow \mathbb{C}P^2$ is the projectivisation map.

Proof. In the special case where $Q = \mathbf{H}^2_{\mathbb{R}}$, the identity matrix satisfy these conditions. In general, let $\mathbf{Q}$ be a lift to $\mathbb{C}^{2,1}$ of Q, and A be a matrix of SU(2,1) mapping $\mathbb{R}^3$ to $\mathbf{Q}$. The matrix $A\bar{A}^{-1}$ satisfies the above conditions. □

Remark 2. Let σ_1 and σ_2 be real symmetries, with lifts M_1 and M_2 given by Proposition 2. The product $\sigma_1\sigma_2$ is a holomorphic isometry, and lifts to the matrix $M_1\overline{M}_2$. Similarly, if h is a holomorphic isometry lifting to H, the conjugation $h\sigma_1 h^{-1}$ lifts to $HM_1\overline{H^{-1}}$.

The isometry type of the product of two real symmetries is directly related to the relative position of their mirrors. The following Lemma is due to Falbel and Zocca in [9] (see the next section for information about the different isometry types).

Lemma 1. *Let P_1 and P_2 be two real planes, with respective symmetries σ_1 and σ_2. Then*

- *The closures in $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ of P_1 and P_2 are disjoint if and only if the isometry $\sigma_1\sigma_2$ is loxodromic.*
- *The intersection of the closures in $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ of P_1 and P_2 contains exactly one boundary point if and only if $\sigma_1\sigma_2$ is parabolic.*
- *The intersection of the closures in $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ of P_1 and P_2 contains one point of $\mathbf{H}^2_{\mathbb{C}}$ if and only if $\sigma_1\sigma_2$ is elliptic.*

2.3 Classification of isometries.

Let A be a holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$. It is said to be *elliptic* (resp. *parabolic*, resp. *loxodromic*) if it has a fixed point inside $\mathbf{H}^2_{\mathbb{C}}$ (resp. a unique fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$, resp. exactly two fixed points on $\partial\mathbf{H}^2_{\mathbb{C}}$). This exhausts all the possibilities.
Note that there is still a small ambiguity among elliptic elements. An elliptic isometry will be called a *complex reflection* if one of its lifts to SU(2,1) has two equal eigenvalues, else, it will be said to be *regular elliptic*.
As in the case of PSL(2,$\mathbb{R}$), there is an algebraic criterion to determine the type of an isometry according to the trace of one of its lifts to SU(2,1). An element of PU(2,1) admits three lifts to SU(2,1) which are obtained one from another by multiplication by a cube root of 1. Therefore its trace is well-defined up to multiplication by a cube root of 1.

Proposition 3. *Let f be the polynomial given by $f(z) = |z|^4 - 8Re(z^3) + 18|z|^2 - 27$, and h be a holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$.*

- *The isometry h is loxodromic if and only if $f(\operatorname{tr} h)$ is positive.*
- *The isometry h is regular elliptic if and only $f(\operatorname{tr} h)$ is negative.*
- *If $f(h) = 0$, then h is either parabolic or a complex reflection.*

Proof. Note that f is invariant under multiplication of z by a cube root of 1. The polynomial f is the resultant of χ and χ', where χ is the characteristic polynomial of a lift of h to SU(2,1). See [15] (chapter 6) for details. □

Remark 3. The function f of Proposition 3 may be written in real coordinates as

$$f(x+iy) = y^4 + y^2\left(x+6-3\sqrt{3}\right)\left(x+6+3\sqrt{2}\right) + (x+1)(x-3)^3,$$

with $x, y \in \mathbb{R}$. It is then a direct consequence that if $\operatorname{Re}(\operatorname{tr}(h)) > 3$, then h is loxodromic.

Remark 4. It is a direct consequence of the definition of SU(2,1) that the set of eigenvalues of a matrix $A \in$SU(2,1) is invariant under the transformation $z \longmapsto 1/\bar{z}$. We again refer the reader to [15] (chapter 6).

Loxodromic isometries The following facts about loxodromic isometries will be needed later.

Proposition 4. *Let $h \in$PU(2,1) be a loxodomic isometry. Then h is conjugate in PU(2,1) to an isometry given by the matrix in SU(2,1)*

$$\mathbf{D}_\lambda = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \bar{\lambda}/\lambda & 0 \\ 0 & 0 & 1/\bar{\lambda} \end{bmatrix} \text{ with } \lambda \in \mathbb{C}, |\lambda| \neq 1. \tag{8}$$

Proof. Since PU(2,1) acts doubly transitively on the boundary of $\mathbf{H}^2_{\mathbb{C}}$, the isometry h is conjugate to a loxodromic isometry fixing the two points ∞, and $[0,0]$. These two points lift respectively to the vectors $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.As a consequence of this, any lift of h tu SU(2,1) (written in the canonical basis) must be diagonal, and as a consequence of Remark 4, has the form given above. □

The family $\{\mathbf{D}_t, t > 0\}$ defines a 1-parameter subgroup of PU(2,1) containing only matrices with real trace greater or equal to 3.

Definition 2. Let R_γ the 1-parameter subgroup of PU(2,1) given by $g_\gamma^{-1}\{\mathbf{D}_t, t > 0\}g_\gamma$, where γ is a geodesic in $\mathbf{H}^2_{\mathbb{C}}$ and g_γ is an isometry mapping the geodesic γ to the geodesic connecting ∞ and $[0,0]$.

The subgroup R_γ does not depend on the choice of g_γ. However, the parametrisation of R_γ depends on this choice. This small ambiguity will not be important in the rest of the paper.

Remark 5. Let us give another characterisation of R_γ. An isometry A belongs to R_γ if and only if for any real plane P containing γ, A preserves P and the two connected components of $P \setminus \gamma$. Indeed, we may normalise the situation in such a way that P is $\mathbf{H}^2_{\mathbb{R}}$ and γ is the geodesic connecting the two points with Heisenberg coordinates $[0,0]$ and ∞, in which case $R_\gamma = (\mathbf{D}_t)_{t>0}$. The two connected components of $\mathbf{H}^2_{\mathbb{R}} \setminus \gamma$ are C^+ and C^-, where, in horospherical coordinates, $C^+ = \{(x,0,u), x > 0 \text{ and } u > 0\}$ and $C^- = \{(x,0,u), x < 0 \text{ and } u > 0\}$. Now, any isometry preserving $\mathbf{H}^2_{\mathbb{R}}$, and fixing both $[0,0]$ and ∞ lifts to SU(2,1) as the diagonal matrix $\text{diag}(t, 1, 1/t)$ with real t. It is then a straightforward computation to check that such an isometry preserves the connected components C^+ and C^- if and only if t is positive, that is, if it belongs to $(\mathbf{D}_t)_{t>0}$.

Parabolic isometries. There are two main types of parabolic isometries: they can be either *unipotent* or *screw-parabolic*.

1. Heisenberg translations are unipotent parabolics. They are conjugate to isometries associated to one of the above matrices $T_{[z,t]}$ in SU(2,1), that correspond to those unipotent parabolics fixing $\infty \in \partial\mathbf{H}^2_{\mathbb{C}}$.

$$T_{[z,t]} = \begin{bmatrix} 1 & -\bar{z}\sqrt{2} & -|z|^2 + it \\ 0 & 1 & z\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } z \in \mathbb{C}, t \in \mathbb{R}.$$

   There are two PU(2,1)-conjugacy classes of Heisenberg translations. The first one contains *vertical* translations, which correspond to $z = 0$. In this case $T_{[0,t]} - Id$ is nilpotent of order 2. These isometries preserve a complex line. The parabolic elements $T_{[0,t]}$ preserves the complex line $\mathbf{H}^1_{\mathbb{C}}$. The second conjugacy class contains *horizontal* translation. This is when $z \neq 0$, in which case $T_{[z,t]} - Id$ is nilpotent of order 3. These isometries preserve a real plane, which is $\mathbf{H}^2_{\mathbb{R}}$ in the case where z is real and t vanishes.

2. Screw-parabolic isometries are conjugate to a product $h \circ r$, where h is a vertical Heisenberg translation and r a complex reflection about the invariant complex line of h.

3 Real ideal triangles.

3.1 Ideal triangles

An ideal triangle is an oriented triple of boundary points of $\mathbf{H}^2_{\mathbb{C}}$.

Definition 3. Let (p_1, p_2, p_3) be an ideal triangle. The quantity

$$\mathbb{A}(p_1, p_2, p_3) = \arg\left(-\langle \mathbf{p}_1, \mathbf{p}_2\rangle\langle \mathbf{p}_2, \mathbf{p}_3\rangle\langle \mathbf{p}_3, \mathbf{p}_1\rangle\right) \tag{9}$$

does not depends on the choice of the lifts of the p_i's, and is called the Cartan invariant of the ideal triangle (p_1, p_2, p_3).

The Cartan invariant classifies the ideal triangles, as stated in the following Proposition (see [15] chapter 7 for a proof).

Proposition 5. *The Cartan invariant enjoys the following properties*

1. *Two ideal triangles are identified by an element of PU(2,1) if and only if they have the same Cartan invariant.*
2. *Two ideal triangles are identified by a antiholomorphic isometry of* $\mathbf{H}^2_{\mathbb{C}}$ *if and only if they have opposite Cartan invariants.*
3. *An ideal triangle has Cartan invariant* $\pm\pi/2$ *(resp.* 0*) if and only if it is contained in a complex line (resp. a real plane).*

Definition 4. We will call any ideal triangle contained in a real plane a *real ideal triangle.*

Since the three points are contained in a real plane we will as well refer to the 2-simplex determined by three points on the boundary of a real plane as a real ideal triangle.

Remark 6. Up to isometry, there is a unique real ideal triangle, as shown by Proposition 5. If Δ and Δ' are two real ideal triangles, there are exactly two isometries mapping Δ to Δ', φ and ψ. One of them (say φ) is holomorphic and the other antiholomorphic. More precisely, denoting by σ the real symmetry about the real plane containing Δ, $\psi = \varphi \circ \sigma$.

3.2 The invariant of a pair of adjacent ideal real triangles.

We say that two real ideal triangles are *adjacent* if they have a common edge. All the pairs of real ideal triangles we consider are **ordered**.

Lemma 2. *Let* τ_1 *and* τ_2 *be two adjacent ideal real triangles, sharing a geodesic* γ *as an edge. There exists a unique complex number* $\mathbb{C} \setminus \{-1, 0\}$ *such that the ordered pair of real triangles* (τ_1, τ_2) *is PU(2,1)-equivalent to the ordered pair of ideal real triangles* (τ_0, τ_z) *given by the Heisenberg coordinates of its vertices by*

$$\tau_0 = (\infty, [-1, 0], [0, 0]) \quad \textit{and} \ \tau_z = (\infty, [0, 0], [z, 0]) \tag{10}$$

Proof. As shown by Proposition 5 and remark 6, there exists a unique holomorphic isometry h mapping τ_1 to τ_0 and γ to the geodesic connecting ∞ to $[0, 0]$. The isometry h maps the triangle τ_2 to an ideal triangle of which vertices are a priori given in Heisenberg coordinates by ∞, $[0, 0]$ and $[z, t]$ with $z \in \mathbb{C}$ and $t \in \mathbb{R}$. Using relation (5), where $u = 0$ sinve we are on the boundary, we lift the latter three points to the three vectors

$$\mathbf{m}_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{m}_{0,0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{m}_{z,t} = \begin{bmatrix} -|z|^2 + it \\ z\sqrt{2} \\ 1 \end{bmatrix}.$$

The triple product is given by

$$\langle \mathbf{m}_\infty, \mathbf{m}_{0,0} \rangle \langle \mathbf{m}_{0,0}, \mathbf{m}_{z,t} \rangle \langle \mathbf{m}_{z,t}, \mathbf{m}_\infty \rangle = -|z|^2 - it.$$

Using (9), we obtain $\mathbb{A}(\infty, [0,0], [z,t]) = \tan\left(t/|z|^2\right)$. Hence the Cartan invariant of $h(\tau_2)$ vanishes if and only if $t = 0$. ☐

Remark 7. Notice that the special cases where z is respectively equal to 0 and -1 correspond respectively to the degenerate cases where one of the triangles has two collapsed vertices, and the case when the two triangles have the same set of vertices.

Definition 5. Let (τ_1, τ_2) be a pair of adjacent real ideal triangles. We will call the complex number z associated to it by Lemma 2 the invariant of the pair (τ_1, τ_2), and denote it by $\mathtt{Z}(\tau_1, \tau_2)$.

Remark 8. Let (a, b, c) be a real ideal triangle, and z be a complex number different from 0 and -1. From Proposition 2, we see that there exists a unique point d in $\partial\mathbf{H}^2_{\mathbb{C}}$ such that (a, c, d) is a real ideal triangle, and $\mathtt{Z}\left((a, b, c), (a, c, d)\right) = z$.

Remark 9. It is possible to give another description of the invariant $\mathtt{Z}$ of a pair of ideal triangles. Let p_1, p_2, p_3 and p_4 be four points in $\partial\mathbf{H}^2_{\mathbb{C}}$, such that $\tau_1 = (p_1, p_2, p_3)$ and $\tau_2 = (p_3, p_4, p_1)$ are two real ideal triangles. Let C_{13} be the (unique) complex line containing p_1 and p_3. Neither p_2 nor p_4 belong to C_{13}, since the two corresponding ideal triangles are real. The complex line C_{13} lifts to $\mathbb{C}^3$ as a complex plane. Let $\mathbf{c}_{13}$ be a vector in $\mathbb{C}^{2,1}$ Hermitian orthogonal to this complex plane. Then $C_{13} = \mathbf{P}\left(\mathbf{c}_{\mathbf{13}}^{\perp}\right)$. Let $\mathbf{p}_i$ be a lift of p_i for $i = 1, 2, 3, 4$. The invariant $\mathtt{Z}(\tau_1, \tau_2)$ is given by

$$\mathtt{Z}(\tau_1, \tau_2) = -\frac{\langle \mathbf{p}_4, \mathbf{c}_{13}\rangle\langle \mathbf{p}_2, \mathbf{p}_1\rangle}{\langle \mathbf{p}_2, \mathbf{c}_{13}\rangle\langle \mathbf{p}_4, \mathbf{p}_1\rangle} \tag{11}$$

The above quantity does not depend on the various choices of lifts we made. This definition is similar to the one of the complex cross-ratio of Koranyi and Reimann (see [21]). To check that this formula is valid, it is sufficient to check it on the special case $p_1 = \infty$, $p_2 = [-1, 0]$, $p_3 = [0, 0]$ and $p_4 = [z, 0]$. In this case, the choice $\mathbf{c}_{13} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$ is convenient. This invariant is similar to the one used by Falbel in [7], although the form (11) is not used there. Note that (11) shows that $\mathtt{Z}$ is preserved by holomorphic isometries.

Lemma 3. *Let (τ_1, τ_2) be a pair of adjacent real ideal triangles, with $\mathtt{Z}(\tau_1, \tau_2) = z$, and f be an antiholomorphic isometry. Then $\mathtt{Z}(f(\tau_1), f(\tau_2)) = \bar{z}$.*

Proof. Let σ be the symmetry about the real plane containing τ_1. The isometry $f \circ \sigma$ is holomorphic, and therefore preserves the invariant of pairs of adjacent real ideal triangles. As a consequence, it is sufficient to show that $\mathtt{Z}(\sigma(\tau_1), \sigma(\tau_2)) = \bar{z}$. We can normalise the situation to the reference pair (τ_0, τ_z) given by (10). In this case, the real symmetry σ is just the complex conjugation. It fixes the three points ∞, $[-1, 0]$ and $[0, 0]$, and maps the point $[z, 0]$ to $[\bar{z}, 0]$. ☐

Proposition 6. *Let $\tau_1 = (a, b, c)$ and $\tau_2 = (a, c, d)$ be two adjacent real ideal triangles. There exists a unique real symmetry σ such that $\sigma(a) = c$ and $\sigma(b) = d$.*

Proof. It is sufficient to prove that such a real symmetry exists and is unique for standard case where the two triangles are τ_0 and τ_z. More precisely, we have to show that for any $z \in \mathbb{C}$ and, there exists a unique real symmetry σ_z such that

$$\sigma_z([-1,0]) = [z,0] \text{ and } \sigma_z(\infty) = [0,0]. \tag{12}$$

If there existed two such symmetries, their product would be a holomorphic isometry having four fixed points on $\partial \mathbf{H}^2_{\mathbb{C}}$, not belonging to the boundary of a complex line. Therefore this product would be the identity. Therefore such a symmetry is unique if it exists. Writing $z = xe^{i\alpha}$, the matrix

$$M_{x,\alpha} = \begin{bmatrix} 0 & 0 & x \\ 0 & e^{i\alpha} & 0 \\ 1/x & 0 & 0 \end{bmatrix} \tag{13}$$

is such that $M_{x,\alpha}\overline{M_{x,\alpha}} = 1$, and the real symmetry associated to it satisfies to (12). □

Definition 6. We call the involution provided by Proposition 6 the symmetry of the pair (Δ_1, Δ_2) and denote it by $\sigma_{\Delta_1,\Delta_2}$.

Remark 10. As a direct consequence of Lemma 3 and Proposition 6, we see that for any pair (τ_1, τ_2) of adjacent real ideal triangles,

$$\mathtt{Z}(\tau_1, \tau_2) = \overline{\mathtt{Z}(\tau_2, \tau_1)}.$$

Remark 11. Let us consider the special case where $\mathtt{Z}$ is real. Going back to Lemma 2, we see that if z is real, the two triangles τ_0 and τ_z are both contained in the standard real plane $\mathbf{H}^2_{\mathbb{R}}$ since their vertices all have real coordinates. In horospherical coordinates, this real plane is nothing but a copy of the upper half-plane. Therefore z is positive if and only if the two triangles are in the same connected component of of $\mathbf{H}^2_{\mathbb{R}} \setminus \gamma$, where γ is the common geodesic edge of the two triangles. Applying Lemma 2, we obtain in general that

- $\mathtt{Z}(\tau_1, \tau_2)$ is real if and only if τ_1 and τ_2 lie in a common real plane P.
- $\mathtt{Z}(\tau_1, \tau_2)$ is positive (resp. negative) if and only if τ_1 and τ_2 lie in opposite (resp. the same) connected components of $P \setminus \gamma$.

Remark 12. When four points $(p_i)_{i=1\cdots 4}$ belong to the boundary of the standard real plane $\mathbf{H}^2_{\mathbb{R}}$, the invariant $\mathtt{Z}((p_1, p_2, p_3), (p_1, p_3, p_4))$ is the classical cross-ratio in the upper-half plane, as can be checked from the embedding of the upper half plane in $\mathbf{H}^2_{\mathbb{C}}$ given by (6) in section 2.2.

We will need the following in section 5.2.

Proposition 7. *Let $\tau = (a, b, c)$ be an ideal real triangle, and γ the geodesic connecting a and c. Let τ_1 and τ_2 be two other real ideal triangles adjacent to τ along γ. Assume moreover that the invariants of the pairs (τ, τ_1) and (τ, τ_2) satisfy*

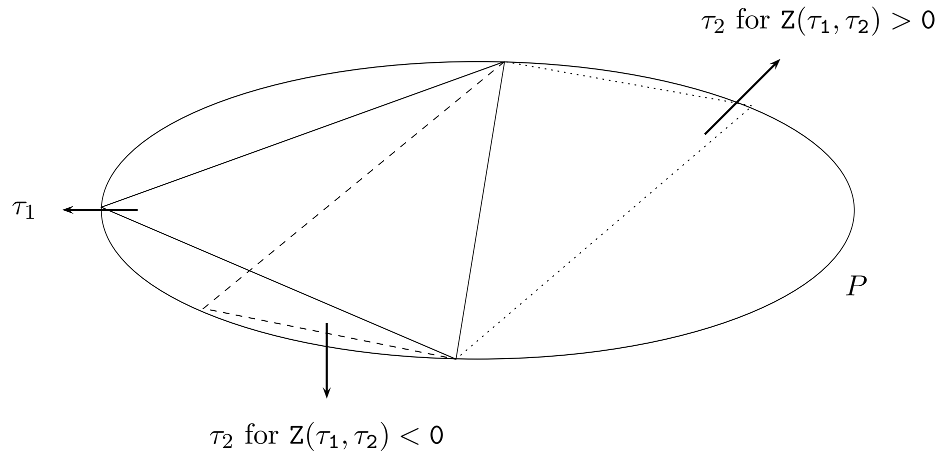


Figure 1: When $\mathtt{Z}(\tau_1,\tau_2)$ is real.

$$\frac{\mathtt{Z}(\tau,\tau_1)}{\mathtt{Z}(\tau,\tau_2)} \in \mathbb{R}_{>0}. \tag{14}$$

Call d_i the vertex of τ_i different from a and c, and Q_i the mirror of the real symmetry σ_i given by the Proposition 6, such that $\sigma_i(a) = c$ and $\sigma_i(b) = d_i$. Then there exists a unique element $g \in R_\gamma$ such that $g(Q_1) = Q_2$.

Proof. We may normalise the situation so that

$$a = \infty,\ b = [-1, 0],\ c = [0, 0], d_1 = [z_1, 0] \text{ and } d_2 = [z_2, 0], \tag{15}$$

where $z_i = \mathtt{Z}(\tau, \tau_\mathtt{i})$. In this normalised situation, the two real symmetries associated to d_1 and d_2 are σ_{z_1} and σ_{z_2}. As in 13, they correspond to the matrices M_{z_i} given by

$$M_{z_i} = \begin{bmatrix} 0 & 0 & |z_i| \\ 0 & z_i/|z_i| & 0 \\ 1/|z_i| & 0 & 0 \end{bmatrix} \tag{16}$$

The one parameter subgroup R_γ corresponds to $(\mathbf{D}_t)_{t>0}$, with $\mathbf{D}_t$ as in Proposition 4). Conjugating M_{z_1} by $\mathbf{D}_t$ yields

$$\mathbf{D}_t M_{z_1} \mathbf{D}_{1/t} = \begin{bmatrix} 0 & 0 & |z_1|t^2 \\ 0 & \dfrac{z_1}{|z_1|} & 0 \\ \dfrac{1}{|z|t^2} & 0 & 0 \end{bmatrix}. \tag{17}$$

Because of (14), we have $z_1/|z_1| = z_2/|z_2|$, and the only possiblity is $t^2 = z_2/z_1$, which leads to a unique value for t since it is positive. $\square$

Remark 13. Keeping the notation and assumptions of Proposition 7, it is an easy exercice to check that in this case, the four points a, b, d_1 and d_2 belong to a common real plane, which is preserved by the isometry g. It is done by going back to the standard case of Lemma 2, and looking at τ_0, τ_{z_1} and τ_{z_2}.

4 The bending theorem

4.1 Notation, Definitions

We denote by $\Sigma = \Sigma_g \setminus \{x_1, \cdots, x_n\}$ an oriented surface of genus g with n deleted points such that $2-2g-n<0$. We denote by π_1 its fundamental group, given by the presentation

$$\pi_1 = \langle a_1, b_1, \dots a_g, b_g, c_1, \dots c_n \mid \prod_i [a_i, b_i] \prod_j c_j = 1 \rangle,$$

where the c_i's are homotopy classes of loops around the deleted points. The universal cover of Σ is an open disk with a π_1-invariant family of boundary points which may be thought of as the lifts of the x_i's. This family is called the *Farey set* of Σ, and we will denote it by $\mathcal{F}_\infty$. On way to understand $\mathcal{F}_\infty$ is to endow Σ with a finite area hyperbolic structure. In this situation, $\mathcal{F}_\infty$ is the set of (parabolic) fixed points of the c_i's and their conjugates. If moreover the holonomy of this hyperbolic structure is a subgroup of PSL(2,$\mathbb{Z}$), we obtain the classical Farey set $\mathbb{Q} \cup \infty$ in the Poincaré upper half-plane.

Recall that an ideal triangulation of Σ is a decomposition

$$\Sigma = \bigcup_\alpha \Delta_\alpha,$$

where each Δ_α is homeomorphic to a triangle of which vertices have been removed, and such that $\alpha \neq \beta \Rightarrow \mathring{\Delta}_\alpha \cap \mathring{\Delta}_\beta = \emptyset$. It is a classical facts using Euler characteristic that any ideal triangulation of a surface of genus g with p deleted points has $4g-4+2p$ triangles and $6g-6+3p$ edges.

Definition 7. Let T be a an ideal triangulation of Σ, and $\hat{T}$ be the associated triangulation of $\hat{\Sigma}$. We will call $\mathbf{H}^2_{\mathbb{C}}$*-realization bent along* T, or T*-bent realization* of $\mathcal{F}_\infty(\Sigma)$ any pair (ϕ, ρ) such that

- ρ is a representation $\pi_1(\Sigma) \longrightarrow \mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$
- $\phi : \mathcal{F}_\infty(\Sigma) \longrightarrow \partial \mathbf{H}^2_{\mathbb{C}}$ is a $(\pi_1(\Sigma), \rho)$-equivariant mapping.
- for any face Δ of $\hat{T}$ with vertices a, b, and c, the three points $\phi(a)$, $\phi(b)$ and $\phi(c)$ are contained in the boundary of a real plane.

The group PU(2,1) acts on the set of T-bent realizations of $\mathcal{F}_\infty$ by $g\cdot(\phi, \rho) = \big(g \circ \phi, g\rho g^{-1}\big)$. We will denote by $\mathcal{BR}_T$ the set of $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$-classes of T-bent realizations for this action.

Definition 8. Let T be an ideal triangulation of Σ. We will call *modified dual graph* of T and denote by $\Gamma(T)$ the graph obtained from the dual graph of T as follows:

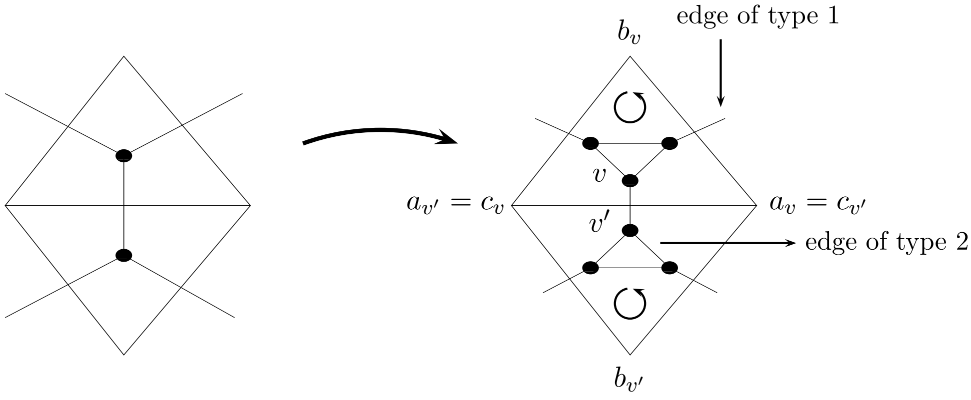


Figure 2: The triangulation, the modified dual graph and the labelling of vertices

- The vertices of $\Gamma(T)$ are the combinations $1/3x + 2/3y$, where x and y are adjacent vertices of the dual graph.
- Two vertices v and v' of $\Gamma(T)$ are connected by an edge if and only if they fall in one of the following two cases.
  - $v = 1/3x + 2/3y$ and $v' = 2/3x + 1/3y$ for some adjacent vertices of the dual graph. In this case the edge connecting v and v' is said to be of type 1.
  - $v = 1/3x + 2/3y$ and $v' = 1/3z + 2/3y$ where xz and yz are edges of the dual graph sharing an endpoint. In this case the edge vv' is of type 2.

  See figure 2.

We define similarly $\Gamma(\hat{T})$, the modified dual graph of $\hat{T}$, which is the lift of $\Gamma(T)$ to the universal cover of Σ. We will refer to these two modified dual graphs as Γ and $\hat{\Gamma}$ whenever it is clear from the context which triangulation we are dealing with. Edges of type 1 and 2 of $\hat{\Gamma}$ are defined similarly as for Γ.

Note that an edge of Γ is of type 1 (resp. type 2) if and only if it intersects an edge of T (resp. no edge of T). The orientation of Σ induces an orientation of edges of type 2 of Γ and $\hat{\Gamma}$.

Definition 9. Let v be a vertex of $\hat{\Gamma}$ and Δ be the unique face of $\hat{T}$ containing v. The orientation of Σ induces an orientation of the edges of Δ, and we will call a_v the ending vertex of the edge of Δ closest to v. We will then call b_v and c_v the two other vertices of Δ, in such a way that the triple (a_v, b_v, c_v) is positively oriented.

Since three vertices of $\hat{\Gamma}$ are contained in Δ, there are three possible labellings of the vertices of a given Δ.

Definition 10. A *bending decoration* of an ideal triangulation T is an application $\mathtt{D} : e(T) \longrightarrow \mathbb{C} \setminus \{-1, 0\}$ defined on the set of unoriented edges of T.

It follows from Remark 7 in section 3.2, that the cases where the invariant $\mathtt{Z}$ of a pair of real ideal triangles equals 0 or -1 correspond to degenerate pairs of triangles. More precisely, $\mathtt{Z}(\tau_1, \tau_2) = 0$ if and only if τ_2 has two identical vertices and -1 if and only if the two triangles are equal. We do not consider these degenerate cases.
We will often refer to the function $\arg(\mathtt{D})$ as the *angular part* of the bending decoration. There is an action of $\mathbb{Z}/2\mathbb{Z}$ on the set of bending decorations of T which is given by the complex conjugation: if $\mathtt{D}$ is a bending decoration of T, the decoration $\overline{\mathtt{D}}$ is given by $\overline{\mathtt{D}}(e) = \overline{\mathtt{D}(e)}$ for any edge e of T.

Definition 11. For any ideal triangulation T of Σ, we denote by $\mathcal{BD}_T$ the set of bending decorations of T, and by $\mathcal{BD}_T^*$ the quotient of $\mathcal{BD}_T$ by the action of $\mathbb{Z}/2\mathbb{Z}$ given above.

The set of bending decoration $\mathcal{BD}_T$ of T is thus a copy of $(\mathbb{C} \setminus \{-1, 0\})^{|e(T)|}$. Prior to proving theorem 2, we introduce the following two isometries of $\mathbf{H}^2_{\mathbb{C}}$.

Definition 12. For any $z \in \mathbb{C} \setminus \{0, 1\}$, we will call σ_z the real symmetry defined by its lift $M_z \in$ U(2,1) (see proposition 2), and $\mathcal{E}$ the isometry given by its lift to SU(2,1), where

$$M_z = \begin{bmatrix} 0 & 0 & |z| \\ 0 & z/|z| & 0 \\ |z|^{-1} & 0 & 0 \end{bmatrix} \text{ and } \mathcal{E} = \begin{bmatrix} -1 & \sqrt{2} & 1 \\ -\sqrt{2} & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{18}$$

(We identify $\mathcal{E}$ and its lift).

The symmetry σ_z acts on the complex hyperbolic space by $\sigma_z(m) = \mathbf{P}\left(M_z \bar{\mathbf{m}}\right)$ (see Proposition 2). In Heisenberg coordinates, its action on the boundary $\partial \mathbf{H}^2_{\mathbb{C}}$ is given by

$$\sigma_z([w, t]) = \left[\frac{z\bar{w}}{|w|^4 + t^2} \left(|w|^2 - it\right), \frac{t|z|^2}{|w|^4 + t^2} \right].$$

From this we see that

$$\sigma_z(\infty) = [0, 0], \sigma_z([-1, 0]) = [z, 0].$$

The isometry $\mathcal{E}$ is elliptic of order 3 and permutes cyclically the three points ∞, $[-1, 0]$ and $[0, 0]$.

4.2 Bent $\mathbf{H}^2_{\mathbb{C}}$-realizations : proof of Theorem 2.

We are now going to prove that there is a bijection between $\mathcal{BD}_T^*$ and $\mathcal{BR}_T$.

Proof of theorem 2. We will associate to any bending decoration in $\mathcal{BR}_T$ a unique pair of PU(2,1)-classes of T-bent realizations of $\mathcal{F}_\infty$, which represent the same $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$–class and correspond to conjugate bending decorations. We will first associate to any vertex v of the modified dual graph a bent realization (ϕ_v, ρ_v) of $\mathcal{F}_\infty$ by using v as a basepoint.

We will see a posteriori that we obtain this way two PU(2,1)-classes of realization which correspond to the same Isom($\mathbf{H}^2_{\mathbb{C}}$)-class.

Step 1: Definition of the mapping ϕ_v. We would like to interpret the complex numbers $\mathtt{D}(e)$ as invariants of pairs of real ideal triangles, and use it in order to construct ϕ_v reccursively. Each edge e belongs to two faces of $\hat{T}$, say Δ_1 and Δ_2, and we have to chose whether we see $\mathtt{D}(e)$ as $\mathtt{Z}(\Delta_1, \Delta_2)$ or as $\mathtt{Z}(\Delta_2, \Delta_1) = \overline{\mathtt{Z}(\Delta_1, \Delta_2)}$. We do it using a bicoloration of $\hat{T}$.

Let v be a vertex of $\hat{\Gamma}$ and Δ_v be the face containing v. Label by a_v, b_v and c_v the vertices of the face of the triangulation v belongs to, as in definition 9.

- Attribute to the face Δ_v the colour white, and define $\phi_v(a_v) = \infty$, $\phi_v(b_v) = [-1, 0]$ and $\phi_v(c_v) = [0, 0]$.
- Colour all the faces of $\hat{T}$ in black or white from the one containing v by following the rule that two triangles sharing an edge have opposite colour.
- Define the images of all the other points of $\mathcal{F}_\infty$ recursively according to the following principle: if an edge e separates two faces Δ_w (white) and Δ_b (black), then the number z associated to the edge e is interpreted as the invariant of the (ordered) pair $\mathtt{Z}(\phi_v(\Delta_w), \phi_v(\Delta_b))$. If $\phi_v(\Delta_w)$ is already constructed, this defines $\phi_v(\Delta_b)$ unambiguously, shown by Remark 8.

Step 2: Definition of the representation ρ_v. For any $\gamma \in \pi_1$, we have to define an isometry g_γ such that $\phi_v(\gamma \cdot m) = g_\gamma \phi_v(m)$ for any m in $\mathcal{F}_\infty$. In particular, such an isometry must map the reference triangle $(\infty, [-1, 0], [0, 0])$ to the ideal real triangle $(\phi_v(\gamma \cdot a_v), \phi_v(\gamma \cdot b_v), \phi_v(\gamma \cdot c_v))$. This can be done in two ways, as shown by Remark 6, using either a holomorphic or an antiholomorphic isometry. We define $\rho_v(\gamma)$ according to the following rule (recall that v belongs to a white triangle).

- If $\gamma \cdot v$ belongs to a white triangle, define $\rho_v(\gamma)$ to be the unique holomorphic isometry mapping $(\infty, [-1, 0], [0, 0])$ to $(\phi_v(\gamma \cdot a_v), \phi_v(\gamma \cdot b_v), \phi_v(\gamma \cdot c_v))$.
- If $\gamma \cdot v$ belongs to a black triangle, choose the antiholomorphic one.

Step 3: ϕ_v is (π_1, ρ_v)-equivariant. If $\rho_v(\gamma)$ is holomorphic, it preserves the invariant $\mathtt{Z}$. As a consequence of the definition of ϕ_v and ρ_v, the identity $\phi_v(\gamma \cdot m) = \rho_v(\gamma)\phi_v(m)$ holds for any m, and for any γ such that $\rho_v(\gamma)$ is holomorphic. If $\rho_v(\gamma)$ is antiholomorphic, then it transforms the invariants of real ideal triangle from z to $\bar{z}$, as seen in Lemma 3. The equivariance property in this case is a direct consequence of the choice made in the construction of ϕ_v to interpret the decoration as $\mathtt{Z}(\phi_v(\Delta_w), \phi_v(\Delta_b))$.

Step 4: Description of the class of the realization (ϕ_v, ρ_v). Let us compare first the classes of T-bent realizations associated to two vertices v and v' of an edge e of $\hat{\Gamma}$.

1. Assume first that v and v' belong to different faces Δ and Δ' of the triangulation, that is, e is of type 1. These two faces have opposite colours. Then e intersects an

edge of $\hat{T}$, which is decorated by some complex number z. Call d_v the vertex of Δ' which is not a vertex of Δ, and the vertices of Δ a_v, b_v and c_v as in Definition 9. Then, according to their definitions, ϕ_v and $\phi_{v'}$ satisfy to

$$\phi_v(a_v) = \infty \quad , \quad \phi_v(b_v) = [-1, 0] \quad , \quad \phi_v(c_v) = [0, 0] \quad \text{and} \quad \phi_v(d_v) = [z, 0]$$

$$\phi_{v'}(a_v) = [0, 0] \quad , \quad \phi_{v'}(b_v) = [z, 0] \quad , \quad \phi_{v'}(c_v) = \infty \quad \text{and} \quad \phi_{v'}(d_v) = [-1, 0].$$

The antiholomorphic involution σ_z (definition 12), is the unique isometry exchanging ∞ and $[0, 0]$ on one hand, and $[-1, 0]$ and $[z, 0]$. Therefore we see $\phi_{v'} = \sigma_z \circ \phi_v$, and $\rho_{v'} = \sigma_z \rho_v \sigma_z$, that is $(\phi_{v'}, \rho_{v'}) = \sigma_z \cdot (\phi_v, \rho_v)$. In this case, the two realizations are in the same isometry class, but not the same PU(2,1)-class.

2. By examining similarly what happens when v and v' are connected by an edge of type 2, that is, if they belong to a common face of $\hat{T}$, we see that $(\phi_v, \rho_v) = \mathcal{E} \cdot (\phi_{v'}, \rho_{v'})$ if the orientation induced on e by the orientation of Σ is $v \to v'$, and $(\phi_v, \rho_v) = \mathcal{E}^{-1} \cdot (\phi_{v'}, \rho_{v'})$ in the opposite case. The two realizations have the same holomorphic class in this case.

If v and v' are arbitrary vertices of $\hat{T}$, belonging to the triangles Δ_v and $\Delta_{v'}$ of $\hat{T}$, colour the faces of $\hat{T}$ starting from Δ_v. The facts *1* and *2* above imply that

- if Δ_v and $\Delta_{v'}$ have the same colour for this choice of coloration, then (ϕ_v, ρ_v) and $(\phi_{v'}, \rho_{v'})$ correspond to the same PU(2,1)-class of T-bent realization,
- if not, then (ϕ_v, ρ_v) and $(\phi_{v'}, \rho_{v'})$ correspond to the same Isom($\mathbf{H}^2_{\mathbb{C}}$)-class, but have opposite PU(2,1)-classes.

Indeed, if Δ_v and $\Delta_{v'}$ have the same colour if and only if any simplicial path connecting v and v' contains an even number of edges of type 1. Since the PU(2,1)-class changes every time an edge of type 1 is used, this shows the above assertion.

Step 5: Passing from $\mathtt{D}$ to $\overline{\mathtt{D}}$. We have so far associated to $\mathtt{D}$ a pair of PU(2,1)-classes of T-bent realizations. The choice of a starting vertex v of $\hat{\Gamma}$ determines a coloration of the faces of $\hat{T}$. Call r_w the class corresponding to white triangles for this choice of coloration, and r_b the one corresponding to black triangles. If we keep the same starting vertex v but construct the classes associated to the decoration $\overline{\mathtt{D}}$, the new equivariant mapping $\psi_v : \mathcal{F}_\infty \longrightarrow \partial\mathbf{H}^2_{\mathbb{C}}$ is defined recursively from

$$\psi_v(a_v) = \infty \quad , \quad \psi_v(b_v) = [0, 0] \quad , \quad \psi_v(c_v) = [-1, 0] \quad \text{and} \quad \psi_v(d_v) = [\bar{z}, 0].$$

As a consequence, we see that $\psi_v = \sigma \circ \phi_v$, where σ is the complex conjugation. The corresponding holonomy representation are conjugate by σ. Therefore the change $\mathtt{D} \longrightarrow \overline{\mathtt{D}}$ induces the permutation $(r_w, r_b) \longrightarrow (r_b, r_w)$.

Step 6: The reverse operation: decorating a triangulation from a T-bent realization. Let $r = (\phi, \rho)$ be a T-bent realization of $\mathcal{F}_\infty$ in $\partial\mathbf{H}^2_{\mathbb{C}}$. Since r is bent along T we obtain by definition a family of real ideal triangles by connecting $\phi(m)$ and $\phi(n)$ each time m and n are connected by an edge of $\hat{T}$. If e is an edge of $\hat{T}$ belonging to two triangles Δ and Δ'. As before colorating the the faces of $\hat{T}$ gives a way to associate to e a complex number z, which is $\mathtt{Z}(\Delta, \Delta')$ if Δ is white and Δ' is black, and $\overline{\mathtt{Z}}(\Delta, \Delta')$ in the other case. There is an order 2 ambiguity: if we start with a given real ideal triangle, and obtain this way a decoration $\mathtt{D}$, starting with an adjacent triangle will produce the decoration $\overline{\mathtt{D}}$. □

4.3 Explicit computation of the representations

Definition 13. For any oriented edge ν of Γ, let A_ν be the isometry defined as follows (see Definition 12).

1. If ν is of type one and intersects an edge e of $\hat{T}$, then A_ν is the real symmetry $\sigma_{\mathtt{D}(e)}$.
2. If ν is of type two, then if it is positively oriented with respect to the orientation of Σ, $A_\nu = \mathcal{E}$, else $A_\nu = \mathcal{E}^{-1}$.

Proposition 8. *Let T be an ideal triangulation of Σ, with a bending decoration, v and v' be two vertices of Γ, and $p_{v,v'} = s_1 \cdots s_k$ be a simplicial path connecting them. Call r_v and $r_{v'}$ the T-bent realizations associated to v and v', and $B_{v,v'}$ be the isometry $A_{s_1} \cdots A_{s_k}$. Then $B_{v,v'}$ satisfies to*

$$r_v = B_{v,v'} \cdot r_{v'}.$$

Proof. This is a direct reccursion using the second step of the proof of Theorem 2. □

We now compute the representation in terms of the bending decoration.

Proposition 9. *Let γ be a homotopy class of loop on Σ, and v be a vertex of Γ. We may represent γ as a simplicial path starting at v consisting of a sequence $e_1 \cdots e_k$ of oriented edges of Γ. Associate to γ the isometry $B_{v,\gamma\cdot v} = A_{e_1} \cdots A_{e_n}$. Then*

1. *The isometry $B_{v,\gamma\cdot v}$ does not depend on the choice of the simplicial loop representing γ.*
2. *The mapping $\gamma \longmapsto B_{v,\gamma\cdot v}$ is equal to the representation ρ_v.*

Proof. 1. It is a classical fact Σ, equipped with an ideal triangulation can be retracted onto the dual graph of T. Therefore any loop l on Σ can be homotoped to a sequence of edges of the dual graph. Once a basepoint is fixed this loop corresponds to a sequence of edges of the dual graph of $\hat{T}$, the triangulation of $\hat{\Sigma}$ coming from T. But the dual graph of $\hat{T}$ is a tree and therefore the sequence of edges representing l is unique (if we assume that two consecutive edges are distinct). Passing from the dual graph to the modified dual graph, we lose this uniqueness property. Indeed, let Δ be a triangle of $\hat{T}$, crossed by this unique sequence of edges of the dual graph. Let v_1, v_2 and v_3 be the three vertices of $\hat{\Gamma}$ belonging to Δ. Then the original loop

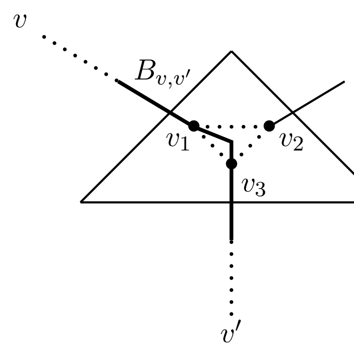


Figure 3: Passing from the dual graph to the modified dual graph.

can be homotoped to the simplicial path $v_1 \to v_2 \to v_3$ or to $v_1 \to v_3$ (see Figure 3). However the isometries associated to these two sequences of edges are $\mathcal{E}$ and $\mathcal{E}^{-2}$ or $\mathcal{E}^{-1}$ and $\mathcal{E}^2$, according to the orientation. Since $\mathcal{E}$ has order three this does not change the contribution of this part of the path to $B_{v,\gamma\cdot v}$.

2. To prove the second assertion, we have to show that

   (a) $B_{v,\gamma\cdot v}$ maps the triple $(\phi_v(\gamma\cdot a_v), \phi_v(\gamma\cdot b_v), \phi_v(\gamma\cdot c_v))$ to the triple $(\infty, [-1,0], [0,0])$

   (b) $B_{v,\gamma\cdot v}$ is holomorphic if and only if v and $\gamma\cdot v$ lie in triangles having the same colour.

   We already know from Proposition 8 that $\phi_{\gamma\cdot v} = A_{e_1}\cdots A_{e_n}\phi_v = B_{v,\gamma\cdot v}\phi_v$. As a consequence, the isometry $A_{v,\gamma\cdot v}$ maps the triple $(\phi_v(\gamma\cdot a_v), \phi_v(\gamma\cdot b_v), \phi_v(\gamma\cdot c_v))$ to the triple $(\phi_{\gamma\cdot v}(\gamma\cdot a_v), \phi_{\gamma\cdot v}(\gamma\cdot b_v), \phi_{\gamma\cdot v}(\gamma\cdot c_v))$, which is by definition $(\infty, [-1,0], [0,0])$. This shows the first part.

   Now, the isometry A_e attached to an edge e is antiholomorphic if and only if the edge e is of type 1, that is, if e passes from a triangle to another. The isometry $B_{v,\gamma}$ is therefore holomorphic if and only if the simplicial path corresponding to γ contains an even number of type 1 edges. Since the colour of the triangle passes from black to white or vice versa at each edge of type 1, we see that the isometry $B_{v,\gamma\cdot v}$ is holomorphic if and only if the first and last triangles have the same colour.

□

Remark 14. Many authors prefer using the notion of *fatgraph* rather than the modified dual graph we are using here. The two notions are equivalent, and the small ambiguity here in the uniqueness of the simplical loop representing a homotopy class, is replaced when one uses fatgraphs by the notion of left and right-turns.

4.4 When is the representation in PU(2,1)?

4.4.1 Representations in PU(2,1) and bipartite triangulations

Definition 14. Let T be an ideal triangulation of Σ, and F be the set of faces of T. The triangulation T is said to be bipartite if there exist to subset of F, F_1 and F_2 such that

1. $F = F_2 \bigcup F_2$

2. If a face Δ belongs to F_i, then its three neighbours belong to F_{i+1}, where the indices are taken modulo 2.

Remark 15. An ideal triangulation is bipartite if and only if it is possible to colour its faces in two colours, black and white, in such a way that any white (resp. black) face has three black (resp. white) neighbours. For this reason, we will refer to black or white triangles. Note that a triangulation is bipartite if and only if its dual graph is.

Remark 16. If T is an ideal triangulation of Σ, then its lift $\hat{T}$ to $\hat{\Sigma}$ is always bipartite, if we keep the same definition of bipartiteness for $\hat{T}$. However, this bipartite structure of $\hat{T}$ will project to a bipartite structure on T if and only if it is π_1-invariant, that is, if and only if for any $\gamma \in \pi_1$ and any triangle Δ of $\hat{T}$, the two triangles Δ and $\gamma \cdot \Delta$ have the same colour.

Proposition 10. *Let $(T, \mathtt{D})$ be an ideal triangulation of Σ equipped with a bending decoration, and let $\rho : \pi_1(\Sigma) \longrightarrow Isom(\mathbf{H}^2_{\mathbb{C}})$ represent the $Isom(\mathbf{H}^2_{\mathbb{C}})$-class of representation of π_1 in $Isom(\mathbf{H}^2_{\mathbb{C}})$ associated to $\mathtt{D}$ by theorem 2. Then the following two statements are equivalent.*

1. *The image of ρ is contained in PU(2,1).*

2. *The triangulation T is bipartite.*

Proof.
- We first prove that the bipartiteness is necessary. Pick a vertex v of Γ to be the basepoint. Let $\nu_1 \cdots \nu_k$ be a simplicial loop based at v representing a homotopy class $\gamma \in \pi_1$. Every ν_l of type 1 (resp. type 2) contributes to $\rho_v(\gamma)$ by an antiholomorphic (resp. holomorphic) isometry. Hence $\rho_v(\gamma)$ is holomorphic if and only if ν_l is of type 1 for an even number of indices l. The number of colour changes is equal to the number of edges of type 1, and is even since γ is a loop. Thus $\rho_v(\gamma)$ is holomorphic.

- Assume now that $\rho(\gamma)$ is holomorphic for any $\gamma \in \pi_1$. Pick a homotopy class, and represent it by a simplicial loop γ based at a vertex v belonging to a face Δ_v of T. Attribute to Δ_v the colour white. We can colour every triangles intersected by γ by changing the colour every time an edge of type 1 is taken by γ. Since $\rho(\gamma)$ is holomorphic, the colour of Δ_v is well-defined (there are an even number of colour changes). We have to check now that if two simplicial loops γ_1 and γ_2 based at v intersect at a vertex $w \in \Delta_w$, then they define the same colour for Δ_w. Write these two loops

$$\gamma_1 = \nu_1^1 \cdots \nu_{k_1}^1 \text{ and } \gamma_2 = \nu_1^2 \cdots \nu_{k_2}^2.$$

  Let γ_i' one of the two subpathes of γ_i connecting v to w. Then $\gamma_{12} = \gamma_1' \gamma_2'^{-1}$ is a loop based at v, and $\rho_v(\gamma_{12})$ is holomorphic. Therefore the number of edges of type 1 in γ_{12} is even. As a consequence, the numbers of edges of type 1 in γ_1' and γ_2' have the same parity and the colour of Δ_w is well-defined.

□

4.4.2 Existence of bipartite triangulations

This section is devoted to the proof of the following proposition.

Proposition 11. *Let $\Sigma_{g,p}$ be a Riemann surface of genus g with $p > 0$ deleted (or marked) points, such that $2 - 2g - p < 0$. Then $\Sigma_{g,p}$ admits a bipartite ideal triangulation.*

Proof. We prove this proposition recursively, starting with the sphere with three marked points and the torus with one marked point.

Both the 1-marked point torus and the 3-marked points sphere admit ideal triangulations consisting of two triangles, and the result is clear in these two cases (see Figure 4). We prove the result from these two cases by describing a reccursion process increasing the genus of the surface by one or adding one puncture to the surface, and respecting the bipartiteness of the triangulation. We take the point of view that any triangulated surface is obtained from a triangulated polygon with identifications of the external edges.

First, the bipartite triangulation of the surface corresponds to a bipartite triangulation of the polygon, compatible with the identification of external edges. By this we mean that if two external edges are identified, then one of them should belong to a black triangle, and the other to a white one. We will denote respectively by F, E and V the sets of faces, edges and vertices of the triangulation.

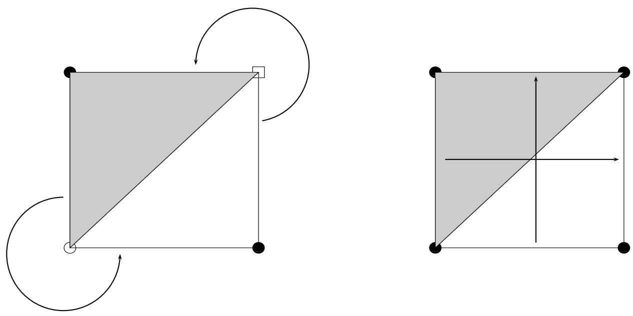
- **Increasing the genus** (see Figure 5). Pick an internal edge of the triangulated polygon, cut along it to open the polygon and insert four new triangles as on figure 5. Identify the new external edges created this way as indicated on figure 5. During this process, 4 new triangles were created, as well as 6 new edges and no new vertex. As a consequence, the Euler characteristic of the compactified surfaces changes from $\chi = |V| - |E| + |F|$ to $\chi' = |V| - (|E| + 6) + (|F| + 4) = \chi - 2$. Since no new vertex was created, the genus has increased by 1. The bicoloration of the new polygon is compatible with the gluing. Therefore the corresponding triangulation of the surface is also bipartite.

- **Increasing the number of punctures**(see Figure 6). The method is the same, inserting this time two new triangles, as indicated on figure 6. This time the transformation changes $|V|$ to $|V| + 1$, $|E|$ to $|E| + 3$ and $|F|$ to $|F| + 2$, and preserves χ. As a consequence, the genus of the surface does not change, and we have introduced a new deleted point on the surface.

□

4.5 Loops around holes

Let T be a bipartite ideal triangulation of Σ and x be a vertex of it. As seen in section 4.2, it is possible to associate to any bending decoration of T a class of T-bent realisation $[\phi, \rho]$. We are going now to analyse the isometry type of images of peripheral curves in terms of the bending decoration.



The 3 -marked points sphere The 1-marked point torus

Figure 4: Bipartite ideal triangulations for surfaces of Euler characteritic -1

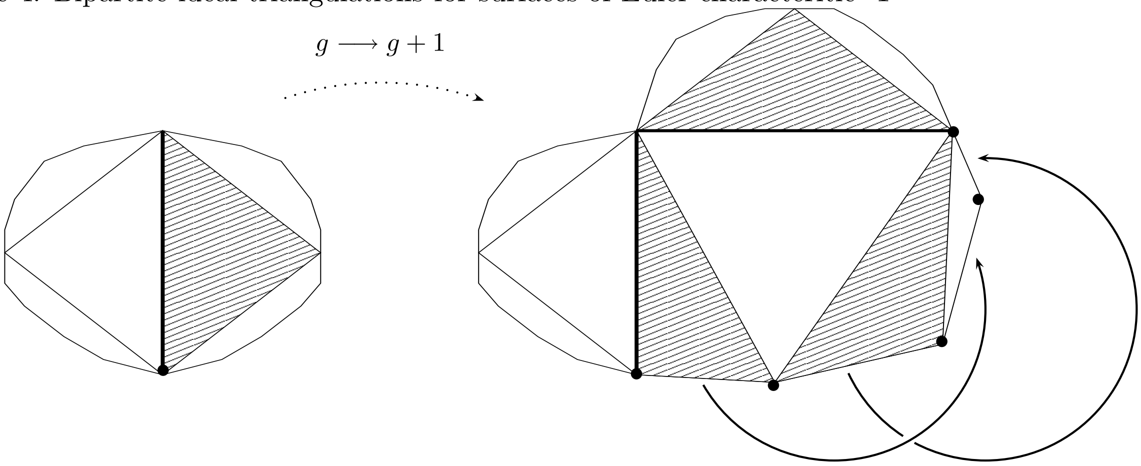


Figure 5: Increasing the genus

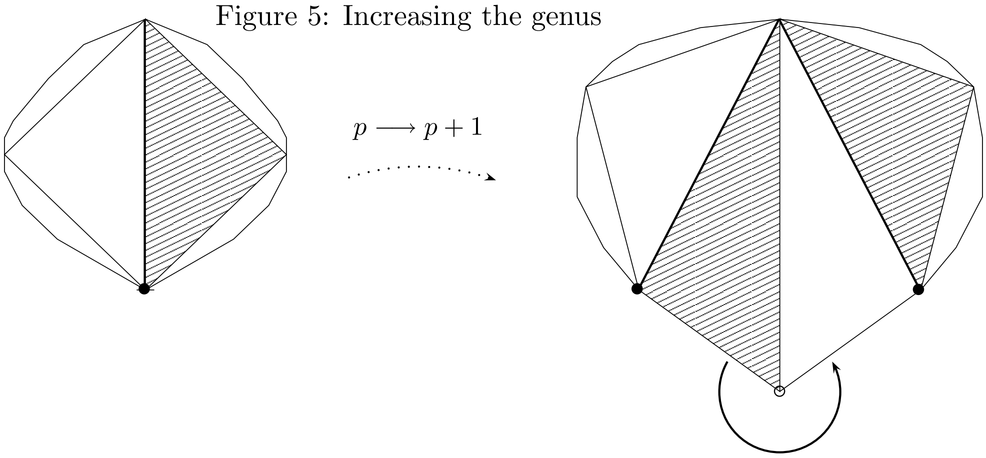


Figure 6: Increasing the number of marked points

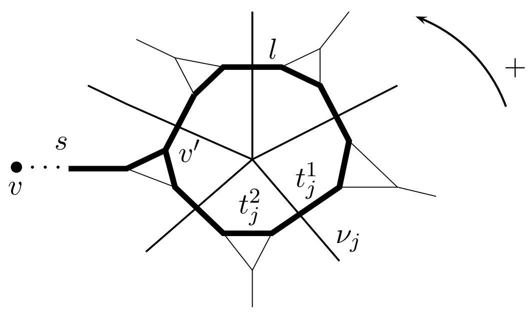


Figure 7: Loop around a vertex of the triangulation

Definition 15. Let T be an ideal triangulation of Σ, $\mathtt{D}$ a bending decoration of T, x one of the points deleted from Σ, and $\{\nu_1, \cdots, \nu_k\}$ be the set of edges of T having x as an endpoint. We will say that $\mathtt{D}$ is *balanced* at x whenever the following condition is satisfied

$$\prod_{i=1}^{k} |\mathtt{D}(\nu_i)| = 1.$$

Notice that when T is bipartite, the number of edges of T having x as a vertex is even.

Proposition 12. *Let T be a bipartite ideal triangulation of Σ, $\mathtt{D}$ be a bending decoration of T and $r = (\phi, \rho)$ be the associated realisation of $\mathcal{F}_\infty$ in $\partial\mathbf{H}^2_{\mathbb{C}}$. Let c be a homotopy class of loop surrounding x and no other puncture. Then*

1. *The holomorphic isometry $\rho(c)$ is loxodromic if and only if $\mathtt{D}$ is not balanced at x.*
2. *If $\mathtt{D}$ is balanced at x, then the isometry $\rho(c)$ is either parabolic or a complex reflection.*

Proof. Pick a vertex v of the modified dual graph of T, Γ, and represent the class $[\rho]$ by the representation ρ_v, as in 9. Let v' be a vertex of Γ belonging to one of the edges of Γ intersecting one of the ν_j's (see Figure 7). The homotopy class c is represented by a simplicial loop sls^-1, where s in a simplicial path connecting v to v', and l is a simplicial loop enclosing p, based at v' such that $l = t_1^2 t_1^1 \cdots t_j^2 t_j^1 \cdots t_k^2 t_k^1$, where t_j^i is an edge of type i of $\hat{\Gamma}$ intersecting ν_j (see figure 7). Then, according to Proposition 9, we see that $\rho_v(c)$ is conjugate to the product

$$\sigma_{z_1} \circ \mathcal{E}^\epsilon \circ \sigma_{z_2} \circ \mathcal{E}^\epsilon \circ \cdots \circ \sigma_{z_{2k}} \circ \mathcal{E}^\epsilon, \tag{19}$$

where $z_j = \mathtt{D}(\nu_j)$ and $\epsilon = 1$ (resp. -1) when the orientation of c coincide with (resp. is opposite to) the one of the surface. The involution σ_z being antiholomorphic, the isometry (19) products lifts to U(2,1) as the product of matrices (see remark 2)

$$M_{z_1}\mathcal{E}\overline{M_{z_2}}\mathcal{E}\cdots\overline{M_{z_{2k-1}}}\mathcal{E}M_{z_{2k}}\mathcal{E} = M_{z_1}\mathcal{E}M_{\bar{z}_2}\mathcal{E}\cdots M_{\bar{z}_{2k-1}}\mathcal{E}M_{z_{2k}}\mathcal{E} = \prod_{j=1}^{2p} M_{z_j^+}\mathcal{E}, \tag{20}$$

where z_j^+ is z_j for odd j and $\bar{z}_j$ for even j. For any z, that the matrix $M_z\mathcal{E}$ is proportional to the element of SU(2,1) given by

$$M_z\mathcal{E} \underset{\mathrm{SU}(2,1)}{\sim} \begin{bmatrix} w & 0 & 0 \\ -\sqrt{2}\bar{w}/w & \bar{w}/w & 0 \\ -1/\bar{w} & \sqrt{2}/\bar{w} & 1/\bar{w} \end{bmatrix} \text{ where } w = \bar{z}^2/z. \tag{21}$$

As a consequence, the product (20) has diagonal coefficients $\pi = \prod_{i=1}^{2p} w_i^+$, $\bar{\pi}/\pi$ and $1/\bar{\pi}$.

The matrix (21), and therefore $\rho(c_i)$ corresponds to a loxodromic isometry if and only if the product π has modulus different from 1, that is, if $\prod_{j=1}^{2k} |z_j| = \prod_{j=1}^{2k} |\mathtt{D}(\nu_j)| \neq 1$ (notice that $|w| = |z|$ in (21)). If π has modulus 1, then the isometry associated to the above matrix represents either a parabolic isometry (if it is not semi-simple) or a complex reflection (if it is semi-simple).

In the case where $\epsilon = -1$ is delt with in the same way, with the only difference that $M_z\mathcal{E}^{-1}$ is upper triangle instead of lower triangle. □

5 The discreteness theorem

5.1 First part of the proof.

The main goal of this section is to focus on those representations associated to a special kind of bending decorations of the triangulation, which we call *regular*, and obtain some discreteness results in this case.

Definition 16. Let T be a triangulation of Σ. We will say that a bending decoration $\mathtt{D}$ of T is *regular* if there exists $\theta \in [-\pi, \pi[$ such that for all edges e of T, $\arg(\mathtt{D}(e)) = \theta$.

Recall that $x_1, \cdots, x_n$ are the points "deleted from Σ", and that c_i denotes the class of peripheral loop surrounding x_i, in the presentation of $\pi_1(\Sigma)$.

Theorem 4. *Let T be a bipartite ideal triangulation of Σ, $\theta \in]-\pi, \pi[$ be a real number. and $\mathtt{D}$ be a regular bending decoration of T with angular part equal to θ. Let ρ be a representative of the (unique) Isom($\mathbf{H}^2_{\mathbb{C}}$)-class of representations associated to $\mathtt{D}$. Then*

- *For any index i, $\rho(c_i)$ is parabolic if and only if $\mathtt{D}$ is balanced at x_i.*
- *The representation ρ does not preserve any totally geodesic subspace of $\mathbf{H}^2_{\mathbb{C}}$, unless $\theta = 0$, in which case it is $\mathbb{R}$-Fuchsian.*
- *As long as $\theta \in [-\pi/2, \pi/2]$, the representation ρ is discrete and faithful.*

The first two parts of theorem 3 follows from what we already know about bent representations. We prove them now, and postpone the proof of the last part of the result to section 5.3.

Proof of parts 1 and 2 of theorem 3. 1. To prove the first part of the theorem, let us go back to the proof of proposition 12. Consider c, one of the homotopy classes of loops around the holes, surroundind the deleted point x. Without loss of generality, we may assume that c is positively oriented with respect to Σ. The fact the decoration is regular implies that this time, $\rho(c)$ is conjugate to the isometry given by the following product of matrices (the assumption on the orientation of c implies that $\epsilon = 1$ in the proof of proposition 12).

$$M_{r_1 e^{i\theta}}\mathcal{E} M_{r_2 e^{-i\theta}}\mathcal{E} \cdots M_{r_{2k} e^{-i\theta}}\mathcal{E} = \prod_{j=1}^{2k} M_{r_j e^{(-1)^{j+1} i\theta}}\mathcal{E}, \text{ where } z_j = r_j e^{i\theta}. \tag{22}$$

Let us be more precise about this product. First, by a direct computation, we see that

$$M_{r_1 e^{i\theta}}\mathcal{E} M_{r_2 e^{-i\theta}}\mathcal{E} = \begin{bmatrix} r_1 r_2 & 0 & 0 \\ -\sqrt{2}(r_2 e^{i\theta}+1) & 1 & 0 \\ * & \dfrac{\sqrt{2}}{r_1 r_2}(r_2 e^{-i\theta}+1) & \dfrac{1}{r_1 r_2} \end{bmatrix}. \tag{23}$$

Notice next that if we multiply (23) on the right by a lower triangle matrix L, the coefficients with indices $(2,1)$ and $(3,2)$ of the new matrix $M_{r_1 e^{i\theta}}\mathcal{E} M_{r_2 e^{-i\theta}}\mathcal{E} L$ are independant of the $*$ coefficient above. Using this fact, it is a straightforward reccursion using to check that the product (22) has the form

$$\begin{bmatrix} \prod_{j=1}^{2k} r_j & 0 & 0 \\ -A\sqrt{2} & 1 & 0 \\ * & -\bar{A}\sqrt{2}\prod_{j=1}^{2k} r_j^{-1} & \prod_{j=1}^{2k} r_j^{-1} \end{bmatrix}, \tag{24}$$

where $z_j = r_i e^{i\theta}$ and

$$A = 1 + \underbrace{\sum_{p=1}^{k-1} \prod_{j=2p+1}^{2k} r_j}_{A_1} + e^{i\theta} \underbrace{\sum_{p=1}^{k} \prod_{j=2p}^{2k}}_{A_2} r_j.$$

Recall that $2k$ is the number of edges adjacent to x. See Remark 17 below.

The latter matrix corresponds to a loxodromic element if and only if it has one eigenvalue of modulus greater than 1, that is if and only if the product $\prod_{j=1}^{2k} r_j$ is different from 1, (i.e.) D is not balanced at x.

Assume now that $\prod_{j=1}^{2p} r_j = 1$. Then the above matrix is either the identity or a unipotent matrix in SU(2,1). If it were the identity, A would to be zero.

- If $e^{i\theta}$ is not real, A is zero if and only if A_1 and A_2 are. The positivity of the x_i's implies that it is not the case.

- If $e^{i\theta}$ is real, then $e^{i\theta} = 1$ since we excluded the case where $\theta = \pi$. Again, the positivity of the x_i's implies that A is not zero in this case.

Therefore the product (22), and thus $\rho(c)$ is unipotent if and only if D is balanced at x.

2. For $\theta \in]-\pi, \pi[\setminus\{0\}$, we know by construction that any two adjacent ideal triangles are not contained in a common totally geodesic subspace of $\mathbf{H}^2_{\mathbb{C}}$. Indeed, they cannot be in a complex line since each of them is real, and Remark 11 implies that they are in a common real plane if and only θ is 0 or π. But each of the vertices of the ideal triangles involved is the fixed point of a conjugate of one of the $\rho(c_i)$'s, all of which are non elliptic as checked above. The result follows then from Lemma 4 below.

□

Remark 17. For the sake of lisibility, let us write down A when $k = 3$, that is if there are 6 edges adjacent to x. In this case:

$$A = 1 + r_5 r_6 + r_3 r_4 r_5 r_6 + e^{i\theta}\left(r_6 + r_4 r_5 r_6 + r_2 r_3 r_4 r_5 r_6\right).$$

Lemma 4. *Let ρ be a representation of $\pi_1(\Sigma)$ in PU(2,1) such that none of the c_i's is mapped to an elliptic isometry which preserves a totally geodesic subspace $\mathcal{V}$ of $\mathbf{H}^2_{\mathbb{C}}$. Then all the fixed points of the $\rho(c_i)$'s belong to $\mathcal{V}$.*

Proof. Call $p_{\mathcal{V}}$ the orthogonal projection onto $\mathcal{V}$, and let c_i be such that $\rho(c_i)$ has a fixed point $m \in \partial\mathbf{H}^2_{\mathbb{C}} \setminus \partial\mathcal{V}$. Since $\rho(c_i)$ is an isometry preserving $\mathcal{V}$, the two geodesic $(m p_{\mathcal{V}}(m))$ and $(m, \rho(c)(p_{\mathcal{V}}(m))$ are both orthogonal to $\mathcal{V}$. They are thus equal, and $\rho(c)$ fixes $p_{\mathcal{V}}(m) \in \mathbf{H}^2_{\mathbb{C}}$, which is absurd since $\rho(c)$ is non-elliptic. □

5.2 Spinal $\mathbb{R}$-surfaces.

In order to prove the third part of the theorem 3, we introduce in this section the main tool we will use.

Definition 17. Let P be an $\mathbb{R}$-plane, and γ a geodesic contained in P. The *spinal* $\mathbb{R}$-*surface* built on γ with respect to P is the hypersurface

$$S_{\gamma,P} = \Pi_P^{-1}(\gamma),$$

where Π_P is the orthogonal projection onto P.

Note that Π_P is well-defined as the orthogonal projection onto a totally geodesic subspace of a negatively curved Riemannian manifold. It is a direct consequence of the definition that any two spinal $\mathbb{R}$-surfaces are isometric, since PU(2,1) acts transitively on the set of pairs (γ, P), where γ is a geodesic contained in a real plane P. It is proved in [26], that if P is a real plane, σ_P the symmetry about P and m a point of $\mathbf{H}^2_{\mathbb{C}}$, a lift to $\mathbb{C}^3$ of the projection of m onto P is given by

$$\frac{1}{|\mathbf{m}|}\mathbf{m} - \frac{\langle \mathbf{m}, \sigma_P(\mathbf{m})\rangle}{|\langle \mathbf{m}, \sigma_P(\mathbf{m})\rangle||\sigma_P(\mathbf{m})|}\sigma_P(\mathbf{m}),$$

where $|\mathbf{m}| = \sqrt{-\langle \mathbf{m}, \mathbf{m}\rangle}$. Note that the above vector is a representant of the midpoint between m and $\sigma_P(m)$. In the special case where P is the standard real plane $\mathbf{H}^2_{\mathbb{R}}$, $\sigma_P(\mathbf{m}) = \bar{\mathbf{m}}$ and $|\mathbf{m}| = |\sigma_P(\mathbf{m})|$, so that we obtain as a lift of $\Pi_{\mathbf{H}^2_{\mathbb{R}}}(m)$ to $\mathbb{C}^3$ the vector

$$\mathbf{m} - \frac{\langle \mathbf{m}, \bar{\mathbf{m}}\rangle}{|\langle \mathbf{m}, \bar{\mathbf{m}}\rangle|}\bar{\mathbf{m}}. \tag{25}$$

The latter expression of the projection extends to $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$. We refer the reader to [26] for more information about this projection.

Example 1. Using the ball-model of $\mathbf{H}^2_{\mathbb{C}}$, $\mathbf{H}^2_{\mathbb{R}}$ is the real disc containing the points with real coordinates. Then the fibre of the orthogonal projection onto $\mathbf{H}^2_{\mathbb{R}}$ over the point $(0,0)$ is the real plane $i\mathbf{H}^2_{\mathbb{R}} = \{(ix_1, ix_2), x_1^2 + x_2^2 < 1\}$.

Remark 18.

- In [25] (p. 185), Mostow defined *spinal surfaces*, which are the inverse images of geodesics by the orthogonal projection onto a complex line instead of a real plane, or equivalently surfaces equidistant from two points in $\mathbf{H}^2_{\mathbb{C}}$. Spinal surfaces are therefore foliated by complex lines. Note that if γ is a geodesic, there exists a unique spinal surface containing it (γ is referred to as its *spine*). In contrast, the set of spinal $\mathbb{R}$-surfaces containing a given geodesic γ is parametrised by a circle S^1, since there is a circle of real planes containing γ.
- Spinal $\mathbb{R}$-surfaces were already used in [32], where they were called $\mathbb{R}$-balls. They were then generalised to *packs* by Parker and Platis in [26]. In their terminology, spinal $\mathbb{R}$-surfaces correspond to *flat packs*. The connection between packs and spinal $\mathbb{R}$-surfaces is given above by lemma 5. See also a discussion in the survey [27].

Proposition 13. *The spinal $\mathbb{R}$-surface $S_{\gamma,P}$ is diffeomorphic to a ball of dimension 3, and is foliated by $\mathbb{R}$-planes. It separates $\mathbf{H}^2_{\mathbb{C}}$ in two connected components which are exchanged by the symmetry about any of the leaves of the foliation.*

Proof. The fibres of the orthogonal projection onto P are $\mathbb{R}$-planes (see for instance [26]). Since $\mathbb{R}$-planes are discs, spinal $\mathbb{R}$-surfaces are diffeomorphic to $\mathbb{R} \times \mathbf{H}^2_{\mathbb{R}}$, that is, a 3-dimensional ball. A spinal $\mathbb{R}$-surface separates $\mathbf{H}^2_{\mathbb{C}}$ in two connected components which are the inverse images of the two connected components of $P \setminus \gamma$ by the orthogonal projection onto P. Let Q be a leaf of $S_{\gamma,P}$. We may normalise so that in the ball model of $\mathbf{H}^2_{\mathbb{C}}$, P is $\mathbf{H}^2_{\mathbb{R}}$, γ connects the two points $(-1,0)$ and $(1,0)$, and $Q = i\mathbf{H}^2_{\mathbb{R}}$. Then the symmetry about Q acts on $\mathbf{H}^2_{\mathbb{R}}$ by $(x_1, x_2) \longmapsto (-x_1, -x_2)$, and the two connected component are exchanged. □

Proposition 14. *Let $\gamma \subset P$ be a geodesic contained in a real plane, and P' be another real plane containing γ. Then one exactly of the following two possibilities occur.*

1. *The real plane P' is contained in $S_{\gamma,P}$.*

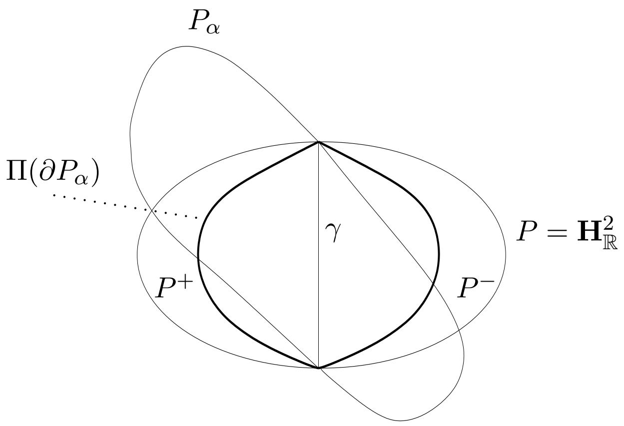


Figure 8: Schematic picture for Proposition 14.

2. *Each of the two connected components of* $\mathbf{H}^2_{\mathbb{C}} \setminus S_{\gamma,P}$ *contains exactly one of the two connected components of* $P' \setminus \gamma$.

Proof. Let us use the ball model of $\mathbf{H}^2_{\mathbb{C}}$. Applying if necessary an element of PU(2,1), we may assume that $P = \mathbf{H}^2_{\mathbb{R}} = (x,y), x^2+y^2<1$ and $\gamma = \{(x,0), x \in]-1,1[\}$. Denote by Π the orthogonal projection onto P, and P^+ (resp. P^-) connected component of P containing points (x,y) with $y>0$ (resp. $y<0$). Any real plane containing γ is the image of P under a rotation of angle α around γ, that is a transformation corresponding to

$$R_\alpha = \begin{bmatrix} e^{-i\alpha/3} & 0 & 0 \\ 0 & e^{2i\alpha_3} & 0 \\ 0 & 0 & e^{-i\alpha/3} \end{bmatrix} \in \mathrm{SU}(2,1), \tag{26}$$

which acts in ball coordinates as $(z_1,z_2) \longmapsto (z_1, e^{i\alpha}z_2)$. Note that R_α fixes pointwise the complex line containing γ. We obtain this way a family of $\mathbb{R}$-planes P_α, such that $(P_\alpha)_\alpha = \big(R_\alpha(\mathbf{H}^2_{\mathbb{R}}))\big)$, and P_α is parametrized by $P_\alpha = \{(x, e^{i\alpha}y), x^2+y^2<1\}$. Note that $P_0 = \mathbf{H}^2_{\mathbb{R}}$.Since $P_{\alpha+\pi} = P_\alpha$, it is only necessary to study the relative position of P_α and $S_{\gamma,P}$ for $\alpha \in]0,\pi[$. Let us pick a point $m = (x, ye^{i\alpha})$ in P_α. Following (25), we see that the projection of m on $\mathbf{H}^2_{\mathbb{R}} = P_0$ is given by the vector

$$\begin{bmatrix} x \\ ye^{i\alpha} \\ 1 \end{bmatrix} - \frac{x^2+y^2e^{2i\alpha}-1}{|x^2+y^2e^{2i\alpha}-1|} \begin{bmatrix} x \\ ye^{-i\alpha} \\ 1 \end{bmatrix}, \tag{27}$$

Specialising (27) for $\alpha = \pi/2$, and using the fact that $x^2-y^2-1 \leqslant x^2+y^2-1 < 0$, we see that the second component of the above vector vanishes and that any point (x, iy) projects onto $(x,0)$. Therefore $P_{\pi/2}$ is contained in $S_{\gamma,P}$.
We examine now the case where $\alpha \neq \pi/2$. Consider a point m on ∂P_α distinct from $(\pm 1, 0)$. The expression (27) becomes (using $x^2+y^2=1$ and $\sin\alpha > 0$):

$$\begin{bmatrix} x \\ ye^{i\alpha} \\ 1 \end{bmatrix} - ie^{\alpha} \begin{bmatrix} x \\ ye^{-i\alpha} \\ 1 \end{bmatrix}. \tag{28}$$

which corresponds after projectivizing and rearranging to the point with coordinates

$$(x, \frac{y\cos\alpha}{1+\sin\alpha}).$$

As a consequence, we see that a point $m = (x, ye^{i\alpha}) \in \partial P_\alpha$ with $y > 0$ (resp. $y < 0$) projects onto P^+ (resp. P^-) if and only if $\alpha \in]0, \pi/2[$ (resp. $\alpha \in]\pi/2, \pi[$). This proves the result for connected components of the boundary of P_α.

Assume now that $\cos\alpha > 0$, and that there exists a point $(x, ye^{i\alpha})$ in P_α with $y > 0$ which projects onto P^-. Then, considering the segment $\{(x, te^{i\alpha}), t \in [y, \sqrt{1-y^2}]\}$, we find a point with coordinates $(x, y'e^{\alpha}) \in P_\alpha$ which projects to a point of of P with vanishing y coordinates, that is a point of γ. Applying if necessary a loxodromic element in R_γ (see Proposition 4 and Definition 2), we may assume that the projection is actually the point $(0, 0)$. Since the fiber of Π above $(0, 0)$ is $i\mathbf{H}^2_{\mathbb{R}}$, this yields $\alpha = \pi/2$, which is absurd. The case where $\cos\alpha < 0$ is done in the same way. This proves the result □

We give now another characterisation of spinal $\mathbb{R}$-surfaces. Recall that if γ is a geodesic, R_γ is the 1-parameter subgroup of PU(2,1) associated to γ. It contains the loxodromic isometries of real trace greater than 3 preserving γ (see Definition 2).

Lemma 5. *Let Q be a real plane, and γ be a geodesic of which endpoints we denote by p and q. Assume that the real symmetry about Q satisfies $\sigma_Q(p) = q$. Then the union $\cup_{g\in R_\gamma} g\cdot Q$ is a spinal $\mathbb{R}$-surface. Conversely, any spinal $\mathbb{R}$-surface may be obtained in this way.*

Proof. We may normalise the situation so that, using the ball model of $\mathbf{H}^2_{\mathbb{C}}$, the points p and q have coordinates $p = (-1, 0)$ and $q = (1, 0)$, and Q is the real plane $i\mathbf{H}^2_{\mathbb{R}}$. The 1-parameter subgroup R_γ preserves the real plane $\mathbf{H}^2_{\mathbb{R}}$ and acts transitively on the geodesic connecting p and q. Since $i\mathbf{H}^2_{\mathbb{R}}$ is the fibre of the orthogonal projection onto $\mathbf{H}^2_{\mathbb{R}}$ above the point $(0, 0)$ which belongs to γ, we see that $\cup_{g\in R_\gamma} g\cdot i\mathbf{H}^2_{\mathbb{R}}$ is the spinal $\mathbb{R}$-surface built on γ with respect to P. □

As said above, spinal surfaces enjoy two equivalent definitions, either as inverse images of geodesics for the orthogonal projection onto complex lines, or as surfaces equidistant from two given points in$\mathbf{H}^2_{\mathbb{C}}$. We have so far given an analogue of the first definition for spinal $\mathbb{R}$-surfaces. The next proposition is more in the flavour of the second one: it is possible to see spinal $\mathbb{R}$-surfaces as natural obects separating two adjacent real ideal triangles, just as spinal surfaces are naturally separating two distinct points. This version of the definition will be of use to understand the geometric meaning of the third part of Theorem 3.

Proposition 15. *Let $\tau = (m_1, m_2, m_3)$ and $\tau' = (m_1, m_3, m_4)$ be two ideal real triangles and γ be the geodesic connecting m_1 and m_3. Assume that the argument of $\mathtt{Z}(\tau, \tau')$ is not π. Then there exists a unique spinal $\mathbb{R}$-surface S built on the geodesic γ having the mirror of $\sigma_{\tau,\tau'}$ as one of its leaves.*

Recall that $\sigma_{\tau,\tau'}$ is the symmetry of the pair (τ, τ') (see Definition 6).

Proof. Let P be the mirror of $\sigma_{\tau,\tau'}$. Applying Lemma 5 to the real plane P and the geodesic γ, we obtain a spinal $\mathbb{R}$-surface having the requested property. If there were another spinal $\mathbb{R}$-surface having the same property, the uniqueness part in Lemma 6 would show that it would have P as a leaf, and contain γ. Thus it would be equal to S by Lemma 5. □

Definition 18. Let τ and τ be two real ideal triangles sharing an edge and such that the argument of $\mathtt{Z}(\tau, \tau')$ is not π. We will call the spinal $\mathbb{R}$-surface given by Proposition 15 the *splitting surface* of τ and τ' and denote it by $\mathrm{Spl}(\tau, \tau')$.

Remark 19. The definition of the splitting surface implies directly that $\mathrm{Spl}(\tau_1, \tau_2){=}\mathrm{Spl}(\tau_2, \tau_1)$.

Proposition 16. *Let τ and τ' be two adjacent ideal triangles such that $\mathtt{Z}(\tau, \tau')$ has argument different from π. Then τ and τ' belong to opposite connected components of $\mathbf{H}^2_{\mathbb{C}} \setminus Spl(\tau, \tau')$.*

Proof. Since two spinal $\mathbb{R}$-surfaces are isometric, we may normalise the situation in such a way that the common geodesic of τ and τ' is in ball coordinates $\gamma = \{(x, 0), x \in]-1, 1[\}$, the splitting surface of τ and τ' is $S_{\gamma,\mathbf{H}^2_{\mathbb{R}}}$, and the symmetry $\sigma_{\tau,\tau'}$ of the pair (τ, τ') is the real symmetry about $i\mathbf{H}^2_{\mathbb{R}}$, which is given in coordinates by

$$(z_1, z_2) \longmapsto (-\bar{z}_1, -\bar{z}_2).$$

We are in the same situation as in the proof of Proposition 14: τ is contained in one of the real planes P_α. Since τ' and τ are exchanged by $\sigma_{\tau,\tau'}$, τ' is contained in the real plane $\sigma_{\tau,\tau'}(P_\alpha)$, which is $P_{-\alpha}$. The result is then a direct application of proposition 14. □

Proposition 17. *The splitting surface associated to a pair of adjacent real ideal triangles is determined by the argument of their* $\mathtt{Z}$*-invariant.*

Proof. Let τ be a real ideal triangle, and γ be one of its edges. Consider τ_1 and τ_2 two real ideal triangles sharing the edge γ with τ such that $\mathtt{Z}(\tau, \tau_j) = x_j e^{i\alpha}$ for $j = 1, 2$. We have to show that the two spinal $\mathbb{R}$-surfaces $\mathrm{Spl}(\tau, \tau_1)$ and $\mathrm{Spl}(\tau, \tau_2)$ coincide.

Call Q_1 and Q_2 the mirrors of the symmetries of the pairs (τ, τ_1) and (τ, τ_2). Proposition 7 provides us a unique isometry g belonging to the 1-parameter subgroup G_γ which maps Q_1 to Q_2. In view of Lemma 5, the result is proved. □

5.3 Proof of the third part of theorem 3

We will prove now that a representation ρ associated to a regular bending decoration $\mathtt{D}$ with anugular part $\theta \in [-\pi/2, \pi/2]$ is discrete and faithful. It is sufficient to prove that for these values of θ, the action of $\rho(\pi_1(\Sigma))$ acts properly discontinuously on some $\rho(\pi_1(\Sigma))$-invariant subset of $\mathbf{H}^2_{\mathbb{C}}$. The following result is the crucial technical point.

Theorem 5. *Let τ be a real ideal triangle with vertices (p_1, p_2, p_3). For $i = 1, 2, 3$, let γ_i be the geodesic $p_{i+1}p_{i+2}$ (indices taken mod. 3). Let τ_1, τ_2 and τ_3 be real ideal triangles, such that*

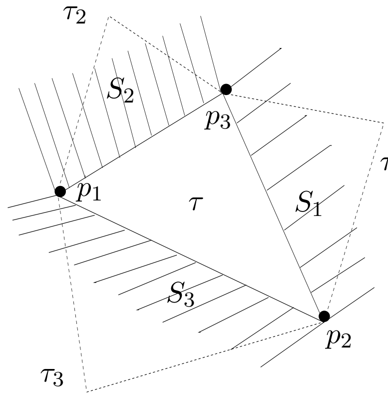


Figure 9: Schematic picture for Theorem 5.

- *For* $i = 1, 2, 3$*,* τ *and* τ_i *are adjacent, and share the geodesic* γ_i *as an edge.*
- *There exists* $\theta \in [-\pi/2, \pi/2]$ *such that arg* $(\mathtt{Z}(\tau, \tau_i)) = \theta$ *for* $i = 1, 2, 3$*.*

Then the three splitting surfaces $S_i = \mathrm{Spl}(\tau, \tau_i)$ *(*$i = 1, 2, 3$*) enjoy the following properties.*

1. *The intersection of* S_i *and* S_{i+1} *in* $\mathbf{H}^2_{\mathbb{C}}$ *is empty.*
2. *The intersection of the closures of* S_i *and* S_{i+1} *in* $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ *is exactly* $\{p_{i+2}\}$*.*

We postpone the proof of theorem 5, and first finish the proof of theorem 3.

proof of part 3 of theorem 3. Let us go back to the reccursive construction of the family of triangles $\{\phi(\Delta), \Delta \text{ face of } \hat{T}\}$.

At the zeroth step, we start from a vertex v of $\hat{\Gamma}$, the lift of the modified dual graph of Σ, which belong to the face Δ_v of $\hat{T}$, and consider the T-bent realization (ϕ_v, ρ_v) of $\mathcal{F}$ associated to v. By construction, Δ_v is mapped by to the real ideal triangle $\phi_v(\Delta_v) = (\infty, [-1, 0], [0, 0])$ (see step 1 in the proof of Theorem 2).

In the first step of induction, the three neighbours τ_1, τ_2 and τ_3 of $\tau = \phi_v(\Delta_v)$ are defined using the bending decoration. Denoting by γ_i the common side of τ and τ_i and by S_i the splitting surface of τ and τ_i, we can apply Theorem 5 and Proposition 16: S_1, S_2 and S_3 are mutually disjoint, and each of the triangles τ, $(\tau_i)_{i=1,2,3}$ lie in a distinct connected component of $\mathbf{H}^2_{\mathbb{C}} \setminus (S_1 \cup S_2 \cup S_3)$ (see Figure 9). The connected component containing τ is a prism, which we denote by $\mathfrak{p}_\tau$.

At the n-th step of induction, we add two new neighbours to each triangle constructed at the $(n-1)$-th step, and construct as many splitting surfaces. Consider a triangle $\tau^{(n)}$ added at the n-th step of induction, and assume it shares a geodesic egde with a triangle $\tau^{(n-1)}$ of the $(n-1)$-th generation, denote by S the corresponding splitting surface. As shown by Proposition 16, S splits $\mathbf{H}^2_{\mathbb{C}}$ in two connected components, one containing $\tau^{(n)}$, the other one containing $\tau^{(n-1)}$. It is clear from Theorem 5 and Proposition 16 that the connected component containing $\tau^{(n-1)}$ contains all the triangles constructed during the previous steps of the recursion. This means that each of the triangles $\phi_v(\Delta)$ is contained in a prism $\mathfrak{p}_\Delta$, and that these prisms satisfy

$$\Delta \neq \Delta' \Rightarrow \overset{\circ}{\mathfrak{p}}_\Delta \cap \overset{\circ}{\mathfrak{p}}_{\Delta'} = \emptyset.$$

Now, if $\gamma \in \pi_1$ maps Δ to Δ', we know that $\Delta \neq \Delta'$, $\rho_v(\gamma)$ maps $\phi_v(\Delta)$ to $\phi_v(\Delta')$. As a consequence, we see that $\rho_v(\pi_1)$ acts properly discontinuously on the union of all prisms, and therefore ρ_v is discrete and faithful. □

We prove now theorem 5.

Proof of theorem 5. (See figure 9).

First step: reduction to a normalised case.
By applying if necessary an isometry, we may assume that τ is the reference real ideal triangle given by $p_1 = \infty$, $p_2 = [-1, 0]$ and $p_3 = [0, 0]$. The isometry $\mathcal{E}$ given in by (18) in Definition 12 cyclically permutes the three latter points, and preserves the invariant $\mathtt{Z}$ of pairs of real ideal triangles since it is holomorphic. The bending decoration being regular, the invariants $\mathtt{Z}(\tau, \tau_i)$ and $\mathtt{Z}(\tau, \tau_j)$ have the same argument. Therefore $\mathcal{E}$ maps τ_i to an ideal $\mathbb{R}$-triangle τ'_{i+1} (indices taken mod. 3) such that $\mathtt{Z}(\tau, \tau_{i+1})$ and $\mathtt{Z}(\tau, \tau'_{i+1})$ have the same argument. As a consequence of Proposition 17, it maps the splitting surface S_i to S_{i+1}, that is, it permutes the three splitting surfaces cyclically. Hence it is enough to prove that the two surfaces S_1 and S_2 satisfy *1* and *2*.

Second step : parametrisation of the symmetries about the leaves of S_2 and S_3.
Let us use the following lifts for the p_i's:

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} -1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \text{ and } \mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{29}$$

We first use Lemma 5 to describe the leaves of S_2. Let q_2 be the third point of τ_2. According to Proposition 17, we may assume that q_2 is any point such that $\mathtt{Z}(\tau, \tau_2)$ has the form $xe^{i\theta}$ with $x > 0$. We make the choice $q_2 = [e^{i\theta}, 0]$. The unique symmetry about a real plane swapping p_1 and p_3, and p_2 and q_2 is given by $\sigma_2(m) = \mathbf{P}(M_2\mathbf{m})$, where M_2 is the matrix

$$M_2^\theta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & e^{i\theta} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The 1-parameter subgroup R_{γ_2} associated to the geodesic connecting p_1 and p_3 is parametrised by the matrices

$$\mathbf{D}_{r_2} = \begin{bmatrix} r_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/r_2 \end{bmatrix} \text{ with } r_2 > 0. \tag{30}$$

We obtain thus the general form M^θ_{2,r_2} of a lift of the symmetry about a leaf of S_1 by conjugating a lift of the involution associated to M^θ_2 by $\mathbf{D}_{r_2}$. Since M^θ_2 stands for a antiholomorphic isometry, this yields (see Remark 2)

$$\begin{aligned} M^\theta_{2,r_2} &= \mathbf{D}_{r_2} M^\theta_2 \overline{\mathbf{D}^{-1}_{r_2}} \\ &= \mathbf{D}_{r_2} M^\theta_2 \mathbf{D}_{1/r_2} \text{ } (\mathbf{D}_{1/r_2} \text{ has real coefficients}) \\ &= \begin{bmatrix} 0 & 0 & r_2^2 \\ 0 & e^{i\theta} & 0 \\ 1/r_2^2 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{31}$$

The general form M^θ_{3,r_3} of a lift of the symmetry about a leaf of S_3 is obtained by conjugating the matrix M^θ_{3,r_3} by the order three elliptic element $\mathcal{E}$:

$$\begin{aligned} M^\theta_{3,r_3} &= \mathcal{E}\mathbf{D}_{r_3} M^\theta_2 \overline{\mathbf{D}^{-1}_{r_3}}\,\overline{E^{-1}} \\ &= \mathcal{E}\mathbf{D}_{r_3} M^\theta_1 \mathbf{D}_{1/r_3}\mathcal{E}^{-1} \\ &= \begin{bmatrix} -r_3^2 & \sqrt{2}\left(e^{i\theta}+r_3^2\right) & \dfrac{1+2e^{i\theta}r_3^2+r_3^4}{r_3^2} \\ \sqrt{2}r_3^2 & e^{i\theta}+2r_3^2 & \sqrt{2}\left(e^{i\theta}+r_3^2\right) \\ r_3^2 & -\sqrt{2}r_3^2 & -r_3^2 \end{bmatrix}. \end{aligned} \tag{32}$$

Third step: proof of the disjunction

Note first that the closures of S_2 and S_3 in $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ both contain the point p_1 as a common end of the geodesics γ_2 and γ_3. Therefore their intersection should at least contain this point. Now, the result will be proved if we show that the closure of any leaf of S_2 is disjoint from the closure or any leaf of S_3. We do this by showing that the product of the symmetries about these leaves is loxodromic as long as $\theta \in [-\pi/2, \pi/2]$ (see Lemma 1). More precisely, we will show that for these values of θ, the isometry associated to the matrix $M^\theta_{2,r_3}\overline{M^\theta_{3,r_3}}$ is loxodromic for any pair $(r_2, r_3) \in \mathbb{R}^2_{>0}$. Using the above matrix form, it is seen that the trace of this matrix is

$$\mathrm{tr} M^\theta_{2,r_2}\overline{M^\theta_{3,r_3}} = 2r_3^2 e^{i\theta} + \frac{2}{r_2^2}e^{-i\theta} + 1 + r_2^2 r_3^2 + \frac{1}{r_2^2 r_3^2} + \frac{r_3^2}{r_2^2}. \tag{33}$$

This yields

$$\begin{aligned}\mathrm{Re}\left(\mathrm{tr} M^{\theta}_{2,r_2}\overline{M^{\theta}_{3,r_3}}\right) &= 2r_3^2\cos\theta + \frac{2}{r_2^2}\cos\theta + 1 + r_2^2r_3^2 + \frac{1}{r_2^2r_3^2} + \frac{r_3^2}{r_2^2} \\ &\geqslant 1 + r_2^2r_3^2 + \frac{1}{r_2^2r_3^2} \text{ while } \cos\theta \geqslant 0 \\ &\geqslant 3 \end{aligned} \tag{34}$$

This implies that the isometry associated to $M^{\theta}_{2,r_3}\overline{M^{\theta}_{3,r_3}}$ is loxodromic as long as $\theta \in [-\pi/2, \pi/2]$ and for any pair $(r_2, r_3) \in \mathbb{R}^2_{>0}$, as shown by Remark 3. As a consequence of Lemma 1, the corresponding leaves of S_2 and S_3 are disjoint. □

6 Remarks and comments

6.1 Embedding of the Teichmüller space I

Let us go focus for a moment on the special case where the bending decosration is positive: for all edge e of T, $\mathtt{D}(e) \in \mathbb{R} > 0$. It this case, all the triangles constructed from$\mathtt{D}$ are contained in the standard real plane $\mathbf{H}^2_{\mathbb{R}}$. As mentionned in Remark 12, the $\mathtt{Z}$-invariant is in this case the classical cross-ratio in the upper half-plane. We recover thus, in $\mathbf{H}^2_{\mathbb{R}} \subset \mathbf{H}^2_{\mathbb{C}}$ the action of the $\rho(\pi_1(\Sigma))$ on the upper half-plane, when ρ is a discrete and faithful representation in PSL(2,$\mathbb{R}$). This corresponds to the embedding PSL(2,$\mathbb{R}$) $\sim$ PO(2,1) as the stabilizer of a real plane. We recover this way the classical *shear coordinates*. Notice also that when $z > 0$, the restriction of the rea symmetry σ_z to $\mathbf{H}^2_{\mathbb{R}}$ is a half-turn. We recover thus also the explicit combinatorial description of classes of discrete and faithful representations in PSL(2,$\mathbb{R}$)given by Fock and Goncharov in [10] by means of elementaries isometries.

Note that the parabolicity criterion for peripheral homotopy classes in [28] or [10] is the same as here.

6.2 Embeddings of the Teichmüller space.

Let us go back for a moment to the case of representations in PSL(2,$\mathbb{R}$), the group of holomorphic isometries of the complex hyperbolic line $\mathbf{H}^1_{\mathbb{C}}$. In this frame, we can define a $\mathbf{H}^1_{\mathbb{C}}$-realisation of the Farey set of a cusped surface Σ as a pair (ϕ, ρ), where $\rho : \pi_1(\Sigma) \longrightarrow$ PSL(2,$\mathbb{R}$) is a discrete and faithful representation and ϕ is a ρ-equivariant mapping from the Farey set to the boundary of the Poincaré disc. Denote by $\mathcal{DF}$ the set of PSL(2,$\mathbb{R}$)-classes of discrete and faithful representations of $\pi_1(\Sigma)$ in PSL(2,$\mathbb{R}$), and by $\mathcal{DF}^+$ the set of PSL(2,$\mathbb{R}$)-classes of $\mathbf{H}^1_{\mathbb{C}}$-realizations of $\mathcal{F}_\infty(\Sigma)$.

Let m be a point of $\mathcal{F}_\infty(\Sigma)$, corresponding to a fixed point of a parabolic c, and let (ϕ, ρ) be a $\mathbf{H}^1_{\mathbb{C}}$-realisation. The ρ-equivariance of ϕ implies that $\phi(m)$ is fixed by $\rho(c)$. Now, ρ being discrete and faithful, $\rho(c)$ is either parabolic or loxodromic. Therefore, if $\rho(c)$ is hyperbolic, $\phi(m)$ may be any of the two fixed points of $\rho(c)$. Consider the projection

$$\begin{aligned} \mathbf{p} \ : \ & \mathcal{DF}^+ \longrightarrow \mathcal{DF} \\ & [(\phi,\rho)] \longmapsto [\rho] \end{aligned} \tag{35}$$

Let $[[1,n]]$ be the set of integers comprised between 1 and n. For any subset $I = \{i_1, \cdots, i_k\}$ of $[[1,n]]$, define

$$\mathcal{P}_I = \left\{[\phi,\rho] \in \mathcal{DF}^+ | \rho(c_i) \text{ is parabolic } \Leftrightarrow i \in I\right\}.$$

Then $\mathcal{DF}^+$ decomposes as the disjoint union

$$\mathcal{DF}^+ = \coprod_{I\subset[[1,n]]} \mathcal{P}_I, \tag{36}$$

and the restriction to $\mathcal{P}_I$ of the projection (35) is $2^{n-|I|}$ to 1. In particular, it is 2^n to 1 when restricted to $\mathcal{P}_\emptyset$, which is the set of realisations associated to totally hyperbolic representations, and it a bijection when restricted to $\mathcal{P}_{[[1,n]]}$, which corresponds to the Teichmüller space.

Once an ideal triangulation T of Σ is fixed, shear coordinates provide a bijection between the set of positive decoration of T (that is, mappings $\mathtt{d} : e(T) \longrightarrow \mathbb{R}_{>0}$), and the set of $\mathcal{DF}^+$. The main tool is the classical cross-ratio, used as a gluing invariant of two ideal triangles in $\mathbf{H}^1_{\mathbb{C}}$. It is also possible to give an explicit representative for a representation associated to a given decoration by use of elementary isometries. This time, the elementary isometries are

$$I_x = \begin{bmatrix} 0 & \sqrt{x} \\ -1/\sqrt{x} & 0 \end{bmatrix},$$

for an edge of type 1 intersecting an edge of T decorated by the positive number x, and

$$E = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix},$$

for a positively oriented edge of type 2. The mechanic of the construction is the same as what we did in section 4, only simplified by the fact that both types of elementary isometries are holomorphic, thus there is no need of colouring faces of $\hat{T}$ in the classical case. This material is classical and exposed for instance in [10]. Notice that if c_j is a peripheral homotopy class around the deleted point x_j, the parabolicity of $\rho(c_j)$ is equivalent to the condition that the associated positive decoration is *balanced at* x (that is, the product of all positive numbers on edges adjacent to x equals 1). Type-preserving representations, and therefore the Teichmüller space of Σ correspond to positive decorations which are balanced at every deleted point of Σ. We calla the decorations simply *balanced.*

Fix a bipartite ideal triangulation T. The set of positive decorations of T is $\mathbb{R}_{>0}^{\sharp e(T)}$. To any real number θ is associated a mapping

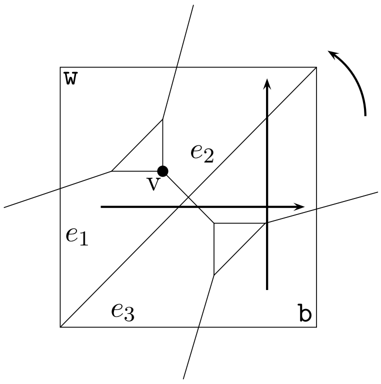


Figure 10: The 1-punctured torus

$$
\begin{aligned}
\psi_\theta \quad : \quad & \mathbb{R}_{>0}^{\sharp e(T)} \longrightarrow \mathcal{BD}_T \\
& \mathtt{d} \longmapsto \mathtt{D} = \mathtt{d}e^{i\theta}.
\end{aligned} \tag{37}
$$

This mapping induces a mapping from $\mathcal{DF}^+$ to $\mathcal{BR}_T$, which maps the realisation associated to $\mathtt{d}$ to the T-bent realisation associated to the regular bending decoration $\mathtt{d}e^{i\theta}$. Restricting this induced mapping to those $\mathbf{H}^1_{\mathbb{C}}$-realisation corresponding to balanced positive decorations (), we can rephrase Theorem 3 as follows.

Theorem 6. *Let $\theta \in [-\pi/2, \pi/2]$ be a real number and T be a bipartite ideal triangulation of Σ. The mapping ψ_θ defined in (37) induces a pair of embeddings ofthe Teichmüller space of Σ in Hom(π_1, PU(2,1))/PU(2,1) of which images contain only classes of discrete, faithful and type-preserving representations.*

Proof. Restricting the mapping $\mathtt{d} \longmapsto \mathtt{d}e^{i\theta}$ to balanced decorations of T produces discrete, faithful and type-preserving representations of $\pi_1(\Sigma)$ with images contained in PU(2,1) since T is bipartite. Once a coloration of the faces of $\hat{T}$ is fixed, we obtain two injective applications by mapping the point in $\mathcal{T}(\Sigma)$ associated to $\mathtt{d}$ to the class of representations associated to $\mathtt{d}e^{i\theta}$ corresponding either to white triangles or to black triangles. These two embeddings are identified by the complex conjugation in $\mathbf{H}^2_{\mathbb{C}}$, and correspond in fact to a single embedding in Hom(π_1,PU(2,1))/Isom($\mathbf{H}^2_{\mathbb{C}}$). □

6.3 Link with previously known families of examples.

In this section, we draw the connection between T-bent realizations and families of examples described in the previous works [8, 19, 32].

The 1-punctured torus. In this case T consists of two triangles, as indicated on figure 10. We will use the vertex v marked on the figure as basepoint. There are two faces, of which colour is indicated by $\mathtt{w}$ and $\mathtt{b}$ on figure 10, and three edges, labelled by e_1, e_2 and e_3 on figure 10. In the case of a regular bending decorations,the decoration is given three positive real numbers x_1, x_2 and x_3 and $\theta \in [0, 2\pi[$ such that the edge e_i is decorated by x_i, θ. Following the results of section 4.3, we see that the identifications between opposite faces of the square correspond to the following holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$. Call A and B the isometries associated respectively to the horizontal and vertical identifications of the opposite sides of the square. Following section 4.3, these isometries are given by

$$\begin{cases} A &= \mathcal{E} \circ \sigma_{x_2,\theta} \circ \mathcal{E}^{-1} \circ \sigma_{x_2,\theta} \\ \\ B &= \sigma_{x_2,\theta} \circ \mathcal{E}^{-1} \circ \sigma_{x_3,\theta} \circ \mathcal{E}. \end{cases} \tag{38}$$

As a consequence, we see that the group $\langle A, B\rangle$ has index two in the group generated by the three real symmetries $I_1 = \mathcal{E} \circ \sigma_{x_1,\theta} \circ \mathcal{E}^{-1}$, $I_2 = \sigma_{x_2,\theta}$ and $I_3 = \mathcal{E}^{-1} \circ \sigma_{x_3,\theta} \circ \mathcal{E}$. The group $\langle I_1, I_2, I_3\rangle$ is an example of a so-called Lagrangian triangle group. This example of bending has been exposed with a different point of view in [32] (see also[31]).

In [32], the discreteness result is stated with an angle $\alpha \in [-\pi/4, \pi/4]$. This angle α is actually half the bending parameter θ we use here. It may be interpreted as an angle between a real ideal triangle Δ and the splitting surface $\mathrm{Spl}(\Delta, \Delta')$, where Δ' is adjacent to Δ. From this point of view, $\mathrm{Spl}(\Delta,\Delta')$ is bisecting the pair (Δ, Δ').

The Toledo invariant and the examples of Gusevskii and Parker The Toledo invariant is a conjugacy invariant defined for representations of fundamental groups of closed surfaces, and for type-preserving representations of cusped surfaces. We refer the reader to [30] and [22, 19] for its definition and main properties. Let us just recall that if ρ is such a representation, then

- if Σ has punctures, then $\mathbf{tol}(\rho)$ is a real number in the interval $[\chi, -\chi]$, where χ is the Euler characteristic of Σ,
- if not, then $\mathbf{tol}(\rho)$ belongs to $2/3\mathbb{Z} \cap [\chi, -\chi]$.

Let (ϕ, ρ) be a T-bent realization of $\mathcal{F}_\infty$, where T is a bipartite triangulation, and Ω be a fundamental domain for the action of $\pi_1(\Sigma)$ on $\tilde{\Sigma}$. We might see Ω as a family of triangles $(\Delta_1, \cdots, \Delta_m)$. Then it follows from [19, 30] that the Toledo invariant $\mathbf{tol}(\rho)$ equals twice the sum of the Cartan invariants of the ideal triangles $\phi(\Delta_i)$. In our particular case, all the triangles are real. We obtain therefore directly the

Proposition 18. *Let (ϕ, ρ) be a T-bent realization of $\mathcal{F}_\infty$, with ρ type-preserving. The Toledo invariant of ρ is equal to zero.*

In [19], Gusevskii and Parker have described for each genus g and number of punctures n a 1-parameter family $(\rho_t)_{t\in[-\chi,\chi]}$ of non PU(2,1)-equivalent discrete, faithful and type-preserving representations of a Riemann surface of genus g with n punctures having the property that the Toledo invariant of ρ_t equals t. This shows that all the possible values of the Toledo invariant for non-compact surfaces are realised by discrete and faithful representation. To prove this result, Gusevskii and Parker start from discrete and faithful representations of the modular group in PU(2,1) and pass to a finite index subgroup

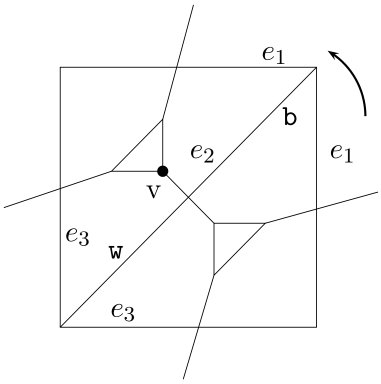


Figure 11: The 3-punctured sphere

using Millington's theorem (see [19]). In their construction, they show that ρ_0 preserves a real plane (this is a so-called $\mathbb{R}$-Fuchsian representation). Therefore ρ_0 is the unique intersection between Gusevskii and Parker's family of representations and our.

The 3-punctured sphere and the examples of Falbel and Koseleff. This time we are using the bipartite triangulation of the 3-punctured sphere showed on figure 11. The representation of the fundamental group associated to the decoration given by $\delta(e_i) = x_i$ and $\alpha(e_i) = \theta_i$ is given by

$$\begin{cases} A &= \mathcal{E}^{-1} \circ \sigma_{x_1,\theta_1} \circ \mathcal{E}^{-1} \circ \sigma_{x_2,\theta_2} \\ B &= \sigma_{x_2,\theta_2} \circ \mathcal{E}^{-1} \circ \sigma_{x_3,\theta_3} \circ \mathcal{E}^{-1}. \\ C &= \mathcal{E} \circ \sigma_{x_3,\theta_3} \circ \mathcal{E}^{-1} \circ \sigma_{x_1,\theta_1} \circ \mathcal{E}, \end{cases} \tag{39}$$

It is easily checked that $ABC = 1$. Using the matrices given in section 4.3, we see that the representation is type preserving if and only if $x_1 = x_2 = x_3 = 1$ and none of the θ_i's is equal to π. When $\theta_1 = \theta_2 = \theta_3 \in [-\pi/2, \pi/2]$, this provides through theorem 3 a 1-parameter family of discrete, faithful and type-preserving representations of the fundamental group of the 3-punctured sphere.
Moreover, it is possible to prove that in the case where $\delta(e_i) = 1$ and $\alpha(e_i) = \theta$ for all i, then there exists three real symmetries s_1, s_2 and s_3 such that $A = s_1 s_2$ and $B = s_2 s_3$. Call Q_i the mirror of s_i. Since A and B are parabolic, the mirrors of the s_i's are mutually asymptotic, that is $Q_i \cap Q_{i+1}$ consists of exactly one point in $\partial \mathbf{H}^2_{\mathbb{C}}$. Therefore these groups belong to the family of groups studied by Falbel and Koseleff in [8]. Note moreover that the discreteness of these groups was not proved in [8], where the focus is on deformations of groups preserving a complex line.

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