

Rank 1 deformations of non-cocompact hyperbolic lattices

Samuel A. Ballas, Julien Paupert, Pierre Will

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Abstract

Let X be a negatively curved symmetric space and Γ a non-cocompact lattice in $\text{Isom}(X)$. We show that small, parabolic-preserving deformations of Γ into the isometry group of any negatively curved symmetric space containing X remain discrete and faithful (the cocompact case is due to Guichard). This applies in particular to a version of Johnson-Millson bending deformations, providing for all n infinitely many non-cocompact lattices in $\text{SO}(n, 1)$ which admit discrete and faithful deformations into $\text{SU}(n, 1)$. We also produce deformations of the figure-8 knot group into $\text{SU}(3, 1)$, not of bending type, to which the result applies.

1 Introduction

This paper concerns an aspect of the deformation theory of discrete subgroups of Lie groups, namely that of non-cocompact lattices in rank 1 semisimple Lie groups. More specifically, we consider the following questions, given a discrete subgroup Γ of a rank 1 Lie group H :

- (1) Does Γ admit any deformations in H ?
- (2) If so, do these deformations have any nice properties (e.g. remain discrete and faithful)?
- (3) What if we replace H with a larger Lie group G ?

Here we call *deformation* of Γ in H any continuous 1-parameter family of representations $\rho_t : \Gamma \rightarrow H$ (for t in some interval $(-\varepsilon, \varepsilon)$) satisfying $\rho_0 = \iota$ (the inclusion of Γ in H), and ρ_t not conjugate to $\rho_{t'}$ for any $t \neq t' \in (-\varepsilon, \varepsilon)$. We say that Γ is *locally rigid* in H if it does not admit any deformations into H .

When H is a semisimple real Lie group without compact factors there are a variety of general local rigidity results which we now outline. Weil proved in [W] that Γ is locally rigid in H if H/Γ is compact and H not locally isomorphic to $\text{SL}(2, \mathbb{R})$. Garland and Raghunathan extended this result to the case where Γ is a non-cocompact lattice in a rank 1 semisimple group H not locally isomorphic to $\text{SL}(2, \mathbb{R})$ or $\text{SL}(2, \mathbb{C})$ (Theorem 7.2 of [GR]).

The exclusion of $\text{SL}(2, \mathbb{R})$ and $\text{SL}(2, \mathbb{C})$ is necessary. Generically, lattices in $H = \text{SL}(2, \mathbb{R})$ admit many deformations in H . The identification of $\text{PSL}(2, \mathbb{R})$ with $\text{Isom}^+(\mathbb{H}_{\mathbb{R}}^2)$ allow us to relate lattices in $\text{SL}(2, \mathbb{R})$ with hyperbolic structures which are in turn parameterized by the classical Teichmüller space when Γ is a surface group. The case of *quasi-Fuchsian* deformations of a discrete subgroup Γ of $H = \text{SL}(2, \mathbb{R})$ into $G = \text{SL}(2, \mathbb{C})$ is also classical, well-studied, and well understood by the Bers simultaneous uniformization theorem [Bers]. In this setting, $\text{PSL}(2, \mathbb{C})$ can be identified with $\text{Isom}^+(\mathbb{H}_{\mathbb{R}}^3)$ and the discrete group $\Gamma \subset \text{SL}(2, \mathbb{C})$ gives rise to a hyperbolic structure on the manifold $M \cong \Sigma \times \mathbb{R}$, where Σ is a hyperbolic surface. Deforming Γ in $\text{SL}(2, \mathbb{C})$ corresponds to deforming the hyperbolic structure on M . Such deformations are abundant and according the Bers simultaneous uniformization can be parameterized by a cartesian product of two copies of the classical Teichmüller space of the surface Σ . Notice that the existence of deformations into G does not violate Weil's result as G/Γ is not compact. This situation can be generalized to the case where $H = \text{SO}^0(n, 1) \cong \text{Isom}^+(\mathbb{H}_{\mathbb{R}}^n)$ and $G = \text{SO}^0(n+1, 1) \cong \text{Isom}^+(\mathbb{H}_{\mathbb{R}}^{n+1})$. In this setting, a lattice Γ in H gives rise to a hyperbolic structure on $M = \mathbb{H}_{\mathbb{R}}^n/\Gamma$. Again, regarding Γ as a subgroup of $\text{SO}^0(n+1, 1)$ gives rise to a hyperbolic structure on $M \times \mathbb{R}$ and deformations of Γ into G correspond to deforming this hyperbolic structure. In this more general setting there is no general theorem that guarantees the existence of deformations of Γ into G . However, when $n = 3$ this deformation problem has been studied by Scannell [Sc], Bart-Scannell [BSc], and Kapovich [Kap] who prove some rigidity results.

Returning to the 3-dimensional case, many non-cocompact lattices Γ in $H = \mathrm{SL}(2, \mathbb{C}) \cong \mathrm{SO}^0(3, 1)$ are known to admit deformations into H . In particular, when Γ is torsion-free Thurston showed that for each cusp there exists a (real) 2-dimensional family of deformations of Γ into H , called *Dehn surgery deformations* (see Section 5.8 of [T]). Geometrically, in each of these families the commuting pair of parabolic isometries generating the corresponding cusp group is deformed to a pair of loxodromic isometries sharing a common axis. In particular these deformations are all non-discrete or non-faithful. If $\mathbb{H}_{\mathbb{R}}^3/\Gamma$ is an orbifold then the existence of deformations depends more subtly on the topology of the cusp cross-sections.

Another case of interest in the context of deformations of geometric structures is that of *projective deformations* of hyperbolic lattices, i.e. deformations of lattices Γ of $H = \mathrm{SO}(n, 1)$ into $G = \mathrm{SL}(n + 1, \mathbb{R})$. When Γ is a torsion-free cocompact lattice in $\mathrm{SO}(n, 1)$ such that the hyperbolic manifold $M = \mathbb{H}_{\mathbb{R}}^n/\Gamma$ contains an embedded totally geodesic hypersurface Σ , Johnson and Millson showed in [JM] that Γ admits a 1-parameter family of deformations into $\mathrm{SL}(n + 1, \mathbb{R})$. They obtained these deformations, called *bending deformations of M along Σ* , by introducing an algebraic version of Thurston's bending deformations of a hyperbolic 3-manifold along a totally geodesic surface. This algebraic version is very versatile, and can be generalized in a variety of ways. For example, the hypothesis that M is compact may be dropped, see [BM]. Furthermore, the construction can be applied to the setting of other Lie groups and will provide us with a rich source of examples that are discussed in Section 4. In addition to deformations constructed via bending there are also instances of projective deformations that do not arise via the previously mentioned bending technique (see [B1, B2, BDL]). On the other hand, despite the existence of these bending examples, empirical evidence compiled by Cooper–Long–Thistlethwaite [CLTII] suggests that the existence of deformations into $\mathrm{SL}(4, \mathbb{R})$ is quite rare for closed hyperbolic 3-manifolds.

In another direction, *complex hyperbolic quasi-Fuchsian* deformations of Fuchsian groups have also been extensively studied (see e.g. [S2], the survey [PP] and references therein). With the above notation, this concerns deformations of discrete subgroups Γ of H into G , with $(H, G) = (\mathrm{SO}(2, 1), \mathrm{SU}(2, 1))$ or $(\mathrm{SU}(1, 1), \mathrm{SU}(2, 1))$. (Recall that the Lie groups $\mathrm{SL}(2, \mathbb{R}), \mathrm{SO}(2, 1), \mathrm{SU}(1, 1)$ are all isomorphic, up to index 2). It turns out that for any $n \geq 2$, by work of Cooper–Long–Thistlethwaite [CLT] there is an intricate relationship between projective deformations and complex hyperbolic deformations of finitely generated subgroups Γ of $\mathrm{SO}(n, 1)$, based on the fact that the Lie algebras of $\mathrm{SL}(n + 1, \mathbb{R})$ and $\mathrm{SU}(n, 1)$ are isomorphic as modules over the $\mathrm{SO}(n, 1)$ group ring. Specifically, they prove:

Theorem 1.1 ([CLT]) *Let Γ be a finitely generated group, and let $\rho : \Gamma \rightarrow \mathrm{SO}^0(n, 1)$ be a smooth point of the representation variety $\mathrm{Hom}(\Gamma, \mathrm{SL}(n + 1, \mathbb{R}))$. Then ρ is also a smooth point of $\mathrm{Hom}(\Gamma, \mathrm{SU}(n, 1))$, and near ρ the real dimensions of $\mathrm{Hom}(\Gamma, \mathrm{SL}(n + 1, \mathbb{R}))$ and $\mathrm{Hom}(\Gamma, \mathrm{SU}(n, 1))$ are equal.*

The primary motivation for this article is to construct examples of complex hyperbolic deformations of real hyperbolic lattices that have nice algebraic and geometric properties. Our main result can be roughly described as providing a sufficient condition for a deformation of a lattice Γ in H into G to continue to be faithful and have discrete image. In what follows, the condition of being *parabolic-preserving* roughly means that parabolic elements remain parabolic, see Definition 2.1 for a more precise statement.

Theorem 1.2 *Let X be a negatively curved symmetric space, S a totally geodesic subspace of X and denote $G = \mathrm{Isom}(X)$, $H = \mathrm{Stab}_G(S)$. Let Γ be a non-cocompact lattice in H , and let ι denote the inclusion of Γ into G . Then any parabolic-preserving representation $\rho : \Gamma \rightarrow G$ sufficiently close to ι is discrete and faithful.*

Remarks:

1. The Dehn surgery deformations of non-cocompact lattices in $\mathrm{SO}(3, 1)$ described above are either indiscrete or non-faithful, showing the necessity of the parabolic-preserving assumption in general.
2. It was pointed out to us by Elisha Falbel that Theorem 1.2 still holds, with the same proof, under the weaker hypothesis that Γ is a subgroup of a non-cocompact lattice in H , with no global fixed point in \bar{S} . If Γ is not itself a lattice, this is equivalent in this context (see [CG]) to saying that Γ is a *thin subgroup* of that lattice, i.e. an infinite-index subgroup with the same Zariski-closure as the lattice.
3. When Γ is a cocompact lattice in H the result is a consequence of the following result of Guichard, as Γ is then convex-cocompact in G .

Theorem 1.3 ([Gui]) *Let G be a semisimple Lie group with finite center, H a rank 1 subgroup of G , Γ a finitely generated discrete subgroup of H and denote $\iota : \Gamma \rightarrow G$ the inclusion map. If Γ is convex-cocompact then ι has a neighborhood in $\text{Hom}(\Gamma, G)$ consisting entirely of discrete and faithful representations.*

We prove Theorem 1.2 in section 2, then apply it in Section 3 to a family of deformations of the figure-8 knot group $\Gamma_8 < \text{SO}(3, 1)$ into $\text{SU}(3, 1)$. Denoting $\Gamma_8 = \pi_1(S^3 \setminus K_8)$ (where K_8 is the figure-8 knot), and $\rho_{\text{hyp}} : \Gamma_8 \rightarrow \text{SO}(3, 1)$ its *hyperbolic representation*, i.e. the holonomy representation of the complete hyperbolic structure on $S^3 \setminus K_8$, we obtain:

Theorem 1.4 *Let Γ_8 be the figure-8 knot group and $\rho_{\text{hyp}} : \Gamma_8 \rightarrow \text{SO}(3, 1)$ its hyperbolic representation. Then there exists a 1-parameter family of discrete, faithful deformations of ρ_{hyp} into $\text{SU}(3, 1)$.*

In Section 4 we apply Theorem 1.2 to a variation of the Johnson-Millson bending deformations, to obtain the following result. As above, given a finite-volume hyperbolic manifold $M = \mathbb{H}_{\mathbb{R}}^n / \Gamma$, we call *hyperbolic representation* of $\Gamma = \pi_1(M)$ the holonomy representation into $\text{SO}(n, 1)$ of the complete hyperbolic structure on M . (This is well-defined up to conjugation by Mostow rigidity).

Theorem 1.5 *For any $n \geq 3$ there exist infinitely many non-commensurable cusped hyperbolic n -manifolds whose corresponding hyperbolic representation admits a 1-parameter family of discrete, faithful deformations into $\text{SU}(n, 1)$.*

Here, two groups $\Gamma, \Gamma' \subset H$ are *commensurable* (in the wide sense) if $\Gamma \cap g\Gamma'g^{-1}$ has finite index in both Γ and $g\Gamma'g^{-1}$ for some $g \in H$. The incommensurability conclusion ensures that in each dimension n the manifolds in Theorem 1.5 are quite distinct in the sense that they are not obtained by taking covering spaces of a single example.

2 Discreteness and faithfulness of parabolic-preserving deformations

In this section we prove Theorem 1.2, stated in the introduction. Our strategy of proof in the non-compact case is to use invariant horospheres, more precisely a variation of what Schwartz ([S1]) called $\rho(\Gamma)$ -invariant neutered space, see Definition 2.2 below.

From E. Cartan's classification of real semisimple Lie groups, any negatively curved symmetric space is a hyperbolic space $\mathbb{H}_{\mathbb{K}}^n$, with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or \mathbb{O} (and $n \geq 2$ if $\mathbb{K} = \mathbb{R}$, $n = 2$ if $\mathbb{K} = \mathbb{O}$). We refer the reader to [CG] for general properties of these spaces and their isometry groups. In particular isometries of such spaces are roughly classified into the following 3 types: elliptic (having a fixed point in X), parabolic (having no fixed point in X and exactly one on $\partial_{\infty}X$) or loxodromic (having no fixed point in X and exactly two on $\partial_{\infty}X$). For our purposes we will need to distinguish between elliptic isometries with an isolated fixed point in X , which we call *single-point elliptic* and elliptic isometries having boundary fixed points, which we call *boundary elliptic*.

Definition 2.1 *Let X be a negatively curved symmetric space, $G = \text{Isom}(X)$, and Γ a subgroup of G . A representation $\rho : \Gamma \rightarrow G$ is called parabolic-preserving if for every parabolic (resp. boundary elliptic) element $\gamma \in \Gamma$, $\rho(\gamma)$ is again parabolic (resp. boundary elliptic).*

Remark 2.1 If Γ_{∞} is a parabolic subgroup of Γ (i.e. a subgroup fixing a point on $\partial_{\infty}X$), then any parabolic-preserving representation of Γ is faithful on Γ_{∞} . Indeed, all elements of $\Gamma_{\infty} \setminus \{\text{Id}\}$ are parabolic or boundary elliptic.

Lemma 2.1 *Let Γ be a discrete subgroup of G containing a parabolic element P ; denote $q_{\infty} = \text{Fix}(P) \in \partial_{\infty}X$ and $\Gamma_{\infty} = \text{Stab}_{\Gamma}(q_{\infty})$. Then, for any parabolic-preserving representation $\rho : \Gamma \rightarrow G$, $\rho(\Gamma_{\infty})$ preserves each horosphere based at $\text{Fix}(\rho(P))$.*

Proof. It is well known that first, parabolic and boundary elliptic isometries with fixed point $q_{\infty} \in \partial_{\infty}X$ preserve each horosphere based at q_{∞} and secondly, in a discrete group of hyperbolic isometries, loxodromic and parabolic elements cannot have a common fixed point. Therefore Γ_{∞} consists of parabolic and possibly boundary elliptic isometries, and likewise for $\rho(\Gamma_{\infty})$ if ρ is parabolic-preserving. The only thing that remains

to be seen is that for any $Q \in \Gamma_\infty$ and parabolic-preserving representation $\rho : \Gamma \rightarrow G$, $\rho(Q)$ fixes $\text{Fix}(\rho(P))$. This follows from the fact that pairs of isometries having a common fixed boundary point can be characterized algebraically. Namely, by the assumption that P is parabolic, P and Q have a common fixed boundary point if and only if the group $\langle P, Q \rangle$ is virtually nilpotent. This property is preserved by any representation of Γ . \square

Definition 2.2 Let X be a negatively curved symmetric space, $G = \text{Isom}(X)$, Γ a subgroup of G and Γ_∞ a subgroup of Γ . We say that a (closed) horoball H_∞ in X is (Γ, Γ_∞) -consistent if the following conditions hold:

$$\left\{ \begin{array}{l} (1) \gamma H_\infty = H_\infty \text{ for all } \gamma \in \Gamma_\infty, \text{ and} \\ (2) \gamma H_\infty \cap H_\infty = \emptyset \text{ for all } \gamma \in \Gamma \setminus \Gamma_\infty. \end{array} \right.$$

Definition 2.3 Given two disjoint horoballs H_1, H_2 in X , we call orthogeodesic for the pair $\{H_1, H_2\}$ the unique geodesic segment with endpoints in the boundary horospheres $\partial H_1, \partial H_2$ and perpendicular to these horospheres; note that it is the unique distance-minimizing geodesic segment between H_1 and H_2 . We will call the set of points of ∂H_1 which are endpoints of a geodesic ray perpendicular to ∂H_1 and intersecting H_2 the shadow of H_2 on H_1 .

Remark 2.2 Since the geodesics orthogonal to ∂H_1 are exactly those geodesics having the vertex v_1 of H_1 as an endpoint, the shadow of H_2 on H_1 is the intersection with ∂H_1 of the geodesic cone over H_2 from v_1 .

Lemma 2.2 Given two disjoint horoballs H_1, H_2 with orthogeodesic $[x_1, x_2]$, the shadow of H_2 on H_1 is the intersection with ∂H_1 of a closed ball centered at x_1 .

Proof. Note that any isometry fixing the geodesic (x_1, x_2) pointwise preserves H_1 and H_2 , hence the shadow of H_2 on H_1 has rotational symmetry around x_1 . The statement follows by observing that this shadow is closed, bounded, and has non-empty-interior, which is clear in the upper half-space model of $\mathbb{H}_\mathbb{R}^n$, and the related Siegel domain models of the other hyperbolic spaces, where horospheres based at the special point ∞ are horizontal slices of the domain and geodesics through ∞ are vertical lines (see [Go] for the complex case and [KP] for the quaternionic case). \square

Proposition 2.1 Let X be a negatively curved symmetric space, $G = \text{Isom}(X)$, Γ a discrete subgroup of G and Γ_∞ a subgroup of Γ . Assume that there exists a (Γ, Γ_∞) -consistent horoball H_∞ in X such that Γ_∞ acts cocompactly on the horosphere ∂H_∞ . Then the set of lengths of orthogeodesics for pairs $\{H_\infty, \gamma H_\infty\}$ (with $\gamma \in \Gamma$) is discrete, and each of its values is attained only finitely many times modulo the action of Γ_∞ .

Proof. First note that for $\gamma_\infty \in \Gamma_\infty$ and $\gamma \in \Gamma \setminus \Gamma_\infty$, $d(H_\infty, \gamma H_\infty) = d(H_\infty, \gamma_\infty \gamma H_\infty)$, and let K be a compact subset of ∂H_∞ whose Γ_∞ -orbit covers ∂H_∞ . The orbit ΓH_∞ is closed, since the single cusp neighborhood $p(H_\infty) = p(\Gamma H_\infty)$ is closed in X/Γ (denoting p the projection map $X \rightarrow X/\Gamma$), hence so is $\Gamma H_\infty \setminus H_\infty$. Then the distance between the compact set K and the closed set $\Gamma H_\infty \setminus H_\infty$ is positive and attained, say by some point $x_0 \in K$ and some point in the horoball $\gamma_0 H_\infty$.

We claim that only finitely many Γ_∞ -orbits of horospheres in $\Gamma H_\infty \setminus H_\infty$ realize this minimum. To see this, it suffices to show that any horosphere based at ∞ intersects finitely many Γ_∞ -orbits of horospheres in ΓH_∞ . Fix a horosphere H'_∞ based at ∞ . Consider a horosphere H in $\Gamma H_\infty \setminus H_\infty$ that intersects H'_∞ , and let B_H be its shadow on H'_∞ . By Lemma 2.2, B_H is (the intersection with H'_∞ of) a closed ball. There exists $r > 0$, depending only on H'_∞ , such that the radius of B_H is at least r . (Indeed r is the radius of the shadow of any horosphere tangent to H'_∞ .)

Now consider two such horospheres H_1 and H_2 and assume they are disjoint. Call x_1 and x_2 the centers of their shadows B_1 and B_2 on H'_∞ . We claim that the distance between x_1 and x_2 is at least r . If not, then $x_1 \in B_2$ and $x_2 \in B_1$. Consider the geodesic σ_1 connecting ∞ to x_1 . Since x_1 is the center of B_1 the intersection $\sigma_1 \cap H_1$ is the geodesic ray connecting the highest point on H_1 to the endpoint of σ_1 which isn't ∞ . As H_1 and H_2 are disjoint, $\sigma_1 \cap H_2$ is a compact geodesic segment contained in $\sigma_1 \setminus (\sigma_1 \cap H_1)$. Permuting the roles of x_1 and x_2 gives the opposite situation on the geodesic σ_2 connecting ∞ to x_2 . Now if we move continuously from

x_1 to x_2 along a curve $x(t)$ and consider the associated pencil of geodesics σ_t connecting ∞ to $x(t)$, we see that there must be a value of t for which $\sigma_t \cap H_1$ and $\sigma_t \cap H_2$ intersect, contradicting disjointness of H_1 and H_2 .

Finally, since K was a compact subset of H'_∞ whose Γ_∞ -translates cover H'_∞ , for any horosphere H in $\Gamma \cdot H_\infty \setminus H_\infty$, we can apply an element of Γ_∞ that maps the center of B_H to a point of K . If H'_∞ meets an infinite number of classes in $\Gamma \cdot H_\infty$ we obtain in this way a sequence of distinct points in K , which must accumulate by compactness of K . But by consistency the corresponding horospheres are disjoint, and the previous discussion tells us that the distance between the centers of their shadows is uniformly bounded from below, a contradiction.

The result follows inductively, repeating the argument after removing the first layer of closest horoballs. \square

Remark 2.3 the hypothesis that the cusp stabilizer acts cocompactly on any horosphere based at the cusp holds for any non-cocompact lattice. In fact, it holds more generally for any discrete group with a maximal rank parabolic subgroup.

Proposition 2.2 *Let X be a negatively curved symmetric space, S a totally geodesic subspace of X and denote $G = \text{Isom}(X)$, $H = \text{Stab}_G(S)$. If Γ is a non-cocompact lattice in H then for any parabolic-preserving representation $\rho : \Gamma \rightarrow G$ sufficiently close to the inclusion $\iota : \Gamma \rightarrow G$, there exists a $(\rho(\Gamma), \rho(\Gamma_\infty))$ -consistent horoball, where Γ_∞ is any cusp stabilizer in Γ .*

Proof. Since Γ is non-cocompact, it contains a parabolic isometry P . Let as above $q_\infty = \text{Fix}(P) \in \partial_\infty S$ and $\Gamma_\infty = \text{Stab}_\Gamma(q_\infty)$. Then there exists a horoball H_∞^S in S , based at q_∞ , which is (Γ, Γ_∞) -consistent (this can be seen by lifting to S an embedded horoball neighborhood of the image of q_∞ in the quotient S/Γ , see e.g. Lemma 2.1 of [S1]). Since S is totally geodesic, H_∞^S is the intersection of S with a horoball H_∞ in X which is $(\iota(\Gamma), \iota(\Gamma_\infty))$ -consistent. Now for any parabolic-preserving representation $\rho : \Gamma \rightarrow G$ and any $\gamma \in \Gamma_\infty$, $\rho(\gamma)H_\infty = H_\infty$ by Lemma 2.1, which is condition (1) of Definition 2.2 for the pair $(\rho(\Gamma), \rho(\Gamma_\infty))$. It follows from Proposition 2.1 that for ρ sufficiently close to ι , the horoballs γH_∞ (with $\gamma \in \Gamma \setminus \Gamma_\infty$) stay disjoint from H_∞ as long as some finite subcollection of them do (note that since S is totally geodesic and the horoballs convex, the distance between 2 horoballs H_∞ and γH_∞ based at points of ∂S is given by their distance in S). Hence condition (2) of Definition 2.2 for the pair $(\rho(\Gamma), \rho(\Gamma_\infty))$ holds for ρ sufficiently close to ι . \square

Proposition 2.3 *Let X be a negatively curved symmetric space, denote $G = \text{Isom}(X)$ and let Γ be a subgroup of G without a global fixed point in \overline{X} . If there exists a (Γ, Γ_∞) -consistent horoball in X for some subgroup Γ_∞ of Γ , then Γ is discrete.*

Proof. First assume for simplicity that Γ does not preserve any proper totally geodesic subspace of X . Then Γ is either discrete or dense in G (Corollary 4.5.1 of [CG]). If Γ is dense in G then the orbit of any point of X is dense in X . But if H_∞ is a (Γ, Γ_∞) -consistent horoball, the orbit of any point of H_∞ is entirely contained in ΓH_∞ , in which case it cannot be dense in X as ΓH_∞ and $X \setminus \Gamma H_\infty$ both have nonempty interior. Therefore Γ must be discrete.

Now if Γ does preserve a strict totally geodesic subspace of X , and if S is the minimal such subspace then by the same argument either Γ is discrete or every orbit of a point of S is dense in S . But the consistent horoball must be based at a point of $\partial_\infty S$ (since it is preserved by all elements of Γ_∞), hence it intersects S along a horoball of S and we conclude as before.

Lemma 2.3 *Let X be a negatively curved symmetric space, denote $G = \text{Isom}(X)$, let Γ be a subgroup of G and Γ_∞ a subgroup of Γ . If $\rho : \Gamma \rightarrow G$ is a parabolic-preserving representation such that there exists a $(\rho(\Gamma), \rho(\Gamma_\infty))$ -consistent horoball, then ρ is faithful.*

Proof. Let $\gamma \in \Gamma \setminus \{\text{Id}\}$. If $\gamma \in \Gamma_\infty$ then $\rho(\gamma) \neq \text{Id}$ by parabolic-preservation, and if $\gamma \in \Gamma \setminus \Gamma_\infty$ then $\rho(\gamma) \neq \text{Id}$ by condition (2) of the definition of $(\rho(\Gamma), \rho(\Gamma_\infty))$ -consistent horoball. \square

Now Theorem 1.2 follows immediately from Propositions 2.2 and 2.3 and Lemma 2.3.

3 Deformations of the figure-8 knot group into $SU(3, 1)$

In this section we construct a family of parabolic-preserving deformations of the hyperbolic representation of the figure-8 knot group into $SU(3, 1)$. Consider $\Gamma_8 = \pi_1(S^3 \setminus K_8)$ where K_8 is the figure-8 knot, and denote $\rho_{\text{hyp}} : \Gamma_8 \rightarrow SO(3, 1)$ the holonomy of the complete hyperbolic structure on $S^3 \setminus K_8$.

Recall that in the presence of a smoothness hypothesis on the relevant representation varieties, Theorem 1.1 implies that the existence of deformations of ρ_{hyp} into $SL(4, \mathbb{R})$ guarantees the existence of deformations of ρ_{hyp} into $SU(3, 1)$. Work of Ballas–Danciger–Lee [BDL] shows that the smoothness hypothesis is guaranteed in the presence of a cohomological condition. Specifically, they prove the following.

Theorem 3.1 ([BDL]) *Let M be an orientable complete finite volume hyperbolic manifold with fundamental group Γ , and let $\rho_{\text{hyp}} : \Gamma \rightarrow SO(3, 1)$ be the holonomy representation of the complete hyperbolic structure. If M is infinitesimally projectively rigid rel boundary, then ρ_{hyp} is a smooth point of $\text{Hom}(\Gamma, SL(4, \mathbb{R}))$ and its conjugacy class is a smooth point of $\chi(\Gamma, SL(4, \mathbb{R}))$.*

Roughly speaking, *infinitesimally projectively rigid rel boundary* is a cohomological condition that says that a certain induced map from the twisted cohomology of M into the twisted cohomology of ∂M is an injection. For a more precise definition, see [HP]. By work of Heusener–Porti [HP], it is known that the figure-8 knot complement is infinitesimally rigid rel boundary, and so we can apply Theorems 3.1 and 1.1 to produce deformations of ρ_{hyp} into $SU(3, 1)$. However, there is no reason why these representations should be parabolic-preserving, and in many cases the deformations will not have this property.

Fortunately, work of the first author (see [B1, B2]) provides a family of deformations of ρ_{hyp} into $SL(4, \mathbb{R})$ whose corresponding deformations into $SU(3, 1)$ are parabolic preserving.

Theorem 3.2 ([B1],[B2]) *Let Γ_8 be the figure-8 knot group. Then there exists a 1-parameter family of discrete, faithful deformations of ρ_{hyp} into $SL(4, \mathbb{R})$.*

The construction of this 1-parameter family can be found in [B1] and ultimately constructs a curve ρ_t of representations of Γ_8 into $SL(4, \mathbb{R})$ containing the hyperbolic representation ρ_{hyp} at $t = 1/2$. In fact, allowing the parameter t to take complex values gives a 1-complex parameter family of representations into $SL(4, \mathbb{C})$. Moreover, it turns out that taking $2t$ to be a unit complex number u gives a 1-parameter family of representations into $SU(3, 1)$. (The reason for this choice of value of the parameter is that the eigenvalues of one of the peripheral elements in $\rho_t(\Gamma_8)$ are 1 and a power of $2t$, see Section 6 of [B2]).

We now give explicit matrices for the generators and Hermitian form for this family, using the presentation and notation of Section 6 of [B2]. There, the following presentation of Γ_8 was used:

$$\Gamma_8 = \langle m, n \mid mw = wn \rangle, \text{ where } w = [n, m^{-1}]. \quad (3.1)$$

The family of representations $\rho_u : \Gamma_8 \rightarrow SL(4, \mathbb{C})$ is defined by $\rho_u(m) = M_u$ and $\rho_u(n) = N_u$, where:

$$M_u = \begin{pmatrix} 1 & 0 & 1 & u/2 - 1 \\ 0 & 1 & 1 & u/2 \\ 0 & 0 & 1 & (u+1)/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad N_u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2(1+\bar{u}) & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad (3.2)$$

When $|u| = 1$, the group $\rho_u(\Gamma_8)$ preserves the Hermitian form H_u on \mathbb{C}^4 given by $H(X, Y) = X^T J_u \bar{Y}$, where:

$$J_u = \begin{pmatrix} 1 + (u + \bar{u})/2 & -1 - (u + \bar{u})/2 & 1 + u & -3 - 2(u + \bar{u}) - \bar{u}^2 \\ -1 - (u + \bar{u})/2 & 1 + (u + \bar{u})/2 & -1 - u & 1 + u \\ 1 + \bar{u} & -1 - \bar{u} & 4 + 2(u + \bar{u}) & -4 - 2(u + \bar{u}) \\ -3 - 2(u + \bar{u}) - u^2 & 1 + \bar{u} & -4 - 2(u + \bar{u}) & 4 + 2(u + \bar{u}) \end{pmatrix} \quad (3.3)$$

Lemma 3.1 *The form H_u has signature $(3, 1)$ for all $u = e^{i\alpha}$ with $|\alpha| < 2\pi/3$, and signature $(2, 2)$ when $\alpha \in \pm(2\pi/3, \pi)$.*

Proof. Computing the determinant of J_u gives:

$$\begin{aligned} \det J_u &= -96 - 83(u + \bar{u}) - 53(u^2 + \bar{u}^2) - 24(u^3 + \bar{u}^3) - 7(u^4 + \bar{u}^4) - (u^5 + \bar{u}^5) \\ &= -96 - 166\cos(\alpha) - 106\cos(2\alpha) - 48\cos(3\alpha) - 14\cos(4\alpha) - 2\cos(5\alpha) \\ &= -4(\cos(\alpha) + 1)^2(2\cos(\alpha) + 1)^3. \end{aligned}$$

The latter function of α is negative for $|\alpha| < 2\pi/3$, and positive for $\alpha \in \pm(2\pi/3, \pi)$; the result then follows by noting that H_u has signature (3,1) when $u = 1$ (corresponding to the hyperbolic representation), and (2,2) for e.g. $u = \pm 3\pi/4$. \square

Lemma 3.2 *The representations ρ_u are pairwise non-conjugate in $\mathrm{SL}(4, \mathbb{C})$.*

Proof. A straightforward computation gives: $\mathrm{Tr} M_u N_u = 6 + u$. \square

Lemma 3.3 *The representations ρ_u are parabolic-preserving.*

Proof. The peripheral subgroup Γ_∞ of Γ_8 is generated by m and $l = ww^{op} = nm^{-1}n^{-1}m^2n^{-1}m^{-1}n$, with the notation of the presentation (3.1) (see [B1]). Now $M_u = \rho_u(m)$ is unipotent for all u , and a straightforward computation (using eg Maple) shows that $L_u = \rho_u(l)$ is non-diagonalizable (with eigenvalues (u, u, u, \bar{u}^3)) for all u , hence parabolic. Since $\Gamma_\infty \simeq \mathbb{Z}^2$, all elements of $\rho_u(\Gamma_\infty) = \langle \rho_u(m), \rho_u(l) \rangle$ will also remain parabolic for u in a neighborhood of 1. \square

The previous result along with Theorem 1.2 has the following immediate corollary

Corollary 3.3 *The representations ρ_u are discrete and faithful for u in some neighborhood of 1 in $\mathrm{U}(1)$.*

It would be interesting to know how far u can get from 1 before discreteness or faithfulness is lost.

4 Bending deformations

In this section we construct additional examples in arbitrary dimensions, proving Theorem 1.5 stated in the introduction. We start with a cusped hyperbolic manifold $M = \mathbb{H}_{\mathbb{R}}^n / \Gamma$, and $\rho_{hyp} : \Gamma \rightarrow \mathrm{SO}(n, 1)$ the hyperbolic representation of $\Gamma = \pi_1(M)$, i.e. the holonomy representation of the complete hyperbolic structure on M . We will construct a one-parameter family of representations $\rho_\theta : \Gamma \rightarrow \mathrm{SU}(n, 1)$, ($\theta \in S^1$) such that $\rho_0 = \rho_{hyp}$, using the bending procedure described by Johnson–Millson ([JM]). Their construction is quite general and allows one to deform representations in a variety of Lie groups. We briefly outline how to use bending to produce families of representations in the complex hyperbolic setting.

Define a Hermitian form H on \mathbb{C}^{n+1} via the formula $H(X, Y) = X^T J_{n+1} \bar{Y}$ where J_{n+1} is the diagonal matrix $\mathrm{Diag}(1, \dots, 1, -1)$, with signature $(n, 1)$. Using this form we produce a projective model for $\mathbb{H}_{\mathbb{C}}^n$ given by

$$\mathbb{H}_{\mathbb{C}}^n = \{[V] \in \mathbb{C}P^n \mid H(V, V) < 0\}$$

Using the splitting $\mathbb{C}^{n+1} = \mathbb{C} \times \mathbb{C}^n$ we can embed $\mathbb{H}_{\mathbb{C}}^{n-1}$ into the $\mathbb{C}P^{n-1}$ corresponding to the second factor. We will refer to this copy of $\mathbb{H}_{\mathbb{C}}^{n-1}$ as H_0 . Using this embedding we can identify $\mathrm{U}(n-1, 1)$ with the intersection of $\mathrm{SU}(n, 1)$ and the stabilizer of the second factor, and we will refer to this subgroup as $\mathrm{U}_0(n-1, 1)$. It is well known that all other copies of $\mathbb{H}_{\mathbb{C}}^{n-1}$ inside $\mathbb{H}_{\mathbb{C}}^n$ are isometric to H_0 , and similarly, all copies of $\mathrm{U}(n-1, 1)$ inside $\mathrm{SU}(n, 1)$ are conjugate to $\mathrm{U}_0(n-1, 1)$.

Let C_{n-1} denote the identity component of the centralizer of $\mathrm{U}_0(n-1, 1)$ in $\mathrm{SU}(n, 1)$. C_{n-1} is a one-dimensional Lie group isomorphic to S^1 and can be written explicitly in block form as

$$C_{n-1} = \left\{ M_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta/n} I_n \end{pmatrix} \mid \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

Let Γ be a lattice in $\mathrm{SO}(n, 1) \subset \mathrm{SU}(n, 1)$; then $M = \mathbb{H}_{\mathbb{R}}^n / \Gamma$ is a finite volume hyperbolic n -orbifold. For simplicity, we will assume that Γ is torsion-free and thus M will be a hyperbolic manifold. Suppose that M contains an embedded orientable totally geodesic hypersurface Σ . By applying a conjugacy of $\mathrm{SO}(n, 1)$ we can assume that $\Sigma = \mathbb{H}_{\mathbb{R}}^{n-1} / \Delta$ where $\mathbb{H}_{\mathbb{R}}^{n-1}$ is thought of as the set of real points of H_0 and Δ is a lattice in $\mathrm{U}_0(n-1, 1) \cap \mathrm{SO}(n, 1)$.

The hypersurface Σ provides a decomposition of Γ into either an amalgamated free product or an HNN extension, depending on whether or not Σ is separating. Using this decomposition we can construct a family $\rho_{\theta} : \Gamma \rightarrow \mathrm{SU}(n, 1)$ such that $\rho_0 = \iota$, where ι is the inclusion of Γ into $\mathrm{SU}(n, 1)$, as follows.

If Σ is separating, then $M \setminus \Sigma$ consists of two connected components M_1 and M_2 , with fundamental groups Γ_1 and Γ_2 respectively. In this case $\Gamma = \Gamma_1 *_{\Delta} \Gamma_2$. The group Γ is generated by $\Gamma_1 \cup \Gamma_2$ and we define

$$\rho_{\theta}(\gamma) = \begin{cases} \iota(\gamma) & \gamma \in \Gamma_1 \\ M_{\theta} \iota(\gamma) M_{\theta}^{-1} & \gamma \in \Gamma_2 \end{cases}$$

on this generating set. Since M_{θ} centralizes Δ we see that the relations coming from the amalgamated product decomposition are satisfied, and so $\rho_{\theta} : \Gamma \rightarrow \mathrm{SU}(n, 1)$ is well defined.

If Σ is non-separating, then $M' = M \setminus \Sigma$ is connected. If we let Γ' be the fundamental group of M' then we can arrive at the decomposition $\Gamma = \Gamma' *_{\iota}$. In this case Γ is generated by $\Gamma' \cup \{t\}$, where t is a free letter and we define ρ_{θ} on generators as

$$\rho_{\theta}(\gamma) = \begin{cases} \iota(\gamma) & \gamma \in \Gamma' \\ M_{\theta} \iota(\gamma) & \gamma = t \end{cases}$$

Again, since M_{θ} centralizes Δ we see that the relations for the HNN extension are satisfied and so $\rho_{\theta} : \Gamma \rightarrow \mathrm{SU}(n, 1)$ is well defined. The representations constructed above are called *bending deformations of Γ along Δ* , or just *bending deformations* if Γ and Δ are clear from context. By work of Johnson–Millson [JM] this path of representations is in fact a deformation of ρ_{hyp} (i.e. the ρ_{θ} are pairwise non-conjugate for small values of θ).

Proof. (proof of Theorem 1.5) We proceed by constructing infinitely many commensurability classes of cusped hyperbolic manifolds containing totally geodesic hypersurfaces. This is done via a well known arithmetic construction (see [Ber]). The rough idea is to look at the group, Γ , of integer points of the orthogonal groups of various carefully selected quadratic forms of signature $(n, 1)$. The quotient $M = \mathbb{H}_{\mathbb{R}}^n / \Gamma$ will be a cusped hyperbolic n -orbifold containing a totally geodesic hypersurface. After passing to a carefully selected cover we can produce our parabolic preserving representations via the bending construction.

We now discuss the details for a specific form and observe that the proof is essentially unchanged if one selects a different form. Let $\hat{\Gamma} = \mathrm{SL}(n+1, \mathbb{Z}) \cap \mathrm{SO}(n, 1)$ and let $\hat{\Delta} = \hat{\Gamma} \cap \mathrm{U}_0(n-1, 1)$. The group $\hat{\Gamma}$ clearly contains unipotent elements and so we see that $\hat{M} = \mathbb{H}_{\mathbb{R}}^n / \hat{\Gamma}$ is a cusped hyperbolic n -orbifold, which contains an immersed totally geodesic codimension-1 suborbifold isomorphic to $\hat{\Sigma} = \mathbb{H}_{\mathbb{R}}^{n-1} / \hat{\Delta}$. By combining work of Bergeron (Théorème 1 of [Ber]) and McReynolds–Reid–Stover (Proposition 3.1 of [MRS]) we can find finite index subgroups $\Gamma \subset \hat{\Gamma}$ and $\Delta \subset \hat{\Delta}$ and corresponding manifolds $M = \mathbb{H}_{\mathbb{R}}^n / \Gamma$ and $\Sigma = \mathbb{H}_{\mathbb{R}}^{n-1} / \Delta$ with the following properties.

- Γ is torsion-free
- Σ is *embedded* in M
- M has only torus cusps.

Each M contains the totally geodesic hypersurface Σ along which we can bend to produce a family ρ_{θ} of representations from Γ into $\mathrm{SU}(n, 1)$. We now show that the representations ρ_{θ} are parabolic-preserving; then by Theorem 1.2 the ρ_{θ} are discrete and faithful for small values of θ .

Lemma 4.1 *The representations $\rho_{\theta} : \Gamma \rightarrow \mathrm{SU}(n, 1)$ obtained by bending Γ along Δ are parabolic-preserving.*

Proof. By construction, we have arranged that Γ is torsion-free, and so there are no elliptic elements to consider. Furthermore, the only parabolic elements of Γ correspond to loops in M that are freely homotopic to one of the torus cusps. We now discuss how such an element is modified when one bends. Let γ be a parabolic

element of Γ and let q_∞ be its fixed point on $\partial_\infty \mathbb{H}_\mathbb{C}^n$. There is a foliation of $\mathbb{H}_\mathbb{C}^n$ by horospheres centered at q_∞ and γ preserves this foliation leafwise. Furthermore, leafwise preservation of this foliation characterizes parabolic isometries of $\mathbb{H}_\mathbb{C}^n$ that fix p_∞ . Thus it suffices to show that $\rho_\theta(\gamma)$ preserves this foliation.

Regard γ as a loop in M based at $x_0 \in \Sigma$ and lift γ to a path $\tilde{\gamma}$ in $\tilde{M} \subset \mathbb{H}_\mathbb{C}^n$ based at \tilde{x}_0 . Let $\tilde{\Sigma}$ be the lift of Σ that contains \tilde{x}_0 . Each time $\tilde{\gamma}$ intersects a lift of Σ to $\mathbb{H}_\mathbb{C}^n$ (counted with orientation) the holonomy is modified by composing with a Heisenberg rotation of angle $\varepsilon_i \theta$ centered at q_∞ that acts as the identity on $\tilde{\Sigma}_i$. Each of these modifications is by an element of $SU(n, 1)$ that leafwise preserves the foliation of horospheres centered at q_∞ , and so $\rho_\theta(\gamma)$ also preserves this foliation leafwise, and is thus parabolic. More specifically, if we let $\varepsilon = \sum_{i=1}^k \varepsilon_i$, then there are two cases. If $\varepsilon = 0$ then $\rho_\theta(\gamma)$ is a unipotent parabolic which is conjugate to γ . If $\varepsilon \neq 0$ then $\rho_\theta(\gamma)$ is an ellipto-parabolic isometry, whose angle of rotation is $\varepsilon \theta$.

See Apanasov [Ap, §4] for a detailed description in the $n = 2$ case. □

Remark 4.1 It is well known, see [T] or more generally [HT], that the complement in S^3 of the figure-8 knot does not contain an embedded totally geodesic hypersurface. Therefore, the deformations produced in Theorem 1.4 are distinct from those produced by Theorem 1.5.

References

- [Ap] B. Apanasov; *Bending deformations of complex hyperbolic surfaces*. J. Reine Angew. Math. **492** (1997), 75–91.
- [B1] S.A. Ballas; *Deformations of noncompact projective manifolds*. Algebr. Geom. Topol. **14** (2014), no. 5, 2595–2625.
- [B2] S.A. Ballas; *Finite volume properly convex deformations of the figure-eight knot*. Geom. Dedicata **178** (2015), 49–73.
- [BDL] S.A. Ballas, J. Danciger, G.-S. Lee; *Convex projective structures on non-hyperbolic three-manifolds*. Preprint 2015, arXiv 1508.0479.
- [BL] S.A. Ballas, D.D. Long; *Constructing thin subgroups commensurable with the figure-eight knot group*. Algebr. Geom. Topol. **15** (2015), no. 5, 3011–3024.
- [BM] S.A. Ballas, L. Marquis; *Properly convex bending of hyperbolic manifolds*. Preprint 2016, arXiv 1609.03046.
- [BSc] A. Bart and K. Scannell; *The generalized cuspidal cohomology problem*. Canad. J. Math. **58** (2006), no. 4, 673–690.
- [Ber] N. Bergeron; *Premier nombre de Betti et spectre du laplacien de certaines variétés hyperboliques*, Enseign. Math. (2). (2000), 109–137.
- [Bers] L. Bers; *Simultaneous uniformization*, Bull. Amer. Math. Soc. (66). (1960), 94–97.
- [CG] S. Chen, L. Greenberg; *Hyperbolic spaces*, in Contributions to Analysis. Academic Press, New York (1974), 49–87.
- [CLT] D. Cooper, D.D. Long and M.B. Thistlethwaite; *Flexing closed hyperbolic manifolds*. Geom. Topol. **11** (2007), 2413–2440.
- [CLTII] D. Cooper, D.D. Long and M.B. Thistlethwaite; *Computing varieties of representations of hyperbolic 3-manifolds into $SL(4, \mathbf{R})$* . J. of Experimental Math. vol. 15 (2006) pp. 291 – 305.
- [GR] H. Garland, M.S. Raghunathan; *Fundamental domains for lattices in $(\mathbb{R}-)$ rank 1 semisimple Lie groups*, Ann. of Math. (2) **92** (1970), 279–326.

- [Go] W.M. Goldman; *Complex Hyperbolic Geometry*. Oxford Mathematical Monographs. Oxford University Press (1999).
- [GPS] M. Gromov, I. Piatetski-Shapiro; *Non-arithmetic groups in Lobachevsky spaces*. Publ. Math. IHES **66** (1987), 93–103.
- [Gui] O. Guichard; *Groupes plongés quasi-isométriquement dans un groupe de Lie*. Math. Ann. **330** (2004), no. 2, 331–351.
- [HT] A. Hatcher and W. Thurston; *Incompressible surfaces in 2-bridge knot complements*. Invent. Math. **79** (1985), no. 2, 225–246.
- [HP] M. Heusener and J. Porti; *Infinitesimal projective rigidity under Dehn filling*. Geom. Topol. **15** (2011), no. 4, 2017–2071.
- [JM] D. Johnson and J. Millson; *Deformation Spaces Associated to Compact Hyperbolic Manifolds*, in Papers in Honor of G.D. Mostow on His Sixtieth Birthday (Roger Howe, ed.). Birkhäuser, Progress in Mathematics **67** (1986), 48–106.
- [Kap] M. Kapovich; *Deformations of representations of discrete subgroups of $SO(3, 1)$* . Math. Annalen. **299** (1994), 341–354.
- [KP] I. Kim, J. R. Parker; *Geometry of quaternionic hyperbolic manifolds*. Math. Proc. Camb. Phil. Soc. **135** (2003), 291–320.
- [PP] J.R. Parker, I.D. Platis; *Complex hyperbolic quasi-Fuchsian groups*, in: Geometry of Riemann Surfaces. London Mathematical Society Lecture Notes **368** (2010) 309–355.
- [MRS] D.B. McReynolds. and A.W. Reid, and M. Stover; *Collisions at infinity in hyperbolic manifolds*, Math. Proc. Cambridge Philos. Soc. (2013), 459–463.
- [Sc] K. Scannell; *Local rigidity of hyperbolic 3-manifolds after Dehn surgery*. Duke Math. J. **114** (2002), no. 1, 1–14.
- [S1] R.E. Schwartz; *The quasi-isometry classification of rank one lattices*. Publ Math. IHES **82** (1995), 133–168.
- [S2] R.E. Schwartz; *Complex hyperbolic triangle groups*, from: Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing (2002) 339–349.
- [T] W.P. Thurston; *The Geometry and Topology of Three-Manifolds*. Electronic version 1.1 (2002). Available at: <http://www.msri.org/publications/books/gt3m/>
- [W] A. Weil; *Discrete subgroups of Lie groups (II)*, Ann. of Math. (2) **75** (1962), 578–602.

SAMUEL BALLAS
DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY
ballas@math.fsu.edu

JULIEN PAUPERT
SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES, ARIZONA STATE UNIVERSITY
paupert@asu.edu

PIERRE WILL
INSTITUT FOURIER, UNIVERSITÉ DE GRENOBLE I
pierre.will@ujf-grenoble.fr