Ex II, 5 8 :

Rappel : - $\nu(A) = \lambda(A) + \mu(A) \quad \forall A$

- $\int_X f d\lambda = \int_X fg \, d\nu \quad \forall f \in L^2(X, \mathcal{A}, \nu)$

- $0 \leq g(x) < 1 \quad \nu$ - p.p.

On pose $f_n = \chi_A \sum_{k=0}^{n} g^k$, alors par le Th. de convergence monotone, on a

$$\int_A h \, d\mu = \int_X \chi_A \sum_{k=1}^{+\infty} g^k \, d\mu$$

$$= \int_A \chi_A \lim_{n \to +\infty} \sum_{k=1}^{n} g^k \, d\mu$$

$$= \lim_{n \to +\infty} \int_A g \cdot f_{n-1} \, d\mu$$

$$= \lim_{n \to +\infty} \left(\int_A g \cdot f_{n-1} \, d\nu - \int_A g \cdot f_{n-1} \, d\lambda \right)$$

$$= \lim_{n \to +\infty} \int_A f_{n-1} \, d\lambda - \lim_{n \to +\infty} \int_A g \cdot f_{n-1} \, d\lambda$$

$$= \lim_{n \to +\infty} \int_A (1-g) f_{n-1} \, d\lambda$$

$$= \int_A (1-g) \lim_{n \to +\infty} f_{n-1} \, d\lambda$$

$$= \int_A \lim_{n \to +\infty} (1 - g^n) \, d\lambda$$

$$= \lambda(A).$$