Exercise sheet 2

Series with positive terms

Exercise 1.

1. Find real numbers a, b, c such that for all $x \in \mathbb{R}$ with $x \neq 0, 1, -1$ we have

$$\frac{1}{x(x^2-1)} = \frac{a}{x-1} + \frac{b}{x} + \frac{c}{x+1}.$$

2. Using the previous relation for $x \in [\![2, n]\!]$, find for all integers $n \geq 2$ a simple expression for

$$S_n = \sum_{j=2}^{n} \frac{1}{j(j^2-1)} = \frac{1}{2(2^2-1)} + \frac{1}{3(3^2-1)} + \cdots + \frac{1}{n(n^2-1)}.$$

3. Deduce from above that $(S_n)_{n\in\mathbb{N}}$ converges and find its limit.

4. Prove that $\sum_{k\geq 2} 1/(k(k^2-1))$ converges and find its sum.

Exercise 2. *Convergence of a series*

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative terms. Prove the series $\sum_{n\geq 1} u_n$ converges if and only if the sequence $(\sum_{k=1}^{n^2} u_k)_{n\,\in\mathbb{N}}$ converges.

Exercise 3. *Convergent or divergent series*

Indicate whether the series associated to following sequences diverge or converge.

1. $u_n = e^n/(n^5+1)$
2. $u_n = 2^n/(3^n n^2)$
3. $u_n = e^{-n}/(4+\sin(n))$
4. $u_n = n/2^n$
5. $u_n = e^{1/n}/(n+1)$
6. $u_n = n\ln(1+1/n^2)$
7. $u_n = n!/n^n$
8. $u_n = ne^{-n}$
9. $u_n = \left(1-\frac{1}{n}\right)^{n^2}$

Exercise 4. *Convergent or divergent series*

Consider the series of the sequences $(u_n)_{n\in\mathbb{N}}$ whose general term is given by:

1. $u_n = \dfrac{1-n\ln(1+1/n)}{\sqrt{n+1}}$,
2. $u_n = \ln(n)/n$,
3. $u_n = (1+1/\sqrt{n})^n$,
4. $u_n = (e^{1/n}-1)/n$,
5. $u_n = ne^{-\sqrt{n}}$,
6. $u_n = \tan(1/n) - 1/n$,
7. $u_n = (1-1/\sqrt{n})^n$,
8. $u_n = \ln(n)/n^\alpha$ (discuss the problem in terms of α),
9. $u_n = n^2\left(\sin(1/n)+\cos(1/n)+\ln^2(1-1/n)-e^{1/n}\right)$, $n \geq 2$.

Decide if they converge or diverge.

Exercise 5. *Computation of sums of series*

Consider the following series:

1. $\sum_{n\in\mathbb{N}} 3^{-n}$,
2. $\sum_{n\geq 3} 2/5^n$,
3. $\sum_{n\in\mathbb{N}} 1/n!$,
4. $\sum_{n\geq 2} 3/(n-1)!$,
5. $\sum_{n\in\mathbb{N}} (n+2)/n!$,
6. $\sum_{n\in\mathbb{N}} (n^2+n+1)/n!$.

Prove that they converge and compute their sums.

Exercise 6.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence with positive terms such that $\sum_{n\in\mathbb{N}} u_n$ converges. Prove that the series

$$\sum_{n\in\mathbb{N}} \frac{u_n}{u_n+1}$$

converges.

Exercise 7. *Comparison between series and integrals*

1. Give an equivalent to $\sum_{k=1}^{n} 1/k^\alpha$ when $0 \leq \alpha < 1$.
2. Give an equivalent to $\sum_{k=n}^{+\infty} 1/k^\alpha$ when $\alpha > 1$.
3. Prove that $\ln(n!) \sim n\ln(n)$ when n tends to $+\infty$.

Exercise 8. *Equivalence of partial sums*

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two positive sequences such that $u_n \sim v_n$ when n tends to $+\infty$.

1. Suppose that the series $\sum_{n\in\mathbb{N}} u_n$ converges. Show that $\sum_{k>n} u_k \sim \sum_{k>n} v_k$ when n tends to $+\infty$.
2. Suppose that the series $\sum_{n\in\mathbb{N}} u_n$ diverges. Show that $\sum_{k=0}^{n} u_k \sim \sum_{k=0}^{n} v_k$ when n tends to $+\infty$.
3. Prove that there exists a constant C such that

$$\sum_{k=1}^{n} \frac{1}{k^2+\sqrt{k}} \underset{n\to+\infty}{=} C - \frac{1}{n} + \mathrm{o}\left(\frac{1}{n}\right).$$

Exercise 9. *Series and decimal expressions*

1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence taking value in $[\![0,9]\!]$. Show that the series

$$\sum_{n\in\mathbb{N}^*} \frac{a_n}{10^n}$$

converges.

2. Prove that if the sequence $(a_n)_{n\in\mathbb{N}^*}$ is periodic (from some point on), then the sum $\sum_{n\in\mathbb{N}^*} a_n/10^n$ is a rational number.

3. For $x \in [0,1[$, let x_1 be the integral part of $10.x$, and define by induction x_{n+1} to be the integral part of
$$10^{n+1}\left(x - \sum_{k=1}^{n} \frac{x_k}{10^k}\right).$$
Prove by induction that $x_n \in [\![0,9]\!]$ and
$$10^{n}\left(x - \sum_{k=1}^{n} \frac{x_k}{10^k}\right) \in [0,1[,$$
for all $n \in \mathbb{N}$.

4. Deduce from the above that $\sum_{n\in\mathbb{N}^*} x_n/10^n$ converges to x. We say that $(x_n)_{n\in\mathbb{N}^*}$ is the **associated sequence of the decimals** of x.

5. Prove that if x is rational, then the associated sequence $(x_n)_{n\in\mathbb{N}^*}$ is periodic from some point on.

Exercise 10. *Another proof of the convergence of Riemann series using block summation*

Let $(u_n)_{\mathbb{N}}$ be a decreasing sequence of real numbers. Set $v_n = 2^n u_{2^n}$.

1. Prove that the series $\sum_{n\in\mathbb{N}} u_n$ et $\sum_{n\in\mathbb{N}} v_n$ converge or diverge simultaneously.

2. Deduce from the previous point that the Riemann series converge.

3. Study the convergence of the series
$$\sum_{n\geq 2} \frac{1}{n\ln(n)} \text{ and } \sum_{n\geq 3} \frac{1}{n\ln(n)\ln(\ln(n))}.$$

Exercise 11. *Series with positive and decreasing general term*

Let $(u_n)_{n\in\mathbb{N}} \in \mathbb{R}_{>0}^{\mathbb{N}}$ be a positive and decreasing sequence such that $\sum_{n\in\mathbb{N}} u_n$ converges.

1. Show that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n > N$ we have $(n-N)u_n \leq \varepsilon$.

2. Prove that nu_n converges to 0 when n tends to $+\infty$.

3. Give an example of a positive sequence $(v_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} v_n$ converges and nv_n doesn't tend to 0 as n goes to $+\infty$.