Real reflections, commutators and cross-ratios in complex hyperbolic space

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October 18, 2013

Abstract

We provide a concrete criterion to determine whether or not two given elements of PU(2,1) can be written as products of real reflections, with one reflection in common. As an application, we show that the Mostow lattices and all known non-arithmetic lattices in PU(2,1) are generated by real reflections.

1 Introduction

A classical way of studying isometries of a symmetric space $\mathcal{S}$ is to decompose them as products of involutions. The most comfortable situation is to have a class $\mathcal{C}$ of isometric involutions with the following properties.

1. Any two involutions in $\mathcal{C}$ are conjugate in $\mathrm{Isom}(\mathcal{S})$.
2. Any element of $\mathrm{Isom}(\mathcal{S})$ can be written as a product $s_1 s_2$, with $s_i \in \mathcal{C}$.

For example, if such a family of involutions exists, describing the fixed points of an isometry $A = s_1 s_2$ of $\mathcal{S}$ amounts to studying the relative position of the fixed point sets of s_1 and s_2 which are totally geodesic subspaces and isometric to one another. If such a family $\mathcal{C}$ exists, one usually says that $\mathrm{Isom}(\mathcal{S})$ has involution length 2 with respect to $\mathcal{C}$. Of course this requirement is too optimistic in general (for example, it fails in Euclidean space of dimension at least 3).

Assuming that a symmetric space has this property, the next question is to decide when two isometries can be decomposed using a common involution. We will call such a pair *decomposable* with respect to $\mathcal{C}$ (the term *linked* is also commonly used, see [BM]). Whenever a pair of isometries (A, B) is decomposable under the form

$$A = s_1 s_2 \text{ and } B = s_2 s_3, \tag{1.1}$$

with $s_i \in \mathcal{C}$, the group generated by A and B has index two in the group $\Gamma = \langle s_1, s_2, s_3 \rangle$. This simplifies the understanding of Γ as it reduces it to the study of the relative position of 3 pairwise isometric totally geodesic subspaces.

One of the most elementary cases is that of the Poincaré disk Δ. There are two classes of involutions in Δ, namely *half-turns* and *reflections*. Viewing Δ as the unit disk in $\mathbb{C}$, half-turns are conjugate in $Isom^+(\Delta)$ to $z \longmapsto -z$ and reflections to $z \longmapsto \overline{z}$. It is a classical fact that $Isom^+(\Delta)$ has involution length 2 with respect to reflections, and 3 with respect to half-turns. Moreover, any pair of orientation-preserving isometries of the Poincaré disk can be decomposed in the form (1.1) with the s_i either all half-turns or all reflections. This makes

*Author partially supported by NSF grant DMS 1007340 and SNF grant 200021-131967/1.

the description of many properties of the group $\langle A,B\rangle$ easier. For instance, the group $\langle A,B\rangle$ is discrete if and only if $\langle s_1,s_2,s_3\rangle$ is.

In [BM], A.Basmajian and B. Maskit have generalized this result to all space forms. In particular, they prove that a space form always has involution length equal to 2, the nature of the involution depending on the dimension of the space.

In this paper we study the question of decomposability in the complex hyperbolic plane, which can be seen via a projective model as the unit ball in $\mathbb{C}^2$ equipped with a PU(2,1)-invariant metric. Indeed, PU(2,1) is the group of holomorphic isometries of $\mathrm{H}^2_{\mathbb{C}}$ and is the identity component of $\mathrm{Isom}(\mathrm{H}^2_{\mathbb{C}})$, the other connected component containing antiholomorphic isometries. In particular, PU(2,1) has index two in $\mathrm{Isom}(\mathrm{H}^2_{\mathbb{C}})$ and in this context holomorphicity plays the role of preservation of orientation for space forms. The class of involutions we will be concerned with consists of antiholomorphic involutions, which are all conjugate in PU(2,1) to the map given in affine ball coordinates by

$$\sigma_0 : (z_1,z_2) \longmapsto (\overline{z_1},\overline{z_2}). \tag{1.2}$$

Clearly, σ_0 fixes pointwise the set of real points of $\mathrm{H}^2_{\mathbb{C}}$, and conjugates of σ_0 by elements of PU(2,1) fix pointwise *real planes*, which are (totally real) totally geodesic embedded copies of the Poincaré disk. We will refer to these antiholomorphic involutions as *real reflections*.

It has been known since Falbel and Zocca (see [FZ]) that the involution length of PU(2,1) with respect to real reflections is 2, and this fact has been generalized to all dimensions by Choi (reference?). The main question we address in this paper is the following.

When is a pair of elements of PU(2,1) decomposable with respect to real reflections?

We will abbreviate this by saying that a pair $(A,B)\in PU(2,1)^2$ is $\mathbb{R}$-*decomposable* (see Definition 2.1). There is no hope that any pair of elements in PU(2,1) is $\mathbb{R}$-decomposable. A rough argument for this is the following dimension count. The group PU(2,1) has dimension 8, thus the product PU(2,1)×PU(2,1) has dimension 16. On the other hand, the set of real planes in $\mathrm{H}^2_{\mathbb{C}}$ has dimension 5 (to see this note that the stabilizer of the set of real points of the ball is PO(2,1) which has dimension 3). As a real reflection is determined by its fixed real plane, the set of triples of real reflections has dimension 15 (and therefore cannot be diffeomorphic to $PU(2,1)\times PU(2,1)$).

This question has been examined in [W2], where it was proved that, under the assumption that A and B are loxodromic, the pair (A,B) is $\mathbb{R}$ decomposable provided that the trace of the commutator $[A,B]$ is real. However, this result was obtained as a byproduct of a classification of pairs of elements of PU(2,1) by traces (namely the data $(\mathrm{Tr}A,\mathrm{Tr}B,\mathrm{Tr}AB,\mathrm{Tr}A^{-1}B)$ determines the pair (A,B) up to PU(2,1)-conjugation modulo an order two indetermination which correspond to the sign of the imaginary part of $\mathrm{Tr}[A,B]$). This approach was less natural than the present one, where we get rid of the assumption that A and B are loxodromic.

Our main result (Theorem 4.1) is the following.

Theorem. *Let $A,B\in$ PU(2,1) be two isometries not fixing a common point in $\overline{\mathrm{H}^2_{\mathbb{C}}}$. Then: the pair (A,B) is $\mathbb{R}$-decomposable if and only if the commutator $[A,B]$ has a fixed point in $\overline{\mathrm{H}^2_{\mathbb{C}}}$ whose associated eigenvalue is real and positive.*

Note that the eigenvalues of $[A,B]$ do not depend on the choice of lifts of A and B to U(2,1). The main tools we use to prove this are the following. First, we connect the cross-ratio of a cycle of points associated to a fixed point of the commutator to the corresponding eigenvalue of the commutator (this is relation 4.1). Secondly we use Proposition 3.1, which asserts that if a quadruple of points has positive cross-ratio, then it admits a anti-holomorphic symmetry. This result is an extension of a result of Goldman (Lemma 7.2.1 in [G]) from quadruples of boundary points to quadruples in $\overline{\mathrm{H}^n_{\mathbb{C}}}$.

A classical and difficult question in complex hyperbolic geometry is to determine the discreteness or non-discreteness of a given finitely generated subgroup Γ of PU(2,1), and to obtain a presentation of Γ. Even when discreteness is known from general results (for instance in the case of arithmetic lattices), finding a presentation of the group is a difficult problem. In fact, the method used most often is to construct a fundamental domain and use the Poincaré Polyhedron theorem. This is a very technical task which requires a detailed understanding of the action of Γ on $\mathrm{H}^2_{\mathbb{C}}$.

There are not so many examples of explicit discrete subgroups of PU(2,1), and most of them are obtained from groups with 2 generators. In this case, the existence of a decomposition as in (1.1) connects an algebraic property of the group (being an index 2 subgroup of a group generated by 3 involutions) to a geometric property (the existence of totally geodesic fixed point sets in a certain configuration). In other words, Γ appears as a *reflection group*. Such decompositions appeared for instance naturally in [FalPar2] or [W3] which studied certain representations of Fuchsian groups in PU(2,1). They were also central in the constructions of fundamental domains for Mostow's lattices ([M]) in [DFP], as well as for the new non-arithmetic lattices obtained in [ParPau], [Pau], [DPP1] and [DPP2]. In all these occurrences, the existence of the real reflections decomposing the generators required some work (part of the detailed geometric construction of the fundamental domains), whereas the concrete criterion given by Theorem 4.1 allows us to easily reprove that the Mostow and Deraux-Parker-Paupert lattices are generated by real reflections.

The paper is organized as follows. We start with some geometric preliminaries in section 2, then study configurations of points and cross-ratios in section 3. Section 4 contains the statement and proofs of our main results, which we then apply to various discrete subgroups of PU(2,1) in section 5.

2 Geometric preliminaries

2.1 Complex hyperbolic space and isometries

The standard reference for complex hyperbolic geometry is [G]. For the reader's convenience we include a brief summary of key definitions and facts. Our main result concerns the case of dimension $n = 2$, but the general setup is identical for higher dimensions so we state it for all $n \geqslant 1$.

Distance function: Consider $\mathbb{C}^{n,1}$, the vector space $\mathbb{C}^{n+1}$ endowed with a Hermitian form $\langle\cdot\,,\cdot\rangle$ of signature $(n, 1)$. Let $V^- = \{Z \in \mathbb{C}^{n,1} | \langle Z, Z\rangle < 0\}$. Let $\pi : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{C}\mathrm{P}^n$ denote projectivization. Define $\mathrm{H}^n_{\mathbb{C}}$ to be $\pi(V^-) \subset \mathbb{C}\mathrm{P}^n$, endowed with the distance d (Bergman metric) given by:

$$\cosh^2 \frac{1}{2} d(\pi(X), \pi(Y)) = \frac{|\langle X, Y\rangle|^2}{\langle X, X\rangle\langle Y, Y\rangle} \tag{2.1}$$

Different choices of Hermitian forms of signature $(n, 1)$ give rise to different models of $\mathrm{H}^n_{\mathbb{C}}$. The two most standard choices are the following. First, when the Hermitian form is given by $\langle Z, Z\rangle = |z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2$, the image of V^- under projectivisation is the unit ball of $\mathbb{C}^n$, seen in the affine chart $\{z_{n+1} = 1\}$ of $\mathbb{C}P^n$. This model is referred to as the *ball model* of $\mathrm{H}^n_{\mathbb{C}}$. Second, when $\langle Z, Z\rangle = 2\mathrm{Re}(z_1\overline{z_{n+1}}) + |z_2|^2 + \cdots + |z_n|^2$, we obtain the so-called *Siegel model* of $\mathrm{H}^n_{\mathbb{C}}$, which generalizes the Poincaré upper half-plane. More details on the Siegel model in dimension 2 will be given in the next section.

Isometries: From (2.1) it is clear that $\mathrm{PU}(n, 1)$ acts by isometries on $\mathrm{H}^n_{\mathbb{C}}$, where $\mathrm{U}(n, 1)$ denotes the subgroup of $\mathrm{GL}(n + 1, \mathbb{C})$ preserving $\langle\cdot, \cdot\rangle$, and $\mathrm{PU}(n, 1)$ its image in $\mathrm{PGL}(n + 1, \mathbb{C})$. In fact, PU($n$,1) is the group of holomorphic isometries of $\mathrm{H}^n_{\mathbb{C}}$, and the full group of isometries is $\mathrm{PU}(n, 1) \ltimes \mathbb{Z}/2$, where the $\mathbb{Z}/2$ factor corresponds to a real reflection (see below). Holomorphic isometries of $\mathrm{H}^n_{\mathbb{C}}$ can be of three types, depending on the number and location of their fixed points. Namely, $g \in \mathrm{PU}(n, 1)$ is :

- *elliptic* if it has a fixed point in $\mathrm{H}^n_{\mathbb{C}}$
- *parabolic* if it has (no fixed point in $\mathrm{H}^n_{\mathbb{C}}$ and) exactly one fixed point in $\partial\mathrm{H}^n_{\mathbb{C}}$
- *loxodromic*: if it has (no fixed point in $\mathrm{H}^n_{\mathbb{C}}$ and) exactly two fixed points in $\partial\mathrm{H}^n_{\mathbb{C}}$

Totally geodesic subspaces and related isometries: A *complex k-plane* is a projective k-dimensional subspace of $\mathbb{C}P^n$ intersecting $\pi(V^-)$ non-trivially (so, it is an isometrically embedded copy of $\mathrm{H}^k_{\mathbb{C}} \subset \mathrm{H}^n_{\mathbb{C}}$). Complex 1-planes are usually called *complex lines*. If $L = \pi(\tilde{L})$ is a complex $(n - 1)$-plane, any $v \in \mathbb{C}^{n+1} - \{0\}$ orthogonal to $\tilde{L}$ is called a *polar vector* of L. Such a vector satisfies $\langle v, v\rangle > 0$, and we will usually normalize v so that $\langle v, v\rangle = 1$.

A *real k-plane* is the projective image of a totally real $(k+1)$-subspace W of $\mathbb{C}^{n,1}$, i. e. a $(k+1)$-dimensional real vector subspace such that $\langle v, w\rangle \in \mathbb{R}$ for all $v, w \in W$. We will usually call real 2-planes simply real planes, or $\mathbb{R}$-planes. Every real n-plane in $\mathrm{H}^n_{\mathbb{C}}$ is the fixed-point set of an isometry of order 2 called a *real reflection* or $\mathbb{R}$-reflection. The prototype of such an isometry is the map given in affine coordinates by $(z_1, ..., z_n) \mapsto (\overline{z_1}, ..., \overline{z_n})$ (this is an isometry provided that the Hermitian form has real coefficients).

An elliptic isometry g is called *regular* if any of its matrix representatives $A \in \mathrm{U}(n, 1)$ has distinct eigenvalues. The eigenvalues of a matrix $A \in U(n, 1)$ representing an elliptic isometry g have modulus one. Exactly one of these eigenvalues has eigenvectors in V^- (projecting to a fixed point of g in $\mathrm{H}^n_{\mathbb{C}}$), and such an eigenvalue will be called *of negative type*. Regular elliptic isometries have an isolated fixed point in $\mathrm{H}^n_{\mathbb{C}}$. A non regular elliptic isometry is called *special*. Among the special elliptic isometries are the following two types (which exhaust all special elliptic types when $n = 2$):

1. A *complex reflection* is an elliptic isometry $g \in \mathrm{PU}(n, 1)$ whose fixed-point set is a complex $(n-1)$-plane. In other words, any lift such an isometry to U(n,1) has n equal eigenvalues, one of which has negative type.

2. A *complex reflection in a point* is an elliptic isometry having a lift with n equal eigenvalues, the remaining one being of negative type. In other words, such an isometry is conjugate to some $\lambda\mathrm{Id} \in U(n)$, where $U(n)$ is the stabilizer of the origin in the ball model. Complex reflections in a point with order 2 are also called *central involutions*; these are the symmetries that give $\mathrm{H}^n_{\mathbb{C}}$ the structure of a symmetric space).

A parabolic isometry is called *strictly parabolic* (or a *Heisenberg translation*) if it has a unipotent lift in $\mathrm{U}(n, 1)$. If not, it is called *screw-parabolic*, and it can be uniquely decomposed as $g = pe = ep$ with p strictly parabolic and e elliptic (see Theorem 2.3 below). In dimensions $n > 1$, strictly parabolic isometries are either *2-step* or *3-step* unipotent, according to whether the minimal polynomial of their unipotent lift is $(X-1)^2$ or $(X-1)^3$ (see section 3.4 of [ChGr]). 2-step unipotents are also called *vertical Heisenberg translations*, and 3-step unipotents are then *non-vertical Heisenberg translations*.

2.2 Models in dimension 2

2.2.1 The ball model of $\mathrm{H}^2_{\mathbb{C}}$

The ball model of $\mathrm{H}^2_{\mathbb{C}}$ arises from the choice of Hermitian form

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It is classical, and we refer the reader to chapter 3 of [G]. We only emphasize the following fact: any elliptic isometry of $\mathrm{H}^2_{\mathbb{C}}$ is conjugate to one given in ball coordinates by $(z_1, z_2) \mapsto (e^{i\alpha}z_1, e^{i\beta}z_2)$ for some $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$. A matrix representative in SU(2,1) for the latter is:

$$E_{(\alpha,\beta)} = \begin{bmatrix} e^{i(2\alpha-\beta)/3} & 0 & 0 \\ 0 & e^{i(2\beta-\alpha)/3} & 0 \\ 0 & 0 & e^{-i(\alpha+\beta)/3} \end{bmatrix}. \tag{2.2}$$

2.2.2 The Siegel model of $\mathrm{H}^2_{\mathbb{C}}$

In the presence of parabolic elements, it is very convenient to use the Siegel domain, as the stabilizer the point at infinity (see below) consists of upper triangle matrices. As mentioned in the previous section, this model corresponds to the Hermitian form given by the matrix:

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

In this model, any point $m \in \mathrm{H}^2_{\mathbb{C}}$ admits a unique lift to $\mathbb{C}^{2,1}$ of the following form, called its *standard lift*:

$$\mathbf{m} = \begin{bmatrix} (-|z|^2 - u + it)/2 \\ z \\ 1 \end{bmatrix} \text{ with } (z,t,u) \in \mathbb{C} \times \mathbb{R} \times]0, \infty[. \tag{2.3}$$

The triple (z,t,u) is called the *horospherical coordinates* of m. The boundary of $\mathrm{H}^2_{\mathbb{C}}$ is the level set $\{u = 0\}$, together with the distinguished point at infinity, given by

$$q_\infty \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The boundary $\partial \mathrm{H}^2_{\mathbb{C}} \setminus \{q_\infty\}$ is a copy of the Heisenberg group $\mathfrak{N}$ of dimension 3, with group law given in $[z,t]$ coordinates by:

$$[z_1, t_1] \cdot [z_2, t_2] = [z_1 + z_2, t_1 + t_2 + 2\mathrm{Im}(z_1 \overline{z_2})]. \tag{2.4}$$

The stabilizer of q_∞ in $\mathrm{SU}(2,1)$ consists of upper triangular matrices, and is generated by the following 3 types of isometries: Heisenberg translations $T_{[z,t]}$ ($(z,t) \in \mathbb{R} \times \mathbb{C}$), Heisenberg rotations R_θ ($\theta \in \mathbb{R}/2\pi\mathbb{Z}$) and Heisenberg dilations D_r ($r > 0$), where:

$$T_{[z,t]} = \begin{bmatrix} 1 & -\overline{z} & -(|z|^2 - it)/2 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \quad R_\theta = \begin{bmatrix} e^{-i\theta/3} & 0 & 0 \\ 0 & e^{2i\theta/3} & 0 \\ 0 & 0 & e^{-i\theta/3} \end{bmatrix} \quad D_r = \begin{bmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/r \end{bmatrix}. \tag{2.5}$$

In Heisenberg coordinates, they correspond respectively to the following transformations

- $T_{[z,t]}$ is the left multiplication by $[z,t]$,
- R_θ is given by $[w,s] \longmapsto [e^{i\theta}w, s]$,
- D_r is the Heisenberg dilation $[w,s] \longmapsto [rw, r^2 s]$.

Note that Heisenberg translations and rotations preserve each horosphere based at q_∞ whereas Heisenberg dilations permute horospheres based at q_∞. We will denote by $\mathrm{Isom}(\mathfrak{N})$ the non-loxodromic stabilizer of q_∞ in $\mathrm{SU}(2,1)$, which is generated by the $T_{[z,t]}$ and R_θ. The notation $\mathrm{Isom}(\mathfrak{N})$ comes from the fact it is the isometry group of the *Cygan metric*, which we will not use here (see [FalPar]). The group $\mathrm{Isom}(\mathfrak{N})$ consists exactly of those matrices of the form:

$$P_{(z,t,\theta)} = T_{[z,t]} R_\theta = \begin{bmatrix} e^{-i\theta/3} & -e^{2i\theta/3}\overline{z} & -e^{-i\theta/3}(|z|^2 - it)/2 \\ 0 & e^{2i\theta/3} & e^{-i\theta/3} z \\ 0 & 0 & e^{-i\theta/3} \end{bmatrix} \tag{2.6}$$

In terms of these parameters, we obtain representatives of the various parabolic conjugacy classes a parabolic map P is conjugate in PU(2,1) to

- $P_{1,0,0} = T_{[1,0]}$ if it is 3-step unipotent,
- $P_{0,1,\theta}$ if it is 2-step unipotent,
- $P_{0,1,\theta}$ for some non zero $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ if it is screw parabolic.

Note that screw and 2-step unipotent parabolic shere the property of having a stable complex line. A direct computation gives the following group law for the non-loxodromic stabilizer of q_∞ in (z,t,θ)-coordinates:

Lemma 2.1. *For all $z_1, z_2 \in \mathbb{C}$, $t_1, t_2 \in \mathbb{R}$ and $\theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}$:*

$$P_{(z_1,t_1,\theta_1)} P_{(z_2,t_2,\theta_2)} = P_{(z_1 + e^{i\theta_1} z_2, t_1 + t_2 + 2\mathrm{Im}(z_1 \bar{z}_2 e^{-i\theta_1}), \theta_1 + \theta_2)}$$

In particular, for all $z \in \mathbb{C}$, $t \in \mathbb{R}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$:

$$P^{-1}_{(z,t,\theta)} = P_{(-ze^{-i\theta}, -t, -\theta)}$$

Corollary 2.1. *Given $z_1, z_2 \in \mathbb{C}$, $t_1, t_2 \in \mathbb{R}$ and $\theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}$:*

$$P_{(z_1,t_1,\theta_1)} \text{ and } P_{(z_2,t_2,\theta_2)} \text{ commute} \iff \left(z_1 + e^{i\theta_1} z_2 = z_2 + e^{i\theta_2} z_1 \text{ and } \mathrm{Im}(z_1 \bar{z}_2 e^{-i\theta_1}) = \mathrm{Im}(z_2 \bar{z}_1 e^{-i\theta_2}) \right).$$

2.3 Invariant objects for parabolics

The Siegel model is very well adapted to describe the action of parabolic isometries. To do so, let us give a few more details on the structure of the boundary of $\mathrm{H}^2_{\mathbb{C}}$. The boundary of $\mathrm{H}^2_{\mathbb{C}}$ is equipped with a contact structure, which is given in Heisenberg coordinates as the kernel of the 1-form

$$\alpha = dt - 2xdy + 2ydx.$$

Viewing the boundary of $\mathrm{H}^2_{\mathbb{C}}$ as the one point compactification of the Heisenberg group $\mathfrak{N} = \mathbb{C} \ltimes \mathbb{R}$ gives to the set of complex lines throught ∞ the stucture of the affine space $\mathbb{C}$. Indeed, for any $z \in \mathbb{C}$ there exists a unique complex line through ∞ which contains ∞ and the point $[z, 0]$ (which is polar to the vector $\begin{bmatrix} \overline{z} & 1 & 0 \end{bmatrix}^T$). This induces a projection $\Pi : \overline{\mathrm{H}^2_{\mathbb{C}}} \setminus \infty \longmapsto \mathbb{C}$ of which fibers are the complex lines through ∞. In the boundary, this projection is just the vertical projection

$$[z, t] \longmapsto [z, 0]. \tag{2.7}$$

A *fan* through ∞ is the inverse image of an affine line in $\mathbb{C}$ by the projection Π. In view of (2.7), if L is an affine line in $\mathbb{C}$, the fan $\Pi^{-1}(L)$ intersects the boundary of $\mathrm{H}^2_{\mathbb{C}}$ along the vertical plane containing L in the Heisenberg group. A general fan is the image of a fan through ∞ by an element of PU(2,1). These objects were defined by Goldman and Parker in [GoP] (see also chapter 4 of [G]). As stated in [GoP], fans enjoy a double foliation, by real planes and complex lines. Let us make this foliation explicit in the case of fans through ∞ in $\mathrm{H}^2_{\mathbb{C}}$.

1. First, the foliation by complex lines is given by the fibers of Π above the affine line L. In the boundary, this foliation correspond to the foliation of the vertical plane above L by vertical lines.

2. The boundary of F is the vertical plane above the affine line $L \subset \mathbb{C}$, then consider a point $m \in \partial F$. The contact plane at m intersects ∂F along an affine line L' in the Heisenberg group which is the boundary of a real plane (see [G, GoP]). Then ∂F is foliated by the family of lines parallel to L' contained in F. In other words, the foliatation of ∂F is obtained by taking all lifts of L to the contact structure. All these lines are boundaries of real planes, and this foliation of ∂F extends inside $\mathrm{H}^2_{\mathbb{C}}$ as a foliation of F by real planes.

Note that an affine line in $\mathfrak{N}$ is the boundary of a real plane in $\mathrm{H}^2_{\mathbb{C}}$ if an only if it there exists m such that L contains m and L is contained in the contact plane at m (see chapter 4 of [G]). In particular, all lines contained in $\mathbb{C}$ through the origin are boundaries of real planes.

Lemma 2.2. *Let $L_{w,k}$ be the affine line in $\mathbb{C}$ parametrized by $L_{w,k} = \{w(s+ik), s \in \mathbb{R}\}$, for some unit modulus w and $k \geq 0$. Then the boundary foliation of the fan above $L_{w,k}$ is given by the lines parametrized in Heisenberg coordinates by $L_{t_0} = \{[w(s+ik), t_0 + 2sk], s \in \mathbb{R}\}$.*

Proof. The lines L_{t_0} all project onto L by the vertical projection. A tangent vector to L_{t_0} is given by $[\mathrm{Re}(w), \mathrm{Im}(w), 2k]$. Evaluating the 1-form α on this vector at the point of parameter $s = 1$, and using $|w| = 1$ shows that the line L_{t_0} is in the kernel of α at this point, thus in the contact plane. $\square$

Our interest in fans comes from the following fact.

Proposition 2.1. *Let P be a 3-unipotent parabolic isometry of $\mathrm{H}^2_{\mathbb{C}}$. There exists a unique fan F_P through ∞ such that*

1. *The fan F_P is stable under P.*

2. *Every leaf of the foliation of F_P by real planes is stable under P.*

Moreover, when P fixes ∞ and thus corresponds to left-translation by $[z, t]$ in Heisenberg coordinates, the fan F_P is the one above the affine line $L_{w,k}$, where $w = z/|z|$, and $k = t/(4|z|)$.

When P is the 3-step unipotent parabolic, we will refer to the fan F_P as *the invariant fan of P.*

Proof. Assume first that $P = T_{[1,0]}$ the left translation by $[1,0]$. It acts on the Heisenberg group as

$$[z,t] \longmapsto [z+1, t-2\mathrm{Im}(z)].$$

Therefore every vertical plane $\mathrm{Im}(z) = k$ is globally preserved. The real foliation of the fan corrsponding to the vertical plane $\mathrm{Im}(z) = k$ is given by the family of lines $L_{t_0} = \{[s+ik, t_0+2sk], s \in \mathbb{R}\}$. Since

$$[1,0][s+ik, t_0+2sk] = [s+1+ik, t_0+2(s-1)k],$$

we see that the real foliation of the vertical plane $\mathrm{Im}(z) = k$ is preserved by P if and only if $k = 0$, which is the result for this special P. The first part of the proposition is obtained by using the fact that any 3-step unipotent parabolic is conjugate in PU(2,1) to $T_{[1,0]}$. To check the last part, write $w = z/|z|$ and $k = t/(4|z|)$, then by a direct calculation

$$\begin{aligned}[z,t][w(s+ik, t_0+2sk] &= [w(|z|+s+ik), t_0+t+2sk-2|z|k] \\ &= [w(|z|+s+ik), t_0+2(|z|+s)k],\end{aligned}$$

which proves that the real leaves of the fan above $L_{w,k}$ are preserved. □

In fact the proof of Proposition 2.1 gives us more information, which we summarize in the following corollary.

Corollary 2.2. *Let P be a 3-step unipotent parabolic.*

1. *A real plane is stable under P if and only if it is a leaf of the real foliation of its invariant fan.*
2. *P is characterized by its invariant fan F, and its restriction to F.*

Proposition 2.2. *Let P be a parabolic isometry in PU(2,1).*

1. *If P is screw parabolic or 2-step unipotent, it has a unique invariant complex line.*
2. *If P is 3-step unipotent parabolic, then there exists a unique fan $\mathcal{F}$ centred at the fixed point of P which is stable by P and such that every leaf of the foliation of $\mathcal{F}$ by real planes is stable by P.*

The following lemma is a direct consequence of the previous results.

Lemma 2.3. *Let P_1 and P_2 be two parabolic isometries fixing ∞. Then P_1 and P_2 commute if and only if one of the following possibilities occurs.*

1. *Both P_1 and P_2 are either 2-step unipotent or screw parabolics with the same stable complex line.*
2. *Both P_1 and P_2 are 3-step unipotent with the same fixed point and their invariant fans interesect $\partial \mathrm{H}^2_{\mathbb{C}}$ along parallel vertical planes.*

Proof. We only prove the second part, the first one being classical. Two 3-step unipotent map fixing ∞ are respectively conjugate to the left translations by $[w,s]$ and $[z,t]$. Using (2.4), we see that these two translations commute if and only if $z\overline{w} \in \mathbb{R}$, which is equivalent to saying that their invariant fans are parallel. □

2.4 Eigenvalues and traces

The following classification of conjugacy classes in $\mathrm{U}(n,1)$ is due to Chen–Greenberg (Theorem 3.4.1 of [ChGr], where the real and quaternionic cases are treated as well):

Theorem 2.3 (Chen–Greenberg)**.** *(a) An elliptic element is semisimple, with eigenvalues of norm 1. Two elliptic elements are conjugate in $\mathrm{U}(n,1)$ if and only if they have the same eigenvalues and the same eigenvalue of negative type.*

(b) A loxodromic element is semisimple, with exactly $n-1$ eigenvalues of norm 1. Two loxodromic elements are conjugate in $\mathrm{U}(n,1)$ if and only if they have same eigenvalues.

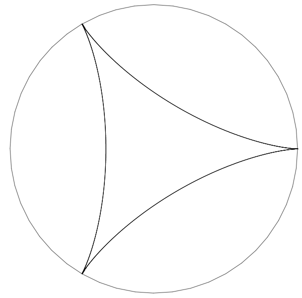


Figure 1: The null-locus of the polynomial f inscribed in the circle of radius 3 centered at the origin

(c) *A parabolic element is not semisimple, and all its eigenvalues have norm 1. It has a unique decomposition $g = pe = ep$ with p strictly parabolic and e elliptic. Two parabolic elements are conjugate in* $\mathrm{U}(n,1)$ *if and only if their strictly parabolic and elliptic components are conjugate.*

(d) *There are 2 classes of strictly parabolic elements for $n > 1$, the vertical Heisenberg translations and the non-vertical Heisenberg translations.*

Note that two elliptic elements may be conjugate in $\mathrm{GL}(n+1,\mathbb{C})$ but not in $\mathrm{U}(n,1)$ (if they have the same eigenvalues but different eigenvalues of negative type). It will be useful for future reference to have matrix representatives in SU(2,1) of the different conjugacy classes of isometries when $n = 2$.

1. Any elliptic isometry of $\mathrm{H}^2_{\mathbb{C}}$ is conjugate to one given in the ball model by the matrix $E_{(\alpha,\beta)}$ given in (2.2).
2. Any loxodromic isometry of $\mathrm{H}^2_{\mathbb{C}}$ is conjugate to one given in the Siegel model by the matrix $D_r R_\theta$ where D_r and R_θ are as in (2.5).
3. Any parabolic isometry of $\mathrm{H}^2_{\mathbb{C}}$ is conjugate to one given in the Siegel model by the matrix $P_{(z,t,\theta)} = T_{[z,t]}R_\theta$. The isometry associated to $P_{(z,t,\theta)}$ is unipotent if and only if $\theta = 0$ and 2-step unipotent if and only if $\theta = 0$ and $z = 0$.

As in the classical case of the Poincaré disc, the isometry type of an isometry is closely related to the trace of a lift to SU(2,1). The characteristic polynomial of a matrix A in SU(2,1) is given by:

$$\chi_A(X) = X^3 - z \cdot x^2 + \overline{z} \cdot x - 1, \text{ where } z = \mathrm{Tr}A. \tag{2.8}$$

Computing its discriminant, we obtain:

$$f(z) = Res(\chi_A, \chi'_A) = |z|^4 - 8Re(z^3) + 18|z|^2 - 27. \tag{2.9}$$

This function provides the following classification of holomorphic isometries via the trace of their lifts to SU(2,1) (see ch. 6 of [G]), which is analogous to the classical SL(2,$\mathbb{C}$) case. Denote by C_3 the set of cube roots of unity in $\mathbb{C}$.

Theorem 2.4 (Goldman)**.** *Let A be a matrix in $SU(2,1)$ and $g \in PU(2,1)$ be the associated isometry of $H^2_{\mathbb{C}}$. Then:*

- *g is regular elliptic* $\iff$ $f(\mathrm{Tr}(A)) < 0$.
- *g is loxodromic* $\iff$ $f(\mathrm{Tr}(A)) > 0$.
- *g is special elliptic or screw-parabolic* $\iff$ $f(\mathrm{Tr}(A)) = 0$ *and* $\mathrm{Tr}(A) \notin 3C_3$.
- *g is unipotent or the identity* $\iff$ $\mathrm{Tr}(A) \in 3C_3$.

The null-locus of the polynomial f can be seen in Figure 1.

We now focus on the special case of elements of SU(2,1) having real trace.

Proposition 2.3. *Let $A \in \mathrm{SU}(2,1)$ satisfy $\mathrm{Tr}A \in \mathbb{R}$. Then A has an eigenvalue equal to 1. More precisely:*

- *If A is loxodromic then A has eigenvalues $\{1, r, 1/r\}$ for some $r \in \mathbb{R} \setminus [-1, 1]$.*
- *If A is elliptic then A has eigenvalues $\{1, e^{i\theta}, e^{-i\theta}\}$ for some $\theta \in [0, \pi]$.*
- *If A is parabolic then A has eigenvalues $\{1, 1, 1\}$ or $\{1, -1, -1\}$*

Proof. If $\mathrm{Tr}A$ is real, then χ_A has real coefficients, therefore the eigenvalue spectrum of A is stable under complex conjugation. The result follows. □

Remark 1. (a) Any loxodromic or parabolic element of SU(2,1) with real trace is conjugate to its inverse; for elliptic isometries this is true under the additional assumption that the eigenvalue 1 has negative type. This follows from Proposition 2.3. An element of a group which is conjugate to its inverse is sometimes called *achiral* or *reversible* in the context of isometry groups. In [GonP], Gongopadhyay and Parker have studied and classified these isometries in $\mathrm{PU}(n, 1)$ for all $n \geqslant 1$.

(b) For later use, let us note that matrix representatives of the conjugacy classes of elements of PU(2,1) having a fixed point in $\overline{\mathrm{H}^2_{\mathbb{C}}}$ with associated positive eigenvalue are given by, in the notation of (2.2) and (2.5):

- $E_{(\theta,-\theta)}$ with $\theta \in \mathbb{R}$ (in the ball model) for elliptic classes,
- D_r with $r > 1$ (in the Siegel model) for loxodromic classes, and
- $P_{(z,t,0)}$ with $z \in \mathbb{C}$ and $t \in \mathbb{R}$ (in the Siegel model) for parabolic classes.

2.5 Antiholomorphic isometries

The following lemma is useful when computing with antiholomorphic isometries (see [FalPau] in the elliptic case, where the corresponding matrix was called a *Souriau matrix* for f). To simplify the statement, we consider a projective model (for example the ball or Siegel model) for which complex conjugation in affine coordinates $\sigma_0 : (z_1, ..., z_n) \mapsto (\overline{z_1}, ..., \overline{z_n})$ is an isometry (i.e. the Hermitian form has real coefficients).

Lemma 2.4. *Let f be an antiholomorphic isometry of $\mathrm{H}^n_{\mathbb{C}}$. Then there exists $M \in \mathrm{U}(n, 1)$ such that for any $m \in \mathrm{H}^n_{\mathbb{C}}$ with lift $\mathbf{m} \in \mathbb{C}^{n+1}$:*

$$f(m) = M \cdot \overline{\mathbf{m}}. \tag{2.10}$$

Any matrix $M \in \mathrm{U}(n, 1)$ with this property will be called a lift of f.

Proof. $f \circ \sigma_0$ is a holomorphic isometry, i.e. corresponds to an element of PU(n,1). Any of its lifts $M \in \mathrm{U}(n, 1)$ satisfies the required property. □

Definition 2.1.

- *Given a real reflection σ and an isometry $A \in \mathrm{PU}(n, 1)$, we say that σ* decomposes *A if $A = \sigma\tau$ for some real reflection τ (equivalently, $A = \tau'\sigma$ where $\tau' = \sigma\tau\sigma$ is also a real reflection).*
- *Given two isometries $A, B \in PU(n,1)$, we say that the pair (A, B) is* $\mathbb{R}$-decomposable *if there exists a real reflection which decomposes both A and B.*

As an application of Lemma 2.4, we obtain a necessary condition on a pair (A, B) to be $\mathbb{R}$-decomposable, namely that the commutator $[A, B]$ must have real trace. Note that the trace of an element of PU(n,1) is not well-defined in general. However, if A and B are in PU(n,1), then the matrix $[\tilde{A}, \tilde{B}]$ does not depend on the choice of lifts $\tilde{A}$ and $\tilde{B}$ made for A and B. This allows us to consider the condition of having real trace for a commutator.

Lemma 2.5. *If a pair $(A, B) \in PU(n,1) \times PU(n,1)$ is $\mathbb{R}$-decomposable, then the commutator $[A, B]$ is the square of an anti-holomorphic isometry.*

Proof. If $A = \sigma_1\sigma_2$ and $B = \sigma_2\sigma_3$ with each σ_i a real reflection, then $[A, B] = (\sigma_1\sigma_2)(\sigma_2\sigma_3)(\sigma_2\sigma_1)(\sigma_3\sigma_2) = (\sigma_1\sigma_3\sigma_2)^2$, where $\sigma_1\sigma_3\sigma_2$ is an antiholomorphic isometry. □

Corollary 2.5. *If a pair $(A, B) \in PU(n,1) \times PU(n,1)$ is $\mathbb{R}$-decomposable, then the commutator $[A, B]$ has real trace.*

Proof. First note that if f_1 and f_2 are antiholomorphic isometries with lifts M_1 and M_2, then $f_1 \circ f_2$ is holomorphic with lift $M_1\overline{M_2} \in \mathrm{U}(n,1)$. Using this fact together with Lemma 2.5, we see that if (A, B) is $\mathbb{R}$-decomposable, then the commutator can be lifted to SU(n,1) as $M\overline{M}$, with $M = M_1\overline{M_3}M_2$, where M_k is a lift of σ_k. As a consequence, we see that

$$\overline{\mathrm{Tr}(M\overline{M})} = \mathrm{Tr}(\overline{M}M) = \mathrm{Tr}(M\overline{M}) \in \mathbb{R}$$

□

Note that this necessary condition holds in any dimension n. However, to obtain a sufficient condition, we will need Lemma 2.4 below, which is false in dimensions $n > 2$.

Lemma 2.6. *Any antiholomorphic isometry of $\mathrm{H}^1_{\mathbb{C}}$ having a fixed point in $\mathrm{H}^1_{\mathbb{C}}$ is a real reflection.*

Proof. Let f be an antiholomorphic isometry of $\mathrm{H}^1_{\mathbb{C}}$ with a fixed point p. Then, for any point $q \neq p$, the angles $(f^{-1}(q), p, q)$ and $(q, p, f(q))$ are opposite (because f is antiholomorphic), therefore $f^{-1}(q) = f(q)$ (because f is an isometry), and f is an involution and fixes pointwise a geodesic. □

Lemma 2.7. *If g is an antiholomorphic isometry of $\mathrm{H}^2_{\mathbb{C}}$ which exchanges two points of $\overline{\mathrm{H}^2_{\mathbb{C}}}$, then g is an $\mathbb{R}$-reflection.*

Proof. Let p and q be the two points exchanged by g. Then g has a fixed point m on the geodesic (pq) (which is the midpoint of the geodesic segment $[pq]$ when p and q are in $\mathrm{H}^2_{\mathbb{C}}$). Then g stabilizes the complex line C spanned by p and q, as well as the complex line C' orthogonal to C at m. By Lemma 2.6, the restrictions of g to C and C' are real reflections, fixing geodesics γ and γ'. Now γ and γ' are geodesics contained in orthogonal complex lines, therefore they span a Lagrangian plane, which is fixed pointwise by g. □

Lemma 2.7 is false as soon as $n \geq 3$. Indeed, consider an element $M \in$ SU(n,1) given in the Siegel model by

$$M = \begin{bmatrix} (0) & & z \\ & U & \\ 1/\bar{z} & & (0) \end{bmatrix}, \tag{2.11}$$

where $z \in \mathbb{C}^\star$ and $U \in U(n-1)$ is a matrix such that $U\overline{U} \neq Id$, that is a non-symmetric matrix in $U(n-1)$ (such matrices only exist when $n > 2$). Then

$$M\overline{M} = \begin{bmatrix} 1 & & (0) \\ & U\overline{U} & \\ (0) & \dots & 1 \end{bmatrix} \neq Id$$

Therefore the antiholomorphic isometry assciated with M is not a real reflection, though it exchanges the two points corresponding to the first and last vectors in the canonical basis of $\mathbb{C}^{n+1}$, which are both on the boundary of $\mathrm{H}^n_{\mathbb{C}}$.

The following proposition describes which elements of PU(2,1) can arise as squares of antiholomorphic isometries.

Proposition 2.4. *Let $A \in PU(2,1)$ be an isometry admitting a lift $\tilde{A}$ to $SU(2,1)$ such that for any fixed point A in $\overline{\mathrm{H}^2_{\mathbb{C}}}$, the corresponding eigenvalue of $\tilde{A}$ is positive. Then A is the square of an antiholomorphic isometry unless A is 2-step unipotent parabolic.*

Proof. Being the square of an antiholomorphic isometry is preserved by conjugation, therefore we only need to prove the result for representatives of each conjugacy class. We first provide matrices M such that $M\overline{M}$ is equal to the lifts listed in Remark 1(b) in the elliptic and loxodromic cases.

1. If A is elliptic the following M satisfies the above condition:

$$M = \begin{bmatrix} 0 & e^{i\theta/2} & 0 \\ e^{-i\theta/2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. If A is loxodromic then we can take:

$$M = \begin{bmatrix} \sqrt{r} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{r} \end{bmatrix}.$$

We now examine the case where A is parabolic, thus unipotent in view of the assumptions. Assume that A is unipotent and that $A = \phi^2$ with ϕ an antiholomorphic isometry. First, ϕ has at least one fixed point in the closure of $\mathrm{H}^2_{\mathbb{C}}$ and ϕ^2 has exactly one fixed point there. This implies that ϕ has only one fixed point, which is the same as that of A. Conjugating by an element of PU(2,1), we can assume that this fixed point is q_∞. Consider a lift M of ϕ to SU(2,1) in the sense of Lemma 2.4. The fact that $M\overline{M}$ is a unipotent map fixing q_∞, which corresponds to the vector $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ implies that M is an upper triangular matrix in SU(2,1) with unit modulus diagonal entries. Therefore M is of the form $P_{(z,t,\theta)}$. Computing $M\overline{M} = P_{(z,t,\theta)}P_{(\bar{z},-t,-\theta)}$, we obtain that A has a lift to SU(2,1) of the form:

$$\begin{bmatrix} 1 & -\left(\overline{z} + ze^{-i\theta}\right) & -\overline{z}\left(z + \overline{z}e^{i\theta}\right) \\ 0 & 1 & \left(z + \overline{z}e^{i\theta}\right) \\ 0 & 0 & 1 \end{bmatrix} \tag{2.12}$$

This matrix can only be 3-step unipotent or the identity. This proves the result as there is only one PU(2,1)-conjugacy class of 3-step unipotents. ☐

2.6 Decomposing isometries

The following Proposition summarizes previously known results characterizing which real reflections decompose a given holomorphic isometry. Parts (e1-e3) are Proposition 2.4 of [FalPau], part (l) is Proposition 4(2) of [W2] (and follows from Proposition 3.1 of [FZ]), and part (p) also follows from Proposition 3.1 of [FZ]:

Proposition 2.5. *Let $A \in PU(2,1)$ and σ a real reflection with fixed $\mathbb{R}$-plane L.*

(e1) If A is a complex reflection with fixed complex line C, then: σ decomposes $A \iff L \cap C$ is a geodesic.

(e2) If A is a complex reflection in a point p_A, then: σ decomposes $A \iff p_A \in L$.

(e3) If A is regular elliptic with fixed point p_A and stable complex lines C_1, C_2, then: σ decomposes $A \iff p_A \in L$ and $L \cap C_i$ is a geodesic for $i = 1, 2$.

(l) If A is loxodromic then: σ decomposes $A \iff \sigma$ exchanges the 2 fixed points of A.

(p) If A is parabolic then: σ decomposes $A \iff \bar{L}$ contains the fixed point of A and σ preserves V, where V is the invariant complex line of A when A is 2-step or screw-parabolic, and V is the invariant fan F_A of A when A is 3-step parabolic (see Proposition 2.2).

Proof. Parts (e1-e3) are Proposition 2.4 of [FalPau], part (l) is Proposition 4(2) of [W2] (and follows from Proposition 3.1 of [FZ]). Let us prove part (p).

Denote by p the fixed point of A. Assume that $\mathbb{A} = \sigma\sigma'$ where σ' is another real reflection. Because both σ and σ' are involutions, either σ and σ' fix p, or there exists $q \neq p$ in $\partial\mathrm{H}^2_{\mathbb{C}}$ such that σ and σ' both swap p and q. In this case, A would fix two distinct points in $\partial\mathrm{H}^2_{\mathbb{C}}$, which is not possible for a parabolic. Thus the fixed real plane of σ contains p. Replacing p by V and using the fact that V is the unique stable complex line or invariant fan of P, we obtain that σ and σ' both preserve V.

Conversely, let σ be such a real reflection.

1. If A is 2-step or screw parabolic, the V is a complex line. The restriction of σ to V is an involution fixing a boundary point of V. As V is a copy of the Poincaré disc, this implies that the restriction $\sigma_{|V}$ is a symmetry about a geodesic, and therefore L intersects $\mathcal{V}$ along a geodesic γ, one of which endpoints is the fixed point of A. Let us call a the other endpoint of γ, and $b = A(a)$. It is a simple exercise to check that the composed isometry $A \circ \sigma$ exchanges a and b. As a consequence $P \circ \sigma$ is an antiholomorphic isometry that exchanges two points of $\partial \mathrm{H}^2_{\mathbb{C}}$. By Lemma 2.7, it is a real reflection and therefore σ decomposes P.

2. If A is 3-step unipotent, we may assume that A is a Heisenberg left translation. The map A acts on the vertical plane ∂F_A as a translation, and Lemma 2.8 below that $\sigma_{||\partial F_A}$ is a euclidean half-turn, about a point m. As a consequence, there exists a unique point m' in ∂F_A such that the restriction of A to ∂F_A is the product of the two half-turns about m and m'. Let Δ' be the line orthogonal to ∂F_A through m'. It is the boundary of a real plane, and we call σ' the real reflection about it. Then $\sigma \circ \sigma'$ fixes ∞ has ∞ as a unique fixed point in $\partial \mathrm{H}^2_{\mathbb{C}}$, and preserves every leaf of the fan F_A. Therefore A and $\sigma \circ \sigma'$ are both 3-step unipotent, have the same invariant fan on which they coincide. They are thus equal.

$\square$

Lemma 2.8. *Let F be a fan through ∞. A real reflection σ preserves F if and only if its fixed real plane intersects the Heisenberg group along an affine line orthogonal to the vertical plane ∂F. In this case, it acts on ∂F as an euclidean half-turn.*

Proof. This is a direct consequence of the fact that a real reflection about a real plane P acts on the Heisenberg group as the half-turn around the line ∂P (see [G], chapter 4). $\square$

Remark 2. The condition that σ preserves $\mathcal{V}$ can be made a little more explicit as follows.

- If $\mathcal{V}$ is a complex line, a real reflection preserve it if and only if its fixed real plane intersect $\mathcal{V}$ along a geodesic.
- If $\mathcal{V}$ is a real plane, then a real reflection preserve it if and only if its fixed real plane intersect $\mathcal{V}$ orthogonally along a geodesic. This means that $\mathcal{V}$ and the fixed real plane of σ are isometric to the pair given the the ball model by $\mathrm{H}^2_{\mathbb{R}} = \{(x_1, x_2), x_i \in \mathbb{R}\}$ and the real plane $\{(x_1, ix_2), x_i \in \mathbb{R}\}$.

We will need to characterize which real reflections decompose a given parabolic isometry, in other words sharpen part (p) above to a necessary and sufficient condition. Normalizing as in section 2.2 the parabolic fixed point to be q_∞ in the Siegel model, any parabolic isometry fixing q_∞ is of the form $P_{(z,t,\theta)}$ given in Equation 2.5. Denoting by L_0 the standard real plane (consisting of vectors with real coordinates), it is well-known that every real plane containing q_∞ is of the form $L_{(w,s,\alpha)} = P_{(w,s,\alpha)} L_0$ for some $w \in \mathbb{C}, s \in \mathbb{R}$ and $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, in other words that $\mathrm{Isom}(\mathfrak{N})$ acts transitively on the set of real planes containing q_∞ (see [FZ]). Denoting by σ_0 the real reflection fixing L_0, and $\sigma_{(w,s,\alpha)}$ the real reflection fixing $L_{(w,s,\alpha)}$, the result is the following:

Lemma 2.9. *Given $(z, t, \theta), (w, s, \alpha) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$:*

$$\sigma_{(w,s,\alpha)} \text{ decomposes } P_{(z,t,\theta)} \iff \zeta \in e^{i\phi/2 \pm \pi/2}\mathbb{R},$$

where: $\zeta = z + e^{i\theta}(w - e^{2i\alpha}\bar{w})$ and $\phi = \theta + 2\alpha$.

Proof. By definition of $\sigma_{(w,s,\alpha)}$ and Lemma 2.1:

$$\sigma_{(w,s,\alpha)} = P_{(w,s,\alpha)} \sigma_0 P^{-1}_{(w,s,\alpha)} = P_{(w,s,\alpha)} \overline{P^{-1}_{(w,s,\alpha)}} \sigma_0 = P_{(w,s,\alpha)} P_{(-e^{i\alpha}\bar{w},s,\alpha)} \sigma_0 = P_{(w - e^{2i\alpha}\bar{w}, 2s - 2\mathrm{Im}(w^2 e^{-2i\alpha}), 2\alpha)} \sigma_0.$$

Therefore: $\sigma_{(w,s,\alpha)}$ decomposes $P_{(z,t,\theta)} \iff P_{(z,t,\theta)} \sigma_{(w,s,\alpha)}$ is an involution $\iff M\overline{M} = \mathrm{Id}$, where: $M = P_{(z,t,\theta)} P_{(w - e^{2i\alpha}\bar{w}, 2s - 2\mathrm{Im}(w^2 e^{-2i\alpha}), 2\alpha)} = P_{(\zeta,u,\phi)}$ with ζ, ϕ as in the statement of the lemma (and the value of u is irrelevant). Then: $M\overline{M} = P_{(\zeta,u,\phi)} P_{(\bar{\zeta},-u,-\phi)} = P_{(\zeta + e^{i\phi}\bar{\zeta}, 2\mathrm{Im}(\zeta^2 e^{-i\phi}), 0)}$ and the result follows by noting that: $\zeta + e^{i\phi}\bar{\zeta} = 0 \iff \zeta^2 \in -e^{i\phi}\mathbb{R}^+ \iff \zeta \in e^{i\phi/2 \pm \pi/2}\mathbb{R}$. $\square$

3 Configurations of points and cross-ratios

3.1 Triples of points

Definition 3.1. *Given a triple (p_1, p_2, p_3) of distinct points in $\mathrm{H}^n_{\mathbb{C}} \cup \partial \mathrm{H}^n_{\mathbb{C}}$ and lifts $P_i \in \mathbb{C}^{n,1}$ of the p_i, the ratio*

$$T(p_1, p_2, p_3) = \frac{\langle P_1, P_2 \rangle \langle P_2, P_3 \rangle \langle P_3, P_1 \rangle}{\langle P_1, P_3 \rangle \langle P_3, P_2 \rangle \langle P_2, P_1 \rangle} \tag{3.1}$$

does not depend on the lifts P_i. We will call $T(p_1, p_2, p_3)$ the triple-ratio of (p_1, p_2, p_3).

Note that $T(p_1, p_2, p_3)$ is also well-defined if 2 or more of the points are equal in $\mathrm{H}^n_{\mathbb{C}}$ (but not in $\partial \mathrm{H}^n_{\mathbb{C}}$).

Observe that holomorphic isometries (elements of PU$(n,1)$) clearly preserve the triple-ratio, whereas for any antiholomorphic isometry g, we have $T(g(p_1), g(p_2), g(p_3)) = \overline{T(p_1, p_2, p_3)}$. The triple-ratio is related to the classical Cartan angular invariant $\mathbb{A}$ (see [C]) and Brehm's shape invariant σ (see [Br]) for triangles as follows.

- The Cartan angular invariant of three points $p_1, p_2, p_3 \in \partial \mathrm{H}^n_{\mathbb{C}}$ is defined as:

  $$\mathbb{A}(p_1, p_2, p_3) = \arg(-\langle P_1, P_2 \rangle \langle P_2, P_3 \rangle \langle P_3, P_1 \rangle).$$

  It relates to the triple-ratio by

  $$T(p_1, p_2, p_3) = e^{2i\mathbb{A}(p_1, p_2, p_3)}.$$

- Brehm's shape invariant σ of three points in $\mathrm{H}^n_{\mathbb{C}}$ is related to the normalized triple product

  $$\tilde{T}(p_1, p_2, p_3) = \frac{\langle P_1, P_2 \rangle \langle P_2, P_3 \rangle \langle P_3, P_1 \rangle}{\langle P_1, P_1 \rangle \langle P_2, P_2 \rangle \langle P_3, P_3 \rangle}.$$

  Namely, $\sigma = -\mathrm{Re}(\tilde{T})$. Note that:

  $$T(p_1, p_2, p_3) = \frac{\tilde{T}(p_1, p_2, p_3)}{\tilde{T}(p_1, p_3, p_2)}.$$

We refer the reader to chapter 7 of [G] for classical properties of the Cartan invariant. Note in particular that the Cartan invariant satisfies $\mathbb{A}(p_1, p_2, p_3) \in [-\pi/2, \pi/2]$, and that $\mathbb{A}(p_1, p_2, p_3) = \pm\pi/2$ (resp. $\mathbb{A}(p_1, p_2, p_3) = 0$) if and only if the three points are contained in a complex line (resp. a real plane). Also, the Cartan invariant classifies triples of pairwise distinct points in $\partial \mathrm{H}^n_{\mathbb{C}}$ up to holomorphic isometries.

The following classification of triples of points in $\mathrm{H}^n_{\mathbb{C}}$ is due to Brehm ([Br]).

Theorem 3.1. *(Brehm) Let (x_1, x_2, x_3), (y_1, y_2, y_3) be two triples of points in $\mathrm{H}^n_{\mathbb{C}}$. There exists $g \in \mathrm{Isom}(\mathrm{H}^n_{\mathbb{C}})$ such that $g(x_i) = y_i$ (for $i = 1, 2, 3$) if and only if:*

- *$d(x_1, x_2) = d(y_1, y_2)$, $d(x_2, x_3) = d(y_2, y_3)$, $d(x_1, x_3) = d(y_1, y_3)$ and $T(x_1, x_2, x_3) = T(y_1, y_2, y_3)$, in which case g is holomorphic, or*
- *$d(x_1, x_2) = d(y_1, y_2)$, $d(x_2, x_3) = d(y_2, y_3)$, $d(x_1, x_3) = d(y_1, y_3)$ and $T(x_1, x_2, x_3) = \overline{T(y_1, y_2, y_3)}$, in which case g is antiholomorphic.*

In fact Brehm's formulation is slightly different as he considers $-\mathrm{Re}(\tilde{T})$ instead of $\tilde{T}$ (so his statement doesn't include our 2 cases). This is equivalent because the norm of $\tilde{T}$ is determined by the 3 side-lengths.

3.2 The complex cross-ratio

The following definition is due in this form to Goldman ([G]) (following Koranyi and Reimann ([KR])) in the case of boundary points:

Definition 3.2. *Let (p_1, p_2, p_3, p_4) be a quadruple of distinct points in $\mathrm{H}^n_{\mathbb{C}} \cup \partial \mathrm{H}^n_{\mathbb{C}}$. The quantity defined by*

$$\mathbf{X}(p_1, p_2, p_3, p_4) = \frac{\langle P_3, P_1 \rangle \langle P_4, P_2 \rangle}{\langle P_4, P_1 \rangle \langle P_3, P_2 \rangle} \tag{3.2}$$

does not depend on the choice of lifts P_i of the p_i's, and is called the complex cross-ratio of (p_1, p_2, p_3, p_4).

Note that $\mathbf{X}(p_1,p_2,p_3,p_4)$ is also well-defined when some of the 4 points coincide, as long as at most 2 of them coincide in $\partial \mathrm{H}^n_{\mathbb{C}}$.

The complex cross-ratio of boundary points has been studied in detail in [G] (pp. 224–228), to which we refer the reader for more details. As for the triple-ratio, it is a direct observation that holomorphic isometries preserve $\mathbf{X}$ whereas antiholomorphic ones change it to its complex conjugate. One of our main tools will be finding conditions under which such a cross-ratio is real, in the spirit of the following result (Theorem 7.2.1 of [G]):

Theorem 3.2 (Goldman). *Let (p_1,p_2,p_3,p_4) be a quadruple of distinct points in $\partial \mathrm{H}^n_{\mathbb{C}}$. Then $\mathbf{X}(p_1,p_2,p_3,p_4)$ is real and positive if and only if there exists a real reflection ϕ such that $\phi : p_1 \leftrightarrow p_2$ and $p_3 \leftrightarrow p_4$.*

Note that, if there exists such a real reflection ϕ, then:

$$\mathbf{X}(p_1,p_2,p_3,p_4) = \overline{\mathbf{X}(\phi(p_1),\phi(p_2),\phi(p_3),\phi(p_4))} = \overline{\mathbf{X}(p_2,p_1,p_4,p_3)} \tag{3.3}$$

Going back to the definition of $\mathbf{X}$, it is straightforward that $\mathbf{X}(p_2,p_1,p_4,p_3) = \mathbf{X}(p_1,p_2,p_3,p_4)$, and we see that the condition that $\mathbf{X} \in \mathbb{R}$ is indeed necessary. In [G] the assumption that $\mathbf{X} > 0$ is omitted, but it must be added for the following reason. $\mathbf{X}$ is related to triple products by:

$$\mathbf{X}(p_1,p_2,p_3,p_4) = \frac{\langle P_1,P_2\rangle\langle P_2,P_3\rangle\langle P_3,P_1\rangle}{\langle P_1,P_2\rangle\langle P_2,P_4\rangle\langle P_4,P_1\rangle} \cdot \frac{|\langle P_4,P_2\rangle|^2}{|\langle P_3,P_2\rangle|^2} \tag{3.4}$$

Because the Cartan invariant belongs to $[-\pi/2,\pi/2]$, we see that if $\mathbf{X}$ is real and negative the two triple products $\langle P_1,P_2\rangle\langle P_2,P_i\rangle\langle P_i,P_1\rangle$ $(i=3,4)$ must have arguments either both equal to $\pi/2$ or both equal to $-\pi/2$. This means that p_3 and p_4 belong to the complex line spanned by p_1 and p_2, and are on the same side of the geodesic (p_1p_2). See Proposition 2 of [KR] and property # 7 on p. 226 of [G]. However, if p_1,p_2,p_3,p_4 are in such a configuration then there cannot exist a real reflection ϕ such that $\phi : p_1 \leftrightarrow p_2$ and $p_3 \leftrightarrow p_4$. Indeed, if a real reflection preserves a complex line then it acts on it by reflection in a geodesic.

The following basic observation will allow us to project orthogonally onto the complex sides of the quadrilateral (p_1,p_2,p_3,p_4).

Lemma 3.1. *Let $p_1,p_2,p_3,p_4 \in \overline{\mathrm{H}^n_{\mathbb{C}}}$ with $p_1 \neq p_2$, and let π_{12} denote orthogonal projection onto the complex line L_{12} spanned by p_1 and p_2. Then: $\mathbf{X}(p_1,p_2,p_3,p_4) = \mathbf{X}\left(p_1,p_2,\pi_{12}(p_3),\pi_{12}(p_4)\right)$.*

Proof. Let c_{12} be a polar vector for L_{12}, normalized so that $\langle c_{12},c_{12}\rangle = 1$. Then π_{12} is the projectivization of the linear projection in $\mathbb{C}^{2,1}$ given by: $\Pi_{12}(z) = z - \langle z,c_{12}\rangle c_{12}$. Then, for any point p in $\mathrm{H}^n_{\mathbb{C}} \cup \partial \mathrm{H}^n_{\mathbb{C}}$:

$$\langle \Pi_{12}(p),p_i\rangle = \langle p - \langle p,c_{12}\rangle c_{12},p_i\rangle = \langle p,p_i\rangle \ (i=1,2).$$

The result follows by substituting p_3 and p_4 in this expression. (Here and whenever it is convenient we will slightly abuse notation by using the same letter for points in $\mathbb{C}P^n$ and their lifts to $\mathbb{C}^{n,1}$; we will however insist that lifts of points inside $\mathrm{H}^n_{\mathbb{C}}$ have norm -1 and lifts of points outside $\overline{\mathrm{H}^n_{\mathbb{C}}}$ have norm 1.) $\square$

3.3 Cross-ratios and real reflections

Recall that the classical cross-ratio of 4 distinct points in $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ is defined by (see for instance [G]):

$$[z_1,z_2;z_3,z_4] = \frac{(z_4-z_1)(z_3-z_2)}{(z_4-z_2)(z_3-z_1)} \tag{3.5}$$

It is invariant under the diagonal action of $\mathrm{PGL}(2,\mathbb{C})$, and is real if and only if the 4 points are cocylic or collinear. Moreover, $[z_1,z_2;z_3,z_4]$ is positive if and only if the pairs (z_1,z_2) and (z_3,z_4) do not separate each other on the common circle/line. The KR cross-ratio generalizes the classical cross-ratio in the following sense:

Lemma 3.2. *If p_1,p_2,p_3,p_4 lie in a common complex line $C \subset \mathbb{C}P^n$, then*

$$\mathbf{X}(p_1,p_2,p_3,p_4) = [\sigma(p_1),\sigma(p_2);p_4,p_3]$$

where σ denotes inversion in the boundary circle of C.

Proof. Applying if necessary an element of PU(n,1), we assume that the complex line C containing the p_i's is the first coordinate axis of $\mathbb{C}^n$ (seen as an affine chart of $\mathbb{C}P^n$) in the ball model. Each point of C has a lift to $\mathbb{C}^{n,1}$ of the form $[z, 0, ..., 0, 1]$, and in these coordinates, σ is given by $z \longmapsto 1/\bar{z}$. We lift each p_i as a vector $P_i = [z_i, 0, ..., 0, 1]^T$ (the standard lift in the ball model), and compute:

$$\mathbf{X}(p_1, p_2, p_3, p_4) = \frac{(z_3\overline{z_1} - 1)(z_4\overline{z_2} - 1)}{(z_4\overline{z_1} - 1)(z_3\overline{z_2} - 1)} = \frac{(z_3 - 1/\overline{z_1})(z_4 - 1/\overline{z_2})}{(z_3 - 1/\overline{z_2})(z_4 - 1/\overline{z_1})} = [\sigma(z_1), \sigma(z_2); z_4, z_3]$$

□

From Lemma 3.1 and 3.2 and the properties of the classical cross-ratio we obtain the following reality condition for $\mathbf{X}$:

Proposition 3.1. *Let $p_1, p_2, p_3, p_4 \in \overline{\mathrm{H}^n_{\mathbb{C}}}$ with $p_1 \neq p_2$, and let π_{12} denote orthogonal projection onto the complex line L_{12} spanned by p_1 and p_2. Then:*

1. *The complex cross-ratio $\mathbf{X}(p_1, p_2, p_3, p_4)$ is real and positive if and only if*
   - *either the points $p_1, p_2, \pi_{12}(p_3), \pi_{12}(p_4)$ are all equidistant from a geodesic γ in L_{12}, with p_1, p_2 on one side of γ and $\pi_{12}(p_3), \pi_{12}(p_4)$ on the other,*
   - *or the p_i are on the boundary of a common complex line and $\{p_1, p_2\}$ does not separate $\{p_3, p_4\}$ on this circle.*
2. *$\mathbf{X}(p_1, p_2, p_3, p_4)$ is real and negative if and only if p_1, p_2, p_3, p_4 are on the boundary of a common complex line and $\{p_1, p_2\}$ separates $\{p_3, p_4\}$ on this circle.*

Note that the second statement with the p_i on the boundary of $\mathrm{H}^n_{\mathbb{C}}$ is one half of Proposition 2 of [KR] and of property 7 on p. 226 of [G]. However the statement of the other half (so, our first statement) is different when some of the p_i are in $\mathrm{H}^n_{\mathbb{C}}$.

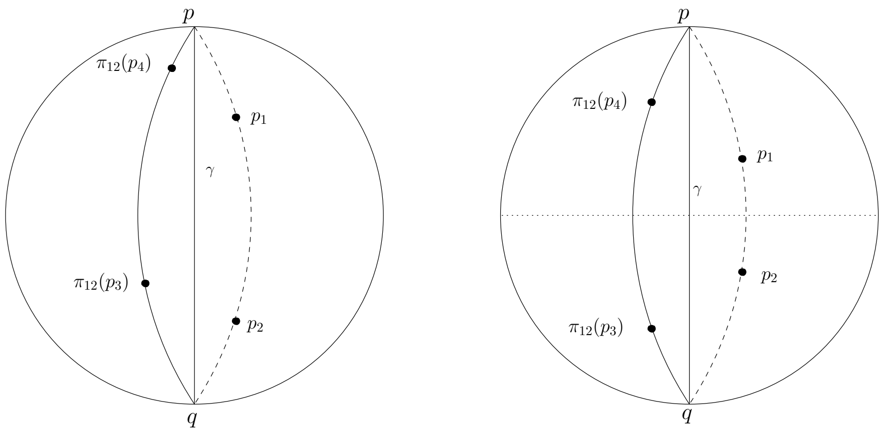
Proof. Normalize as in Lemma 3.2 so that the complex line L_{12} containing p_1 and p_2 is the first coordinate axis in the ball model of $\mathrm{H}^n_{\mathbb{C}}$. Denote by $z_1, \cdots, z_4$ the respective coordinates in this unit disk of the points $p_1, p_2, \pi_{12}(p_3)$, and $\pi_{12}(p_4)$. According to Lemma 3.2 and Proposition 3.1,

$$\begin{aligned} \mathbf{X}(p_1, p_2, p_3, p_4) \in \mathbb{R} \quad &\Longleftrightarrow \quad [1/\overline{z_1}, 1/\overline{z_2}; z_4, z_3] \in \mathbb{R} \\ &\Longleftrightarrow \quad 1/\overline{z_1}, 1/\overline{z_2}, z_4, z_3 \text{ lie on a common circle } C \text{ in } \mathbb{C}P^1 \end{aligned} \tag{3.6}$$

Note that z_1, z_2, z_3, z_4 are in the closed unit disk of $\mathbb{C}$, so that $1/\overline{z_1}$ and $1/\overline{z_2}$ are outside the open unit disk. In particular, either C intersects the unit circle in 2 points p and q, or C is the unit circle.

1. In view of (3.6), if $\mathbf{X}(p_1, p_2, p_3, p_4) < 0$ the two pairs (z_1, z_2) and $(1/\bar{z}_3, 1/\bar{z}_4)$ separate each other on C. The latter remark tells us that this is only possible when C is the unit circle. Therefore p_1, p_2, p_3 and p_4 all belong to the boundary of L_{12} and the pairs (p_1, p_2) and (p_3, p_4) separate each other on C.
2. Assume that $\mathbf{X}(p_1, p_2, p_3, p_4) > 0$.
   (a) If C is the unit circle, then $z_i = 1/\bar{z}_i$ for $i = 1, 2$ and thus $\mathbf{X}(p_1, p_2, p_3, p_4) = [z_1, z_2, z_4, z_3]$, which is positive if and only if $\{p_1, p_2\}$ do not separate $\{p_3, p_4\}$ in C.
   (b) If C intersects the unit circle in 2 points p and q, let γ denote the geodesic whose endpoints are p and q. Then $\pi_{12}(z_3)$ and $\pi_{12}(z_4)$ are on a hypercycle with endpoints p and q (the part of C which is inside the unit disk), and z_1, z_2 are on the image of this hypercycle by reflection in γ (this is the image of the other half of C by inversion in the unit circle), see the left part of Figure 2. Therefore $p_1, p_2, \pi_{12}(p_3), \pi_{12}(p_4)$ are all equidistant from γ, with p_1, p_2 on one side of γ and $\pi_{12}(p_3), \pi_{12}(p_4)$ on the other. □

The following result is the analogue of Theorem 3.2 in the case where the 4 points are inside $\mathrm{H}^2_{\mathbb{C}}$:



$\mathbf{X}(p_1, p_2, p_3, p_4)$ real $\quad$ $\mathbf{X}(p_1, p_2, p_3, p_4)$ real, $d(p_1, p_3) = d(p_2, p_4)$ and $d(p_1, p_4) = d(p_2, p_3)$

Figure 2: Projections of quadruples (p_1, p_2, p_3, p_4) to L_{12}

Theorem 3.3. *Let $p_1, p_2, p_3, p_4 \in \mathrm{H}^2_{\mathbb{C}}$ with $p_1 \neq p_2$ or $p_3 \neq p_4$. There exists a real reflection ϕ such that $\phi : p_1 \leftrightarrow p_2$ and $p_3 \leftrightarrow p_4$ if and only if: $\mathbf{X}(p_1, p_2, p_3, p_4) > 0$, $d(p_1, p_3) = d(p_2, p_4)$ and $d(p_1, p_4) = d(p_2, p_3)$.*

Proof. Let $p_1, p_2, p_3, p_4 \in \mathrm{H}^2_{\mathbb{C}}$ satisfy $\mathbf{X}(p_1, p_2, p_3, p_4) > 0$, $d(p_1, p_3) = d(p_2, p_4)$ and $d(p_1, p_4) = d(p_2, p_3)$. Consider the 2 ordered triples (p_1, p_2, p_3) and (p_2, p_1, p_4); these have equal side-lengths by our assumptions. We now compare the corresponding triple-ratios, $T_1 = T(p_1, p_2, p_3)$ and $T_2 = T(p_2, p_1, p_4)$. Observe that, taking lifts $P_1, ..., P_4$ of $p_1, ..., p_4$ in $\mathbb{C}^{2,1}$:

$$\begin{aligned}
T_2 &= \frac{\langle P_2, P_1\rangle\langle P_1, P_4\rangle\langle P_4, P_2\rangle}{\langle P_1, P_2\rangle\langle P_4, P_1\rangle\langle P_2, P_4\rangle} \\
&= \frac{\langle P_2, P_1\rangle\langle P_3, P_2\rangle\langle P_1, P_3\rangle}{\langle P_1, P_2\rangle\langle P_2, P_3\rangle\langle P_3, P_1\rangle} \cdot \frac{\langle P_2, P_3\rangle\langle P_1, P_4\rangle}{\langle P_2, P_4\rangle\langle P_1, P_3\rangle} \cdot \frac{\langle P_3, P_1\rangle\langle P_4, P_2\rangle}{\langle P_3, P_2\rangle\langle P_4, P_1\rangle} \\
&= \overline{T_1} \cdot \frac{\mathbf{X}(p_1, p_2, p_3, p_4)}{\overline{\mathbf{X}(p_1, p_2, p_3, p_4)}}
\end{aligned} \tag{3.7}$$

Now by assumption $\mathbf{X}(p_1, p_2, p_3, p_4) > 0$, so that $T_2 = \overline{T_1}$. Then, by Theorem 3.1, there exists an antiholomorphic isometry g sending p_1 to p_2, p_2 to p_1 and p_3 to p_4. But such an isometry must be an $\mathbb{R}$-reflection by Lemma 2.7, assuming that $p_1 \neq p_2$. If $p_1 = p_2$ but $p_3 \neq p_4$, the same argument applies, permuting the indices by the permutation (13)(24). $\square$

It may seem that, in the above proof, only the hypothesis that $\mathbf{X}$ is real was necessary. However recall that if $\mathbf{X}$ is negative, the four points are on the boundary of a complex line, and the two pairs $\{p_1, p_2\}$ and $\{p_3, p_4\}$ separate each other. In that case, as observed above, there cannot exist a real reflection ϕ such that $\phi(p_1) = p_2$ and $\phi(p_3) = p_4$.

4 Commutators, decomposable pairs and traces

4.1 Main results

Recall that a pair of holomorphic isometries $(A, B) \in \mathrm{PU}(n, 1)^2$ is said to be $\mathbb{R}$-*decomposable* if there exist 3 $\mathbb{R}$-reflections σ_1, σ_2 and σ_3 such that $A = \sigma_1\sigma_2$ and $B = \sigma_1\sigma_3$. We are now ready to prove our main result.

Theorem 4.1. *Let $A, B \in PU(2,1)$ be two isometries not fixing a common point in $\overline{\mathrm{H}^2_{\mathbb{C}}}$. Then: the pair (A, B) is $\mathbb{R}$-decomposable if and only if the commutator $[A, B]$ has a fixed point in $\overline{\mathrm{H}^2_{\mathbb{C}}}$ whose associated eigenvalue is real and positive.*

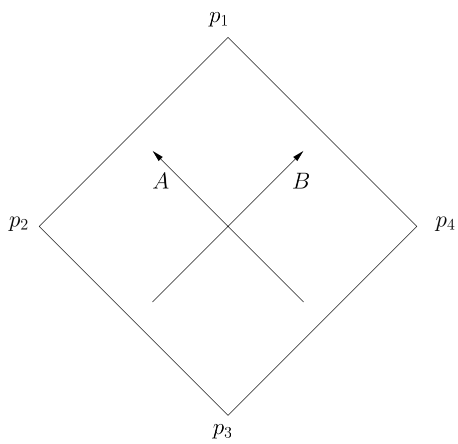


Figure 3: The 4-cycle associated to a fixed point of $[A, B]$

Note that the eigenvalues of elements of $\mathrm{PU}(2,1)$ are not well-defined (up to change of lift in $\mathrm{U}(2,1)$, or even $\mathrm{SU}(2,1)$), but the eigenvalues of a commutator are well-defined (the commutator itself is independent of lifts). Using Goldman's classification of isometries by trace and Proposition 2.3, this criterion can be reduced to the following:

Theorem 4.2. *Let $A, B \in PU(2,1)$ be two isometries not fixing a common point in $\overline{\mathrm{H}^2_{\mathbb{C}}}$. Then (A, B) is $\mathbb{R}$-decomposable if and only if:*

- *$[A, B]$ is loxodromic and $\mathrm{Tr}[A, B] > 3$, or*
- *$[A, B]$ is unipotent, or*
- *$[A, B]$ is elliptic, $\mathrm{Tr}[A, B] \in \mathbb{R}$ and the eigenvalue 1 of $[A, B]$ is of negative type.*

The extra assumption in the elliptic case means that the eigenvalue 1 corresponds to the fixed point of $[A, B]$. The other eigenvalues of $[A, B]$ are then $e^{\pm i\theta}$ for some θ, by the assumption that $\mathrm{Tr}[A, B] \in \mathbb{R}$ and Proposition 2.3.

Proof of Theorem 4.1: Let p_1 be a fixed point of $[A, B]$ in $\overline{\mathrm{H}^2_{\mathbb{C}}}$, P_1 a lift of p_1 in $\mathbb{C}^{2,1}$ and λ_1 the associated eigenvalue, so that: $[A, B]P_1 = \lambda_1 P_1$. Consider the cycle of four points defined as follows: let $P_2 = B^{-1}P_1$, $P_3 = A^{-1}P_2$ and $P_4 = B(P_3)$. First assume for simplicity that these 4 points are all distinct. Then opposite sides of the quadrilateral $(P_1P_2P_3P_4)$ are identified by A and B as on Figure 3. Note that $AP_4 = [A, B]P_1 = \lambda_1 P_1$. Then:

$$\mathbf{X}(p_2, p_4, p_1, p_3) = \frac{\langle P_1, P_2\rangle\langle P_3, P_4\rangle}{\langle P_3, P_2\rangle\langle P_1, P_4\rangle} = \frac{\langle \lambda_1^{-1}AP_4, AP_3\rangle\langle P_3, P_4\rangle}{\langle P_3, P_2\rangle\langle BP_2, BP_3\rangle} = \lambda_1^{-1}\frac{|\langle P_3, P_4\rangle|^2}{|\langle P_3, P_2\rangle|^2}. \tag{4.1}$$

This proves the following:

Lemma 4.1. *The cross ratio $\mathbf{X}(p_2, p_4, p_1, p_3)$ is real and positive if and only if $\lambda_1 \in \mathbb{R}^+$*

For the next step we use Theorem 3.2 or 3.3, depending on whether p_1 (and hence all other p_i's) is on $\partial\mathrm{H}^2_{\mathbb{C}}$ or in $\mathrm{H}^2_{\mathbb{C}}$. In the latter case, by construction of the 4 points we have $d(p_1, p_2) = d(p_3, p_4)$ and $d(p_1, p_4) = d(p_2, p_3)$. Therefore Theorem 3.3 tells us:

Lemma 4.2. *There exists a real reflection ϕ such that $\phi(p_1) = p_3$ and $\phi(p_2) = p_4$ if and only if the cross-ratio $\mathbf{X}(p_2, p_4, p_1, p_3)$ is real and positive.*

The following lemma concludes the proof of Theorem 4.1:

Lemma 4.3. *(A, B) is $\mathbb{R}$-decomposable $\iff$ there exists a real reflection ϕ such that $\phi : p_1 \leftrightarrow p_3$ and $p_2 \leftrightarrow p_4$.*

Indeed, if $A = \sigma_2\sigma_1$ and $B = \sigma_3\sigma_1$ then $\sigma_1 = \phi$ satisfies $\phi : p_1 \leftrightarrow p_3$ and $p_2 \leftrightarrow p_4$. Conversely, if such a ϕ exists then by Lemma 2.7 above, $A \circ \phi$ and $B \circ \phi$ are real reflections. Indeed they are both antiholomorphic, and $A \circ \phi$ (resp. $B \circ \phi$) exchanges p_1 and p_2 (resp. p_1 and p_4). Therefore (A, B) is $\mathbb{R}$-decomposable. □

Finally we examine the case where some of the 4 points $p_1, ..., p_4$ are equal. The cross-ratio appearing in Lemma 4.1 is well-defined as long as no three of the points are on $\partial \mathrm{H}^2_{\mathbb{C}}$ and equal, in which case all four of them would be equal, contradicting the assumption that A and B do not have a common fixed point. The proofs of Lemmas 4.2 and 4.3 carry through as long as $p_1 \neq p_3$ or $p_2 \neq p_4$.

Now if $p_1 = p_3 = p$ and $p_2 = p_4 = q$, then A and B both exchange p and q (if they are distinct). If p and q are in $\mathrm{H}^2_{\mathbb{C}}$ then A and B both fix the midpoint of the segment $[pq]$ which is again assumed not to be the case. If p and q are on $\partial \mathrm{H}^2_{\mathbb{C}}$ then A and B both have a fixed point on the geodesic line (pq) and act as a half-turn on the complex line spanned by p and q. In that case, on one hand by Proposition 2.5(a1-a3), any real plane containing (pq) decomposes A and B, therefore (A, B) is $\mathbb{R}$-decomposable. On the other hand, the commutator $[A, B]$ is loxodromic (because it acts by translation along the geodeesic (pq)) and has an eigenvalue equal to 1 (because A and B have a common eigenvector as they both preserve the complex line spanned by p, q), therefore $[A, B]$ has 3 real and positive eigenvalues. In particular, the conclusion of Theorem 4.1 holds in this case as well. □

4.2 Groups fixing a point

When A and B have a common fixed point in $\overline{\mathrm{H}^2_{\mathbb{C}}}$ the results are the following:

Proposition 4.1. *If $A, B \in PU(2,1)$ have a common fixed point in $\mathrm{H}^2_{\mathbb{C}}$ then (A, B) is $\mathbb{R}$-decomposable.*

Proposition 4.1 is the first part of Theorem 2.1 of [FalPau]; it essentially follows from the fact that, given two complex lines in $\mathbb{C}^2$, there exists a Lagrangian subspace intersecting each of them in a line (see Proposition 2.5 (e1-e3)).

Proposition 4.2. *Let $A, B \in PU(2,1)$ have a common fixed point on $\partial \mathrm{H}^2_{\mathbb{C}}$.*

- *(a) If A or B is loxodromic then: (A, B) is $\mathbb{R}$-decomposable $\iff$ $[A, B] = Id$.*
- *(b) If A and B are not loxodromic, and one of them is a complex reflection or parabolic with 2-step unipotent part then (A, B) is $\mathbb{R}$-decomposable.*
- *(c) If A and B are both 3-step unipotent then: (A, B) is $\mathbb{R}$-decomposable $\iff$ $[A, B] = Id$.*

Note that the 3 parts of Proposition 4.2 cover all cases where A and B have a common fixed point on $\partial \mathrm{H}^2_{\mathbb{C}}$, because screw-parabolic isometries have 2-step unipotent part (this follows from the fact that their elliptic and unipotent parts commute in the classification theorem of Chen-Greenberg, our Theorem 2.2(c)).

Proof of Proposition 4.2 (a): First assume that A and B are both loxodromic, with a common fixed point. Then from Proposition 2.5 (l): (A, B) is $\mathbb{R}$-decomposable $\iff$ A and B have the same fixed points $\iff$ $[A, B] = Id$.

Now assume that one of A, B is loxodromic but not the other, say A is loxodromic and B is not, and that A and B commute. In particular, A has distinct eigenvalues, therefore B must also be diagonalizable by the assumption that A and B commute. Therefore, B is a complex reflection, and by assumption its fixed line contains one of the endpoints of the axis of A. Since A and B commute then the fixed line of B must in fact contain the entire axis. Therefore any real reflection which decomposes A also decomposes B (by Proposition 2.5 (e2) and (l)). In particular, (A, B) is $\mathbb{R}$-decomposable.

Conversely, assume that A is loxodromic and B is not, and that (A, B) is $\mathbb{R}$-decomposable. Then by parts (l) and (p) of Proposition 2.5, B cannot be parabolic so it must be a complex reflection. Denote by C the complex line fixed by B, and by p_A, q_A the fixed points of A. Then by Proposition 2.5 (e2) and (l), there exists a real reflection σ with fixed $\mathbb{R}$-plane L such that σ exchanges p_A and q_A, and such that $L \cap C$ is a geodesic. Then σ preserves $\bar{C}$ which contains one of p_A and q_A, therefore $\bar{C}$ also contains the other, and A and B commute. □

Proof of Proposition 4.2 (b): Normalizing as in section 2.2 the parabolic fixed point to be q_∞ in the Siegel model, any parabolic isometry fixing q_∞ is of the form $P_{(z,t,\theta)}$ given in Equation (2.5). Denoting as earlier by

L_0 the standard real plane (consisting of vectors with real coordinates), recall from section 2.5 that every real plane containing q_∞ is of the form $L_{(w,s,\alpha)} = P_{(w,s,\alpha)}L_0$ for some $w \in \mathbb{C}, s \in \mathbb{R}$ and $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$. Denote by σ_0 the real reflection fixing L_0, and $\sigma_{(w,s,\alpha)}$ the real reflection fixing $L_{(w,s,\alpha)}$. By assumption A and B are of the form $P_{(z,t,\theta)}$, and at least one of them has z-coordinate 0, say $A = P_{(0,t_1,\theta_1)}$ and $B = P_{(z_2,t_2,\theta_2)}$. By lemma 2.9, given $(w, s, \alpha) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$:

$$\sigma_{(w,s,\alpha)} \text{ decomposes } A \iff \zeta_1 \in e^{i\phi_1/2\pm\pi/2}\mathbb{R}$$

$$\sigma_{(w,s,\alpha)} \text{ decomposes } B \iff \zeta_2 \in e^{i\phi_2/2\pm\pi/2}\mathbb{R}$$

where: $\zeta_1 = e^{i\theta_1}(w - e^{2i\alpha}\bar{w})$, $\zeta_2 = z_2 + e^{i\theta_2}(w - e^{2i\alpha}\bar{w})$, $\phi_1 = \theta_1 + 2\alpha$, and $\phi_2 = \theta_2 + 2\alpha$.

Note that, denoting $W = w - e^{2i\alpha}\bar{w}$, the first condition can be realized by taking $W = 0$, and that this in turn can be realized with any value of α, which can therefore be chosen to realize the second condition as well. □

Proof of Proposition 4.2 (c): By assumption A and B are both of the form $P_{(z,t,0)}$ from Equation (2.5) with $z \neq 0$, say $A = P_{(z_1,t_1,0)}$ and $B = P_{(z_2,t_2,0)}$. By lemma 2.9, given $(w, s, \alpha) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$:

$$\sigma_{(w,s,\alpha)} \text{ decomposes } A \iff z_1 + W \in e^{i\alpha+\pi/2}\mathbb{R}$$

$$\sigma_{(w,s,\alpha)} \text{ decomposes } B \iff z_2 + W \in e^{i\alpha+\pi/2}\mathbb{R},$$

where as above $W = w - e^{2i\alpha}\bar{w}$. Note that $W = e^{i\alpha}(e^{-i\alpha}w - e^{i\alpha}\bar{w}) \in e^{i\alpha+\pi/2}\mathbb{R}$ (and any value on this line can be realized), therefore there exists (w, s, α) such that $\sigma_{(w,s,\alpha)}$ simultaneously decomposes A and B if and only if $Arg(z_1) = Arg(z_2)$ mod. π, which is equivalent to $[A, B] = 1$ by Corollary 2.1. □

4.3 Extremal representations: $\mathbb{C}$-Fuchsian punctured torus groups

The case where the cross-ratio $\mathbf{X}(p_2, p_4, p_1, p_3)$ is real and negative corresponds to a rigidity phenomenon, giving the following result which holds in all dimensions:

Proposition 4.3. *If $[A, B]$ has a fixed point in $\overline{\mathrm{H}^n_{\mathbb{C}}}$ whose associated eigenvalue is real and negative, then the group $\langle A, B\rangle$ stabilizes a complex line L in $\mathrm{H}^n_{\mathbb{C}}$. Moreover in that case the corresponding fixed point of $[A, B]$ is on the boundary $\partial\mathrm{H}^n_{\mathbb{C}}$, so that $\langle A, B\rangle$ is a ($\mathbb{C}$-) Fuchsian punctured torus group.*

Proof. Indeed, with the notation of the proof of Theorem 4.1, we have the following variation of lemma 4.1:

$$\mathbf{X}(p_2, p_4, p_1, p_3) \in \mathbb{R}^- \iff \lambda_1 \in \mathbb{R}^-.$$

Now by assumption $\lambda_1 < 0$, so that by Proposition 3.1 p_1, p_2, p_3, p_4 are on the boundary of a common complex line L and the pairs (p_2, p_4) and (p_1, p_3) separate each other. But A sends (p_3, p_4) to (p_2, p_1) and B sends (p_2, p_3) to (p_1, p_4), so A and B each stabilize L. The ideal quadrilateral $(p_1, p_2, p_{,3}, p_4)$ is clearly disjoint from all its images by elements of the group generated by A and B. This proves that $\langle A, B\rangle$ is discrete. Note that in general this quadrilateral is not a fundamental domain for $\langle A, B\rangle$, as its images only tesselate the complex line L if $[A, B]$ is parabolic. See figure 4. □

One can also interpret Proposition 4.3 in terms of the Toledo invariant of the corresponding type-preserving representation of the fundamental group of the once-punctured torus. Given a representation ρ of the fundamental group of a surface Σ into $\mathrm{PU}(n, 1)$, the *Toledo invariant* $\tau(\rho)$ is defined as the integral over Σ of the pull-back of the Kähler 2-form on $\mathrm{H}^n_{\mathbb{C}}$ by an equivariant map $f : \tilde{\Sigma} \longrightarrow \mathrm{H}^n_{\mathbb{C}}$:

$$\tau(\rho) = \int_\Sigma f^\star\omega \tag{4.2}$$

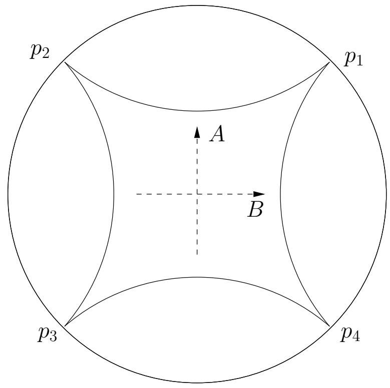


Figure 4: A fundamental fomain for the discrete punctured torus group in the case where $\lambda_1 < 0$

In the case where Σ is non-compact, one should be careful that this integral is well-defined. This is guaranteed for instance by the condition that the map f has finite energy (see [KM]). The existence of a such a finite energy equivariant map is guaranteed by the assumption that all peripheral loops on Σ are mapped to parabolics by the representation ([KM]). This assumption being made, it is possible to use an ideal triangulation of $\Sigma = \bigsqcup_i \Delta_i$ to compute the Toledo invariant, as it is done in [GP]. The result is that the Toledo invariant can be written as

$$\tau(\rho) = \sum_i \int_{f(\Delta_i)} \omega, \tag{4.3}$$

where each integral is computed over any 2-simplex with boundary $f(\partial\Delta_i)$. But the integral of ω over any 2-simplex with boundary $\partial\Delta$ is equal to twice the Cartan invariant of Δ (this is Theorem 7.1.11, page 218 of [G]). As a consequence, we see that for a type-preserving representation of the fundamental group of the once-punctured torus, the Toledo invariant is given by:

$$\tau(\rho) = 2\left(\mathbb{A}(p_1, p_2, p_3) + \mathbb{A}(p_1, p_3, p_4)\right), \tag{4.4}$$

with $p_1, ..., p_4$ defined as previously. On the other hand, taking arguments in equation (3.4) gives:

$$\arg\left(\mathbf{X}(p_2, p_4, p_1, p_3)\right) = \mathbb{A}(p_2, p_4, p_1) - \mathbb{A}(p_2, p_1, p_3) = \mathbb{A}(p_1, p_2, p_4) + \mathbb{A}(p_1, p_2, p_3) \tag{4.5}$$

Therefore, if the cross-ratio $\mathbf{X}(p_2, p_4, p_1, p_3)$ is negative, then $\mathbb{A}(p_1, p_2, p_4)$ and $\mathbb{A}(p_1, p_2, p_3)$ must both be equal to either $\pi/2$ or $-\pi/2$. In view of (4.4), this means that $|\tau(\rho)| = 2\pi$. But the Toledo invariant satisfies the Milnor-Wood inequality:

$$2\pi|\tau(\rho)| \leq 2g - 2 + p, \tag{4.6}$$

where equality holds if and only if the representation is discrete and preserves a complex line. In that case, the representation is called *extremal* (see [T, BIW, BILW]). In the case of the once punctured torus $g = p = 1$, and therefore if $|\tau(\rho)| = 2\pi$ the representation is extremal.

5 Groups generated by real reflections

We now use the criterion from Theorem 4.2 to show that various subgroups of $\mathrm{PU}(2,1)$ are generated by real reflections. More accurately this means that they are the index 2 holomorphic subgroup of a group of isometries generated by real reflections.

5.1 Mostow's lattices and other non-arithmetic lattices in $\mathrm{SU}(2,1)$

Mostow's lattices from [M] (revisited in [DFP]) as well as the new non-arithmetic lattices in $\mathrm{SU}(2,1)$ studied by Deraux, Parker and the first author (see [ParPau], [Pau], [DPP1] and [DPP2]) are all symmetric complex reflection triangle groups. This means that they are generated by 3 complex reflections R_1, R_2 and R_3 which are in a symmetric configuration in the sense that there exists an isometry J of order 3 such that $JR_iJ^{-1} = R_{i+1}$ (with i mod 3). These groups are in fact contained (with index 1 or 3, depending on the parameters) in the group Γ generated by R_1 and J.

It was shown in [DFP] that Mostow's lattices are generated by real reflections, and in [DPP2] this is extended to all symmetric complex reflection triangle groups. In both cases though, one finds an explicit real reflection which decomposes both holomorphic generators R_1 and J, which requires knowing explicit geometric properties of the group. Now the existence of such a real reflection follows immediately from the following consequence of Theorem 4.1:

Proposition 5.1. *If $R \in \mathrm{PU}(2,1)$ is a complex reflection or a complex reflection in a point, then for any $A \in \mathrm{PU}(2,1)$, the pair (R, A) is $\mathbb{R}$-decomposable.*

Proof. First assume that R is a complex reflection and let L denote its fixed complex line. Then $[R, A] = RR'$, where $R' = AR^{-1}A^{-1}$ is a complex reflection conjugate to R^{-1}; denote by L' the complex line fixed by R'. The extensions of L and L' to $\mathbb{C}P^2$ intersect at a unique point $p \in \mathbb{C}P^2$, unless $L = L'$, in which case $[R, A] = \mathrm{Id}$ and the result follows from Theorem 4.2. In general, p is fixed by both R and R', thus by $[R, A]$; we distinguish 2 cases, depending on whether p is in $\overline{\mathrm{H}^2_{\mathbb{C}}}$ or outside of $\overline{\mathrm{H}^2_{\mathbb{C}}}$.

1. First assume that $p \in \overline{\mathrm{H}^2_{\mathbb{C}}}$. Consider lifts of R and R' to SU(2,1) such that the lift of R has eigenvalues $e^{2i\phi}, e^{-i\phi}, e^{-i\phi}$ (where the rotation angle of R is 3ϕ), with p corresponding to a $e^{-i\phi}$-eigenvector, and likewise, the lift of R' has eigenvalues $e^{-2i\phi}, e^{i\phi}, e^{i\phi}$, with p corresponding to a $e^{i\phi}$-eigenvector of R'. Then p is a fixed point of $[R, A] = RR'$ in $\overline{\mathrm{H}^2_{\mathbb{C}}}$ with corresponding eigenvalue 1. The result follows from Theorem 4.1.

2. If p is outside of $\overline{\mathrm{H}^2_{\mathbb{C}}}$, then L and L' are ultraparallel and p is polar to their common perpendicular line, which we denote by $\tilde{L}$. The isometries R and R' act on $\tilde{L}$ by rotation through angle 3ϕ and -3ϕ respectively. It is an elementary fact from plane hyperbolic geometry that the product of two elliptic elements with opposite rotation angles and distinct fixed points must be hyperbolic (it follows for instance from Lemma 7.38.2 page 180 in [Be]). Therefore RR' is loxodromic; moreover its eigenvalue of positive type is 1, and therefore its eigenvalue spectrum must be either $\{r, 1, 1/r\}$ or $\{-r, 1, -1/r\}$ for some $r > 0$. Geometrically, this means that the rotation angle of the loxodromic isometry $[R, A]$ is 0 or π; it is 0 when $L = L'$ (as R and R' have opposite rotation angles) so by continuity it is 0 for any other configuration as well. This means that $[R, A]$ has only positive eigenvalues, and the result follows from Theorem 4.1.

Now assume that R is a complex reflection about a point. As above, we write $[R, A] = RR'$ with $R' = AR^{-1}A^{-1}$. The eigenvalues of R (respectively R') are $\{e^{i\phi}, e^{i\phi}, e^{-2i\phi}\}$ (resp. $\{e^{-i\phi}, e^{-i\phi}, e^{2i\phi}\}$), with $e^{2i\phi}$ (resp. $e^{-2i\phi}$) of negative type. This means that R (resp. R') acts on any complex line through its fixed point as a rotation of angle 3ϕ (resp. -3ϕ). Consider the complex line $\tilde{L}$ spanned by the fixed points of R and R'. The action of R and R' on $\tilde{L}$ is the same as in the second item above, and this leads to the same conclusion. □

5.2 Groups with more than two generators

Applying the criterion from Theorem 4.1 to a 2-generator subgroup of $\mathrm{PU}(2,1)$ is completely straightforward. For subgroups generated by more elements (e.g. Picard modular groups) one needs to be a bit more careful, as being generated by real reflections is stronger than having generators which are pairwise $\mathbb{R}$-decomposable.

The following observation gives a way to bridge this gap, however its hypotheses are in general too restrictive and in practice we will need some more work to show that a given group is generated by real reflections.

Lemma 5.1. *Let Γ be a subgroup of $\mathrm{PU}(2,1)$ generated by $A_1, ..., A_k$. If there exist real reflections $\sigma_1, ..., \sigma_{k+1}$ such that*

(a) $A_i = \sigma_i\sigma_{i+1}$ for $1 \leqslant i \leqslant k$, or

(b) $A_i = \sigma_1\sigma_{i+1}$ for $1 \leqslant i \leqslant k$

then Γ has index 2 in $\hat{\Gamma} = \langle\sigma_1, ..., \sigma_{k+1}\rangle$. In particular such a Γ is generated by real reflections.

Proof. In each case, pairwise products of the σ_i are in Γ, therefore Γ is the index 2 holomorphic subgroup of $\hat{\Gamma}$. ☐

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