Summation polynomials and symmetries for the ECDLP over extension fields

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Background

The Elliptic Curve Discrete Log Problem

E elliptic curve defined over finite field $\mathbb{F}_q$, and $P, Q \in E(\mathbb{F}_q)$.

$$\text{Goal (ECDLP) : compute } x \text{ s.t. } Q = [x]P$$

- If $\mathbb{F}_q$ **prime field**: no known non-generic algorithms in general.
- If $\mathbb{F}_q = \mathbb{F}_{p^n}$ **extension field**: decomposition index calculus (Gaudry/Diem).

Decomposition index calculus

Outline of the attack:

1. Choose a **factor base** $\mathcal{F} \subset E(\mathbb{F}_{q^n})$.
2. Relation search step: look for **decompositions** of the form

$$[a]P + [b]Q = P_1 + \cdots + P_n, \quad P_i \in \mathcal{F}$$

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Step 2 **hopeless** if $\mathcal{F}$ arbitrary subset of $E(\mathbb{F}_{q^n})$.
Only method so far: define $\mathcal{F}$ algebraically, over subfield $\mathbb{F}_q$
$\rightarrow$ Weil restriction structure

Gaudry/Diem's decomposition

- Standard choice is $\mathcal{F} = \{P \in E(\mathbb{F}_{q^n}) : x(P) \in \mathbb{F}_q\}$
  $\rightarrow$ algebraic curve in the Weil restriction of E seen as a dim. n abelian variety over $\mathbb{F}_q$
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- Still not obvious to find decompositions. Main tool: description of the addition law on E with Semaev polynomials.

Semaev polynomials

Semaev summation polynomials

For all $k \geqslant 2$, there exists $S_k \in \mathbb{F}_{q^n}[X_1, \dots, X_k]$ irreducible s.t.

$$S_k(a_1, \dots, a_k) = 0 \iff \exists P_i \in E(\overline{\mathbb{F}_q}),\ x(P_i) = a_i \text{ and } \sum_i P_i = \mathcal{O}$$

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Degree 2^{k-2} in each variable $\rightarrow$ hard to compute for $k \geqslant 5$

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- Decomposition try for $R = [a]P + [b]Q$: solve

$$S_{n+1}(x_1, \ldots, x_n, x(R)) = 0 \text{ with } x_i \in \mathbb{F}_q$$

Restriction of scalar $\rightsquigarrow$ resolution of multivariate polynomial system defined over $\mathbb{F}_q$ with n variables/equations, total degree $n\,2^{n-2}$.

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This is the hardest part.

Natural improvements

- Factor base $\mathcal{F} = \{P \in E(\mathbb{F}_{q^n}) : x(P) \in \mathbb{F}_q\}$ is **invariant** by $-$:

$$P \in \mathcal{F} \Leftrightarrow -P \in \mathcal{F}$$

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$\rightarrow$ less relations needed and faster linear algebra

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Computation of decompositions still slow if $n \leqslant 4$, intractable if $n \geqslant 5$

Our contribution

Main idea

Replace x by arbitrary rational map $\varphi : E \to \mathbb{F}_{q^n}$ in definition of factor base:

$$\mathcal{F} = \{P \in E(\mathbb{F}_{q^n}) : \varphi(P) \in \mathbb{F}_q\}$$

Implies ability to define and compute associated summation polynomials.

Useful generalization?

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Useful generalization? **Yes!**

If φ well-chosen:

- $\mathcal{F}$ can have more invariance properties $\to$ further reduction of its size
- associated summation polynomial have more symmetries $\to$ easier to compute and faster decompositions

Summation polynomials

Theorem

For any rational map $\varphi : E \to \mathbb{F}_{q^n}$ *and* $k \geqslant 3$*, there exists a unique (up to constant)* $P_{\varphi,k} \in \mathbb{F}_{q^n}[X_1, \ldots, X_k]$*, irreducible, symmetric, s.t.*

$$P_{\varphi,k}(a_1, \ldots, a_k) = 0 \Longleftrightarrow \exists P_i \in E(\overline{\mathbb{F}_q}),\ \varphi(P_i) = a_i \text{ and } \sum_i P_i = \mathcal{O}$$

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"Projection of the group law on φ"

$\deg_{X_i} P_{\varphi,k}$ proportional to $(\deg \varphi)^k$ in general, and also for all interesting cases so far
→ computation tractable only if $\deg \varphi$ and k small.

Computation of summation polynomials

First method: Riemann-Roch

Observation

$P_1 + \cdots + P_k = \mathcal{O} \Leftrightarrow \exists f \in \bar{\mathbb{F}}_q(E)$ s.t. $\mathrm{div}(f) = (P_1) + \cdots + (P_k) - k(\mathcal{O})$
Function f in Riemann-Roch space $\mathcal{L}(k(\mathcal{O}))$.

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1. Write equation of E in terms of φ and a 2nd var. w (usually x or y)
2. Compute basis of $\mathcal{L}(k(\mathcal{O})) = \langle 1, f_2(\varphi, w), \ldots, f_k(\varphi, w)\rangle$ and consider $f = f_k(\varphi, w) + \lambda_{k-1} f_{k-1}(\varphi, w) + \cdots + \lambda_1$

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Steps 2-3 similar to Nagao's method for higher genus decomposition attacks

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4. Equate coeff. of F with elementary sym. polynomials $e_1, \ldots, e_k$ and compute Gröbner basis of these k equations wrt. elimination order.
5. The Gröbner basis contains $P_{\varphi,k}$ symmetrized, i.e. expressed in terms of $e_1, \ldots, e_k$

Computation of summation polynomials

Second method: induction and resultants

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$$P_1 + \cdots + P_k = \mathcal{O} \Leftrightarrow \exists Q \in E \text{ s.t. } \begin{cases} P_1 + \cdots + P_j + Q = \mathcal{O} \\ P_{j+1} + \cdots + P_k - Q = \mathcal{O} \end{cases}$$

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Assume for simplicity $\varphi(P) = \varphi(-P)\ \forall P \in E$. Then

$$P_1 + \cdots + P_k = \mathcal{O}$$
$$\Updownarrow$$
$$P_{\varphi,j+1}(\varphi(P_1), \ldots, \varphi(P_j), X) \text{ and } P_{\varphi,k-j+1}(\varphi(P_{j+1}), \ldots, \varphi(P_k), X)$$

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$$P_{\varphi,k}(X_1, \ldots, X_k) = \mathrm{Res}(P_{\varphi,j+1}(X_1, \ldots, X_j, X), P_{\varphi,k-j+1}(X_{j+1}, \ldots, X_k, X))$$

Computation by induction still requires to know $P_{\varphi,3}$.

Action of small torsion points

Fact: many elliptic curves only have *near-prime* cardinality
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Free relations

Let $T \in E(\mathbb{F}_{q^n})$ point of small order ℓ, $\tau_T : E \rightarrow E$ translation-by-T map.
Suppose $\mathcal{F}$ invariant by τ_T, i.e. $P \in \mathcal{F}$ iff $P + T \in \mathcal{F}$.

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Then each decomposition yields many more:

$$\begin{aligned} R &= P_1 + \cdots + P_n \\ &= (P_1 + T) + (P_2 + [\ell - 1]T) + \cdots + P_n \\ &= (P_1 + T) + (P_2 + T) + (P_3 + [\ell - 2]T) + \cdots + P_n \\ &= \ldots \end{aligned}$$

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Consequences

- Pro: size of factor base $\mathcal{F}$ can be effectively divided by ℓ
- Con: decreases the probability that a random R can be decomposed
- **Main advantage:** big speed-up in computation of summation polynomials and point decomposition

Equivariant morphisms

Goal: factor base $\mathcal{F} = \{P : \varphi(P) \in \mathbb{F}_q\}$ invariant by τ_T, $T \in E[\ell]$

First idea

Look for *invariant* $\varphi : E \to \mathbb{F}_{q^n}$, i.e.

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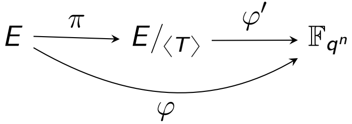

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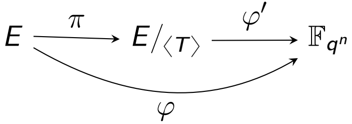

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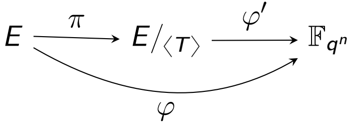

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$\Rightarrow$ equivalent decompositions on E with φ and on $E_{/\langle T \rangle}$ with φ', but
no use of torsion on the latter!

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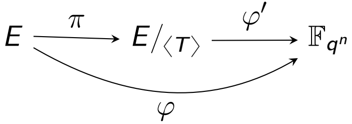
But then φ factorizes through quotient isogeny $E \to E/_{\langle T \rangle}$:



Same summation polynomials: $P_{\varphi,n} = P_{\varphi',n}$
$\Rightarrow$ equivalent decompositions on E with φ and on $E_{/\langle T \rangle}$ with φ', but
no use of torsion on the latter!

Equivariant morphisms

Goal: factor base $\mathcal{F} = \{P : \varphi(P) \in \mathbb{F}_q\}$ invariant by τ_T, $T \in E[\ell]$

Better idea

Look for *equivariant* $\varphi : E \to \mathbb{F}_{q^n}$, i.e. $\exists$ rational map $f : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ s.t.

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- So $f^{(\ell)} = f \circ \cdots \circ f = Id$
- Invariance of $\mathcal{F}$ requires stability by f of $\mathbb{F}_q$, or rather $\mathbb{P}^1(\mathbb{F}_q)$

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Fact

$$\varphi \text{ strictly equivariant wrt. translation by } T \in E[\ell] \ \Rightarrow \ \ell \mid \deg \varphi$$

Two-torsion in char 2: morphism

$E : y^2 + xy = x^3 + ax^2 + b$ ordinary elliptic curve over binary field $\mathbb{F}_{q^n}$.
Non-trivial 2-torsion point is $T_2 = (0, b^{1/2})$.

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Factor base can be effectively divided by 4 $\to \#\mathcal{F} \approx q/4$

Two-torsion in char 2: summation polynomials

Since $P_1 + \cdots + P_k = (P_1 + T_2) + (P_2 + T_2) + P_3 + \cdots + P_k = \ldots$,
we have $P_{\varphi,k}(X_1, \ldots, X_k) = P_{\varphi,k}(X_1 + 1, X_2 + 1, X_3, \ldots, X_k) = \ldots$

$\rightarrow$ invariant if even number of $+1$ added.

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Proposition

- *$P_{\varphi,k}$ invariant under affine action of the group $G_2 = (\mathbb{Z}/2\mathbb{Z})^{k-1} \rtimes \mathfrak{S}_k$.*
- *Invariant ring $\mathbb{F}_{q^n}[X_1, \ldots, X_k]^{G_2}$ free algebra, generated by*

$$\begin{aligned} e_1 &= X_1 + \cdots + X_k \\ s_2 &= Y_1 Y_2 + \cdots + Y_{k-1} Y_k \\ &\vdots \\ s_k &= Y_1 \ldots Y_k \end{aligned}$$

where $Y_i = X_i^2 + X_i$.

Two-torsion in char 2: results [FHJRV, Eurocrypt 2014]

Writing down $P_{\varphi,k}$ in terms of invariant generators $e_1, s_2, \ldots, s_k$ makes a **huge** difference:

k		3	4	5	6	7	8
Semaev polynomials	nb of monomials	3	6	39	638	–	–
	timings	0 s	0 s	26 s	725 s	–	–
$P_{\varphi,k}$	nb of monomials	2	3	9	50	2 247	470 369
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Computations for $k = 4$ to 7 in two steps:

1. take resultant of partially symmetrized summation polynomials
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Resultant too large for $k = 8$ case → dedicated interpolation technique

Two-torsion in char 2: results [FHJRV, Eurocrypt 2014]

Target: IPSEC Oakley curve, defined over $\mathbb{F}_{2^{31\times 5}}$.
Cardinality is 12 times a 151-bit prime → can use 2-torsion point.

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"$n-1$" approach: only known approach before this work. Estimated timing for one relation is $\approx$ 37 years (but easy to distribute).

With additional symmetries: $\approx$ 5.5 hr for one relation.

Still too slow for ECDLP resolution, but threatens non-standard problems e.g. oracle-assisted static Diffie-Hellman.

Two-torsion in odd char: morphism

$E: y^2 = c\,x(x-1)(x-\lambda)$ elliptic curve over $\mathbb{F}_{q^n}$ in twisted Legendre form.
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  - Either only use first invariance (from $\varphi(P+T_0) = -\varphi(P)$).
    Then $P_{\varphi,k}$ belongs to explicit invariant ring $\rightarrow$ results as in char. 2 case.
  - Or consider invariant *rational fraction*

$$Q_{\varphi,k}(X_1,\ldots,X_k) = \frac{P_{\varphi,k}(X_1,\ldots,X_k)}{(X_1\ldots X_k)^{2^{k-3}}}$$

and work with invariant fields instead.

Two-torsion in odd char: summation polynomials (2)

Proposition

- *$Q_{\varphi,k}$ is invariant under action of the group* $G_4 = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{k-1} \rtimes \mathfrak{S}_k$.
- *Invariant field $\mathbb{F}_{q^n}(X_1, \ldots, X_k)^{G_4}$ has explicit generators* $w_0, w_1, \sigma_1, \ldots, \sigma_{k-2}$.

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FYI:

$\sigma_i =$ i-th elementary symmetric polynomial in $X_1^2 + X_1^{-2}, \ldots, X_k^2 + X_k^{-2}$

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$w_0 = \sum_{i=0}^{\lfloor k/2 \rfloor} s_{2i}/(X_1 \cdots X_k)$, $\quad w_1 = \sum_{i=1}^{\lfloor (k-1)/2 \rfloor} s_{2i+1}/(X_1 \cdots X_k)$, where

s_i = i-th elementary symmetric polynomial in $X_1^2, \ldots, X_k^2$ (and $s_0 = 1$).

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  However in our case $Q_{\varphi,k}$ is **polynomial** in our choice of invariant generators
  $\rightarrow$ inductive computation with partially symmetrized resultants OK.

Two-torsion in odd char: results (1)

k	3	4	5	6
Semaev polynomials	5	36	940	–
$P_{\varphi,k}(s_1,\ldots,s_{k-1},e_k)$	5	13	182	4125
$Q_{\varphi,k}(\sigma_1,\ldots,\sigma_{k-2},w_0,w_1)$	3	6	32	396

Comparison of number of monomials for:

- Semaev polynomials, symmetrized wrt. the action of $\mathfrak{S}_k$
- $P_{\varphi,k}$ symmetrized wrt. the action of only one 2-torsion point
- $Q_{\varphi,k}$ symmetrized wrt. the action of the full 2-torsion

Note: less sparse than in char. 2

Two-torsion in odd char: results (2)

Target: random curve over OEF $\mathbb{F}_{(2^{31}+413)^5}$, with full 2-torsion and near-prime cardinality.

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With full 2-torsion: $\approx$ 2.5 days for one relation.

Equivariance for higher order torsion

Let G be a subgroup of $E(\mathbb{F}_{q^n})$.
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Strict equivariance $\Rightarrow$ injective homomorphism $G \to \mathrm{PGL}_2(\mathbb{F}_q)$
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Theorem

The only possible subgroups are:

- $G = E[2]$, *plus invariance wrt.* $[-1]$
- $G = \langle T \rangle \subset E[\ell]$, *plus equivariance wrt.* $[-1]$, *with either*
  - $\ell | q-1$
  - $\ell | q+1$
  - $\ell = char(\mathbb{F}_q)$

Case $\ell | q - 1$

If φ equivariant for $\langle T_\ell \rangle \subset E[\ell]$, we can always assume that

$$\varphi(P + T_\ell) = \zeta\varphi(P), \quad \zeta \in \mu_\ell^*(\mathbb{F}_q).$$

So $\varphi(P + T)/\varphi(P)$ independent of P if $T \in \langle T_\ell \rangle$

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Sounds familiar? **Pairings** are not far away...

Cartier pairing

Let ψ be the ℓ-isogeny $E \to E/_{\langle T_\ell \rangle}$. Then there exists a pairing on $\ker \psi \times \ker \hat{\psi} \simeq \langle T_\ell \rangle \times E[\ell]/_{\langle T_\ell \rangle}$.

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Let $T \in \langle T_\ell \rangle$ and $\overline{T'} \in E[\ell]/_{\langle T_\ell \rangle}$. Let $g_{T'}$ the function with divisor

$$\psi^*((\psi(T')) - (\mathcal{O})) = \sum_{i=1}^{\ell}(T' + [i]T_\ell) - \sum_{i=1}^{\ell}([i]T_\ell).$$

Then $e_\psi(T, \overline{T'}) = g_{T'}(P + T)/g_{T'}(P)$ is independent of $P \in E$.

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Because $T_\ell \in E(\mathbb{F}_{q^n})$ and $\ell | q-1$, function $g_{T'}$ is defined over $\mathbb{F}_{q^n}$.

Equivariant morphism for $\ell | q-1$

If T_ℓ, T' generate $E[\ell]$ then $g_{T'} : E \to \mathbb{P}^1$ is a strictly equivariant morphism.

To get equivariance wrt. $[-1]$, set $\varphi(P) = \dfrac{g_{T'}(P)}{g_{T'}(-P)}$ (at least if ℓ odd), so $\varphi(-P) = 1/\varphi(P)$.

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Proposition

This construction essentially yields all morphisms $E \to \mathbb{P}^1$ equivariant wrt. to translation by a ℓ-torsion point.

Case $\ell | q + 1$ is very similar, except that the action on $\mathbb{P}^1$ is less nice than $z \mapsto \zeta z$.

Summation polynomial and invariant ring

Assume $\varphi(P + T_\ell) = \zeta\varphi(P)$ and $\varphi(-P) = 1/\varphi(P)$.
As in the 2-torsion case, we have:

Proposition

- *$P_{\varphi,k}$ invariant under linear action of the group $G_\ell = (\mathbb{Z}/\ell\mathbb{Z})^{k-1} \rtimes \mathfrak{S}_k$.*
- *Invariant ring $\mathbb{F}_{q^n}[X_1, \ldots, X_k]^{G_\ell}$ free algebra, generated by*

$$s_1 = Y_1 + \ldots + Y_k,\ \ldots\ ,\ s_{k-1} = Y_1 \ldots Y_{k-1} + \ldots + Y_2 \ldots Y_k,\ e_k = X_1 \ldots X_k$$

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$$s_1 = Y_1 + ... + Y_k,\ \ ...\ ,\ s_{k-1} = Y_1...Y_{k-1} + ... + Y_2...Y_k,\ e_k = X_1...X_k$$

where $Y_i = X_i^\ell$.

Equivariance wrt. $[-1]$ more difficult to take into account: replacing polynomials by rational fractions gives no simplification.

Still allows to reduce size of factor base by 2.

Example

For $\ell = 3$ $(\ell | q-1)$, and $E : y^2 = x^3 + (x+a)^2$, the point $T = (0, a)$ has order 3.

The equivariant morphism is given by

$$\varphi(x,y) = \frac{\sqrt{3}y + i(x+3a)}{-\sqrt{3}y + i(x+3a)}.$$

Then the corresponding third summation polynomial is

$$\begin{aligned}P_{\varphi,3}(s_1,s_2,e_3) = & -27e_3^6+27s_1e_3^4+27s_2e_3^4-81e_3^5-9s_2^2e_3^2+54s_1e_3^3+54s_2e_3^3 \\ & -81e_3^4+s_1^3+3s_1^2s_2+3s_1s_2^2+s_2^3-9s_1^2e_3+27s_1e_3^2+27s_2e_3^2-27e_3^3 \\ & +\delta(12s_1^2e_3^3-(27a-16)(s_1^2e_3^2+s_2^2e_3)-(54a+16)(s_1s_2e_3^2+s_1s_2e_3)+12s_2^2),\end{aligned}$$

$\delta = 9/(27a-4)$.

Case $\ell = p$

If φ equivariant for $\langle T_p \rangle = E[p]$, we can always assume that

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Sounds familiar? Easy DLP in order p subgroup $\rightarrow$ **anomalous attack.**

Equivariant morphism for $\ell = p$

Let $T_p \in E[p]$ and $g(x) = \prod_{i=1}^{(p-1)/2} (x - x([i]T_p))$

($g \leftrightsquigarrow p$-th root of p-th division polynomial).

Proposition

There exists $\lambda \in \mathbb{F}_{q^n}$ such that the map $\varphi(x, y) = \frac{yg'(x)}{g(x)}$ satisfies the equivariance properties

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If φ can be computed efficiently for p large, gives a q-adic independent version of the anomalous attack.

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where $Y_i = X_i^p - X_i$.

Equivariance wrt. $[-1]$ more difficult to take into account: invariant ring is no longer a free algebra.

Still allows to reduce size of factor base by 2.

Example

For $p = 3$, and $E : y^2 = x^3 + (x + a)^2$, the point $T = (0, a)$ has order 3.

The equivariant morphism is simply given by

$$\varphi(x, y) = \frac{y}{x}.$$

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$$P_{\varphi,3}(e_1, s_2, s_3) = 2ae_1^6 + e_1^2 s_2^2 + e_1^3 s_3 + 2s_2^3.$$

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Fourth summation polynomial is

$$P_{\varphi,4}(e_1, s_2, s_3, s_4) = s_3^9 + e_1^3 s_3^8 + 120 \text{ other terms}.$$

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  Computations possible only for small values of $\ell > 2$ and n.

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  Con: larger degree for summation polynomials $\rightarrow$ harder decompositions

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- Further developments: more automorphisms ($j = 0$ or 1728), hyperelliptic curves.

Summation polynomials and symmetries for the ECDLP over extension fields

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