Exercise sheet 1

Sequences, comparison of sequences and Taylor polynomials

Exercise 1. *Existence and computations of limits*

Decide whether the following sequences $(u_n)_{n\in\mathbb{N}}$ given by

1. $u_n = 1/(2n+1)$,
2. $u_n = (n+2)/(2n+3)$,
3. $u_n = n^2/(n+1)$,
4. $u_n = (10n^2+1)/(n^3-1)$,
5. $u_n = 1/(\sqrt{n+1}-\sqrt{n})$,
6. $u_n = (n+1)^2/((n+1)^3-n^3)$,
7. $u_n = n^{10}/1.01^n$,

converge or diverge. In the first case, compute the limit.

Exercise 2. *Equivalence, domination and negligibility*

For each of the following pair of sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$, verify whether $u_n \sim v_n$, $u_n = \mathrm{O}(v_n)$, $u_n = \mathrm{o}(v_n)$, $v_n = \mathrm{O}(u_n)$, and/or $v_n = \mathrm{o}(u_n)$ when n tends to $+\infty$ hold/s:

1. $u_n = 2^{-n}$, $v_n = 3^{-n}$;
2. $u_n = 1/n$, $v_n = 1/\sqrt{n}$;
3. $u_n = n^2$, $v_n = 2^n$;
4. $u_n = \cos(1/n)$, $v_n = e^{1/n}$;
5. $u_n = \cos(n)$, $v_n = 1$;
6. $u_n = \ln(n)$, $v_n = \sqrt{n}$;
7. $u_n = \sin(1/n)$, $v_n = 1/n$.

Exercise 3. *Asymptotic expansions*

Give the asymptotic expansion of (u_n) when n grows to infinity, with a remainder in $o(1/n^2)$ for each of the following sequences:

1. $u_n = \frac{1}{2n+1}$;
2. $u_n = \frac{n+1}{3+2n}$;
3. $u_n = \frac{\ln\left(1-\frac{1}{n}+\frac{1}{n^2}\right)}{\sqrt{1-\frac{1}{n}}}$;
4. $u_n = \left(1+\frac{1}{n}\right)^{n+2}$.

Exercise 4. *A few examples*

Give examples of the following situations :

1. an increasing positive sequence not converging to 0;
2. a bounded sequence which is not convergent;

3. a positive sequence which is not bounded and not tending to ∞;
4. a non monotone sequence not converging to 0;
5. two divergent sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ such that the product sequence $(u_nv_n)_{n\in\mathbb{N}}$ is convergent.

Exercise 5. *Limit of a product of sequences*

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be complex sequences. Assume that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are convergent. Prove that the product sequence $(u_nv_n)_{n\in\mathbb{N}}$ also converges and moreover

$$\lim_{n\to\infty} u_nv_n = \left(\lim_{n\to\infty} u_n\right).\left(\lim_{n\to\infty} v_n\right).$$

Exercise 6. *Subsequences*

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers.

1. Show that if $(u_{2n})_{n\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ both converge to the same limit, then $(u_n)_{n\in\mathbb{N}}$ also converges.
2. Show that if the sequences $(u_{2n})_{n\in\mathbb{N}}$, $(u_{2n+1})_{n\in\mathbb{N}}$ and $(u_{3n})_{n\in\mathbb{N}}$ are convergent, then $(u_n)_{n\in\mathbb{N}}$ also converges.

Exercise 7. *Computation of limits using usual functions*

Compute the limit, if it exists, of the following sequences $(u_n)_{n\in\mathbb{N}}$ given by:

1. $u_n = n^4(\ln(1-1/n^2)+1/n^2)$,
2. $u_n = n(e^{2/n}-1)$,
3. $u_n = n!/n^n$,
4. $u_n = \tan(1/n)\cos(2n+1)$,
5. $u_n = (\sqrt{n-3}+i\ln(2n))/\ln(n)$,
6. $u_n = \ln(n^2+3n-2)/\ln(n^{1/3})$.

Exercise 8. *Adjacent sequences*

1. Prove that each of the following pair of sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent.
   (i) $u_n = \sum_{k=1}^n 1/k^2$ and $v_n = u_n + 1/n$.
   (ii) $u_n = \sum_{k=1}^n 1/k^3$ and $v_n = u_n + 1/n^2$.
   (iii) $u_0 = a > 0$, $v_0 = b > a$, $v_{n+1} = (u_n+v_n)/2$ and $u_{n+1} = \sqrt{u_nv_n}$.
2. Define the real sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ by

$$u_n = \sum_{k=0}^n \frac{1}{k!} \text{ and } v_n = u_n + \frac{1}{n!n}.$$

   (i) Show that these sequences are adjacent, with a common limit e (it's a possible definition of e).

(ii) Show that e is not rational.
Hint : Suppose that $e = p/q$ and note that for $n \in \mathbb{N}$ we have the inequalities $n!u_n < n!p/q < n!v_n$. Then choose n such that $n!p/q$ is an integer.

Exercise 9. *Sequences defined recursively*

1. Let $f : [0,1] \to [0,1]$ be a continuous function such that $f(0) = 0$, $f(1) = 1$ and $f(x) < x$ for all $x \in]\,0,1\,[$. Define recursively a sequence $(u_n)_{n\in\mathbb{N}}$ by
$$\begin{cases} u_0 \in [0,1], \\ u_{n+1} = f(u_n), \text{ for all } n \in \mathbb{N}. \end{cases}$$
Show that the sequence $(u_n)_{n\in\mathbb{N}}$ converges and compute its limit.

2. Define recursively a sequence $(v_n)_{n\in\mathbb{N}}$ by
$$\begin{cases} v_0 = \frac{1}{2}, \\ v_{n+1} = \frac{v_n}{2-\sqrt{v_n}}, \text{ for all } n \in \mathbb{N}. \end{cases}$$
Show that the sequence $(v_n)_{n\in\mathbb{N}}$ converges and compute its limit.

Exercise 10. *Cesàro average*

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Define
$$S_n = \frac{u_1 + \ldots + u_n}{n}$$
for all $n \in \mathbb{N}$.

1. Show that if $(u_n)_{n\in\mathbb{N}}$ converges in $\mathbb{C}$, then $(S_n)_{n\in\mathbb{N}}$ converges to the same limit.
2. Give an example of a divergent sequence $(u_n)_{n\in\mathbb{N}}$ such that $(S_n)_{n\in\mathbb{N}}$ converges.
3. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of (strictly) positive real numbers such that u_{n+1}/u_n converges to $\ell \in \mathbb{R}^*$. Show that $(u_n^{1/n})_{n\in\mathbb{N}}$ converges to the same limit.

Exercise 11. *Lim sup and lim inf*

Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. Define sequences $(i_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ by
$$i_n = \inf\{u_k : k \geq n\} \text{ and } s_n = \sup\{u_k : k \geq n\}$$
for all $n \in \mathbb{N}$.

1. Show that both $(i_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ converge. The limit of $(i_n)_{n\in\mathbb{N}}$ is called **limit inferior** or **lower limit** of the sequence $(u_n)_{n\in\mathbb{N}}$, and is denoted by
$$\liminf_{n\to\infty} u_n.$$
The limit of $(s_n)_{n\in\mathbb{N}}$ is called **limit superior** or **upper limit** of the sequence $(u_n)_{n\in\mathbb{N}}$, and is written
$$\limsup_{n\to\infty} u_n.$$

2. Let $(u_{\varphi(n)})$ be a converging subsequence of (u_n). Show that

$$\liminf_{n\to\infty} u_n \leq \lim_{n\to\infty} u_{\varphi(n)} \leq \limsup_{n\to\infty} u_n.$$

3. Show that for any positive integer N and any positive real ϵ, the following set is non empty:

$$\{k > N : u_k \leq \liminf_{n\to\infty} u_n + \epsilon\}.$$

4. Deduce that there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ converging to the limit inferior of $(u_n)_{n\in\mathbb{N}}$ and another subsequence of $(u_n)_{n\in\mathbb{N}}$ converging to the limit superior.

5. Prove that $(u_n)_{n\in\mathbb{N}}$ converges if and only if $\liminf_{n\to\infty} u_n = \limsup_{n\to\infty} u_n$.