Convergence of the wave equation damped on the interior to the one damped on the boundary

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Abstract : *In this paper, we study the convergence of the wave equation with variable internal damping term $\gamma_n(x)u_t$ to the wave equation with boundary damping $\gamma(x)\otimes\delta_{x\in\partial\Omega}u_t$ when $(\gamma_n(x))$ converges to $\gamma(x)\otimes\delta_{x\in\partial\Omega}$ in the sense of distributions. When the domain Ω in which these equations are defined is an interval in $\mathbb{R}$, we show that, under natural hypotheses, the compact global attractor of the wave equation damped on the interior converges in $X=\mathbb{H}^1(\Omega)\times\mathbb{L}^2(\Omega)$ to the one of the wave equation damped on the boundary, and that the dynamics on these attractors are equivalent. We also prove, in the higher dimensional case, that the attractors are lower-semicontinuous in X and upper-semicontinuous in $\mathbb{H}^{1-\varepsilon}(\Omega)\times\mathbb{H}^{-\varepsilon}(\Omega)$.*

Keywords : damped wave equation, boundary damping, attractor, stability, perturbation, Morse-Smale property
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1 Introduction

This article is devoted to the comparison of the dynamics of the wave equation damped in the interior of the domain Ω with the dynamics of the wave equation damped on the boundary of Ω, when the interior damping converges to a Dirac distribution supported by the boundary.

One of the physical motivation is the following. We consider a soundproof room, where carpet covers all the walls. This situation is modeled as follows. Let Ω be a smooth bounded domain of $\mathbb{R}^d$ ($d = 1, 2$ or 3) and let γ be a non-negative function in $\mathbb{L}^\infty(\partial\Omega)$ (the effective dissipation of the carpet at a point of the wall). The propagation of waves in the room is modeled by the wave equation damped in the boundary

$$\left\{\begin{array}{ll} u_{tt}(x,t) = (\Delta - Id)u(x,t) + f(x,u(x,t)) \ , & (x,t) \in \Omega \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu}(x,t) + \gamma(x) u_t(x,t) = 0 \ , & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u,u_t)_{|t=0} = (u_0,u_1) \in \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega) & \end{array}\right. \tag{1.1}$$

Notice that, in this model, the waves are not dissipated in the interior of the room but instantaneously damped at each rebound on the walls. This corresponds to a ponctual dissipation of the form $\gamma(x) \otimes \delta_{x\in\partial\Omega}$, where $\delta_{x\in\partial\Omega}$ is the Dirac function supported by the boundary. Of course, this is an approximation of the reality, as the carpet has some thickness. Thus, we can model more precisely the propagation of waves in the soundproof room by the equation

$$\left\{\begin{array}{ll} u_{tt}(x,t) + \gamma_n(x) u_t(x,t) = (\Delta - Id)u(x,t) + f(x,u(x,t)) \ , & (x,t) \in \Omega \times \mathbb{R}_+ \\ \frac{\partial}{\partial \nu} u(x,t) = 0 \ , & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ (u,u_t)_{|t=0} = (u_0,u_1) \in \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega) & \end{array}\right. \tag{1.2}$$

where γ_n is a bounded function, which is positive on a small neighborhood of $\partial\Omega$ and vanishes elsewhere.

The purpose of this paper is to study the relevance of the model equation (1.1), that is to understand in which sense the dynamics of Equation (1.2) converge to the ones of Equation (1.1) when γ_n converges to $\gamma_\infty = \gamma(x) \otimes \delta_{x\in\partial\Omega}$ in the sense of distributions. This paper is also an opportunity to present in a different way some classical proofs on stability of gradient Morse-Smale systems.

Both equations have been extensively studied, we cite for example [8], [10], [14], [23], [40] and [45] for the wave equation with internal damping (1.2) ; and [9], [11], [29], [31], [32], [44] and [46] for the wave equation with boundary damping (1.1). However, the convergence of the dynamics of Equation (1.2) to these of Equation (1.1) has apparently not yet been studied. The only work in this direction is the convergence of the internal control of the wave equation towards boundary control in the one-dimensional case (see [13]). In this paper, we have chosen to focus on the convergence of the compact global attractor of

(1.2) to the one of (1.1), when they exist, and on the comparison of the respective dynamics on them. Indeed, the compact global attractor, which consists of all the globally bounded solutions on $\mathbb{R}$, is somehow representative of the dynamics of the equation. We note that the study of convergence of attractors for other less regular perturbations is classical ; the main tools can be found for example in [19], [3], [4] and [41].
We introduce the spaces $X = \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega)$ and $X^s = \mathbb{H}^{1+s}(\Omega) \times \mathbb{H}^s(\Omega)$ ($s \in \mathbb{R}$). In the general case, we are able to prove results similar to the following one.

Let Ω be a smooth bounded domain of $\mathbb{R}^2$, let $\gamma_\infty = \delta_{x\in\partial\Omega}$ and $\gamma_n(x) = n$ if $\mathrm{dist}(x, \partial\Omega) < 1/n$ and 0 elsewhere. Let $f \in \mathcal{C}^2(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ be such that $\sup_{x\in\Omega} \limsup_{|u|\to+\infty} \frac{f(x,u)}{u} < 0$ and that there exist two constants $C > 0$ and $p \in \mathbb{R}_+$ so that $|f''_{uu}(x, u)| + |f''_{x,u}(x, u)| < C(1 + |u|^p)$ for $(x, u) \in \Omega \times \mathbb{R}$.

Theorem 1.1. *Let Ω, γ_n, γ_∞ and f be as described above. Then, Equations (1.1) and (1.2) have compact global attractors $\mathcal{A}_\infty$ and $\mathcal{A}_n$ respectively. Moreover, the union of the attractors $(\cup_{n\in\mathbb{N}\cup\{+\infty\}}\mathcal{A}_n)$ is bounded in X and the attractors $(\mathcal{A}_n)$ are upper-semicontinuous at $\mathcal{A}_\infty$ in X^{-s}, for any $s > 0$, that is,*

$$\sup_{U_n\in\mathcal{A}_n} \inf_{U_\infty\in\mathcal{A}_\infty} \|U_n - U_\infty\|_{X^{-s}} \longrightarrow 0 \ .$$

If all the equilibrium points of (1.1) are hyperbolic, the attractors $(\mathcal{A}_n)$ are lower- semicontinuous in X at $\mathcal{A}_\infty$. Moreover, the upper and lower semicontinuity can be estimated in the sense that there exists $\delta > 0$ such that

$$\max\left(\sup_{U_n\in\mathcal{A}_n} \inf_{U_\infty\in\mathcal{A}_\infty} \|U_n - U_\infty\|_{X^{-s}} \ ; \ \sup_{U_\infty\in\mathcal{A}_\infty} \inf_{U_n\in\mathcal{A}_n} \|U_\infty - U_n\|_X\right) \leq \frac{1}{n^\delta} \ .$$

In general, we cannot prove upper-semicontinuity in X because the perturbation is too singular. Let A_n and A_∞ be the linear operators associated respectively with the equations (1.2) and (1.1). The perturbation is not regular in the sense that $e^{A_n t}$ does not converge to $e^{A_\infty t}$ in $\mathcal{L}(X)$. However, we can prove that, in general, A_n^{-1} converges to A_∞^{-1} in $\mathcal{L}(X)$ and that this convergence of the inverses implies convergence of the trajectories in X^{-s} for any initial data in X, and convergence of the trajectories in X if the initial data (u_0, u_1) are bounded in a more regular space X^s ($s > 0$).
The proof of the lower-semicontinuity in X uses as main arguments the gradient structure of (1.1) and (1.2), as well as the convergence of the local unstable manifolds of the equilibria. To prove this property, we identify the local unstable manifolds with local strongly unstable manifolds and show the continuity of these manifolds with respect to the parameter n. Although our perturbation is irregular, we can prove lower-semicontinuity in X due to the regularity of the local unstable manifolds of the equilibria of the limit problem.
The upper-semicontinuity instead cannot be shown in X in general. Indeed, we know that the union $\cup_n\mathcal{A}_n$ is bounded in X, but we do not know if it is bounded in a more regular

space X^s. Thus, for initial data in $\cup_n \mathcal{A}_n$, we are able to compare the trajectories only in the norm of X^{-s}.
To prove upper-semicontinuity in X, we need to bound $\cup_n \mathcal{A}_n$ in X^s for some $s > 0$. The main way to prove this property is to show a uniform decay rate for the semigroups, that is that there exist constants $M > 0$ and $\lambda > 0$ such that, for all $U \in X$ and $t \geq 0$, we have

$$\forall n \in \mathbb{N},\ \|e^{A_n t} U\|_X \leq M e^{-\lambda t} \|U\|_X \ . \tag{1.3}$$

Such estimate is well-known for fixed n. However, the methods for proving the exponential decay for fixed n often give constants M and λ depending on $\|\gamma_n\|_{\mathbb{L}^\infty}$, or are based on a contradiction argument. Thus, they are not adaptable to the proof of a uniform estimate in the case of our irregular perturbation, where $\|\gamma_n\|_{\mathbb{L}^\infty}$ goes to $+\infty$. In dimension two and higher dimension, the uniform bound (1.3) is not known to hold, except for some very particular examples presented here. In the one-dimensional case, we give necessary and sufficient conditions for (1.3) to hold. The proof uses a multiplier method and is inspired by [13] and [23] (other methods are also possible, see the result of [2] in the appendix). Thus, in dimension one, we can show a more precise result, which is typically the following.

Let $\Omega =]0,1[$, $\gamma_\infty = 2\delta_{x=0}$ and $\gamma_n(x) = 2n$ if $x \in]0, \frac{1}{n}[$ and 0 elsewhere. Let $f \in \mathcal{C}^2([0,1] \times \mathbb{R}, \mathbb{R})$ be such that $\sup_{x \in \Omega} \limsup_{|u| \to +\infty} \frac{f(x,u)}{u} < 0$. Notice that we do not choose $\gamma_\infty = \delta_{x=0}$ because, with this dissipation, Equation (1.1) does not satisfy the backward uniqueness property. Without backward uniqueness result, we cannot properly define the Morse-Smale property (see [11] and the remarks preceding Theorem 2.12).

Theorem 1.2. *Let Ω, γ_n, γ_∞ and f be as described above. Then, Equations (1.1) and (1.2) have compact global attractors $\mathcal{A}_\infty$ and $\mathcal{A}_n$ respectively. Moreover, the union of the attractors $(\cup_{n \in \mathbb{N} \cup \{+\infty\}} \mathcal{A}_n)$ is bounded in X^s for $s \in]0, 1/2[$. As a consequence, the attractors $\mathcal{A}_n$ are upper-semicontinuous at $\mathcal{A}_\infty$ in the space X.*
If all the equilibrium points of (1.1) are hyperbolic, then the sequence of attractors $(\mathcal{A}_n)$ is continuous in X in the sense that there exists $\delta > 0$ such that

$$\max \left(\sup_{U_n \in \mathcal{A}_n} \inf_{U_\infty \in \mathcal{A}_\infty} \|U_n - U_\infty\|_X \ ; \ \sup_{U_\infty \in \mathcal{A}_\infty} \inf_{U_n \in \mathcal{A}_n} \|U_\infty - U_n\|_X \right) \leq \frac{1}{n^\delta} \ .$$

In dimension one, we can even go further and compare the dynamics on the attractors $\mathcal{A}_n$ and $\mathcal{A}_\infty$. A part of this comparison is described by the notion of equivalence of phase-diagrams. Let $S(t)$ be a gradient dynamical system which admits a compact global attractor with only hyperbolic equilibrium points. If E and E' are two equilibrium points of $S(t)$, we say that $E \leq E'$ if and only if there exists a trajectory $U(t) \in \mathcal{C}^0(\mathbb{R}, X)$ such that

$$\lim_{t \to -\infty} U(t) = E' \text{ and } \lim_{t \to +\infty} U(t) = E \ .$$

The phase-diagram of $S(t)$ is the above oriented graph on the set of equilibria. Two phase-diagrams are equivalent if there exists an isomorphism between the set of equilibria, which preserves the oriented edges.
It is proved in [19], [37] and [38] that the stability of phase-diagrams is related to the Morse-Smale property. We recall that a gradient dynamical system $S(t)$ has the Morse-Smale property if it has a finite number of equilibrium points which are all hyperbolic and if the stable and unstable manifolds of these equilibria intersect transversally. The result of [19] says that if $S_0(t)$ is a dynamical system, which satisfies the Morse-Smale property, and if $S_\varepsilon(t)$ is a "regular" perturbation of $S_0(t)$ such that the compact global attractors of $S_\varepsilon(t)$ are upper-semicontinuous at $\varepsilon = 0$, then $S_\varepsilon(t)$ satisfies the Morse-Smale property for ε small enough and its phase-diagram is equivalent to the one of $S_0(t)$. Unfortunately, our perturbation is not regular enough for a direct application of [19]. However, using the smoothness of the attractors, we can adapt the proof of [19] to show the following result.

Theorem 1.3. *Let Ω, γ_n, γ_∞ and f be as in Theorem 1.2. If the dynamical system generated by (1.1) satisfies the Morse-Smale property, then, for n large enough, the dynamical system generated by (1.2) satisfies the Morse-Smale property and its phase-diagram is equivalent to the one of (1.1). Moreover, there exists a homeomorphism h defined from $\mathcal{A}_n$ into $\mathcal{A}_\infty$ which maps the trajectories of $S_n(t)_{|\mathcal{A}_n}$ onto the trajectories of $S_\infty(t)_{|\mathcal{A}_\infty}$ preserving the sense of time.*

We notice that (1.1) satisfies the Morse-Smale property for a generic non-linearity f (see [26]). We also enhance that we give a proof of Theorem 1.3 presented in a way, which is different from [19], and, which extensively uses the gradient structure of (1.1) and (1.2).

Of course, in this paper, we do not only consider the particular situations of Theorems 1.2 and 1.1, but more general cases. The general frame, the main hypotheses and the main results are stated in Section 2. The abstract result of convergence for semigroups of contractions and the study of the convergence of the trajectories of Equation (1.1) to those of Equation (1.2) are given in Section 3. Continuity of the local unstable manifolds and of part of the local stable manifolds as well as stability of phase-diagrams are studied in Sections 4 and 5 respectively. In Section 6, we give concrete conditions under which the inequality (1.3) holds. In Section 7, we describe examples of applications. Finally, in the Appendix, we state the above-mentionned result of [2] and study another one-dimensional case.

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2 Setting of the problem and main results

In this section, we first introduce the notation. We immediately prove a first result of convergence, without which nothing can be done. This leads to a condition, which will be implicitely assumed in all what follows. Finally, in the last part of this section, we put together the main hypotheses, which will be used, and state the most important results.

2.1 The abstract frame

We introduce an abstract frame for Equations (1.1) and (1.2). This has two purposes. The first one is to give results, which concern a larger family of equations than (1.1) and (1.2) (for example, other boundary conditions can be chosen). The second advantage of the abstract setting is to gather Equations (1.1) and (1.2) into a common frame, which makes the comparison easier.

Let Ω be a smooth bounded domain of $\mathbb{R}^d$ ($d = 1, 2$ or 3) and let ω_N be a non-empty smooth open subset of $\partial\Omega$. We denote by ω_D the largest open subset of $\partial\Omega \setminus \omega_N$.
If $\omega_D \neq \emptyset$, we set $B = -\Delta_{BC}$ where Δ_{BC} is the Laplacian with Neumann boundary condition on ω_N and Dirichlet condition on ω_D. If ω_N covers the whole boundary, we set $B = -\Delta_N + Id$ where Δ_N is the Laplacian with Neumann boundary condition. In all the cases, B is a positive self-adjoint operator from $D(B)$ into $\mathbb{L}^2(\Omega)$.
Let (λ_k, φ_k) be the set of eigenvalues of B and corresponding eigenvectors normalized in $\mathbb{L}^2(\Omega)$. We denote $D(B^{s/2})$ the Hilbert space

$$D(B^{s/2}) = \left\{ u = \sum c_k \varphi_k \ / \ \|u\|^2_{D(B^{s/2})} = \sum |c_k|^2 \lambda_k^s < +\infty \right\} .$$

We notice that for $s \in [0, 1/2[$, $D(B^{s/2}) = \mathbb{H}^s(\Omega)$ and for $s \in]1/2, 5/4]$, $D(B^{s/2}) = \mathbb{H}^s(\Omega) \cap \{u \in \mathbb{H}^s(\Omega) / u_{|\omega_D} = 0\}$ (see Proposition 2.1). For larger s, the domain of $B^{s/2}$ can be less simple due to the regularity problem induced by mixed boundary conditions. We set

$$X = D(B^{1/2}) \times \mathbb{L}^2(\Omega) \ ,$$

endowed with the product topology. We also set $X^s = D(B^{(1+s)/2}) \times D(B^{s/2})$. Let γ be a non-negative function in $\mathbb{L}^\infty(\omega_N)$, which is positive on an open subset of ω_N. We set $\gamma_\infty(x) = \gamma(x)\delta_{x\in\omega_N}$. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of non-negative functions in $\mathbb{L}^\infty(\Omega)$, which are positive on an open subset of Ω and which converge to γ_∞ in the sense of distributions, that is that

$$\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d), \ \int_\Omega \gamma_n \varphi \longrightarrow \int \gamma_\infty \varphi = \int_{\omega_N} \gamma\varphi \ .$$

For each $n \in \mathbb{N}$, we introduce the linear continuous operator Γ_n, defined from $D(B^{1/2})$ into $D(B^{1/2})$ by $\Gamma_n = B^{-1}(\gamma_n .)$. We also introduce the operator Γ_∞ defined from $D(B^{1/2})$ into

$D(B^{1/2})$ by

$$\forall u\in D(B^{1/2}),\ \Gamma_\infty u \text{ is the solution of } \left\{\begin{array}{ll}(\Delta-\kappa Id)\Gamma_\infty u=0 & \text{on } \Omega\\ \frac{\partial}{\partial\nu}\Gamma_\infty u=\gamma(x)u & \text{on } \omega_N\\ \Gamma_\infty u=0 & \text{on } \omega_D\end{array}\right. , \tag{2.1}$$

where $\kappa=1$ if $\omega_D=\emptyset$ and $\kappa=0$ if not. We remark that

$$\forall n\in\mathbb{N},\ \forall\varphi,\psi\in D(B^{1/2}),\ \ <\Gamma_n\varphi|\psi>_{D(B^{1/2})}=\int_\Omega\gamma_n\varphi\overline{\psi}\ ,$$

and

$$\forall\varphi,\psi\in D(B^{1/2}),\ \ <\Gamma_\infty\varphi|\psi>_{D(B^{1/2})}=\int_{\partial\Omega}\gamma\varphi\overline{\psi}\ .$$

We set

$$s_0=\left\{\begin{array}{ll}s_0=1/2 & \text{for } d=1 \text{ or } d=2\\ s_0=1/4 & \text{for } d=3\end{array}\right. \tag{2.2}$$

Proposition 2.1. *For all $\varepsilon>0$, $s\in[0,s_0[$ and $n\in\mathbb{N}\cup\{+\infty\}$, the operator Γ_n can be extended to a continuous linear operator from $D(B^{\varepsilon+1/4})$ into $D(B^{(1+s)/2})$. In particular, Γ_n is a compact non-negative selfadjoint operator from $D(B^{\varepsilon+1/4})$ into $D(B^{1/2})$.*

Proof : The proposition follows from the regularity properties of the operator B. If $\overline{\omega}_N\cap\overline{\omega}_D=\emptyset$, then the regularity is clear since $D(B^{(1+s)/2})=\{u\in\mathbb{H}^{1+s}(\Omega)/u_{|\omega_D}=0\}$ if $s<1/2$ for any d. If we have mixed boundary conditions with $\overline{\omega}_N\cap\overline{\omega}_D\neq\emptyset$, then the regularity is more difficult to obtain. In dimension $d=2$ (resp. $d=3$), we refer to [16] (resp. [12]). □

For all $n\in\mathbb{N}\cup\{+\infty\}$, let A_n be the unbounded operator defined on X by

$$\forall\begin{pmatrix}u\\v\end{pmatrix}\in X,\ A_n\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}v\\-B(u+\Gamma_n v)\end{pmatrix}\ ,$$

$$D(A_n)=\left\{\begin{pmatrix}u\\v\end{pmatrix}\in X\ \Big/\ v\in D(B^{1/2}) \text{ and } u+\Gamma_n v\in D(B)\right\}\ .$$

We enhance that, if n is finite, A_n is the classical wave operator

$$\forall n\in\mathbb{N},\ A_n=\begin{pmatrix}0 & Id\\ -B & -\gamma_n\end{pmatrix},\ D(A_n)=D(B)\times D(B^{1/2})\ .$$

Using the Hille-Yosida theorem, one shows that the operator A_n generates a linear $\mathcal{C}^0-$semigroup $e^{A_n t}$ of contractions (see [29] for $n=+\infty$, see also [26] for a proof in the given abstract frame). In particular, A_n is dissipative since

$$\forall U=(u,v)\in D(A_n),\ \ <A_nU|U>_X=-<\Gamma_n v|v>_{D(B^{1/2})}\leq 0\ . \tag{2.3}$$

For $U=(u,v)$, we set

$$F(U)=\left(\begin{array}{c}0\\ f(x,u)\end{array}\right)\ . \tag{2.4}$$

We are interested in the convergence of the following family of equations, when n goes to $+\infty$

$$\left\{\begin{array}{l}U_t=A_nU+F(U)\\ U_{|t=0}=U_0\in X\end{array}\right. \ . \tag{2.5}$$

We first introduce conditions so that the above equations are be well-posed.
In the whole paper, we assume that the non-linearity f satisfies the following hypothesis.

(NL) $f\in\mathcal{C}^2(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ and if the dimension is

d=2 there exist $C>0$ and $\alpha\geq 0$ such that

$$|f''_{uu}(x,u)|+|f''_{ux}(x,u)|\leq C(1+|u|^\alpha)\ .$$

d=3 there exist $C>0$ and $\alpha\in[0,1[$ such that

$$|f''_{uu}(x,u)|\leq C(1+|u|^\alpha)\text{ and }|f''_{ux}(x,u)|\leq C(1+|u|^{3+\alpha})\ .$$

Since the regularity of f is not the main purpose of this paper, we choose to state Hypothesis (NL) in a simple but surely too strong way. For example, the condition $f\in\mathcal{C}^2$ could be relaxed to the condition $f\in\mathcal{C}^1$ with Hölder continuous derivatives. We can also assume an exponential growth rate for the non-linearity if $d=2$ (see [21] or [5]). We notice that, for most of our results, weaker hypotheses on f are sufficient. For example, the critical case of a cubic non-linearity in dimension $d=3$ is studied in [27].
To obtain global existence of solutions and existence of a compact global attractor, we also need to assume a dissipative condition for f, for example,

(Diss) $$\sup_{x\in\Omega}\limsup_{|u|\to+\infty}\frac{f(x,u)}{u}<0\ .$$

Classical Sobolev imbeddings (see for example [1]) show that Hypothesis (NL) implies the following properties (see Chapters 4.7 and 4.8 of [17] for a proof).

Lemma 2.2. *Assume that Hypothesis (NL) holds. Then, there exists a positive number p such that for any u, v in $\mathbb{H}^1(\Omega)$, we have*

$$\|f(x,u)-f(x,v)\|_{\mathbb{L}^2}\leq C(1+\|u\|^p_{\mathbb{H}^1}+\|v\|^p_{\mathbb{H}^1})\|u-v\|_{\mathbb{H}^1}\ .$$

Moreover, if $\mathcal{B}$ is a bounded set of $\mathbb{H}^1(\Omega)$, then $\{f(x,u)|u\in\mathcal{B}\}$ and $\{f'_u(x,u)v|(u,v)\in\mathcal{B}^2\}$ are bounded subsets of $\mathbb{H}^\sigma(\Omega)$, where $\sigma\in]0,1[$ when $d=1$ or $d=2$ and $\sigma\in]0,\frac{1-\alpha}{2}[$ when $d=3$. In addition, we have

$$\forall u\in\mathcal{B},\ \|f(x,u)\|_{\mathbb{H}^\sigma}\leq C_\sigma\|u\|_{\mathbb{H}^1}\ \textit{and}\ \|f'_u(x,u)v\|_{\mathbb{H}^\sigma}\leq C_\sigma\|v\|_{\mathbb{H}^1}\ ,$$

where the constant C_σ depends on σ, except if $d = 1$.
In particular, $F : (u, v) \in X \mapsto (0, f(x, u))$ is of class $\mathcal{C}^{1,1}_{loc}(X, X)$ and is a compact and Lipschitz-continuous function on the bounded sets of X.

Using a classical result of local existence (see [39], Chapter 6, Theorem 1.2), we deduce from Hypothesis (NL) that for each $n \in \mathbb{N} \cup \{+\infty\}$, Equation (2.5) generates a local dynamical system $S_n(t)$ on X.

Proposition 2.3. *If f satisfies (NL), then for all $M > 0$ and $K > 0$, there exists a time $T > 0$ such that, for all $n \in \mathbb{N} \cup \{+\infty\}$ and U_0 with $\|U_0\|_X \leq M$, Equation (2.5) has a unique mild solution $U_n(t) = S_n(t)U_0 \in \mathcal{C}^0([0, T], X)$, which satisfies*

$$\forall t \in [0, T],\ \|U_n(t)\|_X \leq M + K\ .$$

Moreover, there exists a constant $C > 0$ such that for all U_0 and U_0' with $\|U_0\|_X \leq M$ and $\|U_0'\|_X \leq M$ we have

$$\forall n \in \mathbb{N} \cup \{+\infty\},\ \forall t \in [0, T],\ \|S_n(t)(U_0 - U_0')\|_X \leq Ce^{Ct}\|U_0 - U_0'\|_X\ .$$

The hypothesis (Diss) implies global existence of trajectories, that is that $S_n(t) : X \longrightarrow X$ are global dynamical systems.

Proposition 2.4. *Assume that f satisfies (NL) and (Diss). Then, for any bounded set $\mathcal{B}$ of X, for any $n \in \mathbb{N} \cup \{+\infty\}$ and for any $U_0 \in \mathcal{B}$, $S_n(t)U_0$ $(t \geq 0)$ is a global mild solution of (2.5) and is uniformly bounded in X with respect to t and U_0.*

Proof : For $U = (u, v) \in X$, we set

$$\Phi(U) = \frac{1}{2}\|U\|_X^2 - \int_\Omega \int_0^u f(x, \zeta)d\zeta\ . \tag{2.6}$$

From (2.3) and the density of $D(A_n)$ in X, we deduce that the functional Φ is non-increasing along the trajectories of the dynamical systems $S_n(t)$ $(n \in \mathbb{N} \cup \{+\infty\})$. Indeed, let $U_0 \in D(A_n)$ and $U(t) = (u(t), v(t)) = S_n(t)U_0$, we have

$$\Phi(U(t_2)) - \Phi(U(t_1)) = \int_{t_1}^{t_2} < A_nU(t)|U(t) >_X dt = -\int_{t_1}^{t_2} < \Gamma_n v(t)|v(t) >_{D(B^{1/2})} \leq 0\ . \tag{2.7}$$

Hypothesis (Diss) implies that there exist two positive constants C and μ such that

$$f(x, u)u \leq C - \mu u^2 \text{ and } \int_0^u f(x, \zeta)d\zeta \leq C - \mu u^2\ . \tag{2.8}$$

So, for any $U_0 \in \mathcal{B}$ and any positive time t such that $S_n(t)U_0$ exists, we have

$$\frac{1}{2}\|S_n(t)U_0\|_X^2 - C \leq \Phi(S_n(t)U_0) \leq \Phi(U_0) \ .$$

Sobolev imbeddings show that $\Phi(U_0)$ is bounded uniformly with respect to $U_0 \in \mathcal{B}$. Thus, the trajectories cannot blow up and are defined and bounded for all times. $\square$

For $U(t) \in \mathcal{C}^0([0,T],X)$, we can also consider the trajectory $V_n(t) = DS_n(U)(t)V_0$ of the linearized dynamical system $DS_n(U)$ along U, that is the solution of

$$\left\{ \begin{array}{l} \partial_t V_n(t) = A_n V_n(t) + F'(U(t))V_n(t) \\ V_n(0) = V_0 \in X \end{array} \right. \tag{2.9}$$

Due to Lemma 2.2, $W \in X \longmapsto F'(U)W$ is locally Lipschitzian and Proposition 2.3 is also valid for $DS_n(U)(t)$. Moreover the trajectories $DS_n(U)(t)V_0$ exist for all $t \in [0,T]$ since $DS_n(U)(t)$ is a linear dynamical system.

2.2 Convergence of the inverses

If the inverses A_n^{-1} do not converge to A_∞^{-1}, then one cannot hope any convergence result, since we cannot even ensure that a part of the spectrum of the operators is continuous when n goes to $+\infty$. That is why, we immediatly show that this convergence holds in natural situations. In the rest of the paper, this convergence of the inverses will be assumed.
A simple calculation shows that A_n is invertible of compact inverse and that A_n^{-1} is given by

$$\forall \begin{pmatrix} u \\ v \end{pmatrix} \in X, \ A_n^{-1}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Gamma_n u - B^{-1}v \\ u \end{pmatrix} \ . \tag{2.10}$$

We present here a typical situation.
Let θ be a bounded open subset of $\mathbb{R}^{d-1}$ with a boundary of class $\mathcal{C}^\infty$. We set $\tilde{\Omega} =]0,1[\times\theta$. Let γ be a nonnegative function in $\mathbb{L}^\infty(\theta)$ and let γ_n be a sequence of nonnegative functions in $\mathbb{L}^\infty(\tilde{\Omega})$, which converges to $\gamma \otimes \delta_{x=0}$ in the sense of distributions, that is that

$$\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d), \ \int_{\tilde{\Omega}} \gamma_n(x,y)\varphi(x,y)dxdy \longrightarrow \int_\theta \gamma(y)\varphi(0,y)dy \ .$$

We assume moreover that

$$\sup_{y\in\theta}\left(\left|\gamma(y) - \int_0^1 \gamma_n(x,y)dx\right| + \int_0^1 \gamma_n(x,y)\sqrt{|x|}dx\right) \longrightarrow 0 \ . \tag{2.11}$$

Notice that Hypothesis (2.11) is always fullfilled in the one-dimensional case $d = 1$. We have the following result.

Theorem 2.5. *Let Ω be a bounded open subset of $\mathbb{R}^d$. Assume that there exists a covering Ω_1,...,Ω_p of Ω such that the description of the dissipations γ_n on Ω_i is $\mathcal{C}^1-$diffeomorphic to the typical situation described previously. Then, there exists a sequence of positive numbers (c_n) converging to zero such that*

$$\forall\varphi \in \mathbb{H}^1(\Omega),\ \ \|(\Gamma_\infty - \Gamma_n)\varphi\|_{D(B^{1/2})} \leq c_n\|\varphi\|_{\mathbb{H}^1}\ . \tag{2.12}$$

As a consequence, A_n^{-1} converges to A_∞^{-1} in $\mathcal{L}(X)$.

Proof : We recall that, on $D(B^{1/2})$, the norms $\|.\|_{D(B^{1/2})}$ and $\|.\|_{\mathbb{H}^1}$ are equivalent. We have to show that, for all φ and ψ in $D(B^{1/2})$, there exists a sequence of positive numbers (c_n) converging to zero such that

$$< (\Gamma_\infty - \Gamma_n)\phi|\overline{\psi} >_{D(B^{1/2})} \leq c_n\|\varphi\|_{\mathbb{H}^1}\|\psi\|_{\mathbb{H}^1}\ ,$$

that is that

$$\int_{\omega_N} \gamma\varphi\psi - \int_\Omega \gamma_n\varphi\psi \leq c_n\|\varphi\|_{\mathbb{H}^1}\|\psi\|_{\mathbb{H}^1}\ . \tag{2.13}$$

Clearly, it is sufficient to prove (2.13) in the typical situation introduced above and with smooth functions. Let φ and ψ be two functions of $\mathcal{C}^\infty(\tilde{\Omega})$, and let

$$I_n = \left|\int_\theta \gamma(y)\varphi(0,y)\psi(0,y)dy\ -\ \int_{\tilde{\Omega}} \gamma_n(x,y)\varphi(x,y)\psi(x,y)dxdy\right|\ .$$

We have $I_n \leq J_n + K_n$, where

$$J_n = \left|\int_\theta \varphi(0,y)\psi(0,y)\left(\gamma(y) - \int_0^1 \gamma_n(x,y)dx\right)dy\right|$$

and

$$K_n = \left|\int_{\tilde{\Omega}} \gamma_n(x,y)(\varphi(x,y)\psi(x,y) - \varphi(0,y)\psi(0,y))dxdy\right|\ .$$

Let

$$d_n = \sup_{y\in\theta}\left(\left|\gamma(y) - \int_0^1 \gamma_n(x,y)dx\right| + \int_0^1 \gamma_n(x,y)\sqrt{|x|}dx\right)\ .$$

Using the control of the norm $\mathbb{L}^2(\theta)$ by the norm $\mathbb{H}^1(\tilde{\Omega})$, we obtain

$$J_n \leq d_n\|\varphi\|_{\mathbb{H}^1}\|\psi\|_{\mathbb{H}^1}\ .$$

For the second term, we write

$$\begin{aligned} K_n\ &\leq \left|\int_\Omega \gamma_n(x,y)\varphi(x,y)(\psi(x,y) - \psi(0,y))dxdy\right| \\ &\quad + \left|\int_\Omega \gamma_n(x,y)\psi(0,y)(\varphi(x,y) - \varphi(0,y))dxdy\right|\ . \end{aligned} \tag{2.14}$$

We deal with the first term of (2.14) by using the Cauchy-Schwarz inequality

$$\begin{aligned}
K_n^1 &= \left| \int_{\tilde\Omega} \gamma_n(x,y)\varphi(x,y)(\psi(x,y)-\psi(0,y))dxdy \right| \\
&= \left| \int_{\tilde\Omega} \gamma_n(x,y)\varphi(x,y)\left(\int_0^x \frac{\partial\psi}{\partial x}(\xi,y)d\xi\right)dxdy \right| \\
&\leq \int_{\tilde\Omega} \gamma_n(x,y)|\varphi(x,y)|\sqrt{x}\left(\int_0^x \left|\frac{\partial\psi}{\partial x}(\xi,y)\right|^2 d\xi\right)^{1/2} dxdy \\
&\leq \int_\theta \left(\int_0^1 \left|\frac{\partial\psi}{\partial x}(\xi,y)\right|^2 d\xi\right)^{1/2} \left(\sup_{\xi\in]0,1[} |\varphi(\xi,y)|\right)\left(\int_0^1 \gamma_n(x,y)\sqrt{x}dx\right)dy \\
&\leq d_n \int_\theta \left(\int_0^1 \left|\frac{\partial\psi}{\partial x}(\xi,y)\right|^2 d\xi\right)^{1/2} \left(\sup_{\xi\in]0,1[} |\varphi(\xi,y)|\right)dy\ .
\end{aligned}$$

Using the control of the $\mathbb{L}^\infty$-norm by the $\mathbb{H}^1$-norm in the one-dimensional space, we find

$$\begin{aligned}
K_n^1 &\leq d_n \int_\theta \left(\int_0^1 \left|\frac{\partial\psi}{\partial x}(x,y)\right|^2 dx\right)^{1/2} \left(\int_0^1 |\varphi(x,y)|^2 + \left|\frac{\partial\varphi}{\partial x}(x,y)\right|^2 dx\right)^{1/2} dy \\
&\leq d_n \left(\int_{\tilde\Omega} \left|\frac{\partial\psi}{\partial x}(x,y)\right|^2 dxdy\right)^{1/2} \left(\int_{\tilde\Omega} |\varphi(x,y)|^2 + \left|\frac{\partial\varphi}{\partial x}(x,y)\right|^2 dxdy\right)^{1/2} \\
&\leq d_n \|\varphi\|_{\mathbb{H}^1}\|\psi\|_{\mathbb{H}^1}\ .
\end{aligned}$$

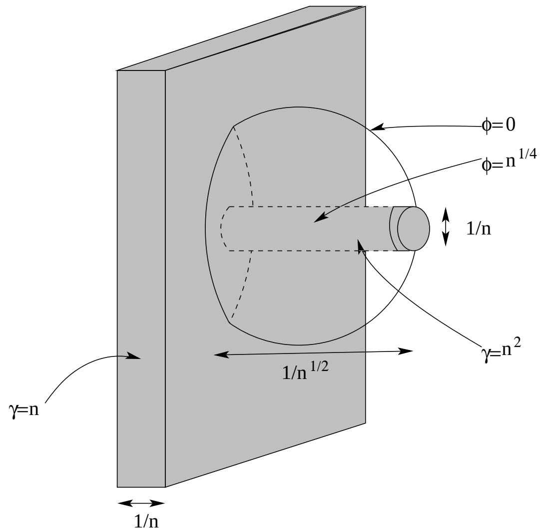
Applying the same argument to the second term of (2.14), we complete the proof of the estimate (2.13).
Thus, we have shown that Γ_n converges to Γ_∞ in $\mathcal{L}(D(B^{1/2}))$. From (2.10) and (2.12), we deduce that A_n^{-1} converges to A_∞^{-1} in $\mathcal{L}(X)$. □

To show that the natural Hypothesis (2.11) is necessary, we give a counter-example to Theorem 2.5 when (2.11) is omitted.
Let $\Omega =]0,1[\times]-1,1[^2$. Let

$$\gamma_n(x,y) = \begin{cases} n & \text{if } 0 \leq x \leq \frac{1}{n} \\ n^2 & \text{if } \frac{1}{n} \leq x \leq \frac{1}{\sqrt{n}},\ |y| < \frac{1}{n} \\ 0 & \text{elsewhere.} \end{cases}$$

We notice that γ_n converges to $\gamma = \delta_{x=0}$ in the sense of the distributions. Let $\varphi_n(x,y)$ be the function with support in the ball B of center $(\frac{1}{2\sqrt{n}},0,0)$ and of radius $R = \frac{1}{2\sqrt{n}}$ with

$\varphi_n(r,\theta) = \frac{1}{2}n^{1/4} - rn^{3/4}$ in it, where $r = ((x - \frac{1}{2\sqrt{n}})^2 + y^2)^{1/2}$.



In the support of φ_n, the norm of the gradient of φ_n is $n^{3/4}$, so $\|\varphi\|_{\mathbb{H}^1} \sim 1$. We have $\int \gamma_\infty |\varphi_n|^2 = 0$ and

$$\int \gamma_n |\varphi_n|^2 \sim \frac{n^2 (n^{1/4})^2}{n^2\sqrt{n}} \sim 1 \ .$$

So Γ_n does not converge to Γ_∞ in $\mathcal{L}(D(B^{1/2}))$.

Using the same arguments as in the proof of Theorem 2.5, we obtain the following property.

Proposition 2.6. *We assume that the same hypotheses as in Theorem 2.5 hold. Let $\frac{1}{2} > s \geq 0$. There exists M independent of n such that*

$$\forall n \in \mathbb{N} \cup \{+\infty\},\ \forall U \in D(A_n),\ \|U\|_{X^s} \leq M \|U\|_{D(A_n)} \ .$$

Proof : Assume that the proposition is not satisfied. Then, there exists a sequence $U_k = (u_k, v_k)$ such that

$$\|U_k\|_{X^s} = 1 \text{ and } \|U_k\|_{D(A_{n_k})} \longrightarrow 0 \ .$$

This implies that $v_k \longrightarrow 0$ in $D(B^{1/2})$ and $Bu_k + \gamma_{n_k} v_k \longrightarrow 0$ in $\mathbb{L}^2(\Omega)$. If we prove that $\gamma_{n_k} v_k \longrightarrow 0$ in $D(B^{(-1+s)/2})$, then we will have $u_k \longrightarrow 0$ in $D(B^{(1+s)/2})$. But the properties $u_k \longrightarrow 0$ in $D(B^{(1+s)/2})$ and $v_k \longrightarrow 0$ in $D(B^{1/2})$ contradict the fact that $\|U_k\|_{X^s} = 1$.
It remains to show that $\gamma_{n_k} v_k \longrightarrow 0$ in $D(B^{(-1+s)/2})$. Let $\varphi \in D(B^{(1-s)/2})$, we have

$$\left| \int_\Omega \gamma_{n_k} v_k \varphi \right|^2 \leq \int_\Omega \gamma_{n_k} |v_k|^2 \int_\Omega \gamma_{n_k} |\varphi|^2 \ . \tag{2.15}$$

As $v_k \longrightarrow 0$ in $D(B^{1/2})$, we have $\int_\Omega \gamma_{n_k}|v_k|^2 \longrightarrow 0$ by Theorem 2.5. In order to prove that $\int_\Omega \gamma_{n_k}|\varphi|^2$ is bounded, we come back to the typical situation introduced before Theorem 2.5. We have

$$\int_\theta \int_0^1 \gamma_{n_k}(x,y)|\varphi(x,y)|^2 dxdy \le \left(\sup_{y\in\theta}\int_0^1 \gamma_{n_k}(x,y)dx\right)\left(\int_\theta |\sup_{x\in[0,1]}\varphi(x,y)|^2 dy\right) .$$

We know that φ is bounded in $D(B^{(1-s)/2})$ and so in $\mathbb{H}^{1-s}(\Omega)$. In the typical case $\tilde{\Omega} =]0,1[\times\theta$, and thus $\mathbb{H}^{1-s}(\tilde{\Omega}) \hookrightarrow \mathbb{L}^2(\theta, \mathbb{H}^{1-s}(]0,1[))$. Using the fact that $\mathbb{H}^{1-s}(]0,1[)$ is embedded in $\mathcal{C}^0(]0,1[)$, we obtain that $\int_\theta |\sup_{x\in[0,1]}\varphi(x,y)|^2 dy < +\infty$. On the other hand, (2.11) implies that $\sup_{y\in\theta}\int_0^1 \gamma_{n_k}(x,y)dx < +\infty$, which implies the proposition. $\square$

2.3 Main hypotheses and results

In this section, we put together all the main hypotheses and theorems.
We recall that $S_n(t)$ denotes the local dynamical system generated by (2.5). In what follows, we will assume that

$$\varepsilon_n = \|A_\infty^{-1} - A_n^{-1}\|_{\mathcal{L}(X)} \longrightarrow 0 . \qquad (2.16)$$

Moreover, we also assume in the whole article that f satisfies Hypothesis (NL). In addition, Hypothesis (Diss) will be assumed when we deal with global results.
In Section 3, we show that the convergence of the inverses implies some weak convergence for the trajectories. The convergence is weak in the sense that, in order to compare $S_n(t)U_0$ with $S_\infty(t)U_0$ in the space X^s, U_0 has to belong to a more regular space $X^{s+\varepsilon}$. For example, we will obtain the following results.

Proposition 2.7. *Assume that Hypothesis (Diss) is satisfied. Let $\mathcal{B}$ be a bounded set of X and $s \in [0,1]$, there exists a positive constant C such that*

$$\forall U \in \mathcal{B},\ \forall t \ge 0,\ \|S_\infty(t)U - S_n(t)U\|_{X^{-s}} \le Ce^{Ct}\varepsilon_n^{s/8} . \qquad (2.17)$$

If $\mathcal{B}^s$ is a bounded set of X^s ($s \in]0, s_0[$), then there exists a positive constant C such that

$$\forall U \in \mathcal{B}^s,\ \forall t \ge 0,\ \|S_\infty(t)U - S_n(t)U\|_X \le Ce^{Ct}\varepsilon_n^\beta , \qquad (2.18)$$

where $\beta = \frac{s^2}{2}$ if d=1 or d=2, and $\beta = \min(\frac{s^2}{2}, \frac{1-\alpha}{4})$ if d=3.

To obtain existence of compact global attractors, we will have to assume that the linear semigroups $e^{A_n t}$ are exponentially decreasing :

(ED) There exists a family of positive constants M_n and λ_n ($n \in \mathbb{N} \cup \{+\infty\}$) such that

$$\|e^{A_n t}\|_{\mathcal{L}(X)} \leq M_n e^{-\lambda_n t} \ .$$

As discussed in the introduction, we will need the uniform version of (ED) in order to obtain uniform regularity of the attractors :

(UED) There exist two positive constants M and λ such that for any $t \geq 0$ and $U \in X$,

$$\forall n \in \mathbb{N}, \ \|e^{A_n t} U\|_X \leq M e^{-\lambda t} \|U\|_X \ .$$

Finally, we introduce hypotheses on the dynamical systems.
The dynamical systems $S_n(t)$ are gradient systems if we show that the function Φ introduced in (2.6) is a strict Lyapounov function. We already know that Φ is not increasing along the trajectories because of (2.7). To prove that Φ is a strict Lyapounov function, it remains to show that, if, for some $n \in \mathbb{N} \cup \{+\infty\}$, U_0 satisfies $\Phi(S_n(t)U_0) = \Phi(U_0)$ for all $t \geq 0$, then U_0 is an equilibrium point, that is $S_n(t)U_0 = U_0$ for all $t \geq 0$. We will assume that this property is fulfilled :

(Grad) the dynamical systems $S_n(t)$ ($n \in \mathbb{N} \cup \{+\infty\}$) are all gradient.

Our last assumption is the following :

(Hyp) All the equilibrium points E of $S_\infty(t)$ are hyperbolic, that is, that the spectrum of $DS_\infty(t)E$ does not intersect the unit circle of $\mathbb{C}$.

A discussion about the hypotheses (ED), (UED) and (Grad) is given in Section 6. We also enhance that Hypothesis (Hyp) is not very restrictive since it is satisfied for a generic non-linearity f (see for example [43] and [7]) or a generic domain Ω (see [24]).

We introduce the distance between a point $U \in X$ and a set $\mathcal{S} \subset X$ as

$$\text{dist}_X(U, \mathcal{S}) = \inf_{V \in \mathcal{S}} \|U - V\|_X \ . \tag{2.19}$$

We also define the Hausdorff distance of two sets $\mathcal{S}_1 \subset X$ and $\mathcal{S}_2 \subset X$ as

$$d_X(\mathcal{S}_1, \mathcal{S}_2) = \max\left(\sup_{U_1 \in \mathcal{S}_1} \text{dist}_X(U_1, \mathcal{S}_2) \ ; \ \sup_{U_2 \in \mathcal{S}_2} \text{dist}_X(U_2, \mathcal{S}_1) \right) \ , \tag{2.20}$$

We denote $\text{dist}_{X^{-s}}$ and $d_{X^{-s}}$ the same notions in the norm $\|.\|_{X^{-s}}$. We have the following theorem.

Theorem 2.8. *We assume that (Diss), (Grad) and (ED) hold. Then, the dynamical system $S_n(t)$, for $n \in \mathbb{N} \cup \{+\infty\}$, has a compact global attractor $\mathcal{A}_n$. Moreover, these attractors are composed by the union of the equilibrium points (denoted by $\mathcal{E}$) and the complete bounded trajectories coming from $\mathcal{E}$, that is that*

$$\mathcal{A}_n = \{U_0 \in X \ / \ \exists U(t) \in \mathcal{C}_b^0(\mathbb{R}, X), \text{ solution of (2.5) such that} \\ U(0) = U_0 \text{ and } \lim_{t \to -\infty} dist_X(U(t), \mathcal{E}) = 0 \ \} \ . \tag{2.21}$$

The set $(\bigcup_n \mathcal{A}_n)$ is bounded in X, and, for any $s \in]0, 1/2[$, the attractors are upper-semicontinuous in X^{-s}, that is

$$\sup_{U_n \in \mathcal{A}_n} dist_{X^{-s}}(U_n, \mathcal{A}_\infty) \longrightarrow 0 \text{ when } n \longrightarrow +\infty \ .$$

Proof : The existence and boundedness of attractors for Equation (2.5) is classical, we briefly recall the outline of the proof. According to Theorem 2.4.6 of [17], $S_n(t)$ has a compact global attractor if $S_n(t)$ is asymptotically smooth and point-dissipative and if the orbits of bounded sets are bounded. Proposition 2.4 implies that the orbits of bounded sets are bounded. Since $e^{A_n t}$ is exponentially decreasing and that the map $F : X \to X$ is compact, $S_n(t)$ is asymptotically smooth (see [17]). The property (2.8) implies that the equilibria $E = (e, 0)$ of (2.5) are bounded independently of n. By LaSalle's principle (see Lemma 3.8.2 of [17]), the gradient structure and the asymptotic smoothness imply that any trajectory is attracted by the set of equilibrium points. Because of the boundedness of the set of equilibria, $S_n(t)$ is point dissipative. Thus $S_n(t)$ has a compact global attractor, which is bounded in X uniformly in n and which, due to the gradient structure, is described by (2.21). For proofs or details about these notions, see [17].
Following the arguments of [18] (see also [41] or [3]), we prove the upper-semicontinuity in X^{-s}. Let $\varepsilon > 0$, as $\mathcal{A}_\infty$ is a global attractor for $S_\infty(t)$ and as the union $\bigcup_n \mathcal{A}_n$ is bounded in X, there exists a time $T > 0$ such that

$$\forall U \in \bigcup_n \mathcal{A}_n, \ \forall t \geq T, \ \mathrm{dist}_X(S_\infty(t)U, \mathcal{A}_\infty) \leq \varepsilon/2 \ . \tag{2.22}$$

As $\mathcal{A}_n$ is uniformly bounded in X, using (2.17), we have that, for n large enough,

$$\forall U_n \in \mathcal{A}_n, \ \|(S_n(T) - S_\infty(T))U_n\|_{X^{-s}} \leq \frac{\varepsilon}{2} \ . \tag{2.23}$$

The estimates (2.22) and (2.23) imply, for n large enough, that

$$\sup_{U_n \in \mathcal{A}_n} \mathrm{dist}_{X^{-s}}(S_n(T)U_n, \mathcal{A}_\infty) \leq \varepsilon \ .$$

As $S_n(T)\mathcal{A}_n = \mathcal{A}_n$, this proves the upper-semicontinuity. □

Remark : The existence of attractors for critical non-linearities (that is cubic-like non-linearities) has been studied in dimension $d = 3$, see for example [14] and [9]. We notice that the above proof shows upper-semicontinuity in X^{-s} for these attractors. See [27] for the lower-semicontinuity.

If we assume a uniform exponential decay for the linear semigroups $e^{A_n t}$, we obtain the upper-semicontinuity in X. Indeed, we have the following regularity result.

Proposition 2.9. *Assume that (Diss), (Grad) and (UED) hold. Then, there exists a constant M such that the attractors $\mathcal{A}_n$ of $S_n(t)$, for $n \in \mathbb{N} \cup \{+\infty\}$, satisfy*

$$\sup_{n\in\mathbb{N}\cup\{+\infty\}} \sup_{U_n\in\mathcal{A}_n} \|U_n\|_{D(A_n)} \leq M \ . \tag{2.24}$$

In particular, the union $\cup_n \mathcal{A}_n$ is bounded in X^s ($s \in]0, 1/2[$).

Proof : If (2.24) holds, then $\cup_n \mathcal{A}_n$ is bounded in X^s as a direct consequence of Proposition 2.6. Thus, we only have to show that (2.24) is satisfied.
It is well-known that, for fixed n, $\mathcal{A}_n$ is bounded in $D(A_n)$. We only have to show that (UED) implies that $\mathcal{A}_n$ is bounded in $D(A_n)$, uniformly with respect to $n \in \mathbb{N} \cup \{+\infty\}$.
We already know that the attractors $\mathcal{A}_n$ are bounded in X by a constant K. Moreover, they are a union of complete trajectories. Let $U(t) = (u, u_t) \subset \mathcal{A}_n$ be such a trajectory, we have

$$U(t) = \int_{-\infty}^{t} e^{A_n(t-s)} F(U(s))ds \ .$$

Notice that this integral has a sense since (UED) holds. Let $\delta > 0$, we write

$$U(t+\delta) - U(t) = \int_{-\infty}^{t} e^{A_n(t-s)}(F(U(s+\delta)) - F(U(s)))ds \ .$$

And so, since (UED) is satisfied, there exist M and λ independent of n such that

$$\|(U(t+\delta) - U(t))\|_X \leq M \int_{-\infty}^{t} e^{-\lambda(t-s)} \|f(x, u(s+\delta)) - f(x, u(s))\|_{\mathbb{L}^2} ds \ . \tag{2.25}$$

Due to the assumption (NL), there exists $\sigma \in]0, 1[$ such that

$$\begin{aligned} \|f(x, u(s+\delta)) - f(x, u(s))\|_{\mathbb{L}^2} &\leq C \|u(s+\delta) - u(s)\|_{\mathbb{H}^\sigma} \\ &\leq C \|u(s+\delta) - u(s)\|_{\mathbb{H}^1}^{\sigma} \|u(s+\delta) - u(s)\|_{\mathbb{L}^2}^{1-\sigma} \end{aligned}$$

The Young inequality implies that, for any $\varepsilon > 0$, there exists a constant C_ε such that

$$\|f(x, u(s+\delta)) - f(x, u(s))\|_{\mathbb{L}^2} \leq \varepsilon \|u(s+\delta) - u(s)\|_{\mathbb{H}^1} + C_\varepsilon \|u(s+\delta) - u(s)\|_{\mathbb{L}^2} \ .$$

As $\|u_t\|_{\mathbb{L}^2}$ is bounded by K, $\|\delta^{-1}(u(s+\delta)-u(s))\|_{\mathbb{L}^2}$ is uniformly bounded. So, combining the above inequality with (2.25), we obtain, for any $t \in \mathbb{R}$,

$$\|\delta^{-1}(U(t+\delta)-U(t))\|_X \leq \varepsilon \frac{M}{\lambda} \sup_{s\in\mathbb{R}} \|\delta^{-1}(U(s+\delta)-U(s))\|_X + \frac{M}{\lambda} C_\varepsilon K \ .$$

Thus, for ε sufficiently small, we get

$$\sup_{s\in\mathbb{R}} \|\delta^{-1}(U(s+\delta)-U(s))\|_X \leq C \ ,$$

where C does not depend on δ or on n. When δ converges to 0, we find that $U(s)$ satisfies $\sup_{s\in\mathbb{R}} \|U_t(s)\|_X \leq C$. Finally, writing $A_n U = U_t - F(U)$, we obtain that U is bounded in $D(A_n)$ by a constant which does not depend on n. ☐

Thus, if we mimic the proof of Theorem 2.8, using (2.18) instead of (2.17), we show the upper-semicontinuity in X.

Theorem 2.10. *We assume that all the hypotheses of Proposition 2.9 hold. Then, the attractors are upper-semicontinuous in X, that is*

$$\sup_{U_n \in \mathcal{A}_n} dist_X(U_n, \mathcal{A}_\infty) \longrightarrow 0 \text{ when } n \longrightarrow +\infty \ .$$

If we assume in addition that all the equilibria are hyperbolic, then we can prove the lower-semicontinuity of attractors. In this case, we can give not only an estimate of the rate of the lower-semicontinuity in X, but also of the upper-semicontinuity in X^{-s}. Notice that we do not need Hypothesis (UED) to obtain the lower-semicontinuity in X.

Theorem 2.11. *We assume that (Diss), (Grad), (ED) and (Hyp) are satisfied. Then, the attractors $\mathcal{A}_n$ are lower-semicontinuous in X.*
Moreover, there exist two positive constants C and δ such that

$$\sup_{U_\infty \in \mathcal{A}_\infty} dist_X(U_\infty, \mathcal{A}_n) \leq C\varepsilon_n^\delta \ . \tag{2.26}$$

and

$$\sup_{U_n \in \mathcal{A}_n} dist_{X^{-s}}(U_n, \mathcal{A}_\infty) \leq C\varepsilon_n^\delta \ . \tag{2.27}$$

Furthermore, if we assume in addition that Hypothesis (UED) holds, then the family of attractors is continuous in X and there exist two positive constants C and δ such that, for any n,

$$d_X(\mathcal{A}_\infty, \mathcal{A}_n) \leq C\varepsilon_n^\delta \ . \tag{2.28}$$

Our last theorem concerns the stability of the phase-diagrams. We have briefly recalled the notion of phase diagrams and its link with the Morse-Smale property in the introduction. First, notice that, in dimension higher than one or if $d = 1$ and $\gamma_\infty = a\delta_{x=0} + b\delta_{x=1}$ with $a = 1$ or $b = 1$, the Morse-Smale property is not relevant. Indeed, in these cases, $e^{A_n t}$ is not a group (see [11] for $d = 1$ and [35] for $d \geq 2$). Thus, we cannot ensure that the backward uniqueness property is satisfied and that the stable sets of equilibria are well-defined manifolds, which is needed to define the transversality (for more details, see [19]). In the cases where we can define the Morse-Smale property, we prove the following theorem in Section 5.

Theorem 2.12. *We assume that $d = 1$, $\Omega =]0, 1[$ and $\gamma_\infty = a\delta_{x=0} + b\delta_{x=1}$ with $a \neq 1$ and $b \neq 1$. We also assume that (Diss) and (UED) are satisfied and that the dynamical system $S_\infty(t)$ satisfies the Morse-Smale property. Then, for n large enough, the dynamical system $S_n(t)$ satisfies the Morse-Smale property and its phase-diagram is equivalent to the one of $S_\infty(t)$.*

We underline that Theorem 2.12 has applications since it is proved in [26] that, if $\Omega =]0, 1[$, $\gamma_\infty = a\delta_{x=0} + b\delta_{x=1}$ with $a \neq 1$ and $b \neq 1$, the Morse-Smale property holds for $S_\infty(t)$, generically with respect to the non-linearity f.

Remark : We can readely adapt the proof of Theorem 3.2 of [36] to show the existence of a homeomorphism h defined from $\mathcal{A}_n$ to $\mathcal{A}_\infty$ which maps the trajectories of $S_n(t)_{|\mathcal{A}_n}$ onto the trajectories of $S_\infty(t)_{|\mathcal{A}_\infty}$ preserving the sense of time. The properties needed to adapt the proof of Theorem 3.2 of [36] are shown in Sections 4 and 5. They namely are the isomorphism of phase-diagrams of Theorem 2.12, the comparison of the local stable and unstable manifolds stated in Theorems 4.7 and 4.13 and the results of Section 5.1.

3 Convergence of the trajectories

3.1 Some abstract results of convergence

The difference between two linear semigroups of contractions can be estimated by the difference between the inverses of the infinitesimal generators.

Proposition 3.1. *Let X be a Hilbert space. Let A_1 and A_2 be two maximal dissipative operators of bounded inverse in $\mathcal{L}(X)$. Then, the operator A_i generates a $\mathcal{C}^0-$semigroup in X and we have, for all $U \in D(A_1)$ and $t \in \mathbb{R}_+$,*

$$\|e^{A_1 t}U - e^{A_2 t}U\|_X \leq \sqrt{\alpha}\left(\sqrt{\alpha} + \sqrt{\alpha + 4t}\right)\|U\|_{D(A_1)} ,$$

where $\alpha = ||A_1^{-1} - A_2^{-1}||_{\mathcal{L}(X)}$.

Proposition 3.1 is a direct consequence of the next proposition. The stronger version, where projections are added, is useful to prove convergence of stable and unstable manifolds of a hyperbolic equilibrium of the dynamical systems, or to estimate convergence of semigroups, which are not defined on the same space.

Proposition 3.2. *Let P_1 and P_2 be two continuous projections on a Hilbert space X. For $i = 1, 2$, let A_i be a linear operator with $D(A_i) \subset P_iX$ and $A_i \in \mathcal{L}(D(A_i), P_iX)$, which is dissipative, invertible and of bounded inverse. Then, A_i generates a $\mathcal{C}^0-$semigroup e^{A_it} on P_iX and for all $U \in D(A_1) \subset P_1X$ and $t \in \mathbb{R}_+$,*

$$\|e^{A_1t}P_1U - e^{A_2t}P_2U\|_X \leq \left(C\alpha + \sqrt{\alpha^2 + 4C^2t\,(\alpha + \beta)}\right) \, \|U\|_{D(A_1)} \ ,$$

where $\alpha = ||A_1^{-1}P_1 - A_2^{-1}P_2||_{\mathcal{L}(X)}$, $\beta = \|A_1^{-1}\|_{\mathcal{L}(P_1X)}\|P_1 - P_2\|_{\mathcal{L}(X)}$ and $C = \max_{i=1,2}\{\|P_i\|_{\mathcal{L}(X)}\}$.

Proof : As the operator A_i is invertible, it is onto P_iX and thus A_i is a maximal operator. Since it is also dissipative on P_iX, it generates a $\mathcal{C}^0-$semigroup e^{A_it} on P_iX, which satisfies

$$\forall U \in X, t \in \mathbb{R}_+, \ \|e^{A_it}P_iU\|_X \leq \|P_iU\|_X \tag{3.1}$$

(see for example [39]). We write that

$$\begin{aligned}\|e^{A_1t}P_1U - e^{A_2t}P_2U\|_X \leq \|e^{A_1t}P_1U - e^{A_2t}(A_2^{-1}P_2A_1U)\|_X \\ + \|e^{A_2t}P_2(A_2^{-1}P_2 - A_1^{-1}P_1)A_1U\|_X \ .\end{aligned} \tag{3.2}$$

Using (3.1), we easily bound the last term of (3.2) by $C\alpha\|U\|_{D(A_1)}$. To estimate the derivative of the first term of (3.2), we set

$$D = \frac{1}{2}\frac{d}{dt}\|e^{A_1t}P_1U - e^{A_2t}(A_2^{-1}P_2A_1U)\|_X^2 \ . \tag{3.3}$$

Since $U \in D(A_1)$ and $A_2^{-1}P_2A_1U \in D(A_2)$, we have

$$D =< A_1e^{A_1t}P_1U - A_2e^{A_2t}(A_2^{-1}P_2A_1U)|e^{A_1t}P_1U - e^{A_2t}(A_2^{-1}P_2A_1U) >_X \ \ ,$$

where $< .|. >_X$ is the scalar product associated with the norm $\|.\|_X$.
We set $V = e^{A_1t}P_1A_1U \in P_1X$ and $W = e^{A_2t}P_2A_1U \in P_2X$. We have

$$\begin{aligned} D &= < V - W|A_1^{-1}V - A_2^{-1}W >_X \\ &= < V - W|A_1^{-1}P_1(V - W) >_X + < V - W|(A_1^{-1}P_1 - A_2^{-1}P_2)W >_X \ . \end{aligned}$$

Since $P_1V = V$ and $P_2W = W$, we obtain

$$D =< P_1(V - W)|A_1^{-1}P_1(V - W) >_X + < (P_1 - P_2)W|A_1^{-1}P_1(V - W) >_X$$

$$+ < V - W|(A_1^{-1}P_1 - A_2^{-1}P_2)W >_X \ .$$

As A_1 is dissipative on P_1X, the first scalar product is nonpositive. Since $\|V\|_X \leq C\|U\|_{D(A_1)}$ and $\|W\|_X \leq C\|U\|_{D(A_1)}$, we obtain $D \leq 2C^2(\alpha+\beta)\|U\|^2_{D(A_1)}$, where α, β and C are as in the statement of the proposition. The integration of (3.3) gives

$$\|e^{A_1t}P_1U - e^{A_2t}(A_2^{-1}P_2A_1U)\|_X \leq \sqrt{\alpha^2 + 4C^2t\,(\alpha+\beta)}\|U\|_{D(A_1)} \ ,$$

Coming back to the estimate (3.2), we finish the proof. □

Corollary 3.3. *Let P_1 and P_2 be two continuous projections on a Hilbert space X. For $i = 1,2$, let A_i be a linear operator with $D(A_i) \subset P_iX$ and $A_i \in \mathcal{L}(D(A_i), P_iX)$. We assume that there exist a constant μ such that $A_i - \mu Id$ is dissipative, invertible and of bounded inverse (which implies that A_i generates a $\mathcal{C}^0-$semigroup). Moreover, we assume that there exist two positive constants M and λ such the semigroup generated by A_i satisfies*

$$\forall t \geq 0, \ \|e^{A_it}\|_{\mathcal{L}(P_iX)} \leq Me^{-\lambda t} \ .$$

Then, for all $\eta \in]0,\lambda[$, there exists M_η, independent of the operator A_i, such that for all $U \in D(A_1) \subset P_1X$ and $t \in \mathbb{R}_+$,

$$\|e^{A_1t}P_1U - e^{A_2t}P_2U\|_X \leq C(\alpha + \sqrt{\alpha+\beta})M_\eta e^{-\eta t}\|U\|_{D(A_1)} \ , \qquad (3.4)$$

where $\alpha = ||(A_1-\mu Id)^{-1}P_1 - (A_2-\mu Id)^{-1}P_2||_{\mathcal{L}(X)}$, $\beta = \|(A_1-\mu Id)^{-1}\|_{\mathcal{L}(P_1X)}\|P_1-P_2\|_{\mathcal{L}(X)}$ and $C = \max\limits_{i=1,2}\{\|P_i\|_{\mathcal{L}(X)}\}$.

Proof : Changing A_i into $A_i - \mu Id$ and λ into $\lambda+\mu$, we can assume that $\mu = 0$. Let $p \in \mathbb{N}^*$, $U \in D(A_1)$ and $t \in \mathbb{R}_+$. We have

$$\|e^{A_1t}P_1U - e^{A_2t}P_2U\|_X \leq \|e^{A_2(1-\frac{1}{p})t}P_2(e^{A_1\frac{t}{p}}P_1 - e^{A_2\frac{t}{p}}P_2)U\|_X$$

$$+ \|e^{A_2(1-\frac{2}{p})t}P_2(e^{A_1\frac{t}{p}}P_1 - e^{A_2\frac{t}{p}}P_2)e^{A_1\frac{t}{p}}P_1U\|_X + ... + \|(e^{A_1\frac{t}{p}}P_1 - e^{A_2\frac{t}{p}}P_2)e^{A_1(1-\frac{t}{p})}P_1U\|_X \ .$$

Using Proposition 3.2, we obtain

$$\|e^{A_1t}P_1U - e^{A_2t}P_2U\|_X \leq pMe^{-\lambda(1-\frac{1}{p})t}\left(C\alpha + \sqrt{\alpha^2 + 4C^2\frac{t}{p}\,(\alpha+\beta)}\right)\ \|U\|_{D(A_1)} \ .$$

Thus, for all $\eta \in]0,\lambda[$ given, we can choose p and M_η large enough such that (3.4) holds. □

Our fourth result concerns the convergence in a weaker norm than the norm of X.

Proposition 3.4. *Let X be a Hilbert space and let A_1 and A_2 be two maximal dissipative operators of bounded inverse in $\mathcal{L}(X)$. Then, for all $U \in X$ and $t \in \mathbb{R}_+$,*

$$\|A_1^{-1}(e^{A_1 t}U - e^{A_2 t}U)\|_X \leq \sqrt{\alpha}\left(3\sqrt{\alpha} + \sqrt{\alpha + 4t}\right)\|U\|_X \ ,$$

where $\alpha = ||A_1^{-1} - A_2^{-1}||_{\mathcal{L}(X)}$.

Proof : We have

$$\|A_1^{-1}(e^{A_1 t}U - e^{A_2 t}U)\|_X \leq \|(e^{A_1 t} - e^{A_2 t})A_1^{-1}U\|_X + \|(A_1^{-1} - A_2^{-1})e^{A_2 t}U\|_X \\ + \|e^{A_2 t}(A_1^{-1} - A_2^{-1})U\|_X \ .$$

We finish the proof by applying Proposition 3.1. $\square$

3.2 Convergence of the trajectories

We recall that $\varepsilon_n = \|A_\infty^{-1} - A_n^{-1}\|_{\mathcal{L}(X)}$ is assumed to converge to zero. In this section, we compare $S_n(t)U_0$ with $S_\infty(t)U_0$ on finite time intervals.
In the previous section, we have seen that the convergence of the linear semigroups $e^{A_n t}$ can be estimated if the initial data are in $D(A_n)$, $n \in \mathbb{N} \cup \{+\infty\}$. Using interpolation arguments, we see that actually less regularity is needed. We recall that s_0 is the positive number defined by (2.2).

Proposition 3.5. *For all $s \in]0, s_0[$, there exists $C > 0$ such that, for all time $T > 0$, for all $t \in [0, T]$ and $U_0 \in X^s$, we have*

$$\forall t \in [0,T], \forall U_0 \in X^s,\ \|(e^{A_\infty t} - e^{A_n t})U_0\|_X \leq C(1 + T^{s^2/2})\varepsilon_n^{s^2/2}\|U_0\|_{X^s} \ . \tag{3.5}$$

Moreover, if the initial data have zero as first component, we can improve the above estimate as follows : for all $s \in [0, 1/2[$, there exists $C > 0$ such that, for all time $T > 0$, for all $t \in [0, T]$ and $(0, v_0) \in X^s$, we have

$$\|(e^{A_\infty t} - e^{A_n t})(0, v_0)\|_X \leq C(1 + T^{s/2})\varepsilon_n^{s/2}\|(0, v_0)\|_{X^s} \ . \tag{3.6}$$

Proof : In this proof, C denotes a generic positive constant, which does not depend on n or T.
If $U_0 = (u_0, v_0) \in D(A_\infty)$, then, using Proposition 3.1, we have

$$\|(e^{A_\infty t} - e^{A_n t})U_0\|_X \leq C(1 + T^{1/2})\varepsilon_n^{1/2}\|U_0\|_{D(A_\infty)} \\ \leq C(1 + T^{1/2})\varepsilon_n^{1/2}(\|u_0 + \Gamma_\infty v_0\|_{D(B)} + \|v_0\|_{D(B^{1/2})}) \ . \tag{3.7}$$

On the other hand, we have

$$\|(e^{A_\infty t} - e^{A_n t})U_0\|_X \leq C\|U_0\|_X \leq C(\|u_0\|_{D(B^{1/2})} + \|v_0\|_{\mathbb{L}^2}) \ . \tag{3.8}$$

Since Γ_∞ is a bounded operator on $D(B^{1/2})$, we have, if $U_0 \in D(B^{1/2}) \times D(B^{1/2})$,

$$\begin{aligned} \|(e^{A_\infty t} - e^{A_n t})U_0\|_X \quad &\leq C(\|u_0\|_{D(B^{1/2})} + \|v_0\|_{D(B^{1/2})}) &(3.9)\\ &\leq C(\|u_0 + \Gamma_\infty v_0\|_{D(B^{1/2})} + \|v_0\|_{D(B^{1/2})}) \ . &(3.10)\end{aligned}$$

Interpolating between (3.7) and (3.10), we obtain

$$\|(e^{A_\infty t} - e^{A_n t})U_0\|_X \leq C(1 + T^{s/2})\varepsilon_n^{s/2}(\|u_0 + \Gamma_\infty v_0\|_{D(B^{(1+s)/2})} + \|v_0\|_{D(B^{1/2})}) \ .$$

Due to Proposition 2.1, if s belongs to $]0, s_0[$, then $\Gamma_\infty v_0$ is in $D(B^{(1+s)/2})$ and we have $\|\Gamma_\infty v_0\|_{D(B^{(1+s)/2})} \leq C\|v_0\|_{D(B^{1/2})}$. Thus,

$$\|(e^{A_\infty t} - e^{A_n t})U_0\|_X \leq C(1 + T^{s/2})\varepsilon_n^{s/2}(\|u_0\|_{D(B^{(1+s)/2})} + \|v_0\|_{D(B^{1/2})}) \ .$$

We interpolate again with (3.8) and we find that, for all $U_0 \in X^s$,

$$\begin{aligned} \|(e^{A_\infty t} - e^{A_n t})U_0\|_X \quad &\leq C(1 + T^{s^2/2})\varepsilon_n^{s^2/2}(\|u_0\|_{D(B^{(1+s)/2})} + \|v_0\|_{D(B^{s/2})}) \\ &\leq C(1 + T^{s^2/2})\varepsilon_n^{s^2/2}\|U_0\|_{X^s} \ . \end{aligned}$$

The proof of (3.6) is similar. Let $(0, v_0) \in D(A_\infty)$. Since $\Gamma_\infty v_0 \in D(B)$, we have that v_0 vanishes on the part of the boundary $\{x \in \omega_N \ / \ \gamma(x) \neq 0\}$. Therefore, $\Gamma_\infty v_0 = 0$ and (3.7) gives that

$$\|(e^{A_\infty t} - e^{A_n t})(0, v_0)\|_X \leq C(1 + T^{1/2})\|v_0\|_{D(B^{1/2})} \ .$$

Interpolating with (3.8), we obtain that (3.6) holds for all $(0, v_0) \in D(A_\infty)$. If $s < 1/2$, the set $\{(u, v) \in D(A_\infty) \ / \ u = 0\}$ is dense in $\{(u, v) \in X^s \ / \ u = 0\}$. Using this density, we conclude that (3.6) holds for all $(0, v_0) \in X^s$. $\square$

Remarks : As noticed in the previous section, if the semigroups $e^{A_n t}$ have a uniform exponential decay rate, then the constant C does not depend on T.
Of course, one can expect that the decay rate $\varepsilon_n^{s^2/2}$ can be replaced by $\varepsilon_n^{s/2}$, when $s < s_0$. To obtain this better decay rate, one has to show that X^s is the interpolated space between X and $D(A_\infty)$, which is not a so easy result.

Proposition 3.4 implies a result similar to the above one.

Proposition 3.6. *For any $s \in [0,1]$ and any positive time T, there exists a positive constant C such that*

$$\forall t \in [0,T], \forall U_0 \in X,\ \|(e^{A_\infty t} - e^{A_n t})U_0\|_{X^{-s}} \leq C(1+T^{s/8})\varepsilon_n^{s/8}\|U_0\|_X\ . \tag{3.11}$$

Proof : Proposition 3.4 implies that

$$\|A_\infty^{-1}((e^{A_\infty t} - e^{A_n t})U_0)\|_X \leq C(1+T^{1/2})\varepsilon_n^{1/2}\|U_0\|_X\ .$$

We set $(\varphi,\psi) = (e^{A_\infty t} - e^{A_n t})U_0$. We have

$$\|\Gamma_\infty\varphi + B^{-1}\psi\|_{D(B^{1/2})} + \|\varphi\|_{\mathbb{L}^2} \leq C(1+T^{1/2})\varepsilon_n^{1/2}\|U_0\|_X\ . \tag{3.12}$$

On the other hand, the dissipativeness of A_n implies that

$$\|\psi\|_{\mathbb{L}^2} + \|\varphi\|_{D(B^{1/2})} \leq C\|U_0\|_X\ . \tag{3.13}$$

Since $\|\varphi\|_{D(B^{\theta/2})} \leq \|\varphi\|_{\mathbb{L}^2}^{1-\theta}\|\varphi\|_{D(B^{1/2})}^{\theta}$, (3.12) and (3.13) give

$$\forall \eta \in [0,1],\ \|\varphi\|_{D(B^{(1-\eta)/2})} \leq C(1+T^{\eta/2})\varepsilon_n^{\eta/2}\|U_0\|_X\ . \tag{3.14}$$

As Γ_∞ is linear continuous from $D(B^{(1-\eta)/2})$ into $D(B^{1/2})$ for all $\eta \in [0,1/2[$, (3.12) and (3.14) imply that

$$\|\psi\|_{D(B^{-1/2})} \leq C(1+T^{1/2})\varepsilon_n^{1/2}\|U_0\|_X + \|\varphi\|_{D(B^{(1-\eta)/2})} \leq C(1+T^{\eta/2})\varepsilon_n^{\eta/2}\|U_0\|_X\ .$$

As $\|\psi\|_{D(B^{-s/2})} \leq \|\psi\|_{D(B^{-1/2})}^{s}\|\psi\|_{\mathbb{L}^2}^{1-s}$, the above inequality and (3.13) yield that

$$\|\psi\|_{D(B^{-s/2})} \leq C(1+T^{\eta s/2})\varepsilon_n^{\eta s/2}\|U_0\|_X\ .$$

The estimate (3.11) follows from the above result for $\eta = 1/4$ and (3.14) for $\eta = s$. □

The comparison of trajectories is based on the following lemma.

Lemma 3.7. *Let $\mathcal{B}$ be a bounded set of X^s, $s \in]0, s_0[$. Let $T > 0$, $M > 0$ and $n_0 \in \mathbb{N}$ be such that, for all $U \in \mathcal{B}$, $n \geq n_0$ (including $n = +\infty$) and $t \in [0,T]$, the integral solution $S_n(t)U \in \mathcal{C}^0([0,T],X)$ of (2.5) exists and satisfies*

$$\|S_n(t)U\|_X \leq M\ .$$

Then, there exists a constant $C = C(M)$ such that

$$\forall U \in \mathcal{B},\ \forall t \in [0,T],\ \|S_\infty(t)U - S_n(t)U\|_X \leq Ce^{CT}\varepsilon_n^{\beta}\ , \tag{3.15}$$

where $\beta = \frac{s^2}{2}$ if d=1 or d=2, and $\beta = \min(\frac{s^2}{2}, \frac{1-\alpha}{4})$ if d=3.

Proof : In this proof, C denotes a positive constant which does not depend on n or T, but may depend on M. We have

$$\|S_\infty(t)U - S_n(t)U\|_X \leq \|(e^{A_\infty t} - e^{A_n t})U\|_X + \int_0^t \|(e^{A_\infty(t-\tau)} - e^{A_n(t-\tau)})F(S_\infty(\tau)U)\|_X d\tau$$

$$+ \int_0^t \|e^{A_n(t-\tau)}(F(S_\infty(\tau)U) - F(S_n(\tau)U))\|_X d\tau \ . \tag{3.16}$$

We bound the three terms of the previous inequality as follows.
Using Proposition 3.5, we have

$$\|(e^{A_\infty t} - e^{A_n t})U\|_X \leq C(1 + T^{s^2/2})\varepsilon_n^{s^2/2} \ .$$

As for $\tau \in [0, T]$, $S_\infty(\tau)U$ is bounded in X, Lemma 2.2 and Proposition 3.5 imply that

$$\int_0^t \|(e^{A_\infty(t-\tau)} - e^{A_n(t-\tau)})F(S_\infty(\tau)U)\|_X \leq C(M)(1 + T^\eta)\varepsilon_n^\eta \ ,$$

with $\eta < 1/4$ if $d = 1$ or $d = 2$, and $\eta = \frac{1-\alpha}{4}$ if $d = 3$. As F is locally Lipschitzian, we have

$$\begin{aligned}\int_0^t \|e^{A_n(t-\tau)}F(S_\infty(\tau)U) - F(S_n(\tau)U)\|_X d\tau \quad &\leq \int_0^t \|F(S_\infty(\tau)U) - F(S_n(\tau)U)\|_X d\tau \\ &\leq C(M) \int_0^t \|S_\infty(\tau)U - S_n(\tau)U\|_X d\tau \ .\end{aligned}$$

We finish the proof by applying Gronwall's lemma to (3.16). □

Remark : In fact, we can show that, if U belongs to X^s for some $s > 0$, then $S_\infty(t)U \in \mathbb{L}^\infty([0, T], X^{s^2})$. Thus, we can prove that (3.15) holds for all $\beta \leq s^2/2$, even if $d = 3$ and if f is cubic-like (see [27]).

We deduce from Lemma 3.7 a stronger result.

Theorem 3.8. *Let $\mathcal{B}$ be a bounded set of X^s, $s \in]0, s_0[$, and let T be a positive time. There exists $M > 0$ such that, for all $U \in \mathcal{B}$ and $t \in [0, T]$, $S_\infty(t)U$ exists and satisfies $\|S_\infty(t)U\|_X \leq M$, if and only if there exists $M' > 0$ such that, for n large enough, $U \in \mathcal{B}$ and $t \in [0, T]$, $S_n(t)U$ exists and satisfies $\|S_n(t)U\|_X \leq M'$.*
Moreover, if one of these equivalent properties is satisfied, then there exists a constant $C = C(M)$ such that, for n large enough,

$$\forall U \in \mathcal{B}, \ \forall t \in [0, T], \ \|S_\infty(t)U - S_n(t)U\|_X \leq Ce^{CT}\varepsilon_n^\beta \ ,$$

where $\beta = \frac{s^2}{2}$ if d=1 or d=2, and $\beta = \min(\frac{s^2}{2}, \frac{1-\alpha}{4})$ if d=3.

Proof : Once the equivalence is proved, the estimate is a consequence of Lemma 3.7. Assume that, for all $U \in \mathcal{B}$ and $t \in [0,T]$, $S_\infty(t)U$ exists and satisfies

$$\|S_\infty(t)U\|_X \leq M \ . \tag{3.17}$$

Assume that there exist sequences $U_k \in \mathcal{B}$, $t_k \in [0,T]$ and $n_k \longrightarrow +\infty$ such that

$$\forall t \in [0, t_k[,\ \|S_{n_k}(t)U_k\|_X < 2M \text{ and } \|S_{n_k}(t_k)U_k\|_X = 2M \ .$$

We have

$$\|S_\infty(t_k)U_k\|_X \geq \|S_{n_k}(t_k)U_k\|_X - \|(S_{n_k}(t_k) - S_\infty(t_k))U_k\|_X \ .$$

For k large enough, applying Lemma 3.7 (with M replaced by $2M$), we find that $\|S_\infty(t_k)U_k\|_X \geq \frac{3}{2}M$, which contradicts (3.17). Thus, for n large enough, for any U in $\mathcal{B}$ and any $t \in [0,T]$, $S_n(t)U$ exists and satisfies $\|S_n(t)U\|_X \leq M' = 2M$.
This proves the "only if" part. The "if" part is shown in the same way. □

The previous theorem together with the density of X^s in X imply the convergence of the trajectories in X for any initial data U in X. However, the convergence is not uniform on a bounded set of X.

Corollary 3.9. *Let U be an initial datum in X and let T be a positive time. Then the mild solution $S_\infty(t)U \in \mathcal{C}^0([0,T],X)$ of (2.5) with $n = \infty$ exists if and only if there exists M such that, for n large enough, the mild solution $S_n(t)U \in \mathcal{C}^0([0,T],X)$ of (2.5) exists and $\|S_n(t)U\|_X \leq M$ for $t \in [0,T]$.*
Moreover, if one of the equivalent properties is satisfied, then

$$\sup_{t\in[0,T]} \|(S_\infty(t) - S_n(t))U\|_X \longrightarrow 0 \ \textit{when } n \longrightarrow +\infty \ . \tag{3.18}$$

In the following theorem, we obtain the convergence of trajectories in X^{-s} for initial data in X. Notice that, contrary to Theorem 3.8, we cannot prove existence of trajectories in $\mathcal{C}^0([0,T],X)$ for n large enough assuming only the existence of trajectories for the limit case $n = \infty$.

Theorem 3.10. *Let $\mathcal{B}$ be a bounded set of X. We assume that there exist $T > 0$, $M > 0$ and $n_0 \in \mathbb{N}$ such that, for all $U \in \mathcal{B}$, $n \geq n_0$ (and also $n = \infty$) and $t \in [0,T]$, the solution $S_n(t)U$ of (2.5) exists in $\mathcal{C}^0([0,T],X)$ and satisfies $\|S_n(t)U\|_X \leq M$. Then, there exists a constant C such that*

$$\forall U \in \mathcal{B}, \forall t \in [0,T],\ \|A_\infty^{-1}(S_\infty(t) - S_n(t))U\|_X \leq Ce^{CT}\varepsilon_n^{1/2} \ . \tag{3.19}$$

Moreover, for any $s \in [0,1]$, there exists a constant C' such that

$$\forall U \in \mathcal{B}, \forall t \in [0,T],\ \|(S_\infty(t) - S_n(t))U\|_{X^{-s}} \leq C'e^{C'T}\varepsilon_n^{s/8} \ . \tag{3.20}$$

Proof : As usual, C denotes a generic positive constant, which may vary from line to line. We recall that A_∞^{-1} is given by (2.10). We set $S_n(t)U = (u_n(t), v_n(t))$. We write

$$\begin{aligned} \|A_\infty^{-1}(S_\infty(t) - S_n(t))U\|_X \quad & \leq \|A_\infty^{-1}(e^{A_\infty t} - e^{A_n t})U\|_X \\ & \quad + \int_0^t \|A_\infty^{-1}(e^{A_\infty(t-\tau)} - e^{A_n(t-\tau)})F(S_n(\tau)U)\|_X d\tau \\ & \quad + \int_0^t \|e^{A_\infty(t-\tau)}A_\infty^{-1}(F(S_\infty(\tau)U) - F(S_n(\tau)U))\|_X d\tau \ . \end{aligned}$$

Using Proposition 3.4, we find

$$\begin{aligned} \|A_\infty^{-1}(S_\infty(t) - S_n(t))U\|_X \quad & \leq C(1+\sqrt{T})\varepsilon_n^{1/2}\|U\|_X \\ & \quad + \int_0^t C(1+\sqrt{T})\varepsilon_n^{1/2}\|f(x, u_n(x,\tau))\|_{\mathbb{L}^2} d\tau \\ & \quad + \int_0^t \|B^{-1/2}(f(x, u_\infty(x,\tau)) - f(x, u_n(x,\tau)))\|_{\mathbb{L}^2} d\tau \ . \end{aligned}$$

Using (NL), we obtain that $\|f(x,u_n)\|_{\mathbb{L}^2}$ is bounded. We next show that

$$I = \|B^{-1/2}(f(x,u_\infty) - f(x,u_n))\|_{\mathbb{L}^2} \leq C\|u_\infty - u_n\|_{\mathbb{L}^2} \ . \tag{3.21}$$

Indeed, if for example the dimension is equal to 3, we have

$$\begin{aligned} I \quad & = \sup_{\|\varphi\|_{D(B^{1/2})}=1} \int_\Omega (f(x,u_\infty) - f(x,u_n))\varphi dx \\ & \leq \sup_{\|\varphi\|_{D(B^{1/2})}=1} C\left|\int_\Omega (1 + |u_\infty|^\alpha + |u_n|^\alpha)|u_\infty - u_n|\varphi dx\right| \\ & \leq \sup_{\|\varphi\|_{D(B^{1/2})}=1} C\|u_\infty - u_n\|_{\mathbb{L}^2}\left(\int_\Omega |\varphi|^6\right)^{1/6}\left(\int_\Omega (1 + |u_\infty|^\alpha + |u_n|^\alpha)^3\right)^{1/3} \end{aligned}$$

Since $\mathbb{H}^1(\Omega)$ (and thus $D(B^{1/2})$) is continuously imbedded in $\mathbb{L}^6(\Omega)$, we obtain (3.21) and we finish the proof of Inequality (3.19) by using Gronwall's lemma .
We enhance that, to obtain (3.20), we cannot directly use Proposition 3.6. This is linked to the fact that A_∞ does not generate a semigroup on X^{-s}. However, we can deduce (3.20) from (3.19) with the same arguments as in the proof of Proposition 3.6. $\square$

With the same arguments, we obtain similar results for the linearized dynamical system.

Proposition 3.11. *Let $U(t) \in \mathcal{C}^0([0,T],X)$. The conclusions of Theorems 3.8 and 3.10 are also valid if $S_n(t)$ is replaced by $DS_n(U)(t)$, the linearized dynamical system defined by (2.9). In particular, let $\mathcal{B}$ be a bounded set of X^s, $s \in]0,s_0[$, and T be a positive time, there exists a positive constant C such that if $U_0 \in \mathcal{B}$ and $U_n(t) \in \mathcal{C}^0([0,T],X)$ is the solution of (2.5) with initial data U_0, then*

$$\forall t \in [0,T],\ \|DS_\infty(U_\infty)(t) - DS_n(U_n)(t)\|_{\mathcal{L}(X^s,X)} \leq Ce^{CT}\varepsilon_n^\beta \ ,$$

where $\beta = \frac{s^2}{2}$ if d=1 or d=2, and $\beta = \min(\frac{s^2}{2}, \frac{1-\alpha}{4})$ if d=3.

4 Comparison of local stable and unstable manifolds

In the previous section, we have proved the convergence of trajectories for a given initial datum. Theorem 3.8 shows that, if we want to study the convergence of orbits for initial data in a bounded set of X, this set must have compactness properties. Thus, it is natural to wonder, in the case where Equation (2.5) has a compact global attractor $\mathcal{A}_n$, if the attractors $\mathcal{A}_n$ converge to $\mathcal{A}_\infty$. The existence, boundedness, regularity and upper-semicontinuity of the attractors have already been discussed in Theorem 2.8, Proposition 2.9 and Theorem 2.10. In this section, we study the convergence of the local unstable manifolds and the convergence of regular parts of the local stable manifolds. Then, we deduce the lower-semicontinuity of the attractors from the convergence of the local unstable manifolds. Notice that the convergence of regular parts of the local stable manifolds is not needed to show the lower-semicontinuity.

We begin by recalling some classical notions. An equilibrium point $E \in X$ is said to be hyperbolic for the dynamical system $S(t)$ if the spectrum of the linearization $DS(E)(1)$ does not intersect the complex unit circle. Let P^u be the spectral projection onto the part of the spectrum of modulus larger than 1, and $P^s = Id - P^u$ the spectral projection onto the part of the spectrum of modulus smaller than 1. If E is hyperbolic, there exist two positive constants λ_u and λ_s and two positive constants M_u and M_s such that

$$\forall t \geq 0,\ \|DS(E)(t)P^s\|_{\mathcal{L}(X)} \leq M_s e^{-\lambda_s t} \ \text{ and } \ \forall t \leq 0,\ \|DS(E)(t)P^u\|_{\mathcal{L}(X)} \leq M_u e^{\lambda_u t} \ .$$

We set $B^u(r) = P^u X \cap B(E,r)$ and $B^s(r) = P^s X \cap B(E,r)$. The following theorem is classical in the theory of dynamical systems (see for example the Appendix of [17]).

Theorem 4.1. *We assume that $S(t)$ is of class $\mathcal{C}^{1,1}$ from X into X and that E is a hyperbolic equilibrium point of $S(t)$. For $r > 0$ small enough, there exists a unique function h^s from $B^s(r)$ into $B^u(M_s r)$, which is of class $\mathcal{C}^{1,1}$, satisfies $h^s(E) = E$ and $Dh^s(E) = 0$. Moreover, its graph $W^s(E,r)$ (called the local stable manifold) satisfies the following properties.*

i) $W^s(E,r) = \{U \in B(E, 2M_s r) \mid P^s U \in B^s(r)$ *and* $\forall t \geq 0,\ S(t)U \in B(E, 2M_s r)\}$,
ii) if $U \in W^s(E,r)$ *then*

$$\limsup_{t \to +\infty} \frac{1}{t} \ln \|S(t)U - E\|_X \leq -\lambda_s \ .$$

There exists also a unique function h^u *from* $B^u(r)$ *into* $B^s(M_u r)$*, which is of class* $\mathcal{C}^{1,1}$*, satisfies* $h^u(E) = E$ *and* $Dh^u(E) = 0$*. Moreover, its graph* $W^u(E,r)$ *(called the local unstable manifold) satisfies the following properties.*
iii) $W^u(E,r) = \{U \in B(E, 2M_u r) \mid P^u U \in B^u(r)$ *and there exists a negative trajectory* $U(t) \in \mathcal{C}^0(]-\infty, 0], X)$ *such that* $\forall t \leq 0,\ U(t) \in B(E, 2M_u r)\}$,
iv) if $U \in W^u(E,r)$ *then there exists a unique negative trajectory* $U(t) \in \mathcal{C}^0(]-\infty,0],X)$ *such that* $U(t) \in B(2M_u r)$ *for any* $t \leq 0$*, and*

$$\limsup_{t \to -\infty} \frac{1}{|t|} \ln \|U(t) - E\|_X \leq -\lambda_u \ .$$

We also introduce some classical definitions and the corresponding notations.

Definition 4.2. *Let* E *be a hyperbolic equilibrium. The dimension of* $P^u X$*, which is also the one of* $W^u(E,r)$*, is called the Morse index of* E *and is denoted by* $m(E)$*.*
We also define the stable and unstable sets of E*, which are not necessarily well-defined manifolds, by* $W^s(E) = \{U \in X \mid \lim_{t \to +\infty} S(t)U = E\}$ *and* $W^u(E) = \{U \in X \mid \exists$ *a negative trajectory* $U(t) \in \mathcal{C}^0(]-\infty,0],X)$ *such that* $\lim_{t \to -\infty} U(t) = E\}$ *respectively.*

4.1 Preliminary results and spectral study

In what follows, we use the notations of Theorem 4.1 with a subscript n for the dependance with respect to n.
Let $E = (e, 0)$ be an equilibrium point of (2.5). We set

$$\forall n \in \mathbb{N} \cup \{+\infty\},\ \tilde{A}_n = A_n + \begin{pmatrix} 0 & 0 \\ f'_u(x, e(x)) & 0 \end{pmatrix} \ .$$

Notice that the linearization of $S_n(t)$ at the equilibrium point E is $DS_n(E)(t) = e^{\tilde{A}_n t}$. We also set, for any $U = (u, v)$ in X,

$$g(U) = \begin{pmatrix} 0 \\ f(x,u) - f'_u(x, e(x))u \end{pmatrix} \ .$$

Equation (2.5) becomes

$$U_t = \tilde{A}_n + g(U) \ . \tag{4.1}$$

When no confusion is possible, we denote $f'_u(x, e)$ by f'_u. Hypothesis (NL) implies the following properties.

Lemma 4.3. *The function g is a compact Lipschitz-continuous function on the bounded sets of X. More precisely, we have*

$$\forall U, U' \in B_X(E,r),\ \|g(U)-g(U')\|_X \le l(r)\|U-U'\|_X \ ,$$

where $l(r)$ is a non-negative and non-decreasing function which tends to 0 when r goes to 0. In addition, g is of class $\mathcal{C}^{1,1}$ and if $\mathcal{B}$ is a bounded set of X, then there exists a positive constant $C = C(\mathcal{B})$ such that

$$\forall U \in \mathcal{B}, \forall V \in X,\ \|g(U)\|_{X^\sigma} \le C\|U\|_X \quad \text{and } \|g'(U)V\|_{X^\sigma} \le C\|V\|_X, \tag{4.2}$$

where $\sigma \in]0,1[$ when $d=1$ or $d=2$ and $\sigma \in]0, \frac{1-\alpha}{2}[$ when $d=3$.
Moreover, $\tilde{A}_n$ and $e^{\tilde{A}_n}$ are compact perturbations of A_n and e^{A_n} respectively.

Proof : The first part of the Theorem is a consequence of Lemma 2.2 and of classical Sobolev imbeddings. In particular, Lemma 2.2 shows that if $u \in \mathbb{H}^1(\Omega)$, then $f'_u(x,e)u \in \mathbb{H}^\sigma(\Omega)$. Thus, the map $(u,v) \mapsto (0, f'_u(x,e)u)$ is compact from X into X and $\tilde{A}_n$ is a compact perturbation of A_n. To show that $e^{\tilde{A}_n}$ is a compact perturbation of e^{A_n}, we remark that if $U_0 \in X$ and $(u(t), u_t(t)) = e^{\tilde{A}_n t}U_0$, then

$$e^{\tilde{A}_n}U_0 = e^{A_n}U_0 + \int_0^1 e^{A_n(1-t)} \begin{pmatrix} 0 \\ f'_u(x,e(x))u(t) \end{pmatrix} dt \ .$$

□

The behaviour of the spectrum of $\tilde{A}_n$ is described in the following proposition.

Proposition 4.4. *Assume that Hypothesis (ED) holds. Let $\lambda \in \mathbb{C}$ be such that the operator $(\tilde{A}_\infty - \lambda Id) \in \mathcal{L}(X)$ is invertible. Then, for n large enough, $(\tilde{A}_n - \lambda Id)$ is also invertible and there exists a positive constant C_λ such that*

$$\|(\tilde{A}_\infty - \lambda Id)^{-1} - (\tilde{A}_n - \lambda Id)^{-1}\|_{\mathcal{L}(X)} \le C_\lambda \varepsilon_n \ .$$

As a consequence, the point spectrum of $\tilde{A}_n$ converges to the one of $\tilde{A}_\infty$ on every bounded set of $\mathbb{C}$. Moreover, if E is a hyperbolic equilibrium point of the dynamical system $S_\infty(t)$, then, for n large enough, it is a hyperbolic equilibrium point of the dynamical system $S_n(t)$ and there exists a positive constant C such that

$$\|P^u_\infty - P^u_n\|_{\mathcal{L}(X)} \le C\varepsilon_n \ . \tag{4.3}$$

In addition, the part of the spectrum of $\tilde{A}_n$ ($n \in \mathbb{N} \cup \{+\infty\}$) with non-negative real part is composed by a finite number of real positive eigenvalues. Finally, the Morse index of E for

$S_n(t)$, which is the number of positive eigenvalues of $\tilde{A}_n$ is equal for n large enough to the Morse index of E for $S_\infty(t)$.

Proof : We denote by $K_\lambda \in \mathcal{L}(D(B^{1/2}))$ the operator $Id + \lambda^2 B^{-1} - B^{-1} f'_u$. A straightforward computation shows that $(\tilde{A}_n - \lambda Id)$ is invertible if and only if $(K_\lambda + \lambda\Gamma_n)$ is invertible in $\mathcal{L}(D(B^{1/2}))$ and in this case

$$(\tilde{A}_n - \lambda Id)^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (K_\lambda + \lambda\Gamma_n)^{-1}(-B^{-1}v - \lambda B^{-1}u + B^{-1}f'_u u - \Gamma_n u) \\ (K_\lambda + \lambda\Gamma_n)^{-1}(-\lambda B^{-1}v + u) \end{pmatrix} .$$

If $(\tilde{A}_\infty - \lambda Id)$ is invertible, then $(K_\lambda + \lambda\Gamma_\infty)$ is invertible in $\mathcal{L}(D(B^{1/2}))$ and we have

$$(K_\lambda + \lambda\Gamma_n) = (K_\lambda + \lambda\Gamma_\infty)(Id - \lambda(K_\lambda + \lambda\Gamma_\infty)^{-1}(\Gamma_\infty - \Gamma_n)) .$$

For n large enough, $\|\lambda(K_\lambda + \lambda\Gamma_\infty)^{-1}(\Gamma_\infty - \Gamma_n))\|_{\mathcal{L}(D(B^{1/2}))} \leq \frac{1}{2}$, and $(K_\lambda + \lambda\Gamma_n)$ is invertible. Moreover,

$$(K_\lambda + \lambda\Gamma_n)^{-1} - (K_\lambda + \lambda\Gamma_\infty)^{-1} = \lambda(K_\lambda + \lambda\Gamma_\infty)^{-1}(\Gamma_\infty - \Gamma_n)$$
$$\times \left(\sum_{k\geq 0} \lambda^k((K_\lambda + \lambda\Gamma_\infty)^{-1}(\Gamma_\infty - \Gamma_n))^k \right) (K_\lambda + \lambda\Gamma_\infty)^{-1} ,$$

and so, for n large enough,

$$\|(K_\lambda + \lambda\Gamma_n)^{-1} - (K_\lambda + \lambda\Gamma_\infty)^{-1}\|_{\mathcal{L}(D(B^{1/2}))} \leq 2\varepsilon_n \|(K_\lambda + \lambda\Gamma_\infty)^{-1}\|^2_{\mathcal{L}(D(B^{1/2}))} .$$

This gives the first assertion of the proposition. It is well-known that this implies the convergence of the point spectrum.
Assume that E is a hyperbolic equilibrium point for the dynamical system $S_\infty(t)$, we want to prove that for n large enough, it is also a hyperbolic equilibrium point for the dynamical system $S_n(t)$. As Hypothesis (ED) holds, the radius of the spectrum of e^{A_n} is strictly less than one. Since, by Lemma 4.3, $e^{\tilde{A}_n}$ is a compact parturbation of e^{A_n}, the radius of the essential spectrum of $e^{\tilde{A}_n}$ is strictly less than one. As a consequence, for each n, there exists $\delta_n > 0$ such that the spectrum of $\tilde{A}_n$ with real part greater than $-\delta_n$ is only composed by a finite number of eigenvalues of finite multiplicity. We next prove that an eigenvalue of $\tilde{A}_n$ with non-negative real part must be real. Then, the proof of the hyperbolicity of E for $S_n(t)$ is reduced to the proof that $\lambda = 0$ is not an eigenvalue of $\tilde{A}_n$. The local convergence of the spectrum of $\tilde{A}_n$ to the one of $\tilde{A}_\infty$, together with the hyperbolicity of E for $S_\infty(t)$, ensure that $\lambda = 0$ is not an eigenvalue of $\tilde{A}_n$, for n large enough.
We finish the proof by showing that the eigenvalues of $\tilde{A}_\infty$ with non-negative real part are real. The proof in the case of $n < \infty$ is similar and even easier.

Let λ be a non-real eigenvalue of $\tilde{A}_\infty$ with eigenvector $(\varphi, \lambda\varphi)$ such that $\|\varphi\|_{\mathbb{L}^2} = 1$. We have

$$\left\{ \begin{array}{l} \lambda^2\varphi = \Delta\varphi - \kappa\varphi + f'_u(x,e)\varphi \\ \frac{\partial\varphi}{\partial\nu} + \lambda\gamma\varphi = 0 \quad \text{on } \omega_N \\ \varphi = 0 \quad \text{on } \omega_D \end{array} \right. \tag{4.4}$$

where ω_N (resp. ω_D) is the part of $\partial\Omega$ where B has Neumann (resp. Dirichlet) boundary conditions, and where $\kappa = 1$ if $\omega_D = \emptyset$ and $\kappa = 0$ in the other case.
Multiplying the first equation by $\overline{\varphi}$ and integrating, we obtain

$$-\|\overrightarrow{\nabla}\varphi\|^2_{\mathbb{L}^2} - \kappa\|\varphi\|^2_{\mathbb{L}^2} + \int_\Omega f'_u|\phi|^2 = \lambda^2\|\varphi\|^2_{\mathbb{L}^2} + \lambda\int_{\omega_N}\gamma|\varphi|^2 \ .$$

Taking the imaginary part and using the fact that $Im(\lambda) \neq 0$, we find

$$Re(\lambda) = -\frac{1}{2}\int_{\omega_N}\gamma|\varphi|^2 \ . \tag{4.5}$$

To prove that $Re(\lambda) < 0$, we argue by contradiction. Assume that $\int_{\omega_N}\gamma|\varphi|^2 = 0$. There exists an open subset ω of the boundary such that $\varphi_{|\omega} \equiv 0$ and Equation (4.4) shows that $\frac{\partial\varphi}{\partial\nu}_{|\omega} \equiv \varphi_{|\omega} \equiv 0$. Let θ be an open connected subset of Ω such that $(\overline{\omega}_N \cap \overline{\omega}_D) \cap \overline{\theta} = \emptyset$, and $\overline{\theta} \cap \omega \neq \emptyset$. The set θ is defined such that it is distant from the points of the boundary where the Neumann boundary condition meets the Dirichlet one. Regularity theorems for problems with mixed boundary conditions imply that e belongs to $\mathbb{H}^2(\theta)$ and so to $\mathbb{L}^\infty(\theta)$ (see [16]). Thus, as φ is a solution of (4.4), φ satisfies in θ

$$\left\{ \begin{array}{l} \lambda^2\varphi = \Delta\varphi + h\varphi \\ \frac{\partial\varphi}{\partial\nu} = \varphi = 0 \quad \text{on } \omega \cap \overline{\theta} \end{array} \right. \tag{4.6}$$

with some additional boundary conditions, where $h = -\kappa Id + f'_u(x, e(x))$ belongs to $\mathbb{L}^\infty(\theta)$. The classical unique continuation property implies that φ identically vanishes on θ and thus on Ω, which is absurd. $\square$

Let E be a hyperbolic equilibrium point. Using the above proposition, we know that there exist two constants μ and η with $0 < \eta < \mu$ such that the spectrum of $\tilde{A}_\infty$ has the following decomposition.

$$\sigma(\tilde{A}_\infty) = \Big(\sigma(\tilde{A}_\infty) \cap \{z \in \mathbb{C}/Re(z) < 0\}\Big) \cup \Big(\sigma(\tilde{A}_\infty) \cap \{z \in \mathbb{C}/Re(z) \geq \mu + 2\eta\}\Big) \ .$$

Proposition 4.4 implies that, for n large enough, we have

$$\sigma(\tilde{A}_n) = \Big(\sigma(\tilde{A}_n) \cap \{z \in \mathbb{C}/Re(z) < 0\}\Big) \cup \Big(\sigma(\tilde{A}_n) \cap \{z \in \mathbb{C}/Re(z) \geq \mu + \eta\}\Big) \ . \tag{4.7}$$

For $n \in \mathbb{N} \cup \{+\infty\}$, we denote by P_n^u the spectral projection onto the space generated by the eigenvectors corresponding to the part of the spectrum of $\tilde{A}_n$ with real part larger than μ. We set $P_n^s = Id - P_n^u$.

Proposition 4.5. *There exist two positive constants M_u and M_s such that*

$$\forall n \in \mathbb{N} \cup \{+\infty\}, \quad \begin{cases} \forall t \geq 0, & \|e^{\tilde{A}_n t} P_n^s\|_{\mathcal{L}(X)} \leq M_s e^{(\mu-\eta)t} \\ \forall t \leq 0, & \|e^{\tilde{A}_n t} P_n^u\|_{\mathcal{L}(X)} \leq M_u e^{(\mu+\eta)t} \end{cases} \tag{4.8}$$

The proof of the above result is based on the following equivalence.

Theorem 4.6. *Let H_n be a sequence of Hilbert spaces. Let D_n be the generator of a $\mathcal{C}^0-$semigroup of contractions $e^{D_n t}$ on H_n, and let $\lambda > 0$. There exist two positive constants ε and C such that*

$$\forall t \geq 0, \ \|e^{D_n t}\|_{\mathcal{L}(H_n)} \leq C e^{-(\lambda+\varepsilon)t} \tag{4.9}$$

if and only if there exists $\varepsilon' > 0$ such that for all $n \in \mathbb{N}$, the spectrum of D_n satisfies $\sigma(D_n) \subset \{z \in \mathbb{C} \ / \ Re(z) < -\lambda - \varepsilon'\}$ and such that we have

$$\exists M > 0 \text{ such that } \sup_{n \in \mathbb{N}} \sup_{\nu \in \mathbb{R}} \|(D_n + (\lambda + i\nu)Id)^{-1}\|_{\mathcal{L}(H_n)} \leq M \ . \tag{4.10}$$

This result is proved in [34]. Although the theorems given in [34] are stated less precisely, it can be deduced from their proofs.

Proof of Proposition 4.5: First, notice that $e^{\tilde{A}_n t}$ is well-defined on $P_n^u X$ even if $t \leq 0$ and that there exists M such that for any $t \leq 0$, $\|e^{\tilde{A}_n t}\|_{\mathcal{L}(P_n^u X)} \leq M e^{(\mu+\eta)t}$, since $P_n^u X$ is a subspace spanned by a finite number of eigenvectors of $\tilde{A}_n$ corresponding to eigenvalues larger than $\mu + \eta$, this number of eigenvectors being independent of n. Thus, the second estimate of (4.8) is a direct consequence of the convergence of P_n^u to P_∞^u. Let $H_n = P_n^s X$ and let $\tilde{D}_n$ be the restriction to H_n of the operator $\tilde{A}_n - \|f_u'\|_\infty Id$. Notice that $\tilde{D}_n$ is a dissipative operator on H_n and thus that $e^{\tilde{D}_n t}$ is a semigroup of contractions. We set $\lambda = \|f_u'\|_\infty - (\mu - \eta)$. If we prove that (4.10) holds for $\tilde{D}_n$, we will obtain that

$$\|e^{\tilde{D}_n t}\|_{\mathcal{L}(X)} \leq M e^{-(\|f_u'\|_\infty - (\mu-\eta))t} \ ,$$

and so that

$$\|e^{\tilde{A}_n t}\|_{\mathcal{L}(X)} \leq M e^{(\mu-\eta)t} \ .$$

Then the first estimate of (4.8) will be a direct consequence of the convergence of P_n^s to P_∞^s.

The spectral condition of Theorem 4.6 is clear due to the definition of H_n and the fact that $\mu - \eta$ is positive. To show that (4.10) holds, we argue by contradiction and assume that there exist sequences (ν_k) and $(n_k) \to +\infty$ such that

$$\|(\tilde{D}_{n_k} + (\lambda + i\nu_k)Id)^{-1}\|_{\mathcal{L}(H_{n_k})} \longrightarrow +\infty \ . \tag{4.11}$$

As E is hyperbolic for $S_\infty(t)$, Proposition 4.4 implies that $|\nu_k| \longrightarrow +\infty$.
Assume that $\nu_k \longrightarrow +\infty$ and that $\nu_k > 0$ (the case $\nu_k \longrightarrow -\infty$ is similar). We set $D_n = A_n - \|f'_u\|_\infty Id$. As $e^{A_n t}$ is a semigroup of contractions, for all $n \in \mathbb{N} \cup \{+\infty\}$, we have that $\|e^{D_n t}\|_{\mathcal{L}(X)} \leq e^{-\|f'_u\|_\infty t}$ and thus Theorem 4.6 show that

$$\exists M > 0, \ \sup_{\nu_k} \|(D_{n_k} + (\lambda + i\nu_k)Id)^{-1}\|_{\mathcal{L}(X)} \leq M \ . \tag{4.12}$$

Let K be the compact operator $(u,v) \in X \mapsto (0, f'_u(x,e(x))u)$. We have

$$(\tilde{D}_{n_k} + (\lambda + i\nu_k)) = (Id + K(D_{n_k} + (\lambda + i\nu_k))^{-1})(D_{n_k} + (\lambda + i\nu_k)) \ . \tag{4.13}$$

A straightforward calculus shows that if $(\varphi_k, \psi_k) = (D_{n_k} + (\lambda + i\nu_k))^{-1}(u,v)$, then

$$-\varphi_k + (\lambda + i\nu_k)\Gamma_{n_k}\varphi_k - (\lambda + i\nu_k)^2 B^{-1}\varphi_k = B^{-1}v - (\lambda + i\nu_k)B^{-1}u + \Gamma_{n_k}u \ .$$

Multiplying by $B\overline{\varphi}_k$ and integrating, we find

$$\begin{aligned} \nu_k^2 \|\varphi_k\|^2_{\mathbb{L}^2} &= \ < \varphi_k - (\lambda + i\nu_k)\Gamma_{n_k}\varphi_k + \Gamma_{n_k}u | \varphi_k >_{D(B^{1/2})} \\ &\quad + < v - (\lambda + i\nu_k)u|\varphi_k >_{\mathbb{L}^2} + (\lambda^2 + 2i\lambda\nu_k)\|\varphi_k\|^2_{\mathbb{L}^2} \ . \end{aligned}$$

So, there exists a positive constant C such that

$${\nu_k}^2 \|\varphi_k\|^2_{\mathbb{L}^2} \leq C(1+\nu_k)\left(\|(u,v)\|_X + \|\varphi_k\|_{D(B^{1/2})}\right)\|\varphi_k\|_{D(B^{1/2})} \ .$$

As (4.12) holds, we have $\|\varphi_k\|_{D(B^{1/2})} \leq M\|(u,v)\|_X$ and so $\|\varphi_k\|_{\mathbb{L}^2} \leq \frac{C}{\sqrt{\nu_k}}\|(u,v)\|_X$. Using (NL), we find that there exists $s \in]0, 1/2[$ such that

$$\|K(D_{n_k} + (\lambda + i\nu_k))^{-1}(u,v)\|_X = \|f'_u(x,e)\varphi_k\|_{\mathbb{L}^2} \leq \frac{C}{\nu_k^s}\|(u,v)\|_X \ ,$$

and so $\|K(D_{n_k} + (\lambda + i\nu_k))^{-1}\|_{\mathcal{L}(X)} \longrightarrow 0$ as $k \longrightarrow +\infty$. Thus, (4.13) implies that $\tilde{D}_{n_k} + (\lambda + i\nu_k)$ is invertible for k large enough and satisfies (4.10) with a constant $\tilde{M}$ independent of ν_k. This contradicts the above assumption (4.11) and proves the proposition. □

4.2 Convergence of the local unstable manifolds

As above, we will use the notations of Theorem 4.1 with a subscript n for the dependance with respect to n. In particular, we recall that $B^u_n(r) = P^u_n X \cap B_X(E,r)$ and $B^s_n(r) = P^s_n X \cap B_X(E,r)$.
The whole section is devoted to the proof of the following theorem.

Theorem 4.7. *Let E be a hyperbolic equilibrium point of the dynamical system $S_\infty(t)$. We assume that the exponential decay (ED) holds. Then, E is a hyperbolic equilibrium point of $S_n(t)$ for n large enough and there exists a radius $r > 0$ such that the function h_n^u and its derivative Dh_n^u are defined in $B_n^u(r)$. In other words, the local unstable manifolds $W_n^u(E, r)$ are defined for n large enough in a neighborhood of E independent of n. Moreover, the decay rate λ_u of Property iv) of Theorem 4.1 and the Lipschitz-constants of h_n^u and Dh_n^u are uniform in n. In addition, there exists a positive constant C such that, for all $\xi \in B_\infty^u(r)$,*

$$\|h_\infty^u(\xi) - h_n^u(P_n^u\xi)\|_X \leq C\varepsilon_n^\beta \quad \text{and} \quad \|Dh_\infty^u(\xi)P_\infty^u - Dh_n^u(P_n^u\xi)P_n^u\|_{\mathcal{L}(X,X)} \leq C\varepsilon_n^\beta \ , \tag{4.14}$$

where β is any number in $]0, 1/8[$ if $d = 1$ or $d = 2$ or any number in $]0, \min(\frac{1}{32}, \frac{1-\alpha}{4})[$ if $d = 3$. In particular, we have that

$$d_X(W_n^u(E, r); W_\infty^u(E, r)) \leq C\varepsilon_n^\beta \ .$$

Til the end of this section, we assume that (ED) holds. For sake of simplicity, we may set without loss of generality that $E = 0$ and $f(x, 0) = 0$. We also assume that $E = 0$ is a hyperbolic equilibrium of the dynamical system $S_n(t)$ and that the spectral decomposition (4.7) holds for any $n \in \mathbb{N} \cup \{+\infty\}$.

The outline of the proof of Theorem 4.7 is as follows. We know that, for each n, there exists a local unstable manifold $W_n^u(E, r_n)$. We will construct, for each $n \in \mathbb{N} \cup \{+\infty\}$, the local strongly unstable manifold $W_n^{su}(E, r_n)$ in $B_X(0, r_n)$, corresponding to the spectral decomposition (4.7). This construction is done with a fixed point theorem, using the method of Lyapounov-Perron (see [17]). We will show that this construction can be made in a ball $B_X(0, r)$ independent of n. Next, we will compare $W_n^{su}(E, r)$ and $W_\infty^{su}(E, r)$, using the continuity of the fixed point with respect to the parameter n. Finally, as $E = 0$ is hyperbolic for each n, and as (4.7) holds, we know that the local strongly unstable manifold $W_n^{su}(E, r)$ is in fact the local unstable manifold $W_n^u(E, r)$ defined in Theorem 4.1.

We introduce the space

$$Y_\mu = \{U \in \mathcal{C}^0(]-\infty, 0], \mathbb{C}) \ / \ \sup_{t\leq 0} \|U(t)\|_X e^{-\mu t} < +\infty\} \ .$$

We endow Y_μ with the norm $\|.\|_\mu$ defined by

$$\|U\|_\mu = \sup_{t\leq 0} \|U(t)\|_X e^{-\mu t} \ .$$

We set $B_\mu(R) = \{U \in Y_\mu \ / \ \|U\|_\mu \leq R\}$. We recall that the integral equation associated to $U_t = \tilde{A}_n U + g(U)$ is

$$U(t) = e^{\tilde{A}_n(t-t_0)}U(t_0) + \int_{t_0}^t e^{\tilde{A}_n(t-s)}g(U(s))ds \ . \tag{4.15}$$

We next prove the following result.

Theorem 4.8. *We assume that the hypotheses of Theorem 4.7 hold. For $r > 0$ small enough, there exists a family $(h_n^u)_{n\in\mathbb{N}\cup\{+\infty\}}$ of functions of class $\mathcal{C}^1$, defined from $B_n^u(r)$ into $B_n^s(M_u r)$, such that $h_n^u(0) = 0$. The graph $W_n^{su}(0,r)$ of h_n^u satisfies*

$$W_n^{su}(0,r) = \{U_0 \in B_X(0,2M_u r) \ / \ P_n^u U_0 \in B_n^u(r) \text{ and there exists } U \in B_\mu(2M_u r) \text{ solution of (4.15) such that } U(0) = U_0\} \ .$$

Moreover, there exists a positive constant $C = C(\beta)$ such that

$$d_X(W_n^{su}(0,r), W_\infty^{su}(0,r)) \leq C\varepsilon_n^\beta \ ,$$

where β is any number in $]0,1/8[$ if $d = 1$ or $d = 2$ or any number in $]0, \min(\frac{1}{32}, \frac{1-\alpha}{4})[$ if $d = 3$.

The proof of this theorem consists of several lemmas.
The solutions of (4.15) are characterized as follows.

Lemma 4.9. *Let $R > 0$ and $U \in B_\mu(R)$. For any $n \in \mathbb{N}\cup\{+\infty\}$, U is a negative trajectory of (4.15) if and only if, for all $t \leq 0$,*

$$U(t) = \int_{-\infty}^{t} e^{\tilde{A}_n(t-s)} P_n^s g(U(s))ds + e^{\tilde{A}_n t} P_n^u \xi - \int_t^0 e^{\tilde{A}_n(t-s)} P_n^u g(U(s))ds \tag{4.16}$$

where $\xi = U(0)$.

Proof : Since the proof is classical, we omit it (see [17]). □

Let $\xi \in X$, we introduce the functional T_n^ξ defined from Y_μ into Y_μ by

$$(T_n^\xi U)(t) = \int_{-\infty}^{t} e^{\tilde{A}_n(t-s)} P_n^s g(U(s))ds + e^{\tilde{A}_n t} P_n^u \xi - \int_t^0 e^{\tilde{A}_n(t-s)} P_n^u g(U(s))ds \ . \tag{4.17}$$

Lemma 4.9 shows that $U(0) \in W_n^{su}(E,r)$ if and only if $T_n^\xi U = U$. It remains to prove that T_n^ξ is a contraction.

Lemma 4.10. *There exists a positive constant r_0, independent of n, such that for all $n \in \mathbb{N}\cup\{+\infty\}$, for all $r \in]0,r_0[$ and $\xi \in X$ with $\|P_n^u\xi\|_X \leq r$, T_n^ξ is defined from $B_\mu(2M_u r)$ into $B_\mu(2M_u r)$. Moreover,*

$$\forall n \in \mathbb{N}\cup\{+\infty\}, \forall U, U' \in B_\mu(2r), \ \|T_n^\xi U - T_n^\xi U'\|_\mu \leq \frac{1}{2}\|U - U'\|_\mu \ .$$

Proof : To see that T_n^ξ maps $B_\mu(2M_u r)$ into $B_\mu(2M_u r)$, we bound the three terms of (4.17). Let $U \in B_\mu(2M_u r)$. We have

$$\begin{aligned} \|e^{-\mu t}\int_{-\infty}^t e^{\tilde{A}_n(t-s)}P_n^s g(U(s))ds\|_X ds \quad &\leq \int_{-\infty}^t e^{-\mu t}M_s e^{(\mu-\eta)(t-s)}l(2M_u r)\|U(s)\|_X ds \\ &\leq M_s l(2M_u r)\int_{-\infty}^t e^{-\eta(t-s)}\|U\|_\mu ds \\ &\leq \frac{M_s}{\eta} l(2M_u r)\|U\|_\mu \end{aligned}$$

Using (4.8), we obtain $\|e^{-\mu t}e^{\tilde{A}_n t}P_n^u\xi\|_X \leq M_u\|\xi\|_X$. To bound the last term, we write

$$\begin{aligned} \|e^{-\mu t}\int_t^0 e^{\tilde{A}_n(t-s)}P_n^u g(U(s))ds\|_X \quad &\leq \int_t^0 e^{-\mu t}e^{(\mu+\eta)(t-s)}M_u l(2M_u r)\|U(s)\|_X ds \\ &\leq M_u l(2M_u r)\int_t^0 e^{\eta(t-s)}\|U\|_\mu ds \\ &\leq \frac{M_u}{\eta} l(2M_u r)\|U\|_\mu \end{aligned}$$

Thus, using the fact that $l(2M_u r) \longrightarrow 0$, we can choose r_1 small enough so that

$$\frac{M_u + M_s}{\eta} l(2M_u r)2M_u r \leq M_u r \ ,$$

and thus T_n^ξ is defined from $B_\mu(2M_u r)$ into $B_\mu(2M_u r)$. The fact that T_n^ξ is a contraction for r small enough is proved by the same way. We will choose $r_0 \in]0, r_1]$ so that T_n^ξ is a contraction with constant of contraction equal to 1/2. $\square$

The previous lemma implies that, if r is small enough, for any $n \in \mathbb{N} \cup \{+\infty\}$ and any $\xi \in B_n^u(r)$ there exists a unique solution $U_n^\xi(t) \in B_\mu(2M_u r)$ of (4.15) such that $P_n^u\xi = P_n^u U_n^\xi(0)$. We define the function h_n^u by

$$h_n^u \ : \ \left(\begin{array}{ccc} B_n^u(r) & \longrightarrow & P_n^s X \\ \xi & \longmapsto & P_n^s U_n^\xi(0) \end{array}\right) .$$

To be more precise, $P_n^s U_n^\xi(0) = \int_{-\infty}^0 e^{-\tilde{A}_n s}P_n^s g(U(s))ds$ and so, the choice of r in the preceding proof implies that $\|P_n^s U_n^\xi(0)\|_X \leq M_u r$. Therefore, h_n^u is defined from $B_n^u(r)$ into $B_n^s(M_u r)$. Moreover, using the same arguments as in the proof of Lemma 4.10, we can show that h_n^u is Lipschitzian. To finish the proof of Theorem 4.8, we show the following two lemmas.

Lemma 4.11. *There exists a positive constant C such that for any $U \in D(A_\infty)$ and $t \le 0$, we have*

$$\left\| \left(e^{(\tilde{A}_n-\mu)t} P_n^u - e^{(\tilde{A}_\infty-\mu)t} P_\infty^u \right) U \right\|_X \le C\varepsilon_n^{1/2} \|U\|_{D(A_\infty)} \ . \tag{4.18}$$

There exists a positive constant C such that, for any $U \in X$ and any $t \le 0$, we have

$$\|(e^{\tilde{A}_n t} P_n^u - e^{\tilde{A}_\infty t} P_\infty^u) g(U(s))\|_X \le C e^{(\mu+\eta/2)t} \varepsilon_n^\beta \|U(s)\|_X \ , \tag{4.19}$$

with β as in Theorem 4.8, and for any $t \ge 0$

$$\|(e^{\tilde{A}_n(t-s)} P_n^s - e^{\tilde{A}_\infty(t-s)} P_\infty^s) g(U(s))\|_X \le C e^{-(\mu-\eta/2)t} \varepsilon_n^\beta \|U(s)\|_X \ . \tag{4.20}$$

Proof : We notice that $-\tilde{A}_n$ is a bounded operator on $P_n^u X$, since $P_n^u X$ is spanned by a finite number of eigenvectors of $-\tilde{A}_n$. This number and the associated eigenvalues being bounded with respect to n, there exists a positive constant C such that for all $n \in \mathbb{N} \cup \{+\infty\}$, $-\tilde{A}_n - C$ is a dissipative operator on $P_n^u X$. We also remark that the operators $(\tilde{A}_n - \|f_u'(x,0)\|_{\mathbb{L}^\infty} Id)$ are dissipative on $P_n^s X$.
Thus, (4.18) is a direct consequence of Corollary 3.3, Propositions 4.4 and 4.5. The estimates (4.19) and (4.20) are proved in the same way, using the regularity property (4.2) of g and interpolation arguments similar to the proof of Proposition 3.5. □

Lemma 4.12. *Let $r \in]0, r_0[$, where r_0 has been defined in Lemma 4.10, and let $\xi \in X$ such that $\|P_\infty^u \xi\|_X \le r$. There exists a positive constant C such that*

$$\|U_\infty^\xi - U_n^\xi\|_\mu \le C\varepsilon_n^\beta \ , \tag{4.21}$$

where β is given in Theorem 4.8. Moreover, if we set, for $n \in \mathbb{N} \cup \{+\infty\}$, $\xi_n = P_n^u \xi$, then

$$\|h_\infty^u(\xi_\infty) - h_n^u(\xi_n)\|_X \le C\varepsilon_n^\beta \ . \tag{4.22}$$

Proof : We have

$$\begin{aligned} \|U_\infty^\xi - U_n^\xi\|_\mu &= \|T_\infty^\xi U_\infty^\xi - T_n^\xi U_n^\xi\|_\mu \\ &\le \|T_n^\xi U_\infty^\xi - T_n^\xi U_n^\xi\|_\mu + \|T_\infty^\xi U_\infty^\xi - T_n^\xi U_\infty^\xi\|_\mu \\ &\le \frac{1}{2}\|U_\infty^\xi - U_n^\xi\|_\mu + \|T_\infty^\xi U_\infty^\xi - T_n^\xi U_\infty^\xi\|_\mu \ , \end{aligned}$$

and thus,

$$\|U_\infty^\xi - U_n^\xi\|_\mu \le 2\|T_\infty^\xi U_\infty^\xi - T_n^\xi U_\infty^\xi\|_\mu \ . \tag{4.23}$$

To simplify the notations, we set $U = U_\infty^\xi$. We have

$$\begin{aligned} T_n^\xi U - T_\infty^\xi U &= (e^{\tilde{A}_n t} P_n^u - e^{\tilde{A}_\infty t} P_\infty^u)\xi - \int_t^0 (e^{\tilde{A}_n(t-s)} P_n^u - e^{\tilde{A}_\infty(t-s)} P_\infty^u) g(U(s)) ds \\ &\qquad + \int_{-\infty}^t (e^{\tilde{A}_n(t-s)} P_n^s - e^{\tilde{A}_\infty(t-s)} P_\infty^s) g(U(s)) ds \\ &= K_1 - K_2 + K_3 \ . \end{aligned} \tag{4.24}$$

To estimate the term $\|T_n^\xi U - T_\infty^\xi U\|_\mu = \sup_{t\leq 0} e^{-\mu t}\|T_n^\xi U - T_\infty^\xi U\|_X$, we proceed as follows.

$$e^{-\mu t}K_1 = \left(e^{(\tilde{A}_n-\mu)t}P_n^u - e^{(\tilde{A}_\infty-\mu)t}P_\infty^u\right)P_\infty^u\xi + e^{-\mu t}e^{\tilde{A}_n t}P_n^u(P_n^u - P_\infty^u)\xi \ . \tag{4.25}$$

As P_∞^u is a projection on a finite number of eigenvalues, $P_\infty^u\xi$ belongs to $D(\tilde{A}_\infty)$ and $\|P_\infty^u\xi\|_{D(\tilde{A}_\infty)} \leq C\|\xi\|_X$. Thus, Lemma 4.11 implies that there exists a positive constant C such that for any $t \leq 0$,

$$\left\|\left(e^{(\tilde{A}_n-\mu)t}P_n^u - e^{(\tilde{A}_\infty-\mu)t}P_\infty^u\right)P_\infty^u\xi\right\|_X \leq C\varepsilon_n^{1/2}\|\xi\|_X \ .$$

For the second term of (4.25), we use (4.8) and (4.3) to get

$$\|e^{-\mu t}e^{\tilde{A}_n t}P_n^u(P_n^u - P_\infty^u)\xi\|_X \leq C\varepsilon_n\|\xi\|_X \ ,$$

and thus, gathering the terms of (4.25), we obtain

$$\|K_1\|_\mu \leq C\varepsilon_n^{1/2}\|\xi\|_X \ .$$

We bound the second term of (4.24) by using (4.19) as follows

$$\begin{aligned}
\|e^{-\mu t}K_2\|_X \quad &= \|e^{-\mu t}\int_t^0 (e^{\tilde{A}_n(t-s)}P_n^u - e^{\tilde{A}_\infty(t-s)}P_\infty^u)g(U(s))ds\|_X \\
&\leq \int_t^0 Ce^{\frac{\eta}{2}(t-s)}e^{-\mu s}\varepsilon_n^\beta\|U(s)\|_X ds \\
&\leq C\varepsilon_n^\beta\|U\|_\mu \int_t^0 e^{\frac{\eta}{2}(t-s)}ds \ \leq \frac{2C}{\eta}\varepsilon_n^\beta \ .
\end{aligned}$$

To bound the third term of (4.24), we use (4.20) :

$$\begin{aligned}
\|e^{-\mu t}K_3\|_X \quad &= \|e^{-\mu t}\int_{-\infty}^t (e^{\tilde{A}_n(t-s)}P_n^s - e^{\tilde{A}_\infty(t-s)}P_\infty^s)g(U(s))ds\|_X \\
&\leq C\varepsilon_n^\beta \int_{-\infty}^t e^{-\frac{\eta}{2}(t-s)}e^{-\mu t}\|U(s)\|_X ds \\
&\leq C\varepsilon_n^\beta\|U\|_\mu \int_{-\infty}^t e^{-\frac{\eta}{2}(t-s)}ds \ \leq \frac{2C}{\eta}\varepsilon_n^\beta \ .
\end{aligned}$$

Due to the decomposition (4.24), the inequality (4.23) and the above bounds of $\|K_i\|_\mu$ $(i = 1, 2, 3)$ imply the estimate (4.21).
The inequality (4.22) is a direct consequence of (4.21) and of (4.3). □

Proof of Theorem 4.7: Lemma 4.12 completes the proof of Theorem 4.8. By Proposition 4.4, for n large enough, E is a hyperbolic equilibrium for $S_n(t)$. Proposition 4.4 together with the decay property (4.8) also imply that there exists a local unstable manifold $W^u_n(E,r)$ which is equal to the strong unstable manifold $W^{su}_n(E,r)$ we have constructed. Thus, the estimate (4.22) of Lemma 4.12 implies the first estimate of (4.14).
It is well-known that, if g is of class $\mathcal{C}^p$, then the mapping $(\xi, U) \longmapsto T^\xi_n U$ is of class $\mathcal{C}^p$ and the fixed point U^ξ_n is a $\mathcal{C}^p$-mapping from $P_n B_X(0,r)$ into Y_μ (see [17]). In particular, we notice that, like in (4.16), we have

$$\begin{aligned} DU^\xi_n\zeta &= e^{\tilde{A}_n t}P^u_n\zeta + \int_{-\infty}^t e^{\tilde{A}_n(t-s)}P^s_n g'(U^\xi_n(s))DU^\xi_n(s)\zeta ds \\ &\quad - \int_t^0 e^{\tilde{A}_n(t-s)}P^u_n g'(U^\xi_n(s))DU^\xi_n(s)\zeta ds \,. \end{aligned}$$

Thus, arguing as in Lemma 4.10, one shows that DU^ξ_n is defined in a ball $P^u_n B_X(0,r)$, where r does not depend on n. Arguing as in Lemma 4.12 and using property (4.3) several times, one shows the convergence of DU^ξ_n towards DU^ξ_∞ as well as the second estimate in (4.14).
Finally, the proof of the fact that the Lipschitz-constants of Dh^u_n is uniform with respect to n is similar to the proof of Lemma 4.10. □

4.3 Convergence of the regular part of the local stable manifolds

We can also study the convergence of the local stable manifolds. Notice that this theorem is not needed for the convergence of the attractors $\mathcal{A}_n$ but will be required for the proof of stability of phase-diagrams (see Theorem 2.12).

Theorem 4.13. *Assume that the uniform exponential decay property (UED) holds. Let E be a hyperbolic equilibrium point of the dynamical system $S_\infty(t)$. Then E is also a hyperbolic equilibrium point of $S_n(t)$ for n large enough. Moreover, there exists $n_0 \in \mathbb{N}$, such that, for $n \geq n_0$, the local stable manifold $W^s_n(E,r)$ satisfies the properties i) and ii) of Theorem 4.1 with positive constants r, M_s and λ_s independent of n and such that, for $n \geq n_0$, $W^s_n(E,r)$ is the graph of a function h^s_n which is of class $\mathcal{C}^{1,1}(B^s_n(r), P^u_n X)$. Furthermore, the Lipschitz-constants of Dh^s_n is bounded uniformly with respect to n.*
In addition, if $\mathcal{B}$ is a bounded set of X^σ ($\sigma \in]0, s_0[$), there exists a positive constant $C = C(\mathcal{B}, \beta)$ such that

$$\forall \xi \in B^s_\infty(r) \cap \mathcal{B},\ \|h^s_\infty(\xi) - h^s_n(P^s_n\xi)\|_X \leq C\varepsilon^\beta_n \,, \tag{4.26}$$

and

$$\forall \xi \in B^s_\infty(r) \cap \mathcal{B},\ \|Dh^s_\infty(\xi)P^s_\infty - Dh^s_n(P^s_n\xi)P^s_n\|_{\mathcal{L}(X^\sigma, X)} \leq C\varepsilon^\beta_n \,, \tag{4.27}$$

where β is any number in $]0, \frac{\sigma^2}{2}[$ if $d = 1$ or $d = 2$ or any number in $]0, \min(\frac{\sigma^2}{2}, \frac{1-\alpha}{4})[$ if $d = 3$. In particular, the regular part of the local stable manifold converges in the following sense :

$$d_X(W_n^s(E,r) \cap \mathcal{B}; W_\infty^s(E,r) \cap \mathcal{B}) \leq C\varepsilon_n^\beta \ . \tag{4.28}$$

Proof : We underline that the important point is the independance of r and λ_s with respect to n. This property is closely linked to Hypothesis (UED). Indeed, assuming the uniform exponential decay (UED), we can improve the estimates (4.8) as follows : there exist positive constants M_s, λ_s and η such that

$$\forall n \in \mathbb{N} \cup \{+\infty\}, \forall t \geq 0, \ \|e^{\tilde{A}_n t} P_n^s\|_{\mathcal{L}(X)} \leq M_s e^{-(\lambda_s+\eta)t} \ . \tag{4.29}$$

The outline of the proof is exactly the same as Theorem 4.7, but here, instead of Y_μ, we consider the space

$$Z_{\tilde{\mu}} = \{U \in \mathcal{C}^0([0,+\infty[,\mathbb{C}) \ / \ \sup_{t\geq 0} \|U(t)\|_X e^{\tilde{\mu}t} < +\infty\} \ ,$$

where $0 < \tilde{\mu} < \lambda_s$ and we remplace T_n^ξ by the functional

$$R_n^\xi : U \in Z_{\tilde{\mu}} \longmapsto \int_t^\infty e^{\tilde{A}_n(t-s)} P_n^u g(U(s)) ds + e^{\tilde{A}_n t} P_n^s \xi - \int_0^t e^{\tilde{A}_n(t-s)} P_n^s g(U(s)) ds \ .$$

We would like to insist on the modifications in the proof of Lemma 4.12. In this proof, we used the fact that, for all $\xi \in X$, $P_\infty^u \xi$ belongs to $D(\tilde{A}_\infty)$, which is not the case of $P_\infty^s \xi$. As a consequence, we cannot prove the convergence of the whole local stable manifold $W_n^s(E,r)$. Fortunately, we only need the convergence of the subset $W_n^s(E,r) \cap \mathcal{B}$. If we choose $\xi \in W_n^s(E,r) \cap \mathcal{B}$, $P_\infty^s \xi = \xi - P_\infty^u \xi$ is bounded in X^σ and the arguments of Lemma 4.12 are valid in our case. In the same way, we can only prove the convergence of the regular part of the tangent spaces and this convergence is shown with the same arguments as the convergence of the tangent spaces of the local unstable manifolds. Finally, notice that the Lipschitz-constants of h_n^s and Dh_n^s are uniform in n because of Estimate (4.29). $\square$

4.4 Lower-semicontinuity and estimates of the convergence.

Proof of Theorem 2.11: The lower-semicontinuity of the attractors follows from the convergence of the local unstable manifolds proved in the previous section. In fact, we can be more precise and prove Estimate (2.26). Proofs of such an estimate of the lower-semicontinuity can be found in [20] and [3]. Although the presentation of these proofs is different, the ideas are the same, in particular the gradient structure is strongly used. We

also underline that the proof of the estimate for the lower-semicontinuity can be made, by using the notion of chain of equilibria that we introduce in Section 5.2.
Hypothesis (Hyp) allows us to prove estimates for the upper-semicontinuity due to the following result. If all the equilibria of $S_\infty(t)$ are hyperbolic, then any bounded set $\mathcal{B}$ of X is exponentially attracted by $\mathcal{A}_\infty$, that is that there exist positive constants M and λ such that

$$\forall t \geq 0,\ \sup_{U\in\mathcal{B}} \operatorname{dist}_X(S_\infty(t)U, \mathcal{A}_\infty) \leq Me^{-\lambda t}\ . \tag{4.30}$$

The proof of this property, and the fact that it implies an estimate of the upper-semicontinuity can be found in [30], [22] or [41]. Once again, the proof of this exponential attraction strongly uses the gradient structure of the dynamical system.

To obtain an estimate of the upper-semicontinuity from (4.30), we modify the proof of Theorem 2.8 as follows. The attracting property (2.22) is replaced by the stronger property

$$\forall U \in \bigcup_n \mathcal{A}_n,\ \operatorname{dist}_X(S_\infty(t)U, \mathcal{A}_\infty) \leq Me^{-\lambda t}\ . \tag{4.31}$$

On the other hand, Theorem 3.10 and the fact that $\cup\mathcal{A}_n$ is bounded in X imply that

$$\forall U_n \in \mathcal{A}_n,\ \|(S_n(t) - S_\infty(t))U_n\|_{X^{-s}} \leq Ce^{Ct}\varepsilon_n^{s/8}\ . \tag{4.32}$$

Replacing t by $-\frac{s}{16C}\ln\varepsilon_n$ in (4.32), which is positive for n large enough, we deduce from (4.31) and (4.32), that

$$\sup_{U_n\in\mathcal{A}_n} \operatorname{dist}_{X^{-s}}(S_n(t)U_n, \mathcal{A}_\infty) \leq Me^{-\lambda t} + Ce^{Ct}\varepsilon_n^{s/8} = M\varepsilon_n^{\frac{\lambda s}{16C}} + C\varepsilon_n^{\frac{s}{16}}$$

This concludes the proof of the inequality (2.27) since $S_n(t)\mathcal{A}_n = \mathcal{A}_n$. □

5 Stability of phase-diagrams

In this section, we prove Theorem 2.12. We assume in the whole section that $\Omega =]0,1[$ and $\gamma_\infty = a\delta_{x=0} + b\delta_{x=1}$, with $a \neq 1$ and $b \neq 1$. We recall that these hypotheses imply that $e^{A_n t}$ is a group of operators for all $n \in \mathbb{N}\cup\{+\infty\}$ and that $S_n(t)$ and $DS_n(t)$ are one to one. Thus, if E is a hyperbolic equilibrium of $S_n(t)$, then the stable and unstable sets $W^s_n(E)$ and $W^u_n(E)$, introduced in Definition 4.2, are well-defined global manifolds of X. We also assume that the hypotheses of Theorem 2.12 hold, that is that Hypotheses (Diss) and (UED) and the Morse-Smale property for $S_\infty(t)$ are satisfied.
Let E_- and E_+ be two equilibria of the dynamical systems $S_n(t)$, we say that $S_n(t)$ admits

a connecting orbit between E_- and E_+ if there exists a complete trajectory $U_n(t)$ ($t \in \mathbb{R}$), solution of Equation (2.5) such that $U_n(t)$ converges to E_- (resp. E_+) when t goes to $-\infty$ (resp. $+\infty$). This orbit is said to be transversal if at any point of it, the manifolds $W^u_n(E_-)$ and $W^s_n(E_+)$ intersect transversally, that is that at each point U_n of the trajectory, the tangent space $T_{U_n}W^s_n(E_+)$ has a closed complement and $T_{U_n}W^u_n(E_-) + T_{U_n}W^s_n(E_+) = X$.

The proof of Theorem 2.12 can be split into the following two lemmas.

Lemma 5.1. *We assume that $\Omega =]0,1[$, $\gamma_\infty = a\delta_{x=0} + b\delta_{x=1}$, with $a \neq 1$ and $b \neq 1$, that $S_\infty(t)$ satisfies the Morse-Smale property and that Hypotheses (Diss) and (UED) hold. Let E_- and E_+ be two hyperbolic equilibria of the dynamical systems $S_n(t)$. If $W^u_\infty(E_-) \cap W^s_\infty(E_+)$ is a manifold of dimension r then, for n large enough, $W^u_n(E_-) \cap W^s_n(E_+)$ is a manifold of dimension r.*

Lemma 5.2. *Assume that the hypotheses of Lemma 5.1 hold. If $\mathcal{O}_n$ is a sequence of connecting orbits for $S_n(t)$ between E_- and E_+, then*

i) $S_\infty(t)$ admits a connecting orbit between E_- and E_+,

ii) there exists a subsequence $\mathcal{O}_{\varphi(n)}$ of $\mathcal{O}_n$ such that, for n large enough, the orbits $\mathcal{O}_{\varphi(n)}$ are transversal.

Remark : We underline that the proof of i) of Lemma 5.2 gives an interesting result even if $S_\infty(t)$ is not a Morse-Smale system. Indeed, the proof shows that there exists a chain of equilibria $E_- = E_0$, $E_1 ... E_p = E_+$ such that $S_\infty(t)$ admits a connecting orbit between E_i and E_{i+1}. The Morse-Smale property is only used to prove that this implies the existence of a connecting orbit between E_- and E_+.

Proof of Theorem 2.12: Lemmas 5.1 and 5.2 imply Theorem 2.12, that is the stability of phase-diagram and the Morse-Smale property. Indeed, the number of equilibrium points of $S_\infty(t)$ (and thus of $S_n(t)$) is finite since they are bounded in $D(A_\infty)$ and are hyperbolic. Thus, Lemmas 5.1 and 5.2 clearly imply the stability of phase-diagrams. The hyperbolicity of equilibria for $S_n(t)$, for n large enough, has been proved in Proposition 4.4. Finally, assume that $S_n(t)$ is not a Morse-Smale system for n large enough, then we can find a sequence of complete bounded trajectories for $S_n(t)$ which are not transversal. Since the number of equilibria is finite, we can assume that the trajectories connect the same equilibria and this contradicts Lemma 5.2. Thus, $S_n(t)$ has the Morse-Smale property for n large enough. $\square$

5.1 Proof of Lemma 5.1

Let E_- and E_+ be two equilibria of $S_\infty(t)$. In Theorems 4.7 and 4.13, we have shown that there exist two radii r_- and r_+ such that the local manifolds $W_n^u(E_-, r_-)$ and $W_n^s(E_+, r_+)$ are well-defined. We denote P_n^{u+} (resp. P_n^{u-}) the projection onto the unstable part of the spectrum of the linearization $\tilde{A}_n$ at the equilibrium point E^+ (resp.E^-). Similarly, $P_n^{s\pm}$ are the projections onto the stable part. We set $B_n^u(E_\pm, r_\pm) = B(E_\pm, r_\pm) \cap P_n^{u\pm}X$ and $B_n^s(E_\pm, r_\pm) = B(E_\pm, r_\pm) \cap P_n^{s\pm}X$. We denote

$$h_n^u : B_n^u(E_-, r_-) \longrightarrow B_n^s(E_-, M_-r_-) \text{ and } h_n^s : B_n^s(E_+, r_+) \longrightarrow B_n^u(E_+, M_+r_+)$$

the functions given in Theorem 4.1, whose graphs are $W_n^u(E_-, r_-)$ and $W_n^s(E_+, r_+)$ respectively.
For any time $T \geq 0$, we introduce the map

$$\Psi_T^n : \left(\begin{array}{ccc} B_\infty^u(E_-, r_-) & \longrightarrow & X \\ \xi & \longmapsto & S_n(T) \circ [Id + h_n^u(.)]P_n^{u-}\xi \end{array} \right) .$$

The union of the ranges $\bigcup_{T \geq 0} R(\Psi_T^n)$ is equal to the unstable manifold $W_n^u(E_-)$. Assume that $S_\infty(t)$ admits a connecting orbit between E_- and E_+, and let U_0 be a point of this trajectory such that $P_\infty^{u-}U_0$ belongs to $B_\infty^u(E_-, r_-)$. There exists a neighborhood θ of $P_\infty^{u-}U_0$ in $B_\infty^u(E_-, r_-)$ such that $\Psi_T^\infty(\theta) \subset B(E_+, r_+)$ for some T large enough. For $n = \infty$ and for any n large enough, we set

$$\Phi^n : \left(\begin{array}{ccc} \theta & \longrightarrow & B_\infty^u(E_+, r_+) \\ \xi & \longmapsto & P_\infty^{u+} \circ [P_n^{u+} - h_n^s(P_n^{s+}.)] \circ \Psi_T^n(\xi) \end{array} \right) .$$

Since, for n large enough, P_∞^{u+} is an isomorphism from $P_n^{u+}X$ onto $P_\infty^{u+}X$, it follows that, by construction, the equality $\Phi^n(\xi) = 0$ is equivalent for n large enough to the existence of a trajectory for $S_n(t)$ between E_- and E_+, which intersects the subset $[Id + h_n^u(.)]P_n^{u-}(\theta)$ of the unstable manifold $W_n^u(E_-, r_-)$.
Using Proposition 3.11 and Theorems 4.7 and 4.13, we obtain the following properties.

Lemma 5.3. *The function Φ^n and the derivatives $D\Psi_T^n$ and $D\Phi^n$ are well-defined for n large enough. Moreover, Ψ_T^n, Φ^n, $D\Psi_T^n$ and $D\Phi^n$ are continuous with respect to $\xi \in \theta$, uniformly in $n \in \mathbb{N} \cup \{+\infty\}$ and converge respectively to Ψ_T^∞, Φ^∞, $D\Psi_T^\infty$ and $D\Phi^\infty$, when n goes to $+\infty$, uniformly in $\xi \in \theta$.*

We recall that $m(E_\pm)$ is the Morse index of $E_\pm$, that is the dimension of the linear unstable space $P_\infty^{u+}X$. As $S_\infty(t)$ and $DS_\infty(t)$ are one-to-one, $\Psi_T^\infty(\theta)$ is an open subset of dimension $m(E_-)$ of $W_\infty^u(E_-)$. By assumption, it has a non-empty transversal intersection with $W_\infty^s(E_+, r_+)$. The classical $\lambda-$lemma (see [38] and [19]) implies that for all $\varepsilon > 0$, we can find T large enough and a submanifold $\tilde{\theta}$ of θ, which contains $P_\infty^{u-}U_0$ and which is of dimension $m(E_+)$, such that $\psi_T^\infty(\tilde{\theta})$ is $\varepsilon - \mathcal{C}^1-$close to $B_\infty^u(E_+, r_+)$. Thus, $P_\infty^{u+} \circ D\psi_T^\infty(P_\infty^{u-}U_0)$ is onto $P_\infty^{u+}X$, and by Lemma 5.3, if θ is choosen small enough, then for any $\xi \in \theta$, $P_\infty^{u+} \circ D\psi_T^\infty(\xi)$ is onto $P_\infty^{u+}X$. As $Dh_\infty^s(E_+) = 0$, if r_+ is small enough, Dh_∞^s is small and $D\Phi^\infty(\xi)$ is onto $P_\infty^{u+}X$, that is that Φ^∞ is a submersion. Using Theorem 2.8 of Chapter one of [15], we see that Φ^∞ is an open function, ie $\Phi^\infty(\theta)$ is a neighborhood of 0. Lemma 5.3 implies that $\Phi^n(\theta)$ is also a neighborhood of 0 for n large enough and that Φ^n is also a submersion. Theorem 2.8 of chapter one of [15] implies that $(\Phi^n)^{-1}(0)$ is a submanifold of θ of dimension $m(E_-) - m(E_+)$. Since $S_\infty(t)$ and $DS_\infty(t)$ are one-to-one, the dimension of the intersection $W_n^u(E_-) \cap W_n^s(E_+)$ is $m(E_-) - m(E_+)$.

5.2 The notion of chain of equilibria

We introduce in this section the notion of chain of equilibria. The ideas behind it are not really new since this notion is close to the one of *family of combined limit trajectories* given in [3] and [4], which was used to show lower-semicontinuity of attractors.
This notion enables us to give a proof of Lemma 5.2, which is different from [19]. In particular, we do not need any result of convergence of the local stable manifolds to prove the property i) of Lemma 5.2. On the other hand, we extensively use the gradient structure, that is that the Lyapounov function Φ given by (2.6) is non-increasing along the trajectories of $S_n(t)$ and that,

$$\text{if, for any } t \geq 0,\ \Phi(S_\infty(t)U) = \Phi(U), \text{ then } U \text{ is an equilibrium point.} \tag{5.1}$$

In the proof of Lemma 5.2, we will use several times the following result. We recall that the upper-semicontinuous in X of the attractors has been shown in Theorem 2.10.

Lemma 5.4. *Assume that the attractors $\mathcal{A}_n$ are upper-semicontinuous in X at $n = +\infty$. For any positive time T and any sequence $(U_n)_{n\in\mathbb{N}}$, such that $U_n \in \mathcal{A}_n$, there exists $U_\infty \in \mathcal{A}_\infty$ and a subsequence (U_{n_k}) of (U_n) satisfying*

$$\sup_{t\in[0,T]} \|S_{n_k}(t)U_{n_k} - S_\infty(t)U_\infty\|_X \longrightarrow 0 \textit{ when } n \longrightarrow +\infty\ .$$

Proof : Due to the upper-semicontinuity of the attractors, there exists a sequence of points $V_n \in \mathcal{A}_\infty$ such that $\|U_n - V_n\|_X \to 0$. As $\mathcal{A}_\infty$ is compact, we can extract a subsequence V_{n_k} which converges to $U_\infty \in \mathcal{A}_\infty$. Proposition 2.3 implies that $\sup_{t\in[0,T]} \|S_{n_k}(t)U_{n_k} - S_{n_k}(t)U_\infty\|_X \longrightarrow 0$. On the other hand, Theorem 3.8 and the regularity of $\mathcal{A}_\infty$ imply that $\sup_{t\in[0,T]} \|S_{n_k}(t)U_\infty - S_\infty(t)U_\infty\|_X \longrightarrow 0$ and the proof is complete. □

To avoid heavy notations, we do not reindex subsequences in what follows. We recall that $\mathcal{E}$ denotes the set of all equilibria. We choose a small enough radius r such that the balls $B(E,2r)$ ($E\in\mathcal{E}$) are disjoint and such that the local stable and unstable manifolds $W^s_\infty(E,2r)$ and $W^u_\infty(E,2r)$ are well-defined. Let E_- and E_+ be two equilibrium points. Assume that for n large enough, $S_n(t)$ has a connecting orbit between E_- and E_+. There exist U^0_n in the local unstable manifold $W^u_n(E_-,r_n)$ ($r_n\le r$) and t_n such that U^0_n converges to E_-, $S_n(t_n)U^0_n$ belongs to $W^s_n(E_+,r'_n)$ ($r'_n\le r$) and $S_n(t_n)U^0_n$ converges to E_+. We introduce the following notion.

Definition 5.5. *Let $\mathcal{O}_n$ be an orbit of $S_n(t)$. A sequence of equilibria $E_-=E_0$, E_1,..., $E_p=E_+$ is called a chain of equilibria of length p for the sequence $(\mathcal{O}_n)$ if there exist $U^0_n\in\mathcal{O}_n$ and $p+1$ sequences of times $0=t^0_n<t^1_n<...<t^p_n=t_n$ such that, if we set $U_n(t)=S_n(t)U^0_n$, then*

$$U_n(t^i_n)\longrightarrow E_i\ ,\ \text{as } n\longrightarrow+\infty$$

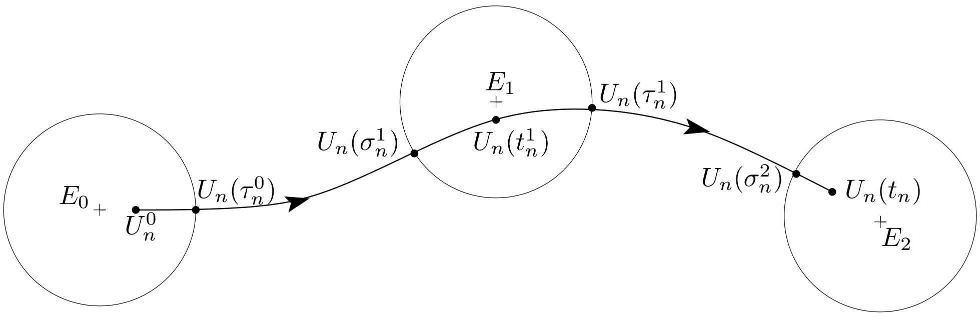
and for all $n\in\mathbb{N}$ and $i<p$, there exists $t\in]t^i_n,t^{i+1}_n[$ such that $U_n(t)$ does not belong to $\cup_{E\in\mathcal{E}}\overline{B(E,r)}$.

If E_i is a chain of equilibria, $U_n(t^i_n)\in B(E_i,r)$ for n large enough and we can assume that this holds for all n. For $i>0$, we denote the time of entrance in $B(E_i,r)$

$$\sigma^i_n=\sup\{t\le t^i_n\mid U_n(t)\notin B(E_i,r)\}\ ,$$

and for $i<p$, we denote the time of exit of $B(E_i,r)$

$$\tau^i_n=\inf\{t\ge t^i_n\mid U_n(t)\notin B(E_i,r)\}\ .$$



We obtain the following result.

Lemma 5.6. *There exist $V_i\in\partial B(E_i,r)\cap W^s_\infty(E_i,2r)\cap\mathcal{A}_\infty$ and $W_i\in\partial B(E_i,r)\cap W^u_\infty(E_i,2r)$ such that, extracting subsequences, we have*

$$U_n(\sigma^i_n)\longrightarrow V_i \text{ and } U_n(\tau^i_n)\longrightarrow W_i\ .$$

Proof : We use Lemma 5.4 with $T=0$ to show that there exists a point $W_i\in\mathcal{A}_\infty$ such that $U_n(\tau^i_n)\longrightarrow W_i$. Due to the definition of τ^i_n, it is clear that $U_n(\tau^i_n)\in\partial B(E_i,r)$ and thus $W_i\in\partial B(E_i,r)$. Assume that there exist a time T and $\tilde{W}_i\in X$ such that $S_\infty(T)\tilde{W}_i=W_i$

and $\tilde{W}_i \notin \overline{B}(E_i, r)$. Using Lemma 5.4, we find that $U_n(\tau_n^i - T) \longrightarrow \tilde{W}_i$, otherwise we contradict the backward uniqueness of $S_\infty(t)$, and thus $U_n(\tau_n^i - T) \notin B(E_i, r)$ for n large enough. If $i = 0$, this contradict the fact that $U_n^0 \in W_n^u(E_-, r_n)$. If $i \geq 1$, we must have $\tau_n^i - T < t_n^i < \tau_n^i$, so we can assume that $t_n^i - (\tau_n^i - T) \longrightarrow s$. Lemma 5.4 shows that $S_\infty(s)\tilde{W}_i = E_i$, which is absurd. We have thus proved that $W_i \in W_\infty^u(E, r)$.
The arguments are similar for σ_n^i. $\square$

The length of a chain of equilibria is bounded, since the number of equilibria is finite and we have the following property.

Lemma 5.7. *If (E_i) is a chain of equilibria, then $i < j$ implies $E_i \neq E_j$.*

Proof : Since the Lyapounov function Φ does not increase along the trajectories of $S_\infty(t)$ and that (5.1) holds, we must have $\Phi(V_j) > \Phi(E_j)$ and $\Phi(W_i) \leq \Phi(E_i)$. Lemma 5.6 and the decay of Φ along the trajectories of $S_n(t)$ imply that $\Phi(E_i) > \Phi(E_j)$. $\square$

Of course, the set of chains of equilibria corresponding to the trajectories $S_n(t)U_n^0$ is not empty as (E_-, E_+) is a trivial chain. So, we can choose a chain of equilibria (E_i) of maximal length since the number of equilibria is finite and since Lemma 5.7 holds.

Lemma 5.8. *If (E_i) is a chain of equilibria of maximal length p, then there exists a finite time T such that*

$$\forall i = 0, .., p-1, \ \sup_{n \in \mathbb{N}} \{\sigma_n^{i+1} - \tau_n^i\} \leq T \ .$$

Proof : Assume that $\sigma_n^{i+1} - \tau_n^i \longrightarrow +\infty$. Let $T_n = \sqrt{\sigma_n^{i+1} - \tau_n^i}$. There exists a sequence of times $s_n \in]\tau_n^i, \sigma_n^{i+1} - T_n[$ such that $\Phi(U_n(s_n)) - \Phi(U_n(s_n + T_n)) \longrightarrow 0$. Indeed, if not, there exists $\varepsilon > 0$ such that for all $s \in]\tau_n^i, \sigma_n^{i+1} - T_n[$ and n large enough, we have $\Phi(U_n(s)) - \Phi(U_n(s+T_n)) > \varepsilon$. If we denote $\lfloor T_n \rfloor$ the largest integer less than T_n, this implies that $\Phi(U_n(\tau_n^i)) - \Phi(U_n(\sigma_n^{i+1})) > \lfloor T_n \rfloor \varepsilon \longrightarrow +\infty$, which is absurd. Using Lemma 5.4, we find that $U_n(s_n)$ converges to $U \in \mathcal{A}_\infty$ and that for all $t \geq 0$, we have $\Phi(U) - \Phi(S_\infty(t)U) = 0$. This means that U is an equilibrium point which contradicts the fact that the length of the chain of equilibria $E_1, ..., E_p$ is maximal. $\square$

We conclude with the following result.

Lemma 5.9. *If (E_i) is a chain of equilibria of maximal length p between E_- and E_+, then, for all $i < p$, $S_\infty(t)$ admits a connecting orbit between E_i and E_{i+1}.*

Proof : We can assume that $\sigma_n^{i+1} - \tau_n^i \longrightarrow T_i$. Using the notation of Lemma 5.6, we have $W_i \in W^u_\infty(E_i)$ and $V_{i+1} \in W^s_\infty(E_{i+1})$. We obtain

$$\begin{aligned}\|S_\infty(T_i)W_i - V_{i+1}\|_X \qquad &\leq \|S_\infty(T_i)W_i - S_\infty(\sigma_n^{i+1} - \tau_n^i)W_i\|_X \\ &\quad + \|S_\infty(\sigma_n^{i+1} - \tau_n^i)W_i - S_n(\sigma_n^{i+1} - \tau_n^i)W_i\|_X \\ &\quad + \|S_n(\sigma_n^{i+1} - \tau_n^i)W_i - S_n(\sigma_n^{i+1} - \tau_n^i)U_n(\tau_n^i))\|_X \\ &\quad + \|U_n(\sigma_n^{i+1}) - V_{i+1}\|_X \ .\end{aligned}$$

Taking the limit when n goes to $+\infty$, we find that $S_\infty(T_i)W_i = V_{i+1}$, which yields a connecting orbit for $S_\infty(t)$ between E_i and E_{i+1}. □

5.3 Proof of Lemma 5.2

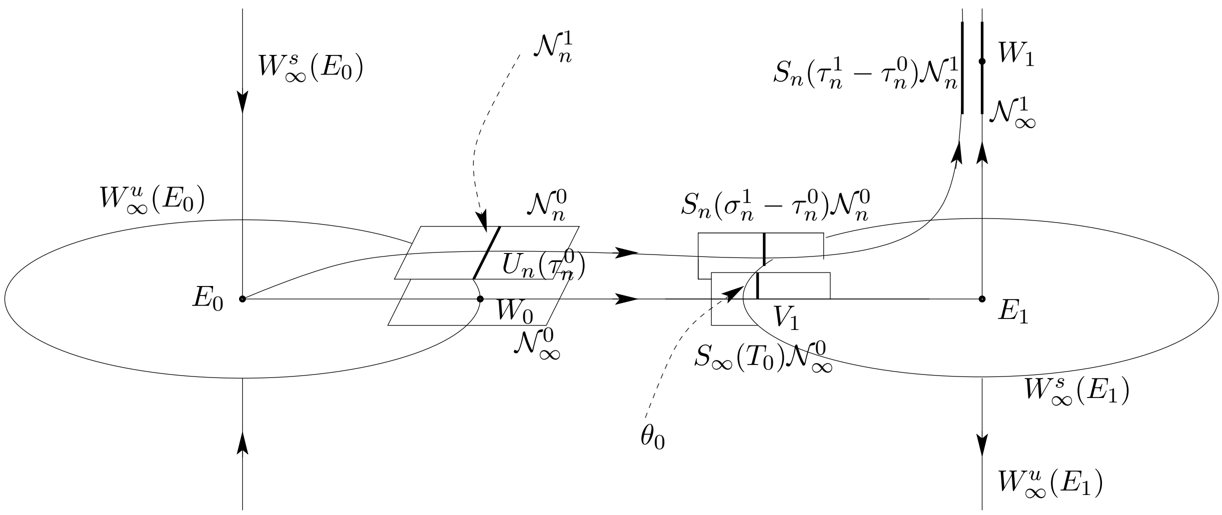
We use the notations of Section 5.2. Assume that there exists a sequence of connecting orbits $\mathcal{O}_n$ for $S_n(t)$ between E_- and E_+. As noticed in the previous section, up to an extraction of a subsequence, there exists a chain of equilibria of maximal length $E_- = E_0, E_1, ..., E_p = E_+$ associated with our sequence $(\mathcal{O}_n)$ of trajectories. Lemma 5.9 shows that $S_\infty(t)$ admits a connecting orbit between E_i and E_{i+1} $(0 \leq i \leq p-1)$. Thus, Property i) of Lemma 5.2 is a direct consequence of the classical cascading property : if $S(t)$ is a Morse-Smale dynamical system which admits a connecting orbit between E_i and E_{i+1} $(0 \leq i \leq p-1)$, then it has a connecting orbit between E_0 and E_p (see for example [38] or [19]).
Next, we prove Property ii). Let θ_1 and θ_2 be two open sets of a Banach space X. We say that two $\mathcal{C}^1-$manifolds $i_1 : \theta_1 \to X$ and $i_2 : \theta_2 \to X$ are $\varepsilon - \mathcal{C}^1-$close if there exists a $\mathcal{C}^1-$diffeomorphism $\varphi : \theta_1 \to \theta_2$, such that $i_1 : \theta_1 \to X$ and $i_2 \circ \varphi : \theta_1 \to X$ are $\varepsilon - \mathcal{C}^1-$maps, that is that $\|i_1 - i_2 \circ \varphi\|_{\mathbb{L}^\infty(\theta_1, X)} < \varepsilon$ and the same for the derivative $\|Di_1 - D(i_2 \circ \varphi)\|_{\mathbb{L}^\infty(\theta_1, X)} < \varepsilon$. We define similarly the $\mathcal{C}^1-$convergence of $\mathcal{C}^1-$manifolds. The classical local $\lambda-$lemma can be extended as follows in our particular frame.

Proposition 5.10. *Let E be a hyperbolic equilibrium point with Morse index $m(E)$. Let $\mathcal{B}$ be a bounded set of X^σ $(\sigma > 0)$. Let q_∞ be a point of $W^s_\infty(E, r) \cap \mathcal{B}$ and let $D_\infty \subset \mathcal{B}$ be a disk of center q_∞, which is transversal to $W^s_\infty(E, r)$ and whose dimension is $m(E)$. Let $(D_n)_{n\in\mathbb{N}}$ be a family of disks with center q_n, bounded in $\mathcal{B}$, and such that D_n $\mathcal{C}^1-$converges to D_∞.*
Then, for all $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and $T > 0$ such that for all $n \geq N$ and $t \geq T$, the connected component of $S_n(t)D_n \cap B_X(E, r)$, to which $S_n(t)q_n$ belongs, is $\varepsilon - \mathcal{C}^1-$close to $W^u_\infty(E, r)$.

Proof : The proof of the proposition is a straightforward adaptation of the proof of the classical $\lambda-$lemma (see for example [37] or [38]). Notice that the proof crucially uses

Hypothesis (UED), which implies that Property ii) of Theorem 4.1 holds uniformly with respect to n, and the fact that the family of disks belongs to a bounded set $\mathcal{B}$ of X^σ ($\sigma > 0$). $\square$

We recall that $\lim U_n(\sigma_n^i) = V_i \in \partial B(E_i, r) \cap W_\infty^s(E_i, 2r) \cap \mathcal{A}_\infty$ and $\lim U_n(\tau_n^i) = W_i \in \partial B(E_i, r) \cap W_\infty^u(E_i, 2r)$. Due to the convergence of the local unstable manifolds proved in Theorem 4.7, there exist a neighborhood $\mathcal{N}_\infty^0$ of W_0 in $W_\infty^u(E_0, 2r)$ and a sequence of neighborhoods $(\mathcal{N}_n^0)$ of $U_n(\tau_n^0)$ in $W_n^u(E_0, 2r)$ such that $\mathcal{N}_n^0$ $\mathcal{C}^1-$converges to $\mathcal{N}_\infty^0$. As, by Lemma 5.8, $\sigma_n^1 - \tau_n^0$ is bounded, we can assume that $\sigma_n^1 - \tau_n^0 \longrightarrow T_0$. Notice that the sequence of manifolds $(\mathcal{N}_n^0)$ is bounded in X^σ for some positive σ and that $\mathcal{N}_n^0$ is finite-dimensional. Thus, Proposition 3.11 implies that the manifold $S_n(\sigma_n^1 - \tau_n^0)\mathcal{N}_n^0$, which contains $U_n(\sigma_n^1)$, $\mathcal{C}^1-$converges in X to the manifold $S_\infty(T_0)\mathcal{N}_\infty^0$, which contains V_1. As $S_\infty(t)$ has the Morse-Smale property, we can find a submanifold θ_0 of $S_\infty(T_0)\mathcal{N}_\infty^0$ of dimension $m(E_1)$ which is tranversal to $W_\infty^s(E_1)$ and which contains V_1. Thus, we can find a submanifold $\mathcal{N}_n^1$ of $\mathcal{N}_n^0$ of dimension $m(E_1)$, which contains $U_n(\tau_n^0)$ and is such that $S_n(\sigma_n^1 - \tau_n^0)\mathcal{N}_n^1$ $\mathcal{C}^1-$converges to θ_0. Using the generalized $\lambda-$lemma of Proposition 5.10, we find that there exists a neighborhood $\mathcal{N}_\infty^1$ of W_1 in $W_\infty^u(E_1, 2r)$ such that $S_n(\tau_n^1 - \tau_n^0)\mathcal{N}_n^1$ $\mathcal{C}^1-$converges to $\mathcal{N}_\infty^1$.



By a finite number of iterations of this process, we obtain that there exists a submanifold $\mathcal{N}_n^{p-1}$ of $\mathcal{N}_n^0$ of dimension $m(E_{p-1})$ such that $S_n(\sigma_n^p - \tau_n^0)\mathcal{N}_n^{p-1}$ $\mathcal{C}^1-$converges to $S_\infty(T_{p-1})\mathcal{N}_\infty^{p-1}$, a neighborhood of V_p in $W_\infty^u(E_{p-1})$. As the union of the attractors $\cup\mathcal{A}_n$ is bounded in X^s for some positive s, there exists a ball $\mathcal{B}$ of X^s such that $S_n(\sigma_n^p - \tau_n^0)\mathcal{N}_n^{p-1} \subset \mathcal{B}$ for all n. The convergence of the regular part of the local stable manifolds (see Theorem 4.13) implies that $W_n^s(E_p, 2r) \cap \mathcal{B}$ $\mathcal{C}^1-$converges to $W_\infty^s(E_p, 2r) \cap \mathcal{B}$. Thus, for n large enough, the dimension of $W_n^s(E_p, 2r) \cap S_n(\sigma_n^p - \tau_n^0)\mathcal{N}_n^{p-1}$ is less than the dimension of $W^s_\infty(E_p,2r)\cap S_\infty(T_{p-1})\mathcal{N}^{p-1}_\infty$. By assumption, $S_\infty(T_{p-1})\mathcal{N}^{p-1}_\infty$ and $W^s_\infty(E_p,2r)$ intersect tranversally and so, a dimensional argument implies that $W^s_n(E_p,2r)$ and $S_n(\sigma^p_n-\tau^0_n)\mathcal{N}^{p-1}_n$ intersect tranversally. As $S_n(\sigma^p_n-\tau^0_n)\mathcal{N}^{p-1}_n$ is a submanifold of $W^u_n(E_0)$, this shows that the orbit $\mathcal{O}_n$ is transversal.

6 Study of the hypotheses

6.1 The one-dimensional case

First, we notice that Hypotheses (ED) and (Grad) are always satisfied in dimension one. Indeed, we have assumed that $\gamma_n\neq 0$ in the space of the measures, which is well-known to imply (ED) and (Grad), even for the case $n=\infty$. Concerning Hypothesis (ED), we refer to [23], [8] and [10] for $n\in\mathbb{N}$; and [11], [29], [32], [44] and [45] for $n=+\infty$. Concerning Hypothesis (Grad), we respectively refer to [21] and [31].
Hypothesis (UED) is the only assumption that we have to verify in dimension one. There exist many methods to prove the exponential decay property for Equation (1.2) when n is fixed. However, the proof of uniform exponential decay for a family of dissipations $(\gamma_n)_{n\in\mathbb{N}}$ is more difficult, especially when the family is not bounded in $\mathbb{L}^\infty$. In the one-dimensional case, we are able to adapt an idea of Haraux (see [23]).

Definition 6.1. *We say that a dissipation γ is effective on the free waves if the following criterium is satisfied.*

(EFW) *There exist a time T and a positive constant C such that, for any $(\varphi_0,\varphi_1)\in X$, the solution of the free wave equation*

$$\left\{\begin{array}{l}\varphi_{tt}+B\varphi=0\\ (\varphi,\varphi_t)_{|t=0}=(\varphi_0,\varphi_1)\in X\end{array}\right. \tag{6.1}$$

satisfies

$$\int_0^T\int_\Omega \gamma(x)|\varphi_t(x,t)|^2dxdt\geq C\|(\varphi_0,\varphi_1)\|^2_X\ . \tag{6.2}$$

The following implication is well-known for n fixed (see [23]). We extend it easily to the case of a family of dissipations.

Proposition 6.2. *If (UED) is satisfied, then the family of dissipations $(\gamma_n)_{n\in\mathbb{N}}\subset\mathbb{L}^\infty(\Omega)$ is uniformly effective on the free waves, that is that the property (EFW) is satisfied for each γ_n, with T and C independent of n.*

Proof : Assume that (UED) is satisfied, then there exists a positive time T, independent of n, such that $\|e^{A_nT}\|^2_{\mathcal{L}(X)} \leq \frac{1}{2}$. Thus, for any $U_0 = (\varphi_0, \varphi_1) \in X$, we have,

$$\int_0^T \int_\Omega \gamma_n |u_t|^2 = \frac{1}{2}(\|U_0\|^2 - \|U(T)\|^2) \geq \frac{1}{4}\|U_0\|^2_X \ , \tag{6.3}$$

where $(u, u_t)(t) = U(t) = e^{A_nt}U_0$, For any $U_0 = (\varphi_0, \varphi_1)$, we denote (φ, φ_t) the solution of the free wave equation (6.1). We set $w = u - \varphi$, which is the solution of the system

$$\left\{ \begin{array}{l} w_{tt} + \gamma_n w_t + Bw = -\gamma_n \varphi_t \\ w(0) = 0 \end{array} \right. .$$

Multipliying by w_t and integrating on $[0, T] \times \Omega$, we obtain

$$\frac{1}{2}\|w(T)\|^2_X + \int_0^T \int_\Omega \gamma_n |w_t|^2 = - \int_0^T \int_\Omega \gamma_n \varphi_t \overline{w}_t \ ,$$

and thus, using Cauchy-Schwartz inequality, we get

$$\int_0^T \int_\Omega \gamma_n |w_t|^2 \leq \int_0^T \int_\Omega \gamma_n |\varphi_t|^2 \ .$$

Finally, (6.3) implies that

$$\|(\varphi_0, \varphi_1)\|^2_X \leq 4 \int_0^T \int_\Omega \gamma_n |u_t|^2 \leq 4 \left(\int_0^T \int_\Omega \gamma_n |w_t|^2 + \int_0^T \int_\Omega \gamma_n |\varphi_t|^2 \right) \leq 8 \int_0^T \int_\Omega \gamma_n |\varphi_t|^2 \ .$$

□

Of course, the interesting question is to know if the uniform effectiveness on the free waves implies (UED). We give here a way of obtaining this implication in dimension one, by using a multiplier method inspired by [13]. This method is of course not the only one. In the appendix, we recall a theorem of [2], which implies the same result. The crucial point in the following results is to obtain the dependence of the constants on $\|\gamma\|_{\mathbb{L}^1}$ and not on $\|\gamma\|_{\mathbb{L}^\infty}$, since our family of dissipations $(\gamma_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^1(]0, 1[)$ but not in $\mathbb{L}^\infty(]0, 1[)$.
To simplify, we work here with $B = -\Delta_N + Id$. The same results are true for other boundary conditions with a similar proof.
First, we use the multipliers method to prove the following estimate.

Proposition 6.3. *Let $\gamma \in \mathbb{L}^1(]0, 1[)$ and $h \in \mathbb{L}^1_t(\mathbb{R}, \mathbb{L}^2_x(]0, 1[))$. Let u be the solution of*

$$\left\{ \begin{array}{l} u_{tt}(x, t) - u_{xx}(x, t) + u(x, t) = h(x, t) \quad (x, t) \in]0, 1[\times \mathbb{R}_+ \\ u_x(0, t) = u_x(1, t) = 0 \\ (u, u_t)_{|t=0} = (u_0, u_1) \in \mathbb{H}^1(]0, 1[) \times \mathbb{L}^2(]0, 1[) \end{array} \right.$$

Then, for all $T>0$, there exists a constant $C=C(T,\|\gamma\|_{\mathbb{L}^1})$ such that

$$\int_0^T\int_0^1 \gamma(x)(|u_x|^2+|u|^2+|u_t|^2)dxdt \leq C\left(\int_0^T\int_0^1 |h|(|u_x|+|u_t|)dxdt+\|u_0\|_{\mathbb{H}^1}^2+\|u_1\|_{\mathbb{L}^2}^2\right) .$$

Proof : We set

$$\rho=\begin{cases} \int_0^x \gamma(\xi)d\xi & 0\leq x\leq 1/2 \\ 2(1-x)\int_0^{1/2}\gamma(\xi)d\xi & 1/2\leq x\leq 1 \end{cases}$$

Notice that $\|\rho\|_{\mathbb{L}^\infty}\leq\|\gamma\|_{\mathbb{L}^1}$ and $\rho(0)=\rho(1)=0$. We have

$$\int_0^T\int_0^1 (u_{tt}-u_{xx}+u)\rho u_x=\int_0^T\int_0^1 h\rho u_x .$$

Using integrations by parts, we find

$$\frac{1}{2}\int_0^T\int_0^1 \rho_x(|u_x|^2+|u_t|^2)=-\left[\int_0^1 \rho u_t u_x dx\right]_0^T+\int_0^T\int_0^1 h\rho u_x+\frac{1}{2}\int_0^T\int_0^1 \rho_x|u|^2dxdt . \quad (6.4)$$

The classical energy argument gives

$$\forall t\in[0,T],\ \int_0^1 (|u_x|^2+|u|^2+|u_t|^2)(t)dx\leq\|u_0\|_{\mathbb{H}^1}^2+\|u_1\|_{\mathbb{L}^2}^2+\int_0^T\int_0^1 |h||u_t|dxdt . \quad (6.5)$$

As ρ_x is bounded in $\mathbb{L}^1(]0,1[)$ by $\|\gamma\|_{\mathbb{L}^1}$ and $\mathbb{H}^1(]0,1[)\hookrightarrow\mathbb{L}^\infty(]0,1[)$, we have

$$\begin{aligned}\int_0^T\int_0^1 \rho_x|u|^2 &\leq CT\|\gamma\|_{\mathbb{L}^1}\sup_{t\in[0,T]}\|u(t)\|_{\mathbb{H}^1}^2 \\ &\leq CT\|\gamma\|_{\mathbb{L}^1}\left(\int_0^T\int_0^1 |h||u_t|dxdt+\|u_0\|_{\mathbb{H}^1}^2+\|u_1\|_{\mathbb{L}^2}^2\right) .\end{aligned}$$

Moreover, ρ is bounded in $\mathbb{L}^\infty$ by $\|\gamma\|_{\mathbb{L}^1}$ and so (6.5) gives

$$\left[\int_0^1 \rho u_t u_x dx\right]_0^T\leq C\|\gamma\|_{\mathbb{L}^1}\left(\int_0^T\int_0^1 |h||u_t|dxdt+\|u_0\|_{\mathbb{H}^1}^2+\|u_1\|_{\mathbb{L}^2}^2\right) .$$

Using the above estimates in (6.4), we find

$$\int_0^T\int_0^1 \rho_x(|u_x|^2+|u|^2+|u_t|^2)dxdt\leq C\left(\int_0^T\int_0^1 |h|(|u_x|+|u_t|)dxdt+\|u_0\|_{\mathbb{H}^1}^2+\|u_1\|_{\mathbb{L}^2}^2\right) . \quad (6.6)$$

On the other hand, since $\rho_x(x)=\gamma(x)$ for $x\in]0,1/2]$ and since $\rho_x(x)$ is bounded by $\|\gamma\|_{\mathbb{L}^1}$ for $x\in]1/2,1[$, we have

$$\int_0^T\int_0^{1/2}\gamma(x)(|u_x|^2+|u|^2+|u_t|^2)dxdt\leq\int_0^T\int_0^1 \rho_x(|u_x|^2+|u_t|^2+|u_t|^2)dxdt$$

$$+ \|\gamma\|_{\mathbb{L}^1} \int_0^T \int_{1/2}^1 (|u_x|^2 + |u_t|^2 + |u_t|^2) dxdt \ . \tag{6.7}$$

The estimates (6.5), (6.6) and (6.7) show that

$$\int_0^T \int_0^{1/2} \gamma(x)(|u_x|^2 + |u|^2 + |u_t|^2) dxdt$$
$$\leq C \left(\|u_{0x}\|^2_{\mathbb{L}^2} + \|u_1\|^2_{\mathbb{L}^2} + \int_0^T \int_0^1 |h|(|u_x| + |u_t|) dxdt \right) \ ,$$

where C depends on $\|\gamma\|_{\mathbb{L}^1}$ and T only.
In order to estimate the integral $\int_0^T \int_{1/2}^1 \gamma(x)(|u_x|^2 + |u|^2 + |u_t|^2) dxdt$, we argue in the same way with ρ taken as follows :

$$\rho = \begin{cases} 2x \int_{1/2}^1 \gamma(\xi) d\xi & 0 \leq x \leq 1/2 \\ \int_x^1 \gamma(\xi) d\xi & 1/2 \leq x \leq 1 \end{cases}$$

□

We obtain the following criterium for the exponential decay.

Theorem 6.4. *Let γ be a nonnegative function of $\mathbb{L}^\infty(]0,1[)$. Assume that (EFW) is satisfied. Then, there exist two positive constants M and λ depending only on the constants C, T introduced in (6.2) and $\|\gamma\|_{\mathbb{L}^1}$ such that, for each initial data $(u_0, u_1) \in \mathbb{H}^1(]0,1[) \times \mathbb{L}^2(]0,1[)$, the solution u of*

$$\begin{cases} u_{tt}(x,t) + \gamma(x) u_t(x,t) = u_{xx}(x,t) - u(x,t) \ , \ (x,t) \in]0,1[\times \mathbb{R}_+ \\ u_x(0,t) = u_x(1,t) = 0 \\ (u, u_t)_{|t=0} = (u_0, u_1) \in \mathbb{H}^1(]0,1[) \times \mathbb{L}^2(]0,1[) \end{cases} \tag{6.8}$$

satisfies

$$\|u(t)\|^2_{\mathbb{H}^1} + \|u_t(t)\|^2_{\mathbb{L}^2} \leq M(\|u_0\|^2_{\mathbb{H}^1} + \|u_1\|^2_{\mathbb{L}^2}) e^{-\lambda t} \ .$$

Proof : We denote the energy $E(t) = \frac{1}{2}(\|u(t)\|^2_{\mathbb{H}^1} + \|u_t(t)\|^2_{\mathbb{L}^2})$. We know that

$$E(0) - E(T) = \int_0^T \int_0^1 \gamma(x) |u_t(x,t)|^2 dxdt \ . \tag{6.9}$$

Let φ be the solution of the wave equation (6.1) with $\varphi_0 = u_0$ and $\varphi_1 = u_1$. We set $v = u - \varphi$, which is the solution of

$$\begin{cases} v_{tt} - v_{xx} + v = -\gamma u_t \\ v_x(0,t) = v_x(1,t) = 0 \\ v(x,0) = 0 \\ v_t(x,0) = 0 \end{cases}$$

Using Proposition 6.3, we obtain

$$\int_0^T\int_0^1 \gamma(x)(|v_x|^2+|v|^2+|v_t|^2)dxdt \leq C\left(\int_0^T\int_0^1 \gamma(x)|u_t|(|v_x|+|v_t|)dxdt\right) ,$$

where C depends on T and $\|\gamma\|_{\mathbb{L}^1}$ only. Thus,

$$\left(\int_0^T\int_0^1 \gamma(x)(|v_x|^2+|v|^2+|v_t|^2)dxdt\right)^2$$

$$\leq C\int_0^T\int_0^1 \gamma(x)|u_t|^2dxdt \ \times \ \int_0^T\int_0^1 \gamma(x)(|v_x|+|v_t|)^2dxdt ,$$

and also, by the Young inequality,

$$\int_0^T\int_0^1 \gamma(x)(|v_x|^2+|v|^2+|v_t|^2)dxdt \leq C\int_0^T\int_0^1 \gamma(x)|u_t|^2dxdt = C(E(0)-E(T)) . \quad (6.10)$$

Finally, using (6.9), (6.10) and the hypothesis (EFW), we obtain

$$\begin{aligned} E(T) \quad & \leq E(0) \leq C\int_0^T\int_0^1 \gamma(x)|\varphi_t(x,t)|^2dxdt \\ & \leq C\left(\int_0^T\int_0^1 \gamma(x)|u_t(x,t)|^2dxdt + \int_0^T\int_0^1 \gamma(x)|v_t(x,t)|^2dxdt\right) \\ & \leq C(E(0)-E(T)) . \end{aligned}$$

The exponential decay of the energy follows from this inequality (see for example [23]). $\square$

In the last part of this section, we give concrete conditions implying the criterium (EFW) uniformly in n. Thus, we obtain examples of one dimensional equations satisfying the Hypothesis (UED). Notice that our method also gives higher dimensional examples where (EFW) is satisfied uniformly in n, but in these cases, we have no proof that (EFW) implies the uniform exponential decay (UED).
We wonder when Hypothesis (UED) is satisfied for the family of equations

$$\left\{\begin{array}{l} u_{tt}(x,t)+\gamma_n(x)u_t(x,t) = u_{xx}(x,t)-u(x,t)+f(x,u) \quad (x,t)\in]0,1[\times\mathbb{R}_+ \\ u_x(0,t)=u_x(1,t)=0 \\ (u,u_t)_{|t=0}=(u_0,u_1)\in \mathbb{H}^1(]0,1[)\times\mathbb{L}^2(]0,1[) \end{array}\right. \quad (6.11)$$

Remark that Proposition 6.2 and Theorem 6.4 imply that, if the semiflow generated by (6.11) satisfies (UED) for a sequence of dissipations γ_n, then the property (UED) also holds for any sequence of dissipations $\tilde{\gamma}_n \geq \gamma_n$. Thus, we may restrict our study to dissipations of the form $\gamma_n(x) = n\chi_{]a_n;a_n+1/n[}$. Next, we show the following lemma, which

replaces the criterium (EFW), concerning the solutions of the free waves, by a criterium on the eigenfunctions of the free waves operator.
We denote by λ_k^2 ($\lambda_k > 0$, $k \in \mathbb{N}^*$) the eigenvalues of B and φ_k the corresponding eigenfunctions normalized by $\|\varphi_k\|_{\mathbb{L}^2} = 1$.

Lemma 6.5. *We assume that γ_∞ is effective on the free waves, that is that (EFW) holds for γ_∞. We also assume that there exist a family of complex numbers (α_k) and an application h defined from $\mathbb{N}^* \times \mathbb{N}^*$ into $\{0,1\}$ such that*

$$\forall k, k' \in \mathbb{N}^*, \quad \int_\Omega \gamma_\infty \varphi_k \overline{\varphi}_{k'} = \alpha_k \overline{\alpha}_{k'} h(k,k') \ .$$

If $h(k,k') = 0$ implies $\int \gamma_n \varphi_k \overline{\varphi}_{k'} = 0$ for all $n \in \mathbb{N}$ and if

$$\inf_{n\in\mathbb{N}} \inf_{k\in\mathbb{N}^*} \frac{1}{|\alpha_k|^2} \int_\Omega \gamma_n |\varphi_k|^2 > 0 \ , \tag{6.12}$$

then the family of dissipation (γ_n) is uniformly effective on the free waves, that is that (EFW) holds uniformly in n.

Proof : For $k \in \mathbb{N}^*$, we set $\lambda_{-k} = -\lambda_k$ and $\varphi_{-k} = \varphi_k$. A solution of (6.1) can be decomposed as follows.

$$\begin{pmatrix} \varphi \\ \varphi_t \end{pmatrix} = \sum_{k\in\mathbb{Z}^*} c_k e^{i\lambda_k t} \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{i\lambda_k}\varphi_k \\ \varphi_k \end{pmatrix} \quad \text{where} \quad \|(\varphi,\varphi_t)_{|t=0}\|_X^2 = \sum_{k\in\mathbb{Z}^*} |c_k|^2 \ .$$

As (6.2) holds for γ_∞, we have that $\int_0^T \int_\Omega \gamma_\infty |\varphi_t|^2 \geq C \|(\varphi,\varphi_t)_{|t=0}\|_X^2$, that is that

$$\sum_{k,k'} c_k \overline{c}_{k'} \frac{e^{i(\lambda_k - \lambda_k')T} - 1}{\lambda_k - \lambda_{k'}} \ \alpha_k \overline{\alpha}_{k'} h(|k|,|k'|) \ \geq \ C \sum_{k\in\mathbb{Z}^*} |c_k|^2 \ , \tag{6.13}$$

where by convention $\frac{e^{i(\lambda_k-\lambda_k')T}-1}{\lambda_k-\lambda_{k'}} = T$ when $\lambda_k = \lambda_k'$. Concerning the dissipation γ_n, we have

$$\begin{aligned} \int_0^T \int_\Omega \gamma_n |\varphi_t|^2 \quad &= \sum_{k,k'} c_k \overline{c}_{k'} \frac{e^{i(\lambda_k - \lambda_k')T} - 1}{\lambda_k - \lambda_{k'}} \int_\Omega \gamma_n \varphi_k \overline{\varphi}_{k'} \\ &= \int_\Omega \sum_{k,k'} (\frac{c_k}{\alpha_k} \sqrt{\gamma_n} \varphi_k) \overline{(\frac{c_{k'}}{\alpha_{k'}} \sqrt{\gamma_n} \varphi_{k'})} \frac{e^{i(\lambda_k - \lambda_k')T} - 1}{\lambda_k - \lambda_{k'}} \alpha_k \overline{\alpha}_k' h(|k|,|k'|) \ . \end{aligned}$$

Notice that (6.13) implies that $\inf|\alpha_k|>0$. Moreover, choosing $(\varphi,\varphi_t)_{|t=0}$ in X^s for s large enough, we have that $\sum(\frac{c_k}{\alpha_k}\|\varphi_k\|_{\mathbb{L}^\infty})^2$ is finite. Thus, due to the inequality (6.13), we obtain

$$\int_0^T\int_\Omega \gamma_n|\varphi_t|^2 \geq \int_\Omega C\sum_{k\in\mathbb{Z}^*}\frac{|c_k|^2}{|\alpha_k|^2}\gamma_n|\varphi_k|^2 \geq C\sum_{k\in\mathbb{Z}^*}\frac{|c_k|^2}{|\alpha_k|^2}\int_\Omega \gamma_n|\varphi_k|^2 \ .$$

Using (6.12) we find that (6.2) holds uniformly in n for any initial data $(\varphi,\varphi_t)_{|t=0}$ in X^s. The density of the space X^s in X then concludes. □

We apply Lemma 6.5 to obtain the following result.

Proposition 6.6. *Let $(a_n)\subset[0,1[$ be a sequence such that $a_n\longrightarrow 0$ when $n\longrightarrow+\infty$. We set*

$$\gamma_n(x)=\begin{cases} n & \text{if } a_n\leq x\leq a_n+\frac{1}{n} \\ 0 & \text{elsewhere} \ . \end{cases}$$

Then the family of equations (6.11) satisfies (UED) if and only if $\sup\{na_n\}<+\infty$.

Proof : We have to verify the hypotheses of Lemma 6.5. In our case, we have $\gamma_\infty=\delta_{x=0}$, $\lambda_k=\sqrt{k^2+1}$ and $\varphi_k=\sqrt{2}\cos(k\pi x)$. Thus, $\int_\Omega\gamma_\infty\varphi_k\overline{\varphi}_{k'}=2$, and we can set $\alpha_k=\sqrt{2}$ and $h\equiv 1$. It is well-known that Equation (6.8) with $\gamma=\gamma_\infty$ generates an exponentially decaying semigroup. So, the criterium (EFW) is satisfied by γ_∞.
To apply Lemma 6.5, it remains to show (6.12). If (6.12) does not hold, then it is clear that (EFW) cannot be satisfied uniformly. Thus, we have to prove that $\sup\{na_n\}<+\infty$ is equivalent to the existence of $\varepsilon>0$ such that

$$\inf_{n\in\mathbb{N}}\inf_{k\in\mathbb{N}^*} n\int_{a_n}^{a_n+\frac{1}{n}}|\cos(k\pi x)|^2dx\geq\varepsilon \ . \tag{6.14}$$

We have

$$n\int_{a_n}^{a_n+\frac{1}{n}}|\cos(k\pi x)|^2dx=\frac{1}{2}\left(1+\frac{n}{k\pi}\sin(\pi\frac{k}{n})\cos(\pi k(2a_n+\frac{1}{n}))\right) \ .$$

Assume that (6.14) is not true, then there exist two sequences (k_p) and (n_p) such that

$$\frac{n_p}{k_p\pi}\sin(\pi\frac{k_p}{n_p})\cos(\pi k_p(2a_{n_p}+\frac{1}{n_p}))\longrightarrow -1 \ .$$

This implies that $\frac{k_p}{n_p}\to 0$ and $2k_pa_{n_p}\longrightarrow 1 \bmod(2)$, and so $|n_pa_{n_p}|\to+\infty$.
Assume now that there exists a subsequence satisfying $|n_pa_{n_p}|\to+\infty$. Let k_p be the smallest integer strictly larger than $\frac{n_p}{2n_pa_{n_p}+1}$. We have

$$\frac{n_p}{k_p\pi}\sin(\pi\frac{k_p}{n_p})\cos(\pi k_p(2a_{n_p}+\frac{1}{n_p}))\longrightarrow -1 \ ,$$

and thus (6.14) is not satisfied. $\square$

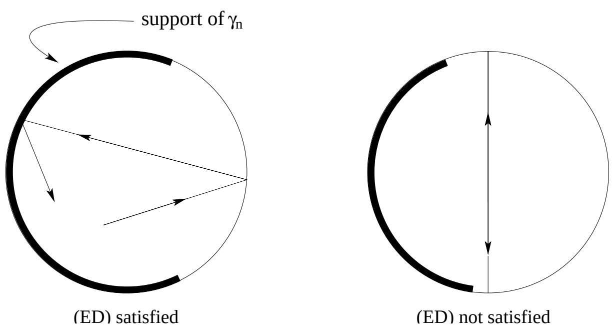
Remark : The same result obviously holds when the Neumann boundary condition at $x = 1$ is replaced by the Dirichlet condition.
We will come back to the case $\sup_n\{na_n\} = +\infty$ in the appendix A.2.

6.2 The two and three-dimensional cases

In dimension higher than one, our hypotheses are less easy to verify. First, Hypotheses (ED) and (Grad) do not always hold. The hypothesis (ED) is equivalent to geometrical conditions on the support of γ_n, which are now well-understood. The case of Hypothesis (UED) is much more difficult and its study in dimension two or higher is still mostly open.

6.2.1 Hypothesis (ED)

It is now well-known that the following geometric condition is equivalent to (ED), see [6]. For each $n \in \mathbb{N} \cup \{+\infty\}$, there is a length L_n such that all geodesics on Ω associated to the operator $\partial^2_{tt} + B$ and of length greater than L_n meet the support of γ_n. In dimension one, the condition is trivially satisfied. In the higher dimensional case, the condition is more restrictive, since, for some examples, there exist geodesics of infinite length, which do not meet the support of γ_n.



6.2.2 Hypothesis (Grad)

Let $n \in \mathbb{N} \cup \{+\infty\}$ be given. Let $U_0 \in X$ be such that for all $t \geq 0$, we have $\Phi(S_n(t)U_0) = \Phi(U_0)$. If $U(t) = S_n(t)U_0 = (u(t), v(t))$, we thus have

$$u_t(t) = v(t) \text{ and } u_{tt} + B(u + \Gamma_n u_t) = f(x, u) \ .$$

We know that

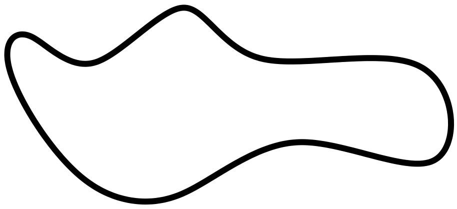
$$\forall t \geq 0,\ \frac{\partial}{\partial t}\Phi(U(t)) =< A_n U|U >= -\int_\Omega \gamma_n |v|^2 = 0\ .$$

Hence, $v = u_t$ satisfies

$$\forall t \geq 0,\ v_{tt} + Bv = f'_u(x,u)v \text{ and } \begin{cases} v = 0 \text{ on } \mathrm{supp}(\gamma_n) \text{ if } n \in \mathbb{N} \\ \text{or} \\ v = 0 \text{ and } \frac{\partial v}{\partial \nu} = 0 \text{ on } \mathrm{supp}(\gamma_\infty) \end{cases} . \tag{6.15}$$

To prove that Hypothesis (Grad) holds, we must show that (6.15) implies that $u_t = v = 0$ on $\Omega \times \mathbb{R}_+$. This unique continuation argument holds under geometrical conditions.

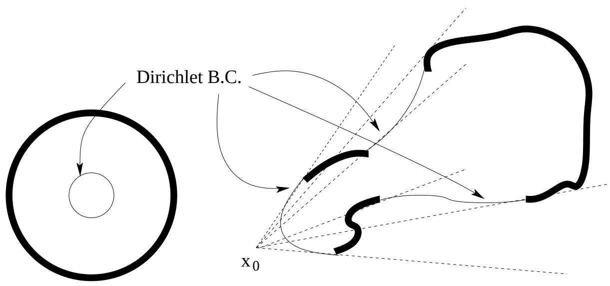
- If the support of γ_n contains a neighborhood of the boundary $\partial\Omega$ for $n \in \mathbb{N}$ and if the support of γ_∞ is equal to $\partial\Omega$, then (Grad) is satisfied (see [42]).



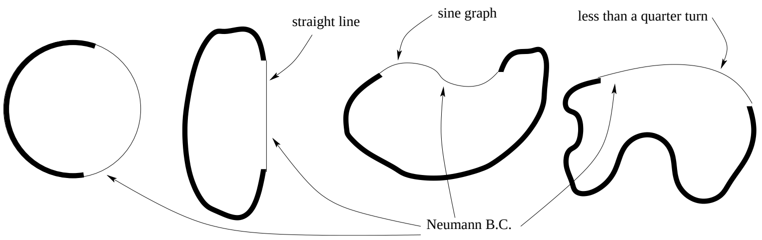
- Assume that $\mathrm{supp}(\gamma_\infty) = \omega_N$, that the support of γ_n contains a neighborhood of ω_N and that there exists a point $x_0 \in \mathbb{R}^d$ such that

$$\{x \in \partial\Omega\ /\ (x - x_0).\nu > 0\} \subset \omega_N\ ,$$

then (Grad) holds (see [28]).



- Let Ω be a domain with a boundary of class $\mathcal{C}^1$. We assume that the support of γ_n includes a neighborhood of $\mathrm{supp}(\gamma_\infty)$ and that the boundary conditions on the whole boundary $\partial\Omega$ are of Neumann type, that is that $\omega_D = \emptyset$. In this case, [33] gives many sufficient conditions for (Grad) to hold. In particular, if Ω is a disk, the fact that the support of γ_∞ covers slightly more than a half circle is sufficient. Other examples are given.



Remark : For all the examples that we give here, one notices that (ED) is satisfied. However, there is no reason that (Grad) implies (ED) in general.

6.2.3 Hypothesis (UED)

The methods, which were used in the one-dimensional case, cannot be generalized to dimensions two or three. For these dimensions, using an energy method, we obtain here a criterium equivalent to the property (UED). However, except for the particular cases where γ_n is uniformly bounded away from 0, it is very difficult to exhibit examples satisfying this criterium.
The following equivalence is very classical. The property (UED) is satisfied if and only if there exist two positive constants T and C such that, for all $U_0 \in X$ and $n \in \mathbb{N}$, if we set $U(t) = (u, u_t)(t) = e^{A_n t}U_0$, then we have

$$\int_0^T \int_\Omega \gamma_n |u_t|^2 \geq C\|U_0\|_X^2 \ . \tag{6.16}$$

We can weaken this criterium as follows.

Proposition 6.7. *The uniform exponential decay property (UED) is satisfied if and only if there exist two positive constants T and C, independent of n, such that, for all $U_0 \in X$ and $n \in \mathbb{N}$, if we set $U(t) = (u, u_t)(t) = e^{A_n t}U_0$, then we have*

$$\int_0^T \int_\Omega \gamma_n |u_t|^2 \geq C \int_0^T \int_\Omega |u_t|^2 \ . \tag{6.17}$$

Proof : The "only if" part is a direct consequence of the classical criterium (6.16). Indeed, the property (UED) implies that there exist two positive constants T and C such that, for all $U_0 \in X$ and $n \in \mathbb{N}$,

$$\int_0^T \int_\Omega \gamma_n |u_t|^2 \geq C\|U_0\|_X^2 = \frac{C}{T}\int_0^T \|U_0\|_X^2 dt \geq \frac{C}{T}\int_0^T \|U(t)\|_X^2 dt \geq C\int_0^T \int_\Omega |u_t|^2 \ .$$

In order to prove the "if" part of the equivalence, we introduce the following functional. Let α be a positive number to be chosen. For all $U = (u, v) \in X$, we set

$$F(U) = \frac{1}{2} \int_\Omega (|u|^2 + 2uv + \alpha|v|^2 + \alpha|B^{1/2}u|^2)dx \ . \tag{6.18}$$

For α large enough, the functional F is clearly equivalent to the energy in the sense that there exists a positive constant μ such that

$$\forall U \in X,\ \frac{1}{\mu}\|U\|_X^2 \leq F(U) \leq \mu\|U\|_X^2 \ . \tag{6.19}$$

Let $U_0 \in D(A_n)$ and $n \in \mathbb{N}$, we set $U(t) = (u, u_t)(t) = e^{A_n t}U_0$. As $U(t) \in \mathcal{C}^1(\mathbb{R}_+, X)$, we can write

$$\begin{aligned}
\partial_t F(U(t)) &= \int_\Omega (uu_t + uu_{tt} + |u_t|^2 + \alpha u_t u_{tt} + \alpha(B^{1/2}u)(B^{1/2}u_t))dx \\
&= \int_\Omega (uu_t - \gamma_n uu_t - (Bu)u + |u_t|^2 - \alpha\gamma_n|u_t|^2 \\
&\qquad - \alpha u_t(Bu) + \alpha(B^{1/2}u)(B^{1/2}u_t))dx \ .
\end{aligned}$$

Thus, for all $\varepsilon > 0$, we have that

$$\partial_t F(U(t)) \leq \int_\Omega \varepsilon(1+\gamma_n)|u|^2 + \frac{1}{\varepsilon}(1+\gamma_n)|u_t|^2 - |B^{1/2}u|^2 + |u_t|^2 - \alpha\gamma_n|u_t|^2 \ .$$

As Γ_n converges to Γ_∞ in $\mathcal{L}(D(B^{1/2}))$, we know that there exists a positive constant C, independent of n, such that, for all $u \in D(B^{1/2})$, $\int_\Omega \gamma_n|u|^2 \leq C\|u\|^2_{D(B^{1/2})}$. Therefore, for ε small enough and α large enough, (6.17) implies the existence of a time T and a positive constant C such that

$$\int_0^T \partial_t F(U(t)) \leq -C\int_0^T F(t) \ .$$

Thus, using the density of $D(A_n)$ in X, we obtain that, for all $U_0 \in X$,

$$F(U_0) - F(e^{A_n T}U_0) \geq C\int_0^T F(e^{A_n t}U_0)dt \ .$$

The inequalities (6.19) and the fact that $e^{A_n t}$ is a contraction imply that, for all $U_0 \in X$ and $k \in \mathbb{N}$,

$$\|U_0\|_X^2 \geq \frac{C}{\mu^2}\int_0^{kT} \|e^{A_n t}U_0\|_X^2 dt \geq \frac{C}{\mu^2}kT\|e^{A_n kT}U_0\|_X^2 \ .$$

For k large enough, we obtain a time T', independent of n, such that $\|e^{A_n T'}U_0\|_X \leq \frac{1}{2}\|U_0\|_X^2$. This is well-known to imply the uniform exponential decay (UED). □

It is difficult to find examples satisfying the criterium (6.17). Indeed, opposite to the criterium (EFW) obtained in the one-dimensional case, (6.17) involves functions $U(t)$, which are solutions of an equation which depends of n. However, Proposition 6.7 gives some very particular examples where (UED) is satisfied in dimension higher than one. This corollary is stated in a way so that it can be applied to a general sequence of dissipations, which satisfies (6.20) only.

Corollary 6.8. *Let $(\gamma_n^0)_{n\in\mathbb{N}}$ be a sequence of non-negative functions in $\mathbb{L}^\infty(\Omega)$. Assume that there exists a positive constant C independent of n such that*

$$\forall u \in D(B^{1/2}),\ \forall n \in \mathbb{N}, \quad \int_\Omega \gamma_n^0(x)|u(x)|^2 dx \le C\|u\|^2_{D(B^{1/2})} \ . \tag{6.20}$$

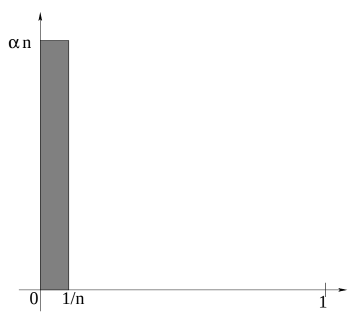
Then, for all $\eta > 0$, the uniform exponential decay property (UED) is satisfied for the sequence of dissipations $\gamma_n = \eta + \gamma_n^0$.

Notice that this result in not a priori trivial, since, as the sequence (γ_n^0) is not necessary bounded in $\mathbb{L}^\infty(\Omega)$, overdamping phenomenas may occur. The fact that $\gamma_n \ge \eta > 0$ seems slightly artificial from a mathematical point of view. However, it is not from the physical point of view since γ_n never really vanishes in the concrete cases. For example, η can be seen as the resistance of air when (2.5) models the propagation of waves in a room.

7 Examples

In this section, we give some examples illustrating our results. For each example, we define Ω, B and γ_n and say if the convergence of the attractors holds in the space X or only in X^{-s} ($s > 0$). Saying that the convergence holds in X^{-s} does not mean that there is no convergence in X. It only means that we are not able to prove it for the moment. Here we do not give explicit non-linearities for which Hypothesis (Hyp) is satisfied.
We recall that we denote by Δ_N the Laplacian with Neumann boundary conditions.

Example 1 :

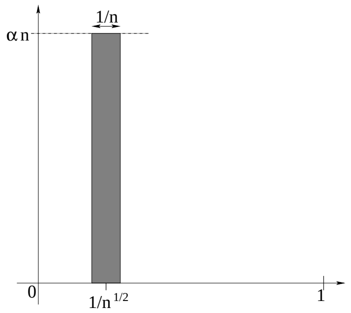


$\Omega =]0,1[$, $B = Id - \Delta_N$, $\alpha > 0$

$$\gamma_n(x) = \begin{cases} \alpha n, & x \in]0, \frac{1}{n}[\\ 0, & \text{elsewhere} \end{cases}, \quad \gamma_\infty = \alpha\delta_{x=0} \ .$$

Convergence in X.

Example 2 :

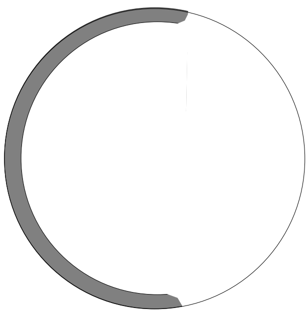


$\Omega =]0,1[$, $B = Id - \Delta_N$, $\alpha > 0$

$$\gamma_n(x) = \begin{cases} \alpha n, & x \in]\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} + \frac{1}{n}[\\ 0, & \text{elsewhere} \end{cases} \quad , \quad \gamma_\infty = \alpha \delta_{x=0} \ .$$

Convergence in X^{-s}.

Example 3 :

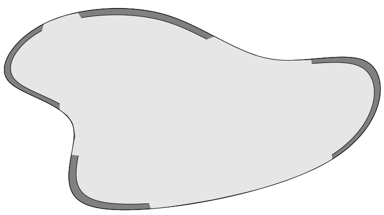


Ω is the disk of $\mathbb{R}^2$, $B = Id - \Delta_N$,
ω is an open subset of $\partial\Omega$ which covers strictly more than half of the circle

$$\gamma_n(x) = \begin{cases} n, & \text{dist}(x,\omega) < \frac{1}{n} \\ 0, & \text{elsewhere} \end{cases} \quad , \quad \gamma_\infty = \delta_{x\in\omega} \ .$$

Convergence in X^{-s}.

Example 4 :

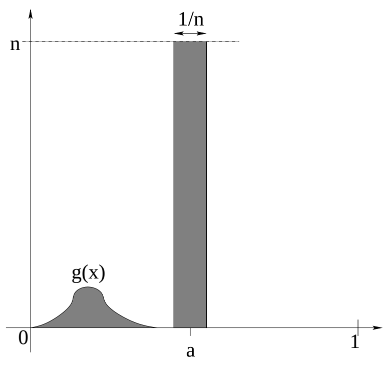


$\Omega \subset \mathbb{R}^2$, $\eta > 0$, ω is any subset of $\partial\Omega$
$B = Id - \Delta_N$

$$\gamma_n(x) = \begin{cases} n, & \text{dist}(x,\omega) < \frac{1}{n} \\ \eta, & \text{elsewhere} \end{cases} \quad , \quad \gamma_\infty = \eta + \delta_{x\in\omega} \ .$$

Convergence in X.

Example 5 : For sake of simplicity, the abstract frame of this paper has not been defined so that this example fits in it. However, all the results given here are valid for this case. Notice that we need the additional dissipation g, since the singular internal dissipation $\delta_{x=a}$ is not sufficient to obtain exponential decay (see [25]), or the gradient structure. We denote by Δ_D the Laplacian with Dirichlet boundary conditions.



$\Omega =]0,1[$, $a \in]0,1[$ and g is a nonnegative function, in $\mathbb{L}^\infty(]0,1[)$ which is positive on an open subset, $B = -\Delta_D$

$$\gamma_n(x) = \begin{cases} g(x) + n, & x \in]a - \frac{1}{2n}, a + \frac{1}{2n}[\\ g(x), & \text{elsewhere} \end{cases}$$

$$\gamma_\infty(x) = g(x) + \delta_{x=a} \ .$$

Convergence in X.

A Appendix

A.1 A result of Ammari and Tucsnak

We have proved in Section 6.1 that if $\Omega =]0,1[$ and if the dissipations γ_n satisfy uniformly the property (EFW), then (UED) is satisfied. We show here how to obtain the same implication with a different method, which has been introduced in [2]. To simplify the notation, we state the results of [2] in our frame.
Let γ be a function in $\mathbb{L}^\infty(\Omega)$. We introduce the following hypothesis

(H) If $\beta > 0$ is fixed and $C_\beta = \{\lambda \in \mathbb{C}/Re(\lambda) = \beta\}$, then the function

$$H(\lambda) = \sqrt{\gamma}\, \lambda(\lambda^2 Id + B)^{-1}\sqrt{\gamma}$$

defined from C_β into $\mathcal{L}(\mathbb{L}^2)$ is bounded and we set

$$M_\beta = \sup_{\lambda \in C_\beta} \|H(\lambda)\|_{\mathcal{L}(\mathbb{L}^2)} < \infty \ .$$

In our case, Theorem 2.2 of [2] can be stated as follows.

Theorem A.1. *Assume that the hypothesis (H) holds and that γ is effective on the free waves, i.e. that (EFW) is satisfied. Then, there exist two positive constants M and λ depending only on the constants C, T introduced in (6.2) and on the family of constants M_β introduced in (H) such that, for any initial data $(u_0, u_1) \in X$, the solution u of*

$$\begin{cases} u_{tt} + \gamma(x) u_t + Bu = 0 \\ (u, u_t)_{|t=0} = (u_0, u_1) \in X \end{cases} \tag{A.1}$$

satisfies

$$\|(u, u_t)(t)\|_X \le M \|(u_0, u_1)\|_X e^{-\lambda t} \ .$$

The idea of the proof of Theorem 2.2 of [2] is to replace the multipliers method by a Laplace transform argument to obtain a result similar to Proposition 6.3.
Theorem 6.4 is then a direct consequence of Theorem A.1 and of the following property.

Proposition A.2. *Let $\Omega =]0,1[$ and $B = -\Delta_N + Id$. For $\gamma \in \mathbb{L}^\infty(]0,1[)$, Hypothesis (H) is satisfied and the bound M_β depends on $\|\gamma\|_{\mathbb{L}^1}$ only.*

Proof : We notice that $\sqrt{\gamma}$ is bounded in $\mathbb{L}^2(]0,1[)$ by $\|\gamma\|_{\mathbb{L}^1}^{1/2}$. So, the operator of multiplication by $\sqrt{\gamma}$ is bounded in $\mathcal{L}(\mathbb{L}^2,\mathbb{L}^1)$ and in $\mathcal{L}(\mathbb{L}^\infty,\mathbb{L}^2)$. It remains to show that, on C_β, the operator $\lambda(\lambda^2 Id + B)^{-1}$ is uniformly bounded in $\mathcal{L}(\mathbb{L}^1,\mathbb{L}^\infty)$.
Let $f \in \mathbb{L}^1(]0,1[)$ and u be the solution of

$$-u_{xx} + u + \lambda^2 u = f \ , \quad u_x(0) = u_x(1) = 0 \ . \tag{A.2}$$

We set $\theta = (-\lambda^2 - 1)^{1/2}$. The solution of (A.2) is given by

$$u(x) = C\cos(\theta x) - \sin(\theta x)\int_0^x \frac{\cos(\theta s)}{\theta} f(s)ds + \cos(\theta x)\int_0^x \frac{\sin(\theta s)}{\theta} f(s)ds \ , \tag{A.3}$$

where

$$C = -\int_0^1 \frac{\sin(\theta s)}{\theta} f(s)ds - \frac{\operatorname{cotg}(\theta)}{\theta}\int_0^1 \cos(\theta s) f(s)ds \ .$$

A direct computation shows that, if $\lambda = \beta + i\mu$, then

$$\operatorname{Im}(\theta) = -\left((\mu^2 - \beta^2 - 1)^2 + (2\mu\beta)^2\right)^{1/4} \sin\left(\frac{1}{2}\operatorname{arctg}\left(\frac{2\mu\beta}{\mu^2 - \beta^2 - 1}\right)\right) \ .$$

Thus, $\operatorname{Im}(\theta) \longrightarrow \mp\beta \neq 0$ when $\mu \longrightarrow \pm\infty$. This implies that $\sin(\theta)$, $\cos(\theta)$, $\operatorname{cotg}(\theta)$ and $\frac{1}{\theta}$ are uniformly bounded on C_β. Since $f \in \mathbb{L}^1(]0,1[)$, (A.3) proves that $u \in \mathbb{L}^\infty$ and so $\lambda(\lambda^2 Id + B)^{-1}$ is uniformly bounded in $\mathcal{L}(\mathbb{L}^1,\mathbb{L}^\infty)$. $\square$

Unfortunately, Theorem A.1 is not applicable in dimension higher than one. Indeed, it is shown in [2] that property (H) implies the following fact. For all $T > 0$, there exists $C > 0$ such that all the solutions φ of the free wave equation (6.1) satisfy

$$\int_0^T \int_\Omega \gamma(x)|\varphi_t|^2 dxdt \leq C\|(\varphi_0,\varphi_1)\|_X^2 \ . \tag{A.4}$$

Let $\gamma_\infty = \delta_{x\in\omega}$ be a dissipation on a part of the boundary. In dimension higher than one, we can imagine a wave travelling along the curve ω for which the left-hand side of the inequality (A.4) is infinite. If (A.4) does not hold for the boundary dissipation, we cannot hope that it holds uniformly for the family $(\gamma_n)_{n\in\mathbb{N}}$ when γ_n converges to γ_∞. The following counter-example illustrates this remark.

Proposition A.3. *Let $\Omega =]0,1[^2$, $B = -\Delta_N + Id$. For all time $T > 0$, there exists a sequence of initial data $(\varphi_0^n, \varphi_1^n) \in X$, with $\|(\varphi_0^n, \varphi_1^n)\|_X = 1$, such that the solutions $\varphi_n(x,y,t)$ of the free wave equation (6.1) satisfy*

$$\int_0^T \int_0^1 |\frac{\partial}{\partial t}\varphi_n(0,y,t)|^2 dydt \longrightarrow +\infty \ ,$$

when $n \longrightarrow +\infty$.

Proof : We choose the decomposition of the initial data on the eigenvectors of the free wave operator as follows. Let

$$\begin{pmatrix} \varphi_0^n \\ \varphi_1^n \end{pmatrix} = \sum_{k=0}^{n-1} \sqrt{\frac{2}{n}} \begin{pmatrix} \frac{1}{i\sqrt{n^6+k^2+1}} \cos(n^3\pi y)\cos(k\pi x) \\ \cos(n^3\pi y)\cos(k\pi x) \end{pmatrix} \ .$$

Notice that $\|(\varphi_0^n, \varphi_1^n)\|_X = 1$. A straightforward calculus gives

$$\int_0^T \int_0^1 |\frac{\partial}{\partial t}\varphi_n(0,y,t)|^2 dydt = 2\sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} \frac{1}{n} \frac{\sin(\sqrt{n^6+k^2+1} - \sqrt{n^6+k'^2+1})T}{\sqrt{n^6+k^2+1} - \sqrt{n^6+k'^2+1}} \ .$$

Since

$$\left|\sqrt{n^6+k^2+1} - \sqrt{n^6+k'^2+1}\right| \leq \frac{|k^2-k'^2|}{2\sqrt{n^6}} \leq \frac{1}{n} \ ,$$

for n large enough, there exists $\varepsilon > 0$ such that

$$\frac{\sin(\sqrt{n^6+k^2+1} - \sqrt{n^6+k'^2+1})T}{\sqrt{n^6+k^2+1} - \sqrt{n^6+k'^2+1}} \geq \varepsilon > 0 \ .$$

And thus,

$$\int_0^T \int_0^1 |\frac{\partial}{\partial t}\varphi_n(0,y,t)|^2 dydt \geq 2n\varepsilon \ .$$

□

A.2 An example of convergence of the attractors in X, when (UED) does not hold

To show the convergence of the attractors $\mathcal{A}_n$ in X, we had to show Proposition 2.9 that is that

$$\exists M \geq 0,\ \forall n \in \mathbb{N},\ \sup_{U_n \in \mathcal{A}_n} \|U_n\|_{D(A_n)} \leq M \ . \tag{A.5}$$

We have shown that Hypothesis (UED) implies the above bound, but, of course, it is not necessary. The purpose here is to give examples where (UED) is not satisfied but where (A.5) holds.
We set $\Omega =]0,1[$ and $B = -\Delta_N + Id$. Let $\alpha > 0$ and $f \in \mathcal{C}^2([0,1]\times\mathbb{R},\mathbb{R})$. We study the family of equations

$$\left\{\begin{array}{ll} u_{tt}(x,t) + \gamma_n u_t(x,t) = u_{xx}(x,t) - u(x,t) + f(x,u(x,t)) & (x,t) \in]0,1[\times\mathbb{R}_+ \\ u_x(0,t) = u_x(1,t) = 0 & t \geq 0 \\ (u,u_t)_{|t=0} = (u_0,u_1) \in \mathbb{H}^1(]0,1[) \times \mathbb{L}^2(]0,1[) & \end{array}\right. \tag{A.6}$$

where, if $n \in \mathbb{N}$

$$\gamma_n(x) = \left\{\begin{array}{ll} n & \text{if } \frac{1}{n^\alpha} \leq x \leq \frac{1}{n^\alpha} + \frac{1}{n} \\ 0 & \text{elsewhere .} \end{array}\right.$$

and $\gamma_\infty(x) = \delta_{x=0}$.
In Proposition 6.6, we proved that (UED) holds if and only if $\alpha > 1$. The purpose of this section is the proof of the following result.

Proposition A.4. *We assume that f satisfies Hypothesis (Diss). The dynamical systems generated by (A.6) admit a compact global attractor $\mathcal{A}_n$. Moreover, if $\alpha > \frac{16}{17}$ then (A.5) holds and the conclusions of Theorem 2.10 are valid.*

In what follows, we assume that $\alpha \in]\frac{16}{17}, 1[$, the case $\alpha \geq 1$ has already been considered in Proposition 6.6. The proof of Proposition A.4 is a consequence of the following two lemmas.

Lemma A.5. *There exist a time T and a constant C such that, for any $(\varphi_0,\varphi_1) \in X$, the solution of the free wave equation*

$$\left\{\begin{array}{l} \varphi_{tt} + B\varphi = 0 \\ (\varphi,\varphi_t)_{|t=0} = (\varphi_0,\varphi_1) \end{array}\right. \tag{A.7}$$

satifies for all $n \in \mathbb{N} \cup \{+\infty\}$

$$\int_0^T \int_\Omega \gamma_n(x)|\varphi_t(x,t)|^2 dxdt \geq C\|(\varphi_0,\varphi_1)\|^2_{X^{1-1/\alpha}} \ . \tag{A.8}$$

Proof : Using the same arguments as those of Lemma 6.5, we see that (A.8) is satisfied uniformly with respect to ε if there exists a positive constant C such that

$$\forall n \in \mathbb{N},\ \forall k \in \mathbb{N},\ \int_\Omega \gamma_n |\cos(k\pi x)|^2 dx = n\int_{n^{-\alpha}}^{n^{-\alpha}+1/n} |\cos(k\pi x)|^2 \geq \frac{C}{k^{2/\alpha-2}} \ , \tag{A.9}$$

that is that
$$\frac{1}{2}\left(1+\frac{n}{k\pi}\sin(\pi\frac{k}{n})\cos(k\pi(\frac{2}{n^\alpha}+\frac{1}{n})\right)\geq\frac{C}{k^{2/\alpha-2}}\ .$$
Assume that the above inequality does not hold. Then, there exist two sequences (k_p) and (n_p) such that
$$\left(1+\frac{n_p}{k_p\pi}\sin(\pi\frac{k_p}{n_p})\cos(k_p\pi(\frac{2}{{n_p}^\alpha}+\frac{1}{n_p})\right)k_p^{2-2/\alpha}\longrightarrow 0\ . \tag{A.10}$$
We must have $\frac{k_p}{n_p}\longrightarrow 0$ and
$$\frac{2k_p}{n_p^\alpha}\longrightarrow 1 \bmod 2\ . \tag{A.11}$$
Thus, for p large enough, we have $0\leq\frac{n_p}{k_p\pi}\sin(\pi\frac{k_p}{n_p})\leq 1-\frac{1}{6}\left(\frac{k_p}{n_p}\right)^2$. This shows that
$$1+\frac{n_p}{k_p\pi}\sin(\pi\frac{k_p}{n_p})\cos(k_p\pi(\frac{2}{{n_p}^\alpha}+\frac{1}{n_p})\geq\frac{1}{6}\left(\frac{k_p}{n_p}\right)^2\ .$$
Using (A.11), we obtain that, for p large enough, $\frac{1}{n_p}>\frac{1}{(4k_p)^{\frac{1}{\alpha}}}$, and thus
$$1+\frac{n_p}{k_p\pi}\sin(\pi\frac{k_p}{n_p})\cos(k_p\pi(\frac{2}{{n_p}^\alpha}+\frac{1}{n_p})\geq\frac{1}{6(4)^{\frac{2}{\alpha}}}\left(\frac{1}{(k_p)^{\frac{1}{\alpha}-1}}\right)^2\ .$$
This is a contradiction to the assumption that (A.9) does not hold. □

The second lemma is a direct adaptation of a theorem of [2].

Lemma A.6. *If $\alpha>\frac{16}{17}$, there exist $\lambda>1$, $s\in]0,1/2[$ and $M>0$ such that*
$$\forall U_0\in X,\ \forall n\in\mathbb{N},\ \|e^{A_nt}U_0\|_X\leq\frac{M}{(1+t)^\lambda}\|U_0\|_{X^s}\ .$$

Proof : The outline of the proof is exactly the same as the one of Theorem 2.4 of [2]. First, notice that we have proved in Proposition A.2 that Hypothesis (H) introduced in Section A.1 is satisfied uniformly in n. Arguing as in [2], with some slight modifications, we show that Lemma A.5 and Proposition 2.6 imply that, for all $\sigma\in]0,1/2[$,
$$\forall t\geq 0,\ \|U(t)\|_X\leq\frac{M}{(1+t)^{\frac{1}{2(1/\theta-1)}}}\|U_0\|_{D(A_n)}\ ,$$

where $\theta = \frac{\sigma}{\sigma+1-1/\alpha}$. With the same interpolation methods as in Proposition 3.5, we obtain

$$\forall t \geq 0,\ \|U(t)\|_X \leq \frac{M}{(1+t)^{\frac{s^2}{2(1/\theta-1)}}} \|U_0\|_{X^s}\ .$$

We end the proof by noticing that we can find $s \in]0,1/2[$ and $\sigma \in]0,1/2[$ such that $\lambda = \frac{s^2}{2(1/\theta-1)} > 1$ is equivalent to $\alpha > \frac{16}{17}$. □

We are now able to prove the proposition.

Proof of Proposition A.4: All we have to prove is that the inequality (A.5) holds. The proof is exactly the same as the one of Proposition 2.9. The only change is the estimate of $e^{A_n(t-\tau)}(F(U(\tau+\delta)) - F(U(\tau)))$ for $\tau \leq t$. Lemma A.6 implies that there exist $s \in]0,1/2[$ and $\lambda > 1$ such that

$$\left\| e^{A_n(t-\tau)}(F(U(\tau+\delta)) - F(U(\tau))) \right\|_X \leq \frac{C}{(1+t-\tau)^\lambda} \|F(U(\tau+\delta)) - F(U(\tau))\|_{X^s}\ .$$

Hypothesis (NL) implies that there exists $\eta \in]0,1[$ such that

$$\|F(U(\tau+\delta)) - F(U(\tau))\|_{X^s} \leq \|u(\tau+\delta) - u(\tau)\|_{\mathbb{H}^\eta}\ .$$

Thus,

$$\begin{aligned} &\left\| e^{A_n(t-\tau)}(F(U(\tau+\delta)) - F(U(\tau))) \right\|_X \\ &\qquad\qquad \leq \frac{C}{(1+t-\tau)^\lambda} \|u(\tau+\delta) - u(\tau)\|_{\mathbb{H}^1}^\eta \|u(s+\delta) - u(s)\|_{\mathbb{L}^2}^{1-\eta}\ . \end{aligned}$$

Using the fact that $\int_{-\infty}^t \frac{d\tau}{(1+t-\tau)^\lambda}$ is finite, we conclude with the same arguments as in Proposition 2.9. □

References

[1] R.A. Adams, *Sobolev spaces*, Pure and Applied Mathematics n^{o}65 (1975), Academic Press.

[2] K. Ammari and M. Tucsnak, *Stabilization of second order evolution equations by a class of unbounded feedbacks*, ESAIM Control, Optimisation and Calculus of Variations n^{o}6 (2001), pp 361-386.

[3] A.V. Babin and M.I. Vishik, *Uniform asymptotic solutions of a singularly perturbed evolutionary equation*, Journal de Mathématiques Pures et Appliquées n^{o}68 (1989), pp 399-455.

[4] A.V. Babin and M.I. Vishik, *Attractors of evolution equations*, Studies in Mathematics and its Applications n^{o}25 (1992), North-Holland.

[5] J.M. Ball, *Global attractors for damped semilinear wave equations*, Discrete and Continuous Dynamical Systems n^{o}10 (2004), pp 31-52.

[6] C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM Journal on Control and Optimization Vol 30 (1992), pp 1024-1065.

[7] P. Brunovský and S-N. Chow, *Generic properties of stationary state solutions of reaction-diffusion equations*, Journal of Differential Equations n^{o}53 (1984), pp 1-23.

[8] G. Chen, S. A. Fulling, F. J. Narcowich and S.Sun, *Exponential decay of energy of evolution equations with locally distributed damping*, SIAM Journal on Applied Mathematics n^{o}51 (1991), pp. 266–301.

[9] I. Chueshov, M. Eller and I. Lasiecka, *On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation*, Communications in Partial Differential Equations n^{o}27 (2002), pp 1901-1951.

[10] S. Cox and E. Zuazua, *The rate at which energy decays in a damped string*, Communication in Partial Differential Equations n^{o}19 (1994), pp. 213-243.

[11] S. Cox and E. Zuazua, *The rate at which energy decays in a string damped at one end*, Indiana University Mathematics Journal n^{o}44 (1995), pp 545-573.

[12] M. Dauge, *Neumann and mixed problems on curvilinear polyhedra*, Integral Equations Operator Theory n^{o}15 (1992), pp 227-261.

[13] C. Fabre and J.P. Puel, *Pointwise controllability as limit of internal controllability for the wave equation in one space dimension*, Portugaliae Mathematica n^{o}51 (1994), pp 335-350.

[14] E. Feireisel and E. Zuazua, *Global attractors for semilinear wave equations with locally distributed nonlinear damping and critical exponent*, Communications in Partial Differential Equations n^{o}18 (1993), pp 1539-1555.

[15] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Text in Mathematics n^{o}14 (1973), Springer-Verlag.

[16] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics n^{o}24 (1985), Pitman.

[17] J.K. Hale, *Asymptotic behavior of dissipative systems*, Mathematical Survey n^{o}25 (1988), American Mathematical Society.

[18] J.K. Hale, X.B. Lin and G. Raugel, *Upper semicontinuity of attractors for approximations of semigroups and partial differential equations*, Mathematics of Computation n^{o}50 (1988), pp 89-123.

[19] J.K. Hale, L. Magalhães and W. Oliva, *An introduction to infinite dimensional dynamical systems*, Applied Mathematical Sciences n^{o}47 (1984), Springer-Verlag. Second edition (2002), *Dynamics in infinite dimensions.*

[20] J.K. Hale and G. Raugel, *Lower semicontinuity of attractors of gradient systems and applications*, Annali di Matematica Pura ed Applicata (IV) n^{o}CLIV (1989), pp 281-326.

[21] J.K. Hale and G. Raugel, *Attractors for dissipative evolutionary equations*, in International Conference on Differential Equations, Vol. 1, 2 (Barcelona 1991) World Sci. Publishing, River Edge, NJ, (1993), pp 3-22.

[22] J.K. Hale and G. Raugel, *Infinite Dimensional Dissipative Systems* manuscript.

[23] A. Haraux, *Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps*, Portugaliae Mathematica n^{o}46 (1989), pp 246-257.

[24] D. Henry *Perturbation of the Boundary for Boundary Value Problems of Partial Differential Operators*, Cambridge University Press, Cambridge UK, to appear.

[25] S. Jaffard, M. Tucsnak and E. Zuazua, *Singular internal stabilization of the wave equation*, Journal of Differential Equations n^{o}145 (1998), pp 184-215.

[26] R. Joly, *Generic transversality property for a class of wave equations with variable damping*, Journal de Mathématiques Pures et Appliquées n^{o}84 (2005), pp 1015-1066.

[27] R. Joly, *Sur la dynamique des équations des ondes amorties*, PhD Thesis (2005).

[28] M. Kazeni and M.V. Klibanov, *Stability estimates for ill-posed Cauchy problems involving hyperbolic equations and inequalities*, Applicable Analysis n^{o}50 (1993), pp 93-102.

[29] V. Komornik and E. Zuazua, *A direct method for the boundary stabilization of the wave equation*, Journal de Mathématiques Pures et Appliquées n^{o}69 (1990), pp 33-45.

[30] I.N. Kostin, *A regular approach to a problem on the attractors of singularly perturbed equations*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) n^{o}181 (1990), English translation in Journal of Soviet Mathematics n^{o}62 (1992), pp 2664-2689.

[31] I.N. Kostin, *Attractor for a semilinear wave equation with boundary damping*, Journal of Mathematical Sciences n^{o}98 (2000), pp. 753-764.

[32] I. Lasiecka and D. Tataru, *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, Differential and Integral Equations n^{o}6 (1993), pp 507-533.

[33] I. Lasiecka, R. Triggiani and X. Zhang, *Nonconservative wave equations with unobserved Neumann B.C. : global uniqueness and observability in one shot*, Contemporary Mathematics n^{o}268 (2000), pp 227-325.

[34] Z. Liu and S. Zheng, *Semigroups associated with dissipative systems*, Chapman and Hall/CRC Resarch Notes in Mathematics n^{o}398 (1999).

[35] A. Majda, *The location of the spectrum for the dissipative acoustic operator*, Indiana University Mathematics Journal n^{o}25 (1976), pp 973-987.

[36] W. Oliva, *Morse-Smale semiflows. Openess and $\mathcal{A}-$stability*, Differential equations and dynamical systems (Lisbon, 2000), pp. 285–307, Fields Institute Communications n^{o}31, American Mathematical Society (2002).

[37] J. Palis, *On Morse-Smale dynamical systems*, Topology n^{o}8 (1969), pp 385-405.

[38] J. Palis and W.de Melo, *Geometric theory of dynamical systems*, Springer-Verlag (1982).

[39] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences n^{o}44 (1983), Springer-Verlag.

[40] J. Rauch, *Qualitative behavior of dissipative wave equations on bounded domains*, Archive of Rational Mechanics and Analysis n^o62 (1976), pp 77-85.

[41] G. Raugel, chapter 17 of *Handbook of dynamical systems* vol.2 (2002), edited by B.Fiedler, Elsevier Science.

[42] A. Ruiz, *Unique continuation for weak solutions of the wave equation plus a potential*, Journal de Mathématiques Pures et Appliquées n^o71 (1992), pp 455-467.

[43] J. Smoller and A. Wasserman, *Generic properties of steady state solutions*, Journal of Differential Equations n^o52 (1984), pp 423-438.

[44] D. Tataru, *Uniform decay rates and attractors for evolution PDE's with boundary dissipation*, Journal of Differential Equations n^o121 (1995), pp. 1-27.

[45] E. Zuazua, *Exponential decay for the semilinear wave equation with locally distributed damping* Communications in Partial Differential Equations n^o15 (1990), pp. 205-235.

[46] E. Zuazua, *Uniform stabilization of the wave equation by nonlinear boundary feedback*, SIAM Journal on Control and Optimization n^o28 (1990), pp. 466-477.