DM 4

1. Soit $n \in \mathbb{N}^*$

$$u_n = \sum_{k=1}^{n} \frac{1}{(n+2k)^3} = \frac{1}{n^3} \sum_{k=1}^{n} \frac{1}{\left(1+2\frac{k}{n}\right)^3}$$

$$= \frac{1}{n^2} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{(1+2k/n)^3}$$

Or $f : x \longmapsto \frac{1}{(1+2x)^3}$ est continue sur $[0,1]$.

Ainsi, $\frac{1}{n} \sum_{k=1}^{n} \frac{1}{(1+2k/n)^3} \xrightarrow[n \to +\infty]{} \int_0^1 \frac{1}{(1+2x)^3} dx = \left[-\frac{1}{4(1+2x)^2} \right]_0^1 = \frac{2}{9}$

D'où $u_n \underset{n \to +\infty}{\sim} \frac{2}{9n^2}$

2. $t = \tan x$

$$\frac{dt}{dx} = \tan'(x) = 1 + \tan^2(x)$$

Ainsi, $\int_0^1 \frac{t^2}{(1+t^2)^2} dt = \int_0^{\pi/4} \frac{\tan^2(x)}{(1+\tan^2(x))^2} (1+\tan^2(x))\, dx$

$$= \int_0^{\pi/4} \frac{\tan^2(x)}{1+\tan^2(x)} dx$$

$\tan(x) = \frac{\sin(x)}{\cos(x)}$

$$= \int_0^{\pi/4} \frac{\sin^2(x)}{\cos^2(x)+\sin^2(x)} dx$$

$\sin^2(x) + \cos^2(x) = 1$

$$= \int_0^{\pi/4} \sin^2(x)\, dx$$

$$= \int_0^{\pi/4} \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^2 dx$$

$$= \int_0^{\pi/4} \frac{e^{2ix} - 2 + e^{-2ix}}{-4} dx$$

$$= \int_0^{\pi/4} \frac{1-\cos(2x)}{2}\,dx$$

$$= \left[\frac{x}{2} - \frac{\sin(2x)}{4} \right]_{x=0}^{x=\pi/4}$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

3. $$\int \frac{\ln(1+t)}{t^2}\,dt = -\frac{\ln(1+t)}{t} + \int \frac{1}{t(1+t)}\,dt$$

$$u'(t) = \frac{1}{t^2} \qquad u(t) = -\frac{1}{t}$$

$$v(t) = \ln(1+t) \qquad v'(t) = \frac{1}{1+t}$$

$$= -\frac{\ln(1+t)}{t} + \int \left(\frac{1}{t} - \frac{1}{1+t}\right)dt$$

$$= -\frac{\ln(1+t)}{t} + \ln(t) - \ln(1+t)$$

$$= -\ln(1+t)\left(1+\frac{1}{t}\right) + \ln(t)$$