Attaques algébriques du problème du logarithme discret sur courbes elliptiques

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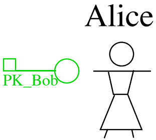
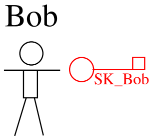
Soutenance de thèse

Asymmetric cryptography

Asymmetric cryptography





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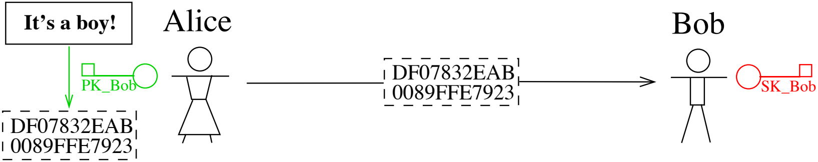

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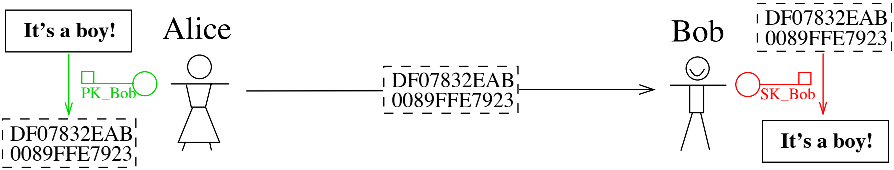

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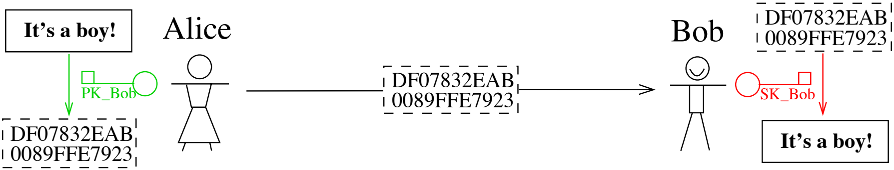

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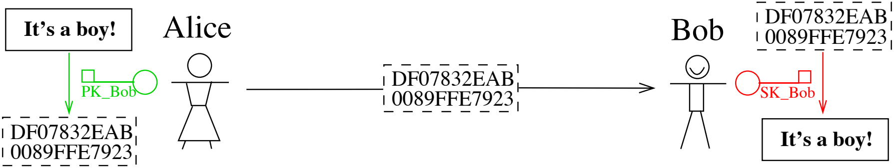

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$\Rightarrow$ asymmetric cryptography relies on **one-way functions**

Main one-way functions currently in use:

- multiplication of two primes (RSA)
- exponentiation in finite groups (Diffie-Hellman, ElGamal)
- evaluation of multivariate polynomial systems (HFE,UOV)

...

The Discrete Logarithm Problem

Let G be a group, $g \in G$ an element of finite order n.
The discrete logarithm of $h \in \langle g \rangle$ is the integer $x \in \mathbb{Z}/n\mathbb{Z}$ such that

$$h = g^x.$$

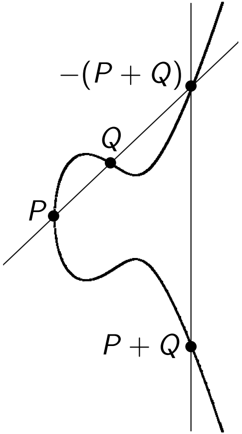
This is a one-way function:

- given g and x, easy to compute $h = g^x$, assuming an efficiently computable group law (*always the case here*)
- computing discrete log much harder in general: best *generic* algorithms in $\tilde{O}(\sqrt{r})$, r largest prime factor of n

DLP: given $g, h \in G$, find x – if it exists – such that $h = g^x$

Elliptic curve DLP

Good candidates for DLP-based cryptosystems:
elliptic curves defined over finite fields



ECDLP: Given $P \in E(\mathbb{F}_q)$ and $Q \in \langle P \rangle$
find x such that $Q = [x]P$

- On $\mathbb{F}_p$ (p prime): in general, no known attack better than generic algorithms
  $\rightarrow$ good security
- On $\mathbb{F}_{p^n}$ (for faster hardware arithmetic): possible to apply *index calculus*
  $\rightarrow$ security reduction in some cases

Part I

Resolution of multivariate polynomial systems

"Solving" polynomial systems

Consider multivariate polynomials $f_1, \ldots, f_m \in \mathbb{K}[X_1, \ldots, X_n]$

$$\begin{cases} f_1(x_1, \ldots, x_n) = 0 \\ \quad\quad\vdots \\ f_m(x_1, \ldots, x_n) = 0 \end{cases} \quad \Leftrightarrow \quad (x_1, \ldots, x_n) \in \mathbb{V}(\langle f_1, \ldots, f_m \rangle)$$

- If $\mathbb{V}(I)$ zero-dimensional, complete resolution makes sense
- Otherwise, goal is to obtain "good" descriptions of $\mathbb{V}(I)$, i.e. special sets of generators of I $\rightarrow$ provided by Gröbner bases

Hard problem in the general case

Main tool: Gröbner bases

$\mathbb{K}[\underline{X}] = \mathbb{K}[X_1, \ldots, X_n]$ polynomial ring, $\mathcal{T}$: set of all monomials

Monomial ordering

$\prec$ is an admissible monomial order if it is a well-founded strict total order on $\mathcal{T}$ such that $m' \prec m'' \Rightarrow m \cdot m' \prec m \cdot m''$

- main orders:
  - lexicographic order (*lex*)
  - graded reverse lexicographic order (*grevlex*)
- allows to define the leading monomial $LM(f)$ of a polynomial f

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Gröbner basis

$(g_1, \ldots, g_s)$ Gröbner basis of I (wrt $\prec$) if

$$I = \langle g_1, \ldots, g_s \rangle \text{ and } \forall f \in I,\ \exists i \text{ s.t. } LM(g_i) | LM(f)$$

Gröbner bases always exist and can be algorithmically computed

Elimination theory and shape lemma

I ideal of $\mathbb{K}[X_1, \ldots, X_n]$, $\quad I_k = I \cap \mathbb{K}[X_k, \ldots, X_n]$ k-th **elimination ideal**

If G is a lex GB of I, then $G \cap \mathbb{K}[X_k, \ldots, X_n]$ is a GB of I_k
$\rightsquigarrow$ lex GB provide "triangular systems"

Shape lemma

Up to a generic linear change of coordinates, the (reduced) lex GB of a 0-dim radical ideal is of the form

$$\big(X_1 - g_1(X_n), \quad \ldots, \quad X_{n-1} - g_{n-1}(X_n), \quad g_n(X_n)\big),$$

where $g_1, \ldots, g_n$ are univariate.

Algorithms for computing Gröbner basis

1. Buchberger (1965): uses critical pairs $(lcm, u_1, f_1, u_2, f_2)$ where $lcm = LM(f_1) \vee LM(f_2)$, $u_i = \frac{lcm}{LT(f_i)}$ to construct new elements of the GB
   - reduction of $u_1 f_1 - u_2 f_2$ modulo current basis very expensive
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4. FGLM (1993): change of order in the 0-dimensional case

Main algorithms

1. F4 algorithm: efficient combination of Buchberger and Lazard
   - fast and simultaneous reductions of several critical pairs: Macaulay-style matrix of polynomials from selected pairs and preprocessing + memorization of previous reductions
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   - elaborate criterion: skip unnecessary reductions
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   - rough complexity estimate: $\tilde{O}\left(\binom{n+d_{max}}{n}^{\omega}\right)$ (based on Lazard)

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- multipurpose algorithms
- what about polynomial systems arising from algebraic cryptanalysis?

A relevant case for cryptanalysis

Several examples from cryptanalysis/index calculus where systems can be considered as "similar":

$V \subset \mathbb{K}^\ell$ algebraic variety

- Parametric family of systems: $F_1, \ldots, F_r \in \mathbb{K}(V)[\underline{X}]$
- Random instance:
  $\{f_1, \ldots, f_r\} = \{F_1(y), \ldots, F_r(y)\} \subset \mathbb{K}[\underline{X}]$ for $y \in V$ random

$\rightarrow$ systems are *similar* if instances of a same parametric family

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$\rightarrow$ systems are *similar* if instances of a same parametric family

Goal: find a technique to solve efficiently many similar systems

A GB algorithm for similar systems

Contribution: the F4Remake algorithm

1. detect the useful critical pairs from F4 computation of a first instance
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When to use F4Remake?

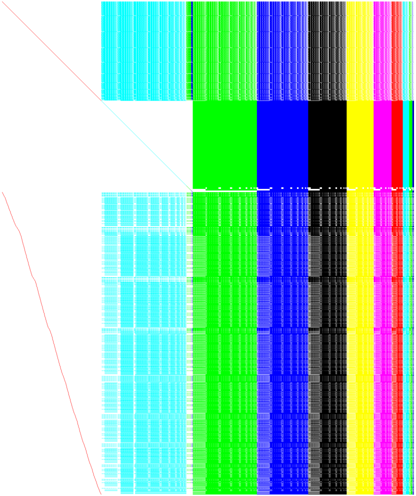
- need to solve many similar systems
- GB computation of one instance is feasible
- computation of "comprehensive Gröbner basis" intractable

Previous works

- GB computations over $\mathbb{Q}[\underline{X}]$ using CRT
- Traverso ('88): "GB traces" for Buchberger's algorithm in the rational case

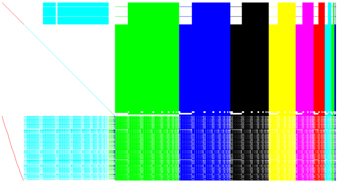
Performances of F4Remake

Example of a matrix of size $1\,539 \times 1\,285$ obtained with F4



Performances of F4Remake

Same matrix with F4Remake is of size $553 \times 1\,043$ (≈ 3.5 times smaller)



Performances of F4Remake

Advantages over F4/F5

- always faster than F4
- same rough complexity upper bound as F5, but computes much less polynomials in practice

F4Remake is a probabilistic algorithm

- heuristic probability of success greater than

$$\left(\prod_{i=1}^{\infty}(1-q^{-i})\right)^{n_{step}} \geq (1-2/q)^{n_{step}}$$

  $\rightarrow$ good probability over large fields
- can also perform well over small fields

Part II

The discrete logarithm problem for curves over extension fields

1. Decomposition index calculus

The index calculus method – basic outline

$(G, +) = \langle P \rangle$ finite abelian group of prime order r, $Q \in G$

1. Choice of a factor base: $\mathcal{F} = \{P_1, \ldots, P_N\} \subset G$
2. Relation search: decompose $[a_i]P + [b_i]Q$ (a_i, b_i random) into $\mathcal{F}$

$$[a_i]P + [b_i]Q = \sum_{j=1}^{N} [c_{ij}]P_j, \text{ where } c_{ij} \in \mathbb{Z}$$

3. Linear algebra: once k relations found ($k \geq N$)
   - construct the matrices $A = \begin{pmatrix} a_i & b_i \end{pmatrix}_{1 \leq i \leq k}$ and $M = (c_{ij})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq N}}$
   - find $v = (v_1, \ldots, v_k) \in \ker({}^t M)$ such that $vA \neq \begin{pmatrix} 0 & 0 \end{pmatrix} \bmod r$
   - compute the solution of DLP: $x = -\left(\sum_i a_i v_i\right) / \left(\sum_i b_i v_i\right) \bmod r$

Application to elliptic curves

No canonical choice of factor base nor natural way of finding decompositions

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What kind of "decomposition" over $E(K)$?

Main idea [Semaev '04]:

- consider decompositions in a fixed number of points of $\mathcal{F}$
$$R = [a]P + [b]Q = P_1 + \cdots + P_m$$
- convert this algebraically by using the $(m+1)$-*th summation polynomial*:
$$f_{m+1}(x_R, x_{P_1}, \ldots, x_{P_m}) = 0$$
$$\Leftrightarrow \exists \epsilon_1, \ldots, \epsilon_m \in \{1, -1\}, R = \epsilon_1 P_1 + \cdots + \epsilon_m P_m$$

Gaudry and Diem (2004)

"Decomposition attack": index calculus on $E(\mathbb{F}_{q^n})$

- Natural factor base: $\mathcal{F} = \{(x,y) \in E(\mathbb{F}_{q^n}) : x \in \mathbb{F}_q\}$
  $\mathcal{F}$ curve in *Weil restriction* $\mathcal{W}$ of $E \rightsquigarrow \#\mathcal{F} \simeq q$
- Relations involve $n = \dim \mathcal{W}$ points: $R = P_1 + \cdots + P_n$
- Restriction of scalars: decompose along a $\mathbb{F}_q$-linear basis of $\mathbb{F}_{q^n}$

$$f_{n+1}(x_R, x_{P_1}, \ldots, x_{P_n}) = 0 \Leftrightarrow \begin{cases} \varphi_1(x_{P_1}, \ldots, x_{P_n}) = 0 \\ \vdots \\ \varphi_n(x_{P_1}, \ldots, x_{P_n}) = 0 \end{cases} \qquad (\mathcal{S}_R)$$

One decomposition trial $\leftrightarrow$ resolution of $\mathcal{S}_R$ over $\mathbb{F}_q$

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One decomposition trial $\leftrightarrow$ resolution of $\mathcal{S}_R$ over $\mathbb{F}_q$

- With "double large prime" variation, overall complexity in
  $\tilde{O}\left(n! 2^{3n(n-1)} q^{2-2/n}\right)$
- Bottleneck: $\deg I(\mathcal{S}_R) = 2^{n(n-1)}$. But most solutions not in $\mathbb{F}_q$

Variant "$n-1$" [Joux-V. '10]

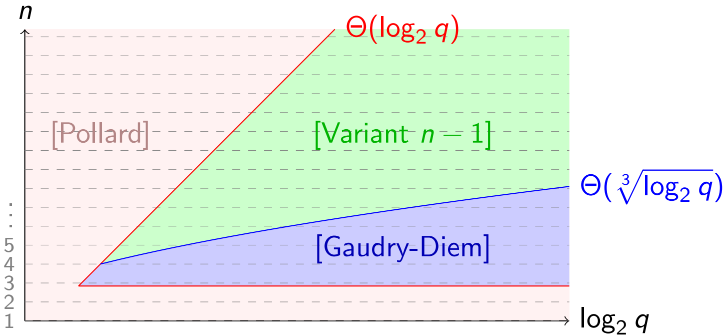
Decompositions into $m = n-1$ points

- compute the n-th summation polynomial (instead of $n+1$-th) with partially symmetrized resultant
- solve $\mathcal{S}_R$ with $n-1$ var, n eq and total degree 2^{n-2}
- $(n-1)!q$ expected numbers of trials to get one relation

Computation speed-up

1. $\mathcal{S}_R$ is overdetermined and $I(\mathcal{S}_R)$ has very low degree (0 or 1 excep.)
   - resolution with a *grevlex* Gröbner basis
   - no need to change order (FGLM)
2. Speed up computations with F4Remake

Comparaison of the three attacks of ECDLP over $\mathbb{F}_{q^n}$



Under some heuristic assumptions, complexity of variant $n-1$ in

$$\tilde{O}\left((n-1)!\left(2^{(n-1)(n-2)}e^n n^{-1/2}\right)^{\omega} q^2\right)$$

Example of application to $E(\mathbb{F}_{p^5})$

Standard 'Well Known Group' 3 Oakley curve

E elliptic curve defined over $\mathbb{F}_{2^{155}}$,
$\#E(\mathbb{F}_{2^{155}}) = 12 \cdot 3805993847215893016155463826195386266397436443$

- $\mathcal{F} = \{P \in E(\mathbb{F}_{2^{155}}) : x(P) \in \mathbb{F}_{2^{31}}\}$
- Decomposition test with variant $n-1$ takes 22.95 ms using F4Remake (on 2.93 GHz Intel Xeon)

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- too slow for complete DLP resolution
- but efficient threat for Oracle-assisted Static Diffie-Hellman Problem (only one relation needed)

Decomposition for Jacobians

$\mathcal{C}$ curve defined over $\mathbb{F}_{q^n}$ of genus g with a unique point $\mathcal{O}$ at infinity

Gaudry's framework

Work with $\mathcal{A} = W_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\mathrm{Jac}_{\mathcal{C}}(\mathbb{F}_{q^n}))$ of dim. ng

- Factor base containing about q elements
  $\mathcal{F} = \{D_Q \in \mathrm{Jac}_{\mathcal{C}}(\mathbb{F}_{q^n}) \ : \ D_Q \sim (Q) - (\mathcal{O}), Q \in \mathcal{C}(\mathbb{F}_{q^n}), x(Q) \in \mathbb{F}_q\}$
- Decomposition search: try to write arbitrary divisor $D \in \mathrm{Jac}_{\mathcal{C}}(\mathbb{F}_{q^n})$ as sum of ng divisors of $\mathcal{F}$

Asymptotic complexity for n, g fixed in $\tilde{O}(q^{2-2/ng})$

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How to check if D can be decomposed?

- Semaev's summation polynomials are no longer available
- use Riemann-Roch based reformulation of Nagao instead

Decomposition for hyperelliptic Jacobians over $\mathbb{F}_{q^n}$

Main difficulty in Nagao's decompositions

Solve a 0-dim quadratic polynomial system of $(n-1)ng$ eq./var. for each divisor $D(=[a_i]D_0 + [b_i]D_1) \in \mathrm{Jac}_{\mathcal{H}}(\mathbb{F}_{q^n})$.

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In practice:

- Decompositions as $D\sim\sum_{i=1}^{ng}((Q_i)-(\mathcal{O}_{\mathcal{H}}))$ are too slow to compute
- Faster alternative [Joux-V.]: compute relations involving only elements of $\mathcal{F}$

$$\sum_{i=1}^{ng+2}((Q_i)-(\mathcal{O}_{\mathcal{H}}))\sim 0$$

The modified relation search

$\mathcal{H}$ hyperelliptic curve of genus g defined over $\mathbb{F}_{q^n}$, $n \geq 2$

- find relations of the form $\sum_{i=1}^{ng+2} ((Q_i) - (\mathcal{O}_{\mathcal{H}})) \sim 0$
- linear algebra: deduce DL of factor base elements up to a constant
- descent phase: compute two Nagao-style decompositions to complete the DLP resolution

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- With Nagao: about $(ng)!\,q$ quadratic polynomial systems of $n(n-1)g$ eq./var. to solve
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Fast resolution

Goal: find a new set of generators of the ideal s.t. each specialization of two variables yields an easy to solve system $\rightarrow$ lex Gröbner basis

A special case: quadratic extensions in odd characteristic

Key point: define $\mathbb{F}_{q^2}$ as $\mathbb{F}_q(t)/(t^2 - \omega)$

Additional structure on the equations: polynomials obtained after restriction of scalars are multi-homogeneous of bidegree $(1, 1)$

$\rightarrow$ variables of the 1st block belong to a one-dimensional variety

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Decomposition method:

1. "specialization": choose a value for the first variables
2. remaining variables lie in a one-dimensional vector space $\rightsquigarrow$ easy to solve system

Further improvement possible by using a sieving technique

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2. remaining variables lie in a one-dimensional vector space $\rightsquigarrow$ easy to solve system

Further improvement possible by using a sieving technique

Much faster to compute decompositions with our variant

$\rightarrow$ about 960 times faster for $(n, g) = (2, 3)$ on a 150-bit curve

Part II

The discrete logarithm problem for curves over extension fields

2. Cover and decomposition

Transfer of the ECDLP via cover maps

Let $\mathcal{W} = W_{\mathbb{F}_{q^n}/\mathbb{F}_q}(E)$ be the Weil restriction of $E_{|\mathbb{F}_{q^n}}$ elliptic curve.
Inclusion of a curve $\mathcal{C}_{|\mathbb{F}_q} \hookrightarrow \mathcal{W}$ induces a **cover map** $\pi : \mathcal{C}(\mathbb{F}_{q^n}) \to E(\mathbb{F}_{q^n})$.

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\mathcal{C}(\mathbb{F}_{q^n}) \arrow[d, "\pi"] & \mathrm{Jac}_{\mathcal{C}}(\mathbb{F}_{q^n}) \arrow[r, "Tr"] & \mathrm{Jac}_{\mathcal{C}}(\mathbb{F}_q) \\
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   - $\tilde{O}(q^{2-2/g})$ if $\mathcal{C}$ is hyperelliptic with small genus g [Gaudry '00]
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The Gaudry-Heß-Smart technique

Construct $\mathcal{C}_{|\mathbb{F}_q}$ and $\pi : \mathcal{C} \to E$ from $E_{|\mathbb{F}_{q^n}}$ and a degree 2 map $E \to \mathbb{P}^1$

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The Gaudry-Heß-Smart technique

Problem: for most elliptic curves, $g(\mathcal{C})$ is of the order of 2^n

A combined attack

Let $E(\mathbb{F}_{q^n})$ elliptic curve such that

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Cover and decomposition attack [Joux-V.]

If n composite, combine both approaches:

1. use GHS on the subextension $\mathbb{F}_{q^n}/\mathbb{F}_{q^d}$ to transfer the DL to $\mathrm{Jac}_{\mathcal{C}}(\mathbb{F}_{q^d})$
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$\longrightarrow$ well adapted for curves defined over some Optimal Extension Fields

The sextic extension case

Comparisons and complexity estimates for 160 bits based on Magma

p 27-bit prime, $E(\mathbb{F}_{p^6})$ elliptic curve with 160-bit prime order subgroup

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$\mathbb{F}_{p^6}/\mathbb{F}_{p}$	intractable	efficient for $\leq 1/p^3$ curves $g = 9$: $\tilde{O}(p^{7/4})$, $\approx 1\,500$ years

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3. Cover and decomposition:
   $\tilde{O}(p^{5/3})$ cost using a hyperelliptic genus 3 cover defined over $\mathbb{F}_{p^2}$
   $\rightarrow$ occurs directly for $1/p^2$ curves and most curves after isogeny walk
   - Nagao-style decomposition: ≈ 750 years
   - Modified relation search: ≈ 300 years

A concrete attack on a 150-bit curve

$E: \; y^2 = x(x-\alpha)(x-\sigma(\alpha))$ defined over $\mathbb{F}_{p^6}$ where $p = 2^{25}+35$, such that $\#E = 4 \cdot 356814156285346166966901450449051336101786213$

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- Previously unreachable curve: GHS gives cover over $\mathbb{F}_p$ of genus 33...
- Complete resolution of DLP in about 1 month
  with cover and decomposition, using genus 3 hyperelliptic cover $\mathcal{H}_{|\mathbb{F}_{p^2}}$

Relation search

- lex GB: 2.7 sec with one core(1)
- sieving: $p^2/(2\cdot 8!) \simeq 1.4\times 10^{10}$ relations in 62 h on 1 024 cores(2)
  $\to$ 960$\times$ faster than Nagao

Linear algebra

- SGE: 25.5 h on 32 cores(2)
  $\to$ fivefold reduction
- Lanczos: 28.5 days on 64 cores(2)
  (200 MB of data broadcast/round)

(Descent phase done in $\sim$ 14 s for one point)

(1) Magma on 2.6 GHz Intel Core 2 Duo

(2) 2.93 GHz quadri-core Intel Xeon 5550

Attaques algébriques du problème du logarithme discret sur courbes elliptiques

Vanessa VITSE

Université de Versailles Saint-Quentin, Laboratoire PRiSM

Soutenance de thèse