Generic Morse-Smale property for the parabolic equation on the circle

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Résumé

Dans cet article, nous démontrons qu'il existe un ensemble générique de non-linéarités f pour lesquelles les équations de réaction-diffusion $u_t = u_{xx} + f(x, u, u_x)$, sur le cercle S^1, ont la propriété de Morse-Smale. Dans [13], Czaja et Rocha avaient montré que toute connexion entre deux orbites périodiques hyperboliques est transverse et qu'il n'existe pas d'orbite homocline à une orbite périodique hyperbolique. Dans [31], nous avons démontré qu'il existe un ensemble générique de non-linéarités f pour lesquelles tous les points d'équilibre et toutes les orbites périodiques sont hyperboliques. Dans ce travail, nous prouvons que toute connexion entre deux points d'équilibre hyperboliques d'indices de Morse distincts ou entre un point d'équilibre et une orbite périodique hyperboliques est transverse. Nous montrons également qu'il existe un ensemble générique de non-linéarités f pour lesquelles il n'existe pas de connexions entre points d'équilibre ayant même indice de Morse. Grâce à la propriété de Poincaré-Bendixson, nous déduisons des propriétés ci-dessus et de l'existence d'un attracteur global compact que, génériquement en la non-linéarité f, l'ensemble non-errant se réduit à un nombre fini de points d'équilibre et d'orbites périodiques hyperboliques. Dans nos démonstrations, les propriétés du nombre de zéros, les dichotomies exponentielles, le comportement asymptotique des solutions des équations linéarisées et évidemment le théorème de Sard-Smale jouent un rôle crucial.

Abstract

In this paper, we show that, for scalar reaction-diffusion equations $u_t = u_{xx} + f(x, u, u_x)$ on the circle S^1, the Morse-Smale property is generic with respect to the non-linearity f. In [13], Czaja and Rocha have proved that any connecting orbit, which connects two hyperbolic periodic orbits, is transverse and that there does not exist any homoclinic orbit, connecting a hyperbolic periodic orbit to itself. In [31], we have shown that, generically with respect to the non-linearity f, all the equilibria and periodic orbits are hyperbolic. Here we complete these results by showing that any connecting orbit between two hyperbolic equilibria with distinct Morse indices or between a hyperbolic equilibrium and a hyperbolic periodic orbit is automatically transverse. We also show that, generically with respect to f, there does not exist any connection between equilibria with the same Morse index. The above properties, together with the existence of a compact global attractor and the Poincaré-Bendixson property, allow us to deduce that, generically with respect to f, the non-wandering set consists in a finite number of hyperbolic equilibria and periodic orbits . The main tools in the proofs include the lap number property, exponential dichotomies and the Sard-Smale theorem. The proofs also require a careful analysis of the asymptotic behavior of solutions of the linearized equations along the connecting orbits.

Key words: Transversality, hyperbolicity, periodic orbits, Morse-Smale, Poincaré-Bendixson, exponential dichotomy, lap-number, genericity, Sard-Smale.

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1 Introduction

In the study of the dynamics of flows or semi-flows generated by systems of ordinary differential or partial differential equations arising in physics or biology, global stability (also called structural stability) is a very important property. Indeed, often, one only knows approximate values of the various coefficients in the equations; or else, in order to numerically determine the solutions of the equations, one introduces a space and (or) time discretized system. Therefore, one often studies a system, which is an approximation of the original dynamical system. If the dynamics of the original system are globally stable, then the qualitative global behaviour of the solutions remains unchanged under small perturbations of the system and the knowledge of the dynamics of this approximate system is sufficient in practice. Unfortunately, in general dynamical systems bifurcation phenomena can take place and thus drastic changes in the dynamics can arise. However, one may hope that such phenomena almost never happen in the considered class of dynamical systems or that the systems, which are robust, are dense or generic in the considered class (see Section 1.2 below for the definition of genericity).

Such structural stability problems first appeared in [2]. In the 1960's and 1970's, they have been extensively studied in the frame of vector fields (and also iterates of diffeomorphisms) on compact smooth manifolds $\mathcal{M}^n$ of finite dimension $n \geq 1$. In this context (see [58]), Smale introduced the notion of Morse-Smale dynamical systems, that is, systems for which the non-wandering set consists only in a finite number of hyperbolic equilibria and hyperbolic periodic orbits and the intersections of the stable and unstable manifolds of equilibria and periodic orbits are all transversal. Palis and Smale have shown that Morse-Smale vector fields on compact manifolds $\mathcal{M}^n$ are structurally stable ([42], [44]). Moreover, the class of Morse-Smale vector fields on a compact manifold $\mathcal{M}^2$ of dimension 2 is generic in the class of all C^1-vector fields ([48]). In the case of the sphere S^2, the proof is a consequence of the genericity of the Kupka-Smale property and of the Poincaré-Bendixson theorem. In the same way, one also shows that the Morse-Smale vector fields are generic in the class of "dissipative" vector fields on $\mathbb{R}^2$. In the case of a general two-dimensional manifold (especially in the non-orientable case), the proof is more delicate. The Morse-Smale property is also generic in the class of all gradient vector fields on a Riemannian manifold $\mathcal{M}^n$, $n \geq 1$. In the simple case of gradient systems (where the non-wandering set is reduced to equilibrium points), this genericity property is an immediate consequence of the genericity of the Kupka-Smale vector fields ([33, 59, 49]). On compact manifolds $\mathcal{M}^n$ of finite dimension $n \geq 3$, the Morse-Smale vector fields are still plentiful. There had been some hope that Morse-Smale systems (and hence stable dynamics) could be generic in the class of general vector fields. But unfortunately, in dimension higher than two, the Morse-Smale vector fields are no longer dense in the set of all vector fields. In particular, transverse homoclinic orbits connecting a hyperbolic periodic orbit to itself may exist, giving rise to chaotic behaviour ([60]). And this cannot be removed by small perturbations.

The above mentioned results give us a hint on what can be expected in the case of dynamical systems generated by partial differential equations (PDE's in short). The results available in infinite dimensions are still rather partial. Like in the case of vector fields on compact manifolds or on $\mathbb{R}^n$, one can define Morse-Smale dynamical systems. W. Oliva [40] (see also [20]) has proved that the Morse-Smale dynamical systems $S(t)$, generated by "dissipative" parabolic equations or more generally by "dissipative in finite time smoothing equations", are structurally stable, in the sense that the restrictions of the flows to the compact global attractors are topologically equivalent under small perturbations. Already, in 1982, he had proved the structural stability of Morse-Smale maps, which in turn implies the stability of Morse-Smale gradient semi-flows generated by PDE's. We recall that a dynamical system $S(t)$ on a Banach space, generated by an evolutionary PDE, is gradient if it admits a strict Lyapunov functional, which implies that the non-wandering set reduces to the set of equilibria.

As in the finite-dimensional case, one expected the density or genericity of the Morse-Smale gradient semi-flows within the set of gradient semi-flows generated by a given class of PDE's. Already, in 1985, D. Henry [23] proved the noteworthy property that the stable and unstable manifolds of two equilibria of the reaction-diffusion equation with separated boundary conditions on the interval $(0, 1)$ intersect transversally (see also [3] for another proof in the case of hyperbolic equilibria). One of the main tools in his proof was the decay property of the zero number (also called Sturm number or lap number; see Section

2.1), which will also be often used in this paper. Since, as shown by Zelenyak in [62], the reaction-diffusion equation on the interval $(0, 1)$ with separated boundary conditions is gradient, the transversality property of Henry implies the first known result of genericity of Morse-Smale systems in a class of PDE's. The scalar reaction-diffusion equation, defined on a bounded domain Ω of $\mathbb{R}^n$, $n \geq 2$, is no longer gradient in general. However, it is gradient if one considers non-linearities $f(x, u)$, depending only on x and the values of the function u (and not of the values of its derivatives). In this class of gradient parabolic equations, Brunovský and Poláčik [8] have shown in 1997 that the Morse-Smale property is generic with respect to the non-linearity $f(x, u)$. Later, the genericity of the gradient Morse-Smale flows in the class of gradient flows generated by the damped wave equations (with fixed damping) defined on any bounded domain has been proved by Brunovský and Raugel in [9] (for the case of variable damping, we refer to [29]).

It must be emphasized that, whereas the proof of the structural stability follows the lines (with some adjustments) of the proof given on compact finite-dimensional manifolds, the proof of the genericity of Morse-Smale property requires other approaches. Indeed, perturbing a semi-flow generated by a PDE in order to make it Morse-Smale has an interest only if one is able to perform it within the same class of equations. In the case of general vector fields on finite-dimensional manifolds, one can perturb the vector field in a local manner with all the freedom one needs. In the case of PDE's, the perturbed equations must remain in the considered class. Therefore, the perturbations are constrained. An analogous problem involving constrained perturbations has been studied in the finite-dimensional case by Robbin [53]. An additional problem arises in the case of PDE's, namely the perturbations could a priori be really non-local. Typically, perturbing the non-linearity changes the semi-flow in a large part of the phase space in a way which is hard to understand.

At first glance, the non-density of the Morse-Smale vector fields on compact manifolds of dimension $n \geq 3$ gave only little hope that infinite-dimensional Morse-Smale semi-flows are dense in some classes of non-gradient PDE's. However, Fiedler and Mallet-Paret ([14]) showed in 1989 that the scalar reaction-diffusion equation (1.1) on S^1 satisfies the Poincaré-Bendixson property (which, as we recalled earlier, played an important role in the proof of density of Morse-Smale vector fields in the set of all dissipative vector fields in $\mathbb{R}^2$). More recently, in 2008, Czaja and Rocha ([13]) proved that, for the scalar reaction-diffusion equation on S^1, the stable and unstable manifolds of hyperbolic periodic orbits always intersect transversally. These results gave us some hope that Morse-Smale dynamics could be generic for scalar reaction-diffusion equations on S^1 since they are generic for two-dimensional vector fields. In 2008, we proved that the equilibria and periodic orbits are hyperbolic, generically with respect to the non-linearity ([31]). The results of Fiedler, Rocha and Wolfrum ([15]) together with the generic hyperbolicity property of [31] imply that the Morse-Smale property is generic in the special class of reaction-diffusion equations with spatially homogeneous non-linearities $f(u, u_x)$.

Here, we complete the global qualitative picture of the scalar reaction-diffusion equations (1.1) on S^1 in the case of a general non-linearity $f(x, u, u_x)$ and conclude the proof of the genericity of the Morse-Smale systems in this class. These results indicate a similarity between scalar reaction-diffusion equations on S^1 and two-dimensional vector fields and take place in a more general correspondence between parabolic equations and finite-dimensional vector fields in any space dimension, as noticed in [32]. For scalar parabolic equations on bounded domains Ω in $\mathbb{R}^d$, $d \geq 2$, the properties of zero number do no longer hold, the Poincaré-Bendixson property fails and the Morse-Smale property is no longer generic. But, the genericity of the Kupka-Smale property still holds like in the case of vector fields in dimension $n \geq 3$, see [7].

1.1 The parabolic equation on the circle: earlier results

In this paper, we consider the following scalar reaction-diffusion equation on S^1,

$$\begin{cases} u_t(x,t) &= u_{xx}(x,t) + f(x, u(x,t), u_x(x,t)) \ , \quad (x,t) \in S^1 \times (0, +\infty) \ , \\ u(x,0) &= u_0(x) \ , \quad x \in S^1 \ , \end{cases} \tag{1.1}$$

where f belongs to the space $C^2(S^1 \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and u_0 is given in the Sobolev space $H^s(S^1)$, with $s \in (3/2, 2)$ (so that $H^s(S^1)$ is continuously embedded into $C^{1+\alpha}(S^1)$ for $\alpha = s - 3/2$).

Eq. (1.1) defines a local dynamical system $S_f(t)$ on $H^s(S^1)$ (see [22] and [50]) by setting $S_f(t)u_0 = u(t)$, where $u(t)$ is the solution of (1.1) (if the dependence in f of the dynamical system is not important, we simply denote $S(t)$ instead of $S_f(t)$).

In order to obtain a global dynamical system, we impose some additional conditions on f, namely we assume that there exist a function $k(.) \in \mathcal{C}^0(\mathbb{R}^+, \mathbb{R}^+)$ and constants $\varepsilon > 0$ and $\kappa > 0$ such that

$$\begin{aligned} &\forall R > 0, \ \forall \xi \in \mathbb{R} \ , \qquad \sup_{(x,u) \in S^1 \times [-R,R]} |f(x,u,\xi)| \leq k(R)(1 + |\xi|^{2-\varepsilon}) \ , \\ &\forall |u| \geq \kappa, \ \forall x \in S^1 \ , \qquad u f(x,u,0) \leq 0 \ . \end{aligned} \tag{1.2}$$

Then, Eq. (1.1) defines a global dynamical system $S_f(t)$ in $H^s(S^1)$ (see [50]). Moreover, $S_f(t)$ admits a compact global attractor $\mathcal{A}_f$, that is, there exists a compact set $\mathcal{A}_f$ in $H^s(S^1)$ which is invariant (i.e. $S_f(t)\mathcal{A}_f = \mathcal{A}_f$, for any $t \geq 0$) and attracts every bounded set of $H^s(S^1)$.

The most interesting part of the dynamics of (1.1) is contained in the attractor $\mathcal{A}_f$. Our purpose is to describe these dynamics, at least for a dense set of nonlinearities f. We introduce the set $\mathfrak{G} = \mathcal{C}^2(S^1 \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ endowed with the *Whitney topology*, that is, the topology generated by the neighborhoods

$$\{g \in \mathfrak{G} \ / \ |D^i f(x,u,v) - D^i g(x,u,v)| \leq \delta(u,v), \ \forall i \in \{0,1,2\}, \ \forall (x,u,v) \in S^1 \times \mathbb{R}^2\} \ , \tag{1.3}$$

where f is any function in $\mathfrak{G}$ and δ is any positive continuous function (see [17]). It is well known that $\mathfrak{G}$ is a Baire space, which means that any countable intersection of open and dense sets is dense in $\mathfrak{G}$ (see [17] for instance). We say that a set is *generic* if it contains a countable intersection of open and dense sets and we say that the parabolic equations on the circle (1.1) satisfy a property generically (with respect to the non-linearity) if this property holds for any f in a generic subset of $\mathfrak{G}$. The notion of genericity is a common notion for defining "large" subsets of Baire spaces, replacing the notion of "almost everywhere" of $\mathbb{R}^d$.

Before describing the properties of (1.1), we recall a few basic notions in dynamical systems, for the reader convenience. For any $u_0 \in H^1(S^1)$, the ω and *α-limit sets* of u_0 are defined respectively by

$$\begin{aligned}\omega(u_0) =& \{v \in H^1(S^1) \,|\, \text{ there exists a sequence } t_n \in \mathbb{R}^+, \text{ such that} \\ & \qquad t_n \underset{n\to+\infty}{\longrightarrow} +\infty \text{ and } S(t_n)u_0 \underset{n\to+\infty}{\longrightarrow} v\} \\ \alpha(u_0) =& \{v \in H^1(S^1) \,|\, \text{ there exist a negative orbit } u(t),\, t \le 0, \text{ with } u(0) = u_0, \\ & \text{and a sequence } t_n \in \mathbb{R}^+, \text{ such that } t_n \underset{n\to+\infty}{\longrightarrow} +\infty \text{ and } u(-t_n) \underset{n\to+\infty}{\longrightarrow} v\}\end{aligned}$$

The *non-wandering set* is the set of points $u_0 \in H^s(S^1)$ such that, for any neighborhood $\mathcal{N}$ of u_0 in $H^s(S^1)$, $S(t)\mathcal{N} \cap \mathcal{N} \neq \emptyset$ for arbitrary large times t. In particular, equilibria and periodic orbits belong to the non-wandering set.
A *critical element* means either an equilibrium point or a periodic solution of (1.1).
Let $e \in H^s(S^1)$ be an equilibrium point of (1.1). As usual, one introduces the linearized operator L_e around the equilibrium e (see Section 2, for the precise definition and the spectral properties of L_e).
We say that e is a *hyperbolic* equilibrium point if the intersection of the spectrum $\sigma(L_e)$ of L_e with the imaginary axis is empty. The *Morse index* $i(e)$ is the (finite) number of eigenvalues of L_e with positive real part (counted with their multiplicities).
If e is a hyperbolic equilibrium point of (1.1), there exists a neighborhood U_e of e such that the local stable and unstable sets

$$\begin{aligned}W^s_{loc}(e) \equiv W^s(e, U_e) =& \{u_0 \in H^s(S^1) \,|\, S_f(t)u_0 \in U_e\,,\, \forall t \ge 0\} \\ W^u_{loc}(e) \equiv W^u(e, U_e) =& \{u_0 \in H^s(S^1) \,|\, S_f(t)u_0 \text{ is well-defined for } t \le 0 \text{ and} \\ & \qquad S_f(t)u_0 \in U_e\,,\, \forall t \le 0\}\ .\end{aligned}$$

are (embedded) C^1-submanifolds of $H^s(S^1)$ of codimension $i(e)$ and dimension $i(e)$ respectively.
We also define the global stable and unstable sets

$$W^s(e) = \{u_0 \in H^s(S^1) \,|\, S_f(t)u_0 \underset{t\to+\infty}{\longrightarrow} e\}\ ,$$
$$W^u(e) = \{u_0 \in H^s(S^1) \,|\, S_f(t)u_0 \text{ is well-defined for } t \le 0 \text{ and } S_f(t)u_0 \underset{t\to-\infty}{\longrightarrow} e\}\ .$$

Since the parabolic equation (1.1), as well as the corresponding adjoint equation, satisfy the backward uniqueness property (see [6]), Theorem 6.1.9 of [22] implies that $W^s(e)$ and $W^u(e)$ are injectively immersed C^1-manifolds in $H^s(S^1)$ of codimension $i(e)$ and dimension $i(e)$ respectively (see also [10] and [18]). We remark that $W^u(e) = \cup_{t\geq 0} S_f(t) W^u(e, U_e)$ is the union of C^1 embedded submanifolds of $H^s(S^1)$ of dimension $i(e)$ (see [13]).

Let next $\Gamma = \{\gamma(x,t) | t \in [0,p]\}$ be a periodic orbit of (1.1) of minimal period p. The linearized equation around Γ defines an evolution operator $\Pi(t,0) : \varphi_0 \in H^s(S^1) \to \Pi(t,0)\varphi_0 = \varphi(t) \in H^s(S^1)$, where $\varphi(t)$ is the solution of the linearized equation. The operator $\Pi(p,0)$ is called the *period map* (see Section 2 for the precise definition of $\Pi(p,0)$ and its spectral properties) .
The periodic orbit Γ or the periodic solution $\gamma(t)$ is *hyperbolic* if the intersection of the spectrum $\sigma(\Pi(p,0))$ with the unit circle in $\mathbb{C}$ is reduced to 1 and 1 is a simple (isolated) eigenvalue. The *Morse index* $i(\Gamma)$ is the (finite) number of eigenvalues of $\Pi(p,0)$ of modulus strictly larger than 1 (counted with their multiplicities).
By [51, Theorem 14.2 and Remark 14.3]) or [21] (see also [18]), if $\gamma(t)$ is a hyperbolic periodic orbit, there exists a small neighborhood U_Γ of Γ in $H^s(S^1)$ such that

$$\begin{aligned} W^s_{loc}(\Gamma) &\equiv W^s(\Gamma, U_\Gamma) = \{u_0 \in H^s(S^1) \,|\, S_f(t)u_0 \in U_\Gamma \,,\, \forall t \geq 0\} \\ W^u_{loc}(\Gamma) &\equiv W^u(\Gamma, U_\Gamma) = \{u_0 \in H^s(S^1) \,|\, S_f(t)u_0 \in U_\Gamma \,,\, \forall t \leq 0\} \end{aligned} \tag{1.4}$$

are (embedded) C^1-submanifolds of $H^s(S^1)$ of codimension $i(\Gamma)$ and dimension $i(\Gamma)+1$ respectively. We also define the global stable and unstable sets

$$\begin{aligned} &W^s(\Gamma) = \{u_0 \in H^s(S^1) \,|\, S_f(t)u_0 \underset{t\to+\infty}{\longrightarrow} \Gamma\} \,, \\ &W^u(\Gamma) = \{u_0 \in H^s(S^1) \,|\, S_f(t)u_0 \text{ is well-defined for } t \leq 0 \text{ and } S_f(t)u_0 \underset{t\to-\infty}{\longrightarrow} \Gamma\} \,. \end{aligned}$$

Again, Theorem 6.1.9 of [22] implies that $W^s(\Gamma)$ and $W^u(\Gamma)$ are injectively immersed C^1-manifolds in $H^s(S^1)$ of codimension $i(\Gamma)$ and dimension $i(\Gamma)+1$ respectively (see also [10] and [18]). We also notice (see Lemma 6.1 of [13] and also [21] or [18]) that $W^u(\Gamma) = \cup_{t\geq 0} S_f(t) W^u_{loc}(\Gamma)$ is a union of embedded submanifolds of $H^s(S^1)$ of dimension $i(\Gamma)+1$.

Let $q^\pm$ be two hyperbolic critical elements. We say that $W^u(q^-)$ and $W^s_{loc}(q^+)$ *intersect transversally* (or are transverse) and we denote it by

$$W^u(q^-) \pitchfork W^s_{loc}(q^+) \,,$$

if, at each intersection point $u_0 \in W^u(q^-) \cap W^s_{loc}(q^+)$, $T_{u_0}W^u(q^-)$ contains a closed complement of $T_{u_0}W^s_{loc}(q^+)$ in $H^s(S^1)$. By convention, two manifolds which do not intersect are always transverse.

We now describe all the known properties of the dynamics of (1.1). First, we mention that in the particular case where $f(x,u,u_x) = f(x,u)$ does not depend on the values of the derivative u_x, the dynamical system $S_f(t)$ generated by (1.1) is gradient, that is, admits a strict Lyapunov functional. In particular, it has no periodic orbits and the non-wandering set is reduced to equilibria. As a direct consequence of [8], the Morse-Smale property is generic with respect to the non-linearity $f(x,u)$.

In the general case where $f(x,u,u_x)$ depends on the three variables, all the two-dimensional dynamics can be realized on locally (non-stable) invariant manifolds of the flow $S_f(t)$ of (1.1) (see [57]) and thus periodic orbits can exist ([5]). Hence, the dynamics can be more complicated. However, like for vector-fields in $\mathbb{R}^2$, the remarkable Poincaré-Bendixson property holds ([14]).

Theorem 1.1. Fiedler-Mallet-Paret (1989) (Poincaré-Bendixson property)
For any $u_0 \in H^s(S^1)$, the $\omega-$limit set $\omega(u_0)$ of u_0, satisfies exactly one of the following possibilities.
i) Either $\omega(u_0)$ consists of a single periodic orbit,
ii) or the $\alpha-$ and $\omega-$limit sets of any $v \in \omega(u_0)$ consist only of equilibrium points.

Theorem 1.1 is the first step towards showing that the non-wandering set generically reduces to a finite number of equilibria and periodic orbits. One of the main ingredients of the proof of Theorem 1.1 is the *Sturm property* (also called zero number or lap number property). More precisely, for any $\varphi \in C^1(S^1)$, we define the zero number $z(\varphi)$ as the (even) number of strict sign changes of φ. If $v(x,t)$ is the solution of a scalar linear parabolic equation on a time interval $I=(0,\tau)$, then, $z(v(\cdot,t))$ is finite, for any $t \in I$, and nonincreasing in t. Moreover, $z(v(\cdot,t))$ drops at $t=t_0$, if and only if $v(\cdot,t_0)$ has a multiple zero (the detailed statement of these properties is recalled in Section 2.1). Sturm property is very specific to the parabolic equation in space dimension one.

The second major step on the way to the proof of the genericity of the Morse-Smale property has been the paper of Czaja and Rocha [13]. Inspired by the transversality results of Fusco and Oliva ([16]) for special classes of vector fields on $\mathbb{R}^n$, Czaja and Rocha have proved the following fundamental and nice transversality properties.

Theorem 1.2. Czaja-Rocha (2008)
1) There does not exist any solution $u(t)$ of (1.1) *converging to a same hyperbolic periodic orbit Γ, as t goes to $\pm\infty$.*
2) Let $\Gamma^{\pm}$ be two hyperbolic periodic orbits. Then,

$$W^u(\Gamma^-) \pitchfork W^s_{loc}(\Gamma^+) \ .$$

Moreover, if the intersection $W^u(\Gamma^-) \cap W^s_{loc}(\Gamma^+)$ is not empty, then $i(\Gamma^-) > i(\Gamma^+)$.

Among other arguments, the proof of Theorem 1.2 ([13]) uses the decay properties of the zero number as well as the filtrations of the phase space with respect to the asymptotic behaviour of the solutions of the linearized equation around an orbit connecting two hyperbolic periodic orbits like in [10].

The above results hold under the assumption of hyperbolicity of the periodic orbits. To complete Theorem 1.2, we have proved in [31] that, generically with respect to the non-linearity f, the periodic orbits are all hyperbolic, which means that the above hyperbolicity assumption is not so restrictive.

Theorem 1.3. Joly-Raugel (2008) *There exists a generic subset $\mathcal{O}_h$ of $\mathfrak{G}$ such that, for any $f \in \mathcal{O}_h$, all the equilibria and the periodic solutions of* (1.1) *are hyperbolic.*

Besides the Sard-Smale theorem (recalled in Appendix A), one of the main ingredients of Theorem 1.3 is again the zero number property of the difference of two solutions of (1.1) or of the solutions of the linearized equations around equilibria or periodic solutions.

1.2 Main new results

In this paper, we prove that, generically with respect to f, the semi-flow $S_f(t)$ generated by Eq. (1.1) on S^1 is Morse-Smale. To this end, we first complete the automatic transversality results of Czaja and Rocha as follows.

Theorem 1.4. Automatic transversality results
1) If e_- and e_+ are two hyperbolic equilibrium points of (1.1) *with different Morse indices, then the unstable manifold $W^u(e_-)$ transversally intersects the local stable manifold $W^s_{loc}(e_+)$.*
2) If Γ (respectively e) is a hyperbolic periodic orbit (respectively a hyperbolic equilibrium point) of (1.1)*, then*

$$W^u(e) \pitchfork W^s_{loc}(\Gamma) \ ,$$

and

$$W^u(\Gamma) \pitchfork W^s_{loc}(e) \ .$$

The proof of Theorem 1.4 is similar to the one of Theorem 1.2. However, the proof of the automatic transversality of the connecting orbits between two hyperbolic equilibria of different Morse indices requires a tricky argument, in addition to those of [13]. We also emphasize that, even if most of the ideas in our proof are basically similar to the ones of [13], they are used in a different way. In fact, as in [13], the basic tools are the same as the ones of [23], [4], [16], namely a careful analysis of the asymptotic behavior of solutions converging to an equilibrium or a periodic orbit (Appendix C) combined with a systematic application of the Sturm properties (recalled in Theorem 2.1).

Theorems 1.2 and 1.4 show that a non-transverse connecting orbit arises only as an orbit connecting two equilibrium points with same Morse index. We call such an orbit a *homoindexed orbit*. Since every two-dimensional flow can be realized in a (locally) invariant manifold of the semi-flow of a parabolic equation on S^1 ([57]), we know that homoindexed orbits, in particular homoclinic orbits, may occur in the flow of (1.1). However, as we prove here, such connecting orbits can be broken generically in f.

Theorem 1.5. Generic non-existence of homoindexed connecting orbits
There exists a generic subset $\mathcal{O}_M \subset \mathcal{O}_h$ of $\mathfrak{G}$ such that, for any $f \in \mathcal{O}_M$, there does not exist any solution $u(t)$ of (1.1) such that $u(t)$ converges, when t goes to $\pm\infty$, to two equilibrium points with the same Morse index. In particular, homoclinic orbits are precluded.

To prove Theorem 1.5, we actually prove the genericity of the transversality of the homoindexed orbits, which at once implies the genericity of non-existence of such orbits. In order to prove the genericity of transversality and to get a meaningful result, we need to perturb (1.1) by arbitrary small perturbations, in such a way that the perturbed semi-flow is still generated by a scalar parabolic equation on S^1. As already mentioned, perturbing the non-linearity acts on the phase-space in a non-local way. In the context of proving generic transversality in the class of gradient scalar-reaction diffusion equations, these problems were first circumvented by Brunovský and Poláčik in [8]. They employed an equivalent formulation of transversality which appeared earlier in [53], [19], [56], but remained almost unnoticed for some time (such formulation has however been used, already in the 1980's, by Chow, Hale and Mallet-Paret [11] in global bifurcation problems of heteroclinic and homoclinic orbits). This equivalent formulation says that 0 is a regular value of a certain mapping Φ (depending on the perturbation parameter), defined on a space of functions of time with values into the state space of the equation. It is noteworthy that, in this formalism, the elements, the image of which are 0, are precisely the trivial and non-trivial connecting orbits. Using this equivalent formulation of the transversality together with the Sard-Smale theorem, given in Appendix A, Brunovský and Poláčik achieved their proof of genericity of the transversality of stable and unstable manifolds of equilibria with respect to the non-linearity.
In the proof of Theorem 1.5, we follow the lines of the one of [8] (see also [9] and [29]), by introducing this equivalent formulation of transversality. But, as in [9], the equivalent regular value formulation of transversality takes place in a space of sequences (obtained by a time discretization of the semi-flow associated with (1.1)). As there, in the application of the Sard-Smale theorem, one returns to the continuous time only in the last step, when one verifies the non-degeneracy condition of the Sard-Smale theorem, which gives rise to a functional condition (see Theorem 5.5). To find a perturbation satisfying this functional condition, we use as a central argument the one-to-one property of homoindexed connecting orbits stated in Proposition 3.6 (which is again a consequence of the Sturm property). Notice that parabolic strong unique continuation properties lead to a weaker version of Proposition 3.6, which is actually sufficient to show the genericity of transversality and holds in any space dimension (see [7]).

Using Theorems 1.1, 1.2, 1.3, 1.4 and 1.5, we finally prove the following genericity of Morse-Smale systems of type (1.1).

Theorem 1.6. Genericity of Morse-Smale property
For any f in the generic set $\mathcal{O}_M$, $S_f(t)$ is a Morse-Smale dynamical system, that is,

1) the non-wandering set consists only in a finite number of equilibria and periodic orbits, which are all hyperbolic.
2) the unstable manifolds of all equilibria and periodic orbits transversally intersect the local stable manifolds of all equilibria and periodic orbits.

Since the second part of Theorem 1.6 is a direct consequence of Theorems 1.2, 1.3, 1.4 and 1.5, it only remains to show that the non-wandering set is trivial. This is done by using the Poincaré-Bendixson Property (Theorem 1.1) together with arguments similar to the ones used for vector fields in $\mathbb{R}^2$.

Remarks:
• We prove here that the sets $\mathcal{O}_M$ and $\mathcal{O}_h$ are generic in $\mathfrak{G}$. The interpretation of these results is that $\mathcal{O}_M$ and $\mathcal{O}_h$ contain "almost all" the functions of $\mathfrak{G}$. The genericity is indeed the most common notion of "almost everywhere" in infinite-dimensional Baire spaces. However, it is not the only one: recently, the notion of prevalence is more and more used, see the review [41] and the references therein. We state our main results using the genericity because this is the most common notion. However, we underline that the sets $\mathcal{O}_M$ and $\mathcal{O}_h$ are not only generic but also prevalent in $\mathfrak{G}$ as shown in [30] (this remark is meaningful since a generic set may be neglectible for the notion of prevalence).
• All the above theorems are stated under the "dissipative" condition (1.2) on the non-linearity f. Actually, Theorems 1.1, 1.2, 1.3 and 1.4 still hold without assuming (1.2) and the proofs are not more involved. Theorem 1.5 is also still true, even if the proof is a little more technical (but not more difficult). However, Theorem 1.6 is not true in general without a dissipative condition on f or more generally without knowing that $S_f(t)$ admits a compact global attractor. Indeed, if $S_f(t)$ has no compact global attractor, already the number of equilibria (and periodic orbits) can be infinite. For the reader convenience and for avoiding unnecessary technicalities, we have chosen to impose the dissipative condition (1.2) in the whole paper.

The paper is organized as follows. In Section 2, we recall the fundamental properties of the zero number on S^1 as well as the useful spectral properties of the linearized equations around equilibrium points or periodic orbits. Section 3 is devoted to relations between the lap number and the Morse indices and to the one-to-one property of homoindexed orbits. In all these results, the zero number plays an important role. While some of these results are primordial in the proof of our main theorems, others are stated for sake of completeness in the description of the properties of Eq. (1.1) on the circle. Section 4 contains the proof of the automatic transversality results stated in Theorem 1.4. In Section 5, we prove Theorem 1.5, that is, the generic non-existence of orbits connecting two hyperbolic equilibria with same Morse index. Section 6 is focused on the study of the non-wandering set and on the proof of Theorem 1.6. The appendices contain the necessary background for reading this paper. In Appendix A, we recall the Sard-Smale theorem in the form used in Section 5.

Appendix B contains the basic definitions and properties of exponential dichotomies and their applications to the functional characterization of the transversality of the trajectories of the parabolic equation. This Appendix plays an important role in the core of the proof of Theorem 1.5. Finally, in the Appendix C, we describe the asymptotics of the solutions of the linearized equations around connecting orbits, which are one of the main ingredients of the proof of Theorem 1.4.

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2 Preliminaries and auxiliary results

In the introduction, we have already seen that (even without Assumption (1.2)) for any $u_0 \in H^s(S^1)$, $s \in (3/2, 2)$, Equation (1.1) admits a local mild solution $u(t) \in C^0([0, \tau_{u_0}), H^s(S^1))$ (see [22]). Moreover, this solution $u(t)$ is classical and belongs to $C^0((0, \tau_{u_0}), H^2(S^1)) \cap C^1((0, \tau_{u_0}), L^2(S^1)) \cap C^\theta((0, \tau_{u_0}), H^s(S^1))$, where $\theta = 1 - s/2$. In addition, the function $u_t(t) : t \in (0, \tau_{u_0}) \mapsto u_t(t) \in H^\ell(S^1)$, $0 \leq \ell < 2$, is locally Hölder-continuous (see [22, Theorem 3.5.2]). Since $u(t)$ is in $C^0((0, \tau_{u_0}), H^2(S^1))$, the term $f(x, u, u_x)$ belongs to $C^0((0, \tau_{u_0}), H^1(S^1))$ and thus $u_{xx} = u_t - f(x, u, u_x)$ is in $C^0((0, \tau_{u_0}), H^1(S^1))$. In particular, $u(t)$ belongs to $C^0((0, \tau_{u_0}), H^3(S^1))$, which is continuously embedded into $C^0((0, \tau_{u_0}), C^2(S^1))$. If moreover the condition (1.2) holds, then $\tau_{u_0} = \infty$ for every u_0.

In the course of this paper, we often need to consider the linearized equation along a bounded trajectory $u(t) \equiv S_f(t)u_0$, $t \in \mathbb{R}$, of Eq. (1.1), that is, the equation

$$v_t = v_{xx} + D_u f(x, u, u_x)v + D_{u_x} f(x, u, u_x)v_x \ , \ t \geq \sigma \ , \quad v(\sigma, x) = v_0 \ , \tag{2.1}$$

where v_0 belongs to $L^2(S^1)$. Since $u(t)$ is in $C^\theta(\mathbb{R}, H^s(S^1))$, the coefficients $D_u f(x, u(t), u_x(t))$ and $D_{u_x} f(x, u(t), u_x(t))$ are locally Hölder-continuous from $\mathbb{R}$ into $C^0(S^1)$. Thus, we deduce from [22, Theorem 7.1.3] that, for any $v_0 \in L^2(S^1)$, for any $\sigma \in \mathbb{R}$, there exists a unique (classical) solution $v(t) \in C^0([\sigma, +\infty), L^2(S^1)) \cap C^0((\sigma, +\infty), H^\ell(S^1))$ of (2.1) such that $v(\sigma) = v_0$, where ℓ is any real number with $0 \leq \ell < 2$. Setting $T_u(t, \sigma)v_0 = v(t)$, we define a family of continuous linear evolution operators on $L^2(S^1)$. We remark that the evolution operator $T_u(t, \sigma)$, $t \geq \sigma$, associated with the trajectory u, is injective and that its range is dense in $H^s(S^1)$.

We complete these generalities by remarking that, in what follows, we sometimes consider the difference $w = u_1 - u_2$ between two solutions $u_1 \in C^0([0, +\infty), H^s(S^1))$ and $u_2 \in C^0([0, +\infty), H^s(S^1))$ of Eq. (1.1). The difference w is a solution of the following linear equation,

$$w_t = w_{xx} + a(x, t)w_x + b(x, t)w \ , \ t \geq \sigma \ , \quad w(\sigma, x) = w_0 \ , \tag{2.2}$$

where

$$\begin{cases} a(x,t) = \int_0^1 f'_{\partial_x u}(x, \theta u_2 + (1-\theta)u_1, \partial_x(\theta u_2 + (1-\theta)u_1))d\theta \\ b(x,t) = \int_0^1 f'_u(x, \theta u_2 + (1-\theta)u_1, \partial_x(\theta u_2 + (1-\theta)u_1))d\theta \end{cases} \tag{2.3}$$

We emphasize that, since u_t is locally Hölder-continuous from $(0,+\infty)$ into $H^\ell(S^1)$, $3/2 < \ell < 2$ and u_x is continuous from $(0,+\infty)$ into $H^2(S^1)$, the coefficients $a(x,t)$ and $b(x,t)$ belong to $C^1(S^1 \times (0,+\infty), \mathbb{R})$.

2.1 The lap number property

The lap number property, or zero number property, is the fundamental property of one-dimensional scalar parabolic equations. We recall that, for any $\varphi \in C^1(S^1)$, the zero number $z(\varphi)$ is defined as the (even) number of strict sign changes of φ.

Theorem 2.1. *1) Let $T > 0$, $a \in \mathcal{C}^1(S^1 \times [0,T], \mathbb{R})$ and $b \in \mathcal{C}^0(S^1 \times [0,T], \mathbb{R})$. Let $v : S^1 \times (0,T) \to \mathbb{R}$ be a classical bounded non-trivial solution of*

$$v_t = v_{xx} + a(x,t)v_x + b(x,t)v \ ,$$

Then, the number $z(v(t))$ of zeros of $v(t)$ is finite and non-increasing in time $t \in [0,T]$ and strictly decreases at $t = t_0$ if and only if $x \mapsto v(x,t_0)$ has a multiple zero.

2) If u and v are two solutions in $C^0([0,T], H^s(S^1))$ of (1.1), then u_t and $u - v$ satisfy the lap number property stated in Statement 1) on the time interval $(0,T]$.

Such kind of results goes back to Sturm [61] in the case where a and b are time-independent. The non-increase of the number of zeros in the time-dependent problems has been obtained in [39] and [37]. The property of strict decay first appeared in [5] in the case of analytic coefficients. It has been generalized in [38] and [4].
We notice that the statement 2) is a direct consequence of the first statement and of the remarks made at the beginning of this section.

2.2 The spectrum of the linearized operators

We recall here the Sturm-Liouville properties of the linearized operators associated to Eq. (1.1). These results mainly come from [4] and [5], together with a property obtained in [31].

Let $e \in H^s(S^1)$ be an equilibrium point of (1.1). We introduce the linearized operator L_e on $L^2(S^1)$, with domain $H^2(S^1)$, defined by

$$L_e v = v_{xx} + D_u f(x,e,e_x)v + D_{u_x} f(x,e,e_x)v_x \ , \tag{2.4}$$

and consider the linearized equation around e, given by

$$\begin{aligned} & v_t(x,t) = L_e v(x,t) \ , \quad (x,t) \in S^1 \times (0,+\infty) \ , \\ & v(x,0) = v_0(x) \ , \quad x \in S^1 \ . \end{aligned} \tag{2.5}$$

The operator L_e is a sectorial operator and a Fredholm operator with compact resolvent. Therefore, its spectrum consists of a sequence of isolated eigenvalues of finite multiplicity. Let $(\lambda_i)_{i\in\mathbb{N}}$ be the spectrum of L_e, the eigenvalues being repeated according to their multiplicity and being ordered so that $\mathrm{Re}(\lambda_{i+1}) \leq \mathrm{Re}(\lambda_i)$.

Proposition 2.2. *The first eigenvalue λ_0 is real and simple and the corresponding eigenfunction $\varphi_0 \in H^2(S^1)$ does not vanish on S^1. The other eigenvalues go by pairs $(\lambda_{2j-1}, \lambda_{2j})$ and $\mathrm{Re}(\lambda_{2j+1}) < \mathrm{Re}(\lambda_{2j})$ for all $j \geq 0$. The pair $(\lambda_{2j-1}, \lambda_{2j})$ consists of either two simple complex conjugated eigenvalues, or two simple real eigenvalues with $\lambda_{2j} < \lambda_{2j-1}$, or a real eigenvalue with multiplicity equal to two. Finally, if φ is a real function belonging to the two-dimensional generalized eigenspace corresponding to $(\lambda_{2j-1}, \lambda_{2j})$, then φ has exactly $2j$ zeros which are all simple.*

Let $\Gamma = \{\gamma(x,t) \,|\, t \in [0,p]\}$ be a periodic orbit of (1.1) of minimal period p. We consider the linearized equation

$$\varphi_t = \varphi_{xx} + D_u f(x,\gamma,\gamma_x)\varphi + D_{u_x} f(x,\gamma,\gamma_x)\varphi_x \ , \quad t \geq \sigma \ , \quad \varphi(x,\sigma) = \varphi_0(x) \ . \tag{2.6}$$

Let $s \in (3/2, 2)$, we introduce the operator $\Pi(t,\sigma) : H^s(S^1) \longrightarrow H^s(S^1)$, defined by $\Pi(t,\sigma)\varphi_0 = \varphi(t)$ where $\varphi(t)$ is the solution of the linearized equation (2.6). The operator $\Pi(p,0)$ is called the *period map*. Due to the regularization properties of the parabolic equation, $\Pi(p,0)$ is compact. Its spectrum consists of zero and a sequence of eigenvalues $(\mu_i)_{i\in\mathbb{N}}$ converging to zero, where we repeat the eigenvalues according to their multiplicity and order them such that $|\mu_{i+1}| \leq |\mu_i|$. Notice that 0 is not an eigenvalue of $\Pi(p,0)$ due to the backward uniqueness property of the parabolic equation. Moreover, we also remark that, since (1.1) is an autonomous equation, 1 is always an eigenvalue of $\Pi(p,0)$ with eigenfunction $\gamma_t(0)$.

Proposition 2.3. *The first eigenvalue μ_0 is real and simple and the corresponding eigenfunction $\varphi_0 \in H^s(S^1)$ does not vanish on S^1. The other eigenvalues go by pairs (μ_{2j-1}, μ_{2j}) and $|\mu_{2j+1}| < |\mu_{2j}|$ for all $j \geq 0$. The pair (μ_{2j-1}, μ_{2j}) consists of either two simple complex conjugated eigenvalues, or two simple real eigenvalues of the same sign, or a real eigenvalue with multiplicity equal to two. In particular, -1 is never an eigenvalue. Finally, if φ is a real function belonging to the two-dimensional generalized eigenspace corresponding to (μ_{2j-1}, μ_{2j}), then φ has exactly $2j$ zeros which are all simple.*

3 Main consequences of the lap number properties on connecting orbits

In this section, we describe several important consequences of the lap number theorem on the properties of orbits connecting hyperbolic critical elements. Most of them will be used in the core of the proofs of Theorems 1.4 and 1.5.

3.1 Relations between Morse indices and lap numbers

In this paragraph, we consider orbits connecting hyperbolic equilibrium points and hyperbolic periodic orbits and give several inequalities involving Morse indices and numbers of zeros. The ideas of these properties were already contained in [13], where they have been proved in the case of connections between two periodic orbits. We complete the results of [13] by considering the cases where equilibrium points are also involved. Actually, some of the proofs are slightly simpler in these cases since, if e is an equilibrium point and $\gamma(t)$ a periodic orbit, $e-\gamma(t)$ is periodic, whereas the difference between two periodic orbits may be only quasiperiodic.

The properties stated in this section have their own interest. In the other parts of this paper, we will use them in the case of a connection between an equilibrium point and a periodic orbit. Therefore, we mainly restrict the proofs to this case. The omitted proofs are similar.

The following theorem corresponds to [13, Theorems 5.2 and 6.2]. We recall that $z(v)$ denotes the number of strict sign changes of the function $x \mapsto v(x)$.

Theorem 3.1. *Let $\gamma(t)$ be a hyperbolic periodic orbit of (1.1) of minimal period p and let $\Gamma = \{\gamma(t),\ t \in [0,p)\}$.*
1) For any $u_0 \in W^s_{loc}(\Gamma)\setminus\Gamma$, there exist $a \in [0,p)$ and $\kappa > 0$ such that $\lim_{t\to\infty} e^{\kappa t}\|S_f(t)u_0 - S_f(t)\gamma(a)\|_{H^s} = 0$. Moreover,

$$z(u_0-\gamma(a)) \geq \begin{cases} i(\Gamma)+1 = 2q & \textit{if } i(\Gamma) = 2q-1, \\ i(\Gamma)+2 = 2q+2 & \textit{if } i(\Gamma) = 2q. \end{cases} \tag{3.1}$$

2) For any $u_0 \in W^u(\Gamma)\setminus\Gamma$, there exist $\tilde{a} \in [0,p)$ and $\tilde{\kappa} > 0$ such that $\lim_{t\to\infty} e^{\tilde{\kappa} t}\|S_f(-t)u_0 - S_f(-t)\gamma(\tilde{a})\|_{H^s} = 0$. Moreover,

$$z(u_0-\gamma(a)) \leq \begin{cases} i(\Gamma)-1 = 2q-2 & \textit{if } i(\Gamma) = 2q-1, \\ i(\Gamma) = 2q & \textit{if } i(\Gamma) = 2q. \end{cases} \tag{3.2}$$

Proof: For sake of completeness, we give a short proof of statement 1) which is slightly different from the one of [13]. The proof of statement 2) is similar.

The first part of the statement 1) is just a reminder of the fact that the local stable manifold of Γ is the union of the local strongly stable manifolds of all the points $\gamma(b)$, $b \in [0,p]$ as explained in Appendix C.3. The second part of 1) directly follows from Corollary C.10. Indeed, let $v(t) = S_f(t)u_0 - \gamma(t+a) \equiv u(t) - \gamma(t+a)$. Then, there exists a complex eigenvalue μ_i of the period map $\Pi(p+a,a)$ with $|\mu_i| < 1$ such that $v(np)$ satisfies one of the asymptotic behaviors (i)-(iv) described in Corollary C.10. If the index $i(\Gamma)$ is equal to $2q-1$, then $\mu_{2q-1} = 1$ and thus $i \geq 2q$, which implies that the number of zeros $z(v(np))$ is at least equal to $2q$ for n large enough. If the index $i(\Gamma)$ is equal to $2q$, then

$\mu_{2q} = 1$ and thus $i \geq 2q+1$, which implies that the number of zeros $z(v(n))$ is at least equal to $2q+2$ for n large enough. Since $v(t)$ is the difference of two solutions of (1.1), Theorem 2.1 shows that these lower bounds on $z(v(np))$ for large $n \in \mathbb{N}$ hold in fact for all $t \in \mathbb{R}$. $\square$

Of course, the corresponding properties are true for hyperbolic equilibrium points. Since their proof is similar to the one of Theorem 3.1 (and even simpler), it is omitted.

Theorem 3.2. *Let $e(x)$ be a hyperbolic equilibrium point of* (1.1).
1) For any $u_0 \in W^s_{loc}(e) \setminus \{e\}$,

$$z(u_0 - e) \geq \begin{cases} i(e)+1 = 2q & \textit{if } i(e) = 2q-1, \\ i(e) = 2q & \textit{if } i(e) = 2q. \end{cases}$$

2) For any $u_0 \in W^u(e) \setminus \{e\}$,

$$z(u_0 - e) \leq \begin{cases} i(e)-1 = 2q-2 & \textit{if } i(e) = 2q-1, \\ i(e) = 2q & \textit{if } i(e) = 2q. \end{cases}$$

The following two lemmas, which are rather simple, are useful in the following sections.

Lemma 3.3. *If e is an equilibrium point of* (1.1) *and $\gamma(t)$ is a periodic solution of* (1.1) *of minimal period $p > 0$, then the zero number $z(e - \gamma(t))$ is constant and thus, for any time t, the function $x \mapsto e(x) - \gamma(x,t)$ has no multiple zero. The same properties hold if one considers the difference between two distinct equilibrium points or two distinct periodic solutions.*

Proof: By Theorem 2.1, the number of zeros of $v(t) = e - \gamma(t)$ is non-increasing and strictly decreases only at the times t where $v(t)$ has a multiple zero. If $v(t)$ has a multiple zero at $t = t_0$, then Theorem 2.1 and the periodicity of v imply that, for any $\varepsilon \in (0,p)$,

$$z(v(t_0 - \varepsilon)) > z(v(t_0 + \varepsilon)) \geq z(v(t_0 + p - \varepsilon)) = z(v(t_0 - \varepsilon)) \ ,$$

which leads to a contradiction. Thus $v(t)$ has no multiple zero. $\square$

Remark: Since, for each t, $v(t)$ has no multiple zero and $H^s(S^1)$ is embedded in $\mathcal{C}^1(S^1)$, there exists a neighborhood $B_{H^s}(v(t), 2\varepsilon_{v(t)})$ on which the zero number is constant. Since the curve $\{e - \gamma(t) \,|\, t \in [0,p]\}$ is compact, there exists a finite covering $\cup_{i=1}^n B_{H^s(S^1)}(e - \gamma(t_i), \varepsilon_i)$ of the set $\{e - \gamma(t) \,|\, t \in \mathbb{R}\}$ in $H^s(S^1)$, on which the zero number in constant.

Lemma 3.4. *Let e^- be a hyperbolic equilibrium of* (1.1) *and let Γ^+ be a hyperbolic periodic orbit of minimal period $p > 0$. Let $u_0 \in W^u(e^-) \cap W^s_{loc}(\Gamma^+)$ and let $u(t)$, $t \in \mathbb{R}$, be the solution of* (1.1) *with $u(0) = u_0$. Let $a \in [0, p)$ be such that $\lim_{t\to+\infty} \|u(t) - \gamma^+(a+t)\|_{H^s} = 0$. Then, for any time $t \in \mathbb{R}$,*

$$z(u(t) - \gamma^+(a + t)) \leq z(\gamma^+(a) - e^-) \leq z(u(t) - e^-) \ .$$

The same property holds for any orbit connecting hyperbolic equilibrium points or hyperbolic periodic orbits.

Proof: We set $v_+(t) = u(t) - \gamma^+(a + t)$ and $v_-(t) = u(t) - e^-$. We notice that the lap number property stated in Theorem 2.1 holds for $v_\pm$. Lemma 3.3 shows that $z(\gamma^+(a) - e^-) = z(\gamma^+(a+t) - e^-)$. Moreover, $\gamma^+(a) - e^-$ has no multiple zeros due to Lemma 3.3 and thus its number of zeros is stable with respect to small enough perturbations in $H^s(S^1)$. For any large t_0, $v_+(t_0)$ is small enough so that

$$z(\gamma^+(a) - e^-) = z(\gamma^+(a + t_0) - e^-) = z(v_+(t_0) + \gamma^+(a + t_0) - e^-) = z(v_-(t_0)) \ .$$

Applying Theorem 2.1, we get that for all $t \leq t_0$, $z(\gamma^+(a) - e^-) \leq z(v_-(t))$.
The inequality $z(v_+(t)) \leq z(\gamma^+(a) - e^-)$, for all $t \in \mathbb{R}$, is proved in a similar way.
The proof is the same in the case of orbits connecting hyperbolic equilibria or hyperbolic periodic orbits. These inequalities have been previously proved in the case of orbits connecting two hyperbolic periodic orbits in [13, Theorem 7.3] □

As a direct consequence of Lemma 3.4, Theorem 3.1 and Theorem 3.2, we obtain the following result.

Corollary 3.5. *Let $e^\pm$ and $\Gamma^\pm$ be hyperbolic equilibria and periodic orbits of Eq.* (1.1).
1) If $W^u(e^-) \cap W^s_{loc}(\Gamma^+) \neq \emptyset$, then

$$i(\Gamma^+) + 1 \leq i(e^-) \ .$$

Moreover, if $i(\Gamma^+) = 2q^+$, $q^+ > 0$, then $i(\Gamma^+) + 2 \leq i(e^-)$.
2) If $W^u(\Gamma^-) \cap W^s_{loc}(e^+) \neq \emptyset$, then

$$i(e^+) \leq i(\Gamma^-) \ .$$

Moreover, if $i(\Gamma^-) = 2q - 1$, $q \geq 1$, then $i(e^+) + 1 \leq i(\Gamma^-)$.
3) If $W^u(e^-) \cap W^s_{loc}(e^+) \neq \emptyset$, then

$$i(e^+) \leq i(e^-)$$

and the equality is possible if and only if $i(e^+) = i(e^-)$ is even.

3.2 One-to-one property of homoindexed orbits

The proposition proved in this section plays a central role in the construction of a suitable perturbation to break homoindexed orbits (see Section 5). Previously, the same one-to-one property has been shown to hold for periodic orbits by Fiedler and Mallet-Paret in [14] (see also [31]). Notice that this one-to-one property can dramatically fail for general trajectories. As already indicated, the proof of this property relies on the decay of the lap-number stated in Theorem 2.1. In space dimension higher than one, there is no equivalent of the lap-number property. However, one can use unique continuation properties to obtain a weaker but useful equivalent of Proposition 3.6, see [7].

Proposition 3.6. *Let $e_\pm$ be two hyperbolic equilibria such that $i(e_-) = i(e_+) = m = 2m'$ is even, and let $u(t)$ be a connecting orbit between e_- and e_+. In the case where $e_- = e_+ = e$, we assume that $u(t) \neq e$. Then the following properties hold:*
1) For any $t \in \mathbb{R}$, for any $x \in S^1$, $(u(x,t), \partial_x u(x,t)) \neq (e_\pm(x), \partial_x e_\pm(x))$.
2) The map $(x,t) \in S^1 \times \mathbb{R} \mapsto (x, u(x,t), \partial_x u(x,t))$ is one to one.

Proof: As in Section 2.2, we denote by $L_{e_\pm}$ the linearized operator around the equilibrium $e_\pm$ and by $\lambda_i^\pm$, $i \geq 0$, its eigenvalues, counted with their multiplicities.

Since $u(t)$ converges to $e_\pm$ when t goes to $\pm\infty$, according to Corollary C.7, there exists an eigenvalue $\lambda_{i^\pm}^\pm$ of $L_{e_\pm}$ such that the asymptotic behaviors of u are given by

$$u(t) = e_\pm + e^{\mathrm{Re}(\lambda_{i^\pm}^\pm)t}\psi_{i^\pm}^\pm(t) + o(e^{\mathrm{Re}(\lambda_{i^\pm}^\pm)t}) \quad \text{when} \quad t \to \pm\infty \tag{3.3}$$

where $\psi_{i^\pm}^\pm$ corresponds to one of the possible asymptotic behaviors (i)-(iv) described at the beginning of Section 4.1. We emphasize that the term $o(e^{\mathrm{Re}(\lambda_{i^\pm}^\pm)t})$ has to be understood in the sense of the $H^s(S^1)$ topology (and thus this term is also small in the C^1-sense). We recall that if $i^\pm = 2q-1$ or $2q$ then $\psi_{i^\pm}^\pm$ has exactly $2q$ zeros which are simple. Finally, notice that since $i(e_-) = i(e_+) = m$ is even, $\lambda_m^\pm < 0 < \lambda_{m-1}^\pm$ are simple real eigenvalues and that $i^- \leq m-1$ and $i^+ \geq m$.

We are now ready to prove the first assertion. We introduce the functions $v_\pm(t) = u(t) - e_\pm$. By the second part of Theorem 2.1, the number of zeros $z(v_\pm)$ is finite and non-increasing and it strictly decreases at some time $\tau_\pm$ if and only if the function $x \mapsto v_\pm(x, \tau_\pm)$ has a multiple zero. But Theorem 3.2 and Lemma 3.4 at once imply that $z(e_+ - e_-) = m = z(v_+(t)) = z(v_-(t))$ for any t and thus that the map $x \mapsto u(x,t) - e_\pm(x)$ has no multiple zero. We deduce that

$$\lambda_{i^-}^- = \lambda_{m-1}^- \ , \quad \psi_{i^-}^- = \varphi_{m-1}^- \ , \quad \lambda_{i^+}^+ = \lambda_m^+ \quad \text{and} \quad \psi_{i^+}^+ = \varphi_m^+ \ , \tag{3.4}$$

where φ_{m-1}^- and φ_m^+ are eigenfunctions corresponding to the simple real eigenvalues λ_{m-1}^- and λ_m^+ respectively.

We next prove by contradiction that the second statement holds. Assume that the map $(x,t) \in S^1 \times \mathbb{R} \mapsto (x, u(x,t), \partial_x u(x,t))$ is not injective. Then there exist x_0, $t_0 \in \mathbb{R}$ and $\tau_0 \in \mathbb{R}$, such that

$$u(x_0, t_0) = u(x_0, t_0 + \tau_0) \ , \quad \partial_x u(x_0, t_0) = \partial_x u(x_0, t_0 + \tau_0) \ .$$

The function $v(x,t) = u(x, t+\tau_0) - u(x,t)$ satisfies $v(x_0, t_0) = 0$ and $\partial_x v(x_0, t_0) = 0$. It is not identically zero since it is a non-trivial connecting orbit. Thus, due to the second part of Theorem 2.1, the zero number $z(v(t))$ is non-increasing and drops strictly at $t = t_0$. The properties (3.3) and (3.4) imply that

$$\begin{aligned} v(t) &= e^{\lambda^-_{m-1} t}(e^{\lambda^-_{m-1}\tau_0} - 1)\varphi^-_{m-1} + o(e^{\lambda^-_{m-1} t}) \ , \quad \text{as } t \to -\infty \\ v(t) &= e^{\lambda^+_m t}(e^{\lambda^+_m \tau_0} - 1)\varphi^+_m + o(e^{\lambda^+_m t}) \ , \quad \text{as } t \to \infty \ . \end{aligned}$$

Thus, $z(v(t)) = m$, for any $t \in \mathbb{R}$, which contradicts the fact that $z(v(t))$ drops at $t = t_0$ and the second statement holds. □

4 Automatic transversality results

This section is devoted to the proof of Theorem 1.4. In Section 4.1, we prove the first statement of Theorem 1.4, whereas in Section 4.2 , we prove the second statement.

4.1 Automatic transversality of heteroindexed orbits connecting two equilibria

Let e_- and e_+ be two hyperbolic equilibrium points of (1.1) with different Morse indices $i(e_-)$ and $i(e_+)$. Following the notations of Section 2.2, we denote $L_{e_\pm}$ the corresponding linearized operators and by $(\lambda_i^\pm, \varphi_i^\pm)$ their set of eigenvalues and generalized eigenfunctions.

Assume that $W^u(e_-) \cap W^s_{loc}(e_+) \neq \emptyset$ (otherwise the intersection is transversal by definition). Notice that Corollary 3.5 and the fact that $i(e_-) \neq i(e_+)$ imply that $i(e^+) < i(e^-)$. Let $u(t)$ be a global solution of (1.1) with $u(0) \in W^u(e_-) \cap W^s_{loc}(e_+)$. Since $W^u(e_-)$ is a finite-dimensional manifold, there exists a finite basis $(v_1^0, \ldots, v_p^0)$ of $T_{u(0)}W^u(e_-) \cap T_{u(0)}W^s_{loc}(e_+)$. Let $v_k(t)$ be the (global) solutions of

$$\partial_t v_k = \partial^2_{xx} v_k + D_u f(x, u, u_x) v_k + D_{u_x} f(x, u, u_x) \partial_x v_k \ , \quad v_k(0) = v_k^0 \ . \tag{4.1}$$

Notice that the solution $v_k(t)$ exists for any $t \in \mathbb{R}$. Corollary C.8 gives all the possible precise asymptotic behaviors of $v_k(t)$ when t goes to $\pm\infty$. For each k, there exist an eigenvalue $\lambda^-_{i_k^-}$ of L_{e_-} with positive real part such that, when $t \to -\infty$, the asymptotic behavior of $v_k(t)$ in $H^s(S^1)$ is described by the following possibilities:

(i) if $\lambda^-_{i^-_k}$ is a simple real eigenvalue with eigenfunction $\varphi^-_{i^-_k}$, then there exists $a_k \in \mathbb{R} - \{0\}$ such that $v_k(t) = a_k e^{\lambda^-_{i^-_k} t} \varphi^-_{i^-_k} + o(e^{\lambda^-_{i^-_k} t}) \equiv \psi^-_{i^-_k}(t) + o(e^{\lambda^-_{i^-_k} t})$.

(ii) If $\lambda^-_{i^-_k} = \lambda^-_{i^-_k+1}$ is a double real eigenvalue with two independent eigenfunctions $\varphi^-_{i^-_k}$ and $\varphi^-_{i^-_k+1}$, then there exist $(a_k, b_k) \in \mathbb{R}^2 - \{(0,0)\}$ such that $v_k(t) = a_k e^{\lambda^-_{i^-_k} t} \varphi^-_{i^-_k} + b_k e^{\lambda^-_{i^-_k} t} \varphi^-_{i^-_k+1} + o(e^{\lambda^-_{i^-_k} t}) \equiv \psi^-_{i^-_k}(t) + o(e^{\lambda^-_{i^-_k} t})$.

(iii) If $\lambda^-_{i^-_k} = \lambda^-_{i^-_k+1}$ is an algebraically double real eigenvalue with eigenfunction $\varphi^-_{i^-_k}$ and with generalized eigenfunction $\varphi^-_{i^-_k+1}$, then there exist $(a_k, b_k) \in \mathbb{R}^2 - \{(0,0)\}$ such that $v_k(t) = (a_k + b_k t) e^{\lambda^-_{i^-_k} t} \varphi^-_{i^-_k} + b_k e^{\lambda^-_{i^-_k} t} \varphi^-_{i^-_k+1} + o(e^{\lambda^-_{i^-_k} t}) \equiv \psi^-_{i^-_k}(t) + o(e^{\lambda^-_{i^-_k} t})$.

(iv) If $\lambda^-_{i^-_k} = \overline{\lambda^-_{i^-_k+1}}$ is a (simple) nonreal eigenvalue with eigenfunction $\varphi^-_{i^-_k} = \overline{\varphi^-_{i^-_k+1}}$, then there exist $(a_k, b_k) \in \mathbb{R}^2 - \{(0,0)\}$ such that

$$\begin{aligned} v_k(t) = e^{\mathrm{Re}(\lambda^-_{i^-_k})t} \Big(& (a_k \cos(\mathrm{Im}(\lambda^-_{i^-_k})t) - b_k \sin(\mathrm{Im}(\lambda^-_{i^-_k})t)) \mathrm{Re}(\varphi^-_{i^-_k}) \\ & - (a_k \sin(\mathrm{Im}(\lambda^-_{i^-_k})t) + b_k \cos(\mathrm{Im}(\lambda^-_{i^-_k})t)) \mathrm{Im}(\varphi^-_{i^-_k}) \Big) + o(e^{\mathrm{Re}(\lambda^-_{i^-_k})t}) \\ \equiv \psi^-_{i^-_k}(t) + o(e^{\mathrm{Re}(\lambda^-_{i^-_k})t}) \ . \end{aligned}$$

The vectors $v_k(t)$ have the same type of behaviors when $t \to +\infty$, provided $\lambda^-_{i^-_k}$ is replaced by an eigenvalue $\lambda^+_{i^+_k}$ of L_{e_+} with negative real part.

Lemma 4.1. *Without loss of generality, we may assume that the behaviors of the functions $v_k(t)$ when t goes to $-\infty$ are different in the sense that the corresponding family $(\psi^-_{i^-_k})$, $1 \le k \le p$, is free and hence generates a finite-dimensional vector space of dimension p.*

Proof: This lemma is a simple consequence of a Gram-Schmidt process. Without loss of generality, we can assume that $\lambda^-_{i^-_1}$ has the smallest real part among the family $(\lambda^-_{i^-_1}, \dots, \lambda^-_{i^-_p})$. If there exists $k > 1$ such that $\mathrm{Re}(\lambda^-_{i^-_k}) = \mathrm{Re}(\lambda^-_{i^-_1})$ with asymptotic behavior of type (i) or with asymptotic behavior of type (ii)-(iv) such that the pairs (a,b) for $v_1(t)$ and $v_k(t)$ are linearly dependent, then we can replace v_k by $v_k - \alpha v_1$ such that the real part of $\lambda^-_{i^-_k}$ increases. Notice that $(v^0_1, \dots, v^0_p)$ is still a basis of $T_{u(0)}W^u(e_-) \cap T_{u(0)}W^s(e_+)$. Assume now that $\lambda^-_{i^-_2}$ has the smallest real part among the real parts of the family $(\lambda^-_{i^-_2}, \dots, \lambda^-_{i^-_p})$ (which can be the same as the real part of $\lambda^-_{i^-_1}$, but not smaller). If there exists $k > 2$ such that $\mathrm{Re}(\lambda^-_{i^-_k}) = \mathrm{Re}(\lambda^-_{i^-_2})$ and $v_k(t)$ has an asymptotic behavior linearly dependent of the one of $v_2(t)$, then we can replace v_k by $v_k - \alpha v_2$ so that the real part of $\lambda^-_{i^-_k}$ increases. We pursue the process until the end to obtain Lemma 4.1. $\square$

Proof of the first statement of Theorem 1.4:
In what follows, we assume that the basis $(v_1^0, \ldots, v_p^0)$ of $T_{u(0)}W^u(e_-) \cap T_{u(0)}W^s(e_+)$ has been chosen as in Lemma 4.1. By definition $W^u(e_-)$ intersects $W^s(e_+)$ transversally if and only if $T_{u(0)}W^u(e_-) + T_{u(0)}W^s(e_+) = H^s(S^1)$. We know that $\dim(T_{u(0)}W^u(e_-)) = i(e_-)$ and $\mathrm{codim}(T_{u(0)}W^s(e_+)) = i(e_+)$. Thus, $T_{u(0)}W^u(e_-) + T_{u(0)}W^s(e_+) = H^s(S^1)$ if and only if

$$\dim\left(T_{u(0)}W^u(e_-) \cap T_{u(0)}W^s(e_+)\right) \leq i(e_-) - i(e_+)\ . \tag{4.2}$$

The proof of the first statement of Theorem 1.4 consists in the careful study of three different cases.

If $i(e_+) = 0$, then $T_{u(0)}W^s(e_+) = H^s(S^1)$ and the transversality trivially holds.

Assume that $i(e_+) = 2q - 1$ is odd, which implies that the eigenvalues of L_{e_+} with negative real part are $(\lambda_i^+)_{i\geq 2q-1}$. By Proposition 2.2, the corresponding generalized eigenfunctions $(\varphi_i^+)_{i\geq 2q-1}$ have all at least $2q$ zeros. Due to Corollary C.8 in Appendix C, each function $v_k(t)$ has at least $2q$ zeros for large t. Since by Theorem 2.1 the number of zeros of $v_k(t)$ is non-increasing, $v_k(t)$ has at least $2q$ zeros for every time $t \in \mathbb{R}$. Applying Corollary C.8 again, we obtain that necessarily $i_k^- \geq 2q - 1$ and that $i(e_-) \geq 2q$. Since Lemma 4.1 states that the asymptotic behaviors of all $v_k(t)$ are different, there are at most $i(e_-) - (2q - 1) = i(e_-) - i(e_+)$ possible asymptotic behaviors.

Assume now that $i(e_+) = 2q \neq 0$ is even. By Proposition 2.2, this means that the pair of eigenvalues $(\lambda^+_{2q-1}, \lambda^+_{2q})$ is a pair of simple real eigenvalues satisfying $\lambda^+_{2q} < 0 < \lambda^+_{2q-1}$. All eigenfunctions corresponding to the eigenvalues of L_{e_+} with negative real part have at least $2q{+}2$ zeros, except φ^+_{2q} which has $2q$ zeros. Arguing as above, we obtain that $i_k^- \geq 2q{-}1$ and that $i(e_-) \geq 2q$, that is, that there are at most $i(e_-) - (2q - 1) = i(e_-) - i(e_+) + 1$ possible asymptotic behaviors for the functions $v_k(t)$ when $t \to -\infty$. Here is the point where the fact that $i(e_-) > i(e_+)$ is crucial. Indeed, this assumption implies that λ^-_{2q} and λ^-_{2q-1} have both positive real parts. Assume that $p = i(e_-) - i(e_+) + 1$. Then, according to Lemma 4.1, all the possible asymptotic behaviors corresponding to the eigenvalues $\lambda^-_{2q-1}, \ldots, \lambda^-_{i(e_-)-1}$ are taken by the functions $v_k(t)$. Without loss of generality, we may assume that $v_1(t)$ and $v_2(t)$ have an asymptotic behavior corresponding to $\psi^-_{2q}(t)$ and $\psi^-_{2q-1}(t)$. Since, due to Theorem 2.1, the number of zeros of $v_k(t)$ is non increasing in time, the asymptotic behaviors of $v_1(t)$ and $v_2(t)$, when $t \to +\infty$, correspond necessarily to λ^+_{2q}. Since λ^+_{2q} is a simple eigenvalue, we can find $\alpha \in \mathbb{R} \setminus \{0\}$ such that $v_1(t) + \alpha v_2(t) = o(e^{\lambda^+_{2q}t})$ when $t \to +\infty$. Therefore, by Corollary C.8, $v_1(t){+}\alpha v_2(t)$ has at least $2q{+}2$ zeros when t tends to infinity. However, when $t \to -\infty$, the asymptotic behavior of $v_1(t) + \alpha v_2(t)$ is determined by $\psi^-_{2q} + \alpha\psi^-_{2q-1}$, which does not identically vanish by Lemma 4.1. By Proposition 2.2,

$\psi_{2q}^- + \alpha\psi_{2q-1}^-$ has exactly $2q$ zeros. Thus, $v_1(t) + \alpha v_2(t)$ has exactly $2q$ zeros for t close to $-\infty$ and we get a contradiction with the lap number property stated in Theorem 2.1. As a consequence, one of the asymptotic behaviors corresponding to ψ_{2q}^- or ψ_{2q-1}^- is not realized by the family $(v_k(t))_{1\leq k\leq p}$. Thus, $p \leq i(e_-) - i(e_+)$ and (4.2) holds.

This concludes the proof of Assertion 1) of Theorem 1.4. □

4.2 Connections involving periodic orbits

This section is devoted to the proof of the second statement of Theorem 1.4. The proof follows the same lines as the ones of the proof of the first part of Theorem 1.4. As we shall see, the proof is even simpler because of the presence of the eigenvalue $\mu = 1$, which implies for example that the dimension of the unstable manifold of a hyperbolic periodic orbit is larger than its Morse index. In particular, the difficulties encountered in the second part of the proof in Section 4.1 (see the case $i(e_+) = 2q$) do not occur.

Theorem 4.2. *1) Let e^- (resp. Γ^+) be a hyperbolic equilibrium point (resp. hyperbolic periodic orbit of period $p^+ > 0$). Then the unstable manifold $W^u(e^-)$ intersects transversally the local stable manifold $W^s_{loc}(\Gamma^+)$.*
2) Let Γ^- (resp. e^+) be a hyperbolic periodic orbit of period $p^- > 0$ (resp. hyperbolic equilibrium point). Then the unstable manifold $W^u(\Gamma^-)$ intersects transversally the local stable manifold $W^s_{loc}(e^+)$.

Proof: We proceed as in Section 4.1. The main ideas behind this proof are the same as those of the proof of the transversality of orbits connecting two hyperbolic periodic orbits of [13].

Assume that $W^u(e^-)\cap W^s_{loc}(\Gamma^+) \neq \emptyset$ (otherwise the intersection is transversal by definition). Let $u(t)$ be a global solution of (1.1) with $u(0) \in W^u(e^-)\cap W^s_{loc}(\Gamma^+)$. By definition $W^u(e^-)$ intersects $W^s_{loc}(\Gamma^+)$ transversally if and only if $T_{u(0)}W^u(e^-) + T_{u(0)}W^s_{loc}(\Gamma^+) = H^s(S^1)$. We know that $\dim(T_{u(0)}W^u(e^-)) = i(e^-)$ and $\operatorname{codim}(T_{u(0)}W^s_{loc}(\Gamma^+)) = i(\Gamma^+)$. Thus, $T_{u(0)}W^u(e^-) + T_{u(0)}W^s_{loc}(\Gamma^+) = H^s(S^1)$ if and only if

$$\dim\left(T_{u(0)}W^u(e^-)\cap T_{u(0)}W^s_{loc}(\Gamma^+)\right) \leq i(e^-) - i(\Gamma^+)\ . \tag{4.3}$$

Since $W^u(e_-)$ is a finite-dimensional manifold, there exists a finite basis $(v_1^0,\ldots,v_p^0)$ of $T_{u(0)}W^u(e_-)\cap T_{u(0)}W^s_{loc}(e_+)$. Let $v_k(t)$ be the global solutions of Eq. 4.1, that is,

$$\partial_t v_k = \partial^2_{xx}v_k + D_u f(x,u,u_x)v_k + D_{u_x}f(x,u,u_x)\partial_x v_k\ ,\quad v_k(0) = v_k^0\ .$$

The basis $(v_1^0,\ldots,v_p^0)$ of $T_{u(0)}W^u(e^-)\cap T_{u(0)}W^s_{loc}(\Gamma^+)$ has the asymptotic behavior described by (i)-(iv) at the beginning of Section 4.1, when t goes to $-\infty$. Without loss of generality, we may assume that this basis has been chosen as in Lemma 4.1. By Corollary C.8, we know that the number of zeros of $v_k(t)$ is at most equal to the Morse index $i(e^-)$. Let $\Gamma^+ = \{\gamma^+(t) | t \in [0, p^+)\}$. Since $u(0)$ belongs to $W^s_{loc}(\Gamma^+)$, there exists $a^+ \in [0, p^+)$ such that $u(0)$ belongs to the local strongly stable manifold of the point $\gamma^+(a^+)$. Thus, when t goes to $+\infty$, the asymptotic behavior of the function $v_k(t)$ is given by Corollary C.11 and corresponds to eigenvalues $\mu^+_{i^+_k}$ of the period map $\Pi^+(p^+ + a^+, a^+)$ (which coincide with the eigenvalues of the period map $\Pi^+(p^+, 0)$), with $|\mu^+_{i^+_k}| < 1$. Hence, $i^+_k > i(\Gamma^+) + 1$. Furthermore, if $i^+_k = 2j$ or $2j - 1$, then, when t is large enough, $v_k(t)$ has exactly $2j$ zeros which are all simple.

If $i(\Gamma^+) = 0$, then $T_{u(0)} W^s_{loc}(\Gamma^+) = H^s(S^1)$ and the transversality trivially holds. We notice that however this situation does not arise, since, as proved by [26], periodic orbits are never stable in the case of Eq. (1.1).

Assume that $i(\Gamma^+) = 2q^+ - 1$ is odd, then for t large enough, each function $v_k(t)$ has at least $2q^+$ zeros. Since by Theorem 2.1 the number of zeros of $v_k(t)$ is non-increasing, $v_k(t)$ has at least $2q^+$ zeros for every time $t \in \mathbb{R}$. Applying Corollary C.8, we obtain that necessarily $i^-_k \geq 2q^+ - 1$ and that $i(e_-) \geq 2q^+$ (we already know this property by Corollary 3.5). Since Lemma 4.1 states that the asymptotic behaviors of all $v_k(t)$ are different, there are at most $i(e^-) - (2q^+ - 1) = i(e^-) - i(\Gamma^+)$ possible asymptotic behaviors when t goes to $-\infty$.

Assume that $i(\Gamma^+) = 2q^+$ is even, then, for t large enough, each function $v_k(t)$ has at least $2q^+ + 2$ zeros and, arguing as above, we obtain that there are at most $i(e^-) - (2q^+ + 1) \leq i(e^-) - i(\Gamma^+)$ possible asymptotic behaviors when t goes to $-\infty$.

In each case, (4.3) is satisfied and the heteroclinic orbit $u(t)$ is transverse.

The proof of the second statement is similar to the one of statement 1) and also to the proof of the first part of Theorem 1.4 given in Section 4.1. Notice that, in the case where $i(e^+) = 2q$ and $i(\Gamma^-) = i(e^+) = 2q$, we do not encounter the difficulty, which arises in Section 4.1, since we have an additional dimension at our disposal. Indeed, if Γ^- is a hyperbolic periodic orbit, then $\dim(W^u(\Gamma^-)) = i(\Gamma^-) + 1$. Therefore, $W^u(\Gamma^-)$ intersects $W^s_{loc}(e^+)$ transversally at $u(0)$ if and only if

$$\dim \left(T_{u(0)} W^u(\Gamma^-) \cap T_{u(0)} W^s_{loc}(e^+)\right) \leq i(\Gamma^-) + 1 - i(e^+) \ . \tag{4.4}$$

Thus, arguing as in Section 4.1, we prove that the intersection $W^u(\Gamma^-) \cap W^s(e^+)$ is transversal, even if $i(\Gamma^-) = i(e^+)$. □

5 Generic non-existence of homoindexed orbits

In the previous sections, we have seen that the unstable manifolds of hyperbolic periodic orbits always intersect transversally the local stable manifolds of hyperbolic periodic orbits or equilibrium points. Likewise, the unstable manifolds of hyperbolic equilibrium points always intersect transversally the local stable manifolds of hyperbolic periodic orbits. In Section 4.1, we have proved that any orbit connecting two hyperbolic equilibrium points e_- and e_+ of different Morse indices $i(e_-)$ and $i(e_+)$ is transverse. In Section 3.1, we have seen that there does not exist any connecting orbit between two hyperbolic equilibrium points with same odd index $i(e_-) = i(e_+) = 2m-1$, $m \geq 1$. Thus, in this section, it remains to show that generically with respect to the non-linearity f, there does not exist any orbit connecting two equilibrium points with same even index $i(e_-) = i(e_+) = 2m$, $m \geq 1$. In the proof of the generic non-existence of homoindexed orbits, we will actually show that generically with respect to f, all the connecting orbits, connecting equilibria with equal even Morse index, are transverse, which precludes the existence of homoindexed orbits. To show this genericity result, we shall use a functional characterization of the transversality of all connecting orbits $\mathcal{C}(e^-(f), e^+(f))$ of (1.1) and apply the Sard-Smale theorem.

5.1 Preliminaries

In the introduction, we have assumed that the conditions (1.2) hold, which imply that Eq. (1.1) defines a global dynamical system $S_f(t)$ in $H^s(S^1)$ given by $S_f(t)u_0 = u(t)$, where $u(t) \in C^1(\mathbb{R}^+, H^s(S^1))$ is the (classical) solution of (1.1). If the conditions (1.2) do not hold, then $S_f(t)$ is only a local dynamical system. At the end of the introduction, we have remarked that the automatic transversality as well as the generic transversality properties are still true, even if Hypothesis (1.2) does no longer hold. For this reason, in this section, we do not take into account this hypothesis.

We recall that $\mathfrak{G}$ denotes the space $C^2(S^1 \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ endowed with Whitney topology (see (1.3)). We fix a non-linearity f_0 in $\mathfrak{G}$ (satisfying or not Hypothesis (1.2)). We assume that f_0 is chosen so that all the equilibria and periodic orbits of the corresponding equation (1.1) are hyperbolic. We also consider the set $\mathcal{C}_0(e_0^-, e_0^+) \equiv \mathcal{C}_{f_0}(e_0^-, e_0^+)$ of all the orbits $u(t) = S_{f_0}(t)u_0$ of (1.1), connecting two (hyperbolic) equilibria $e_0^{\pm}$. Remark that e_0^+ could be equal to e_0^-.

We shall give a functional characterization of the transversality of the connecting orbits $\mathcal{C}(e^-(f), e^+(f))$ of (1.1) for $t \in \mathbb{R}$, which connect equilibria $e^-(f)$ and $e^+(f)$, close to e_0^- and e_0^+, when f belongs to a small enough neighborhood of f_0 in $\mathfrak{G}$. We will show that, even if $\mathcal{C}_0(e_0^-, e_0^+)$ is not a transverse connecting orbit, we can find f as close to f_0 as is wanted so that all the connecting orbits $\mathcal{C}(e^-(f), e^+(f))$ (with norm less than a given constant) are transverse.

Since $\mathfrak{G}$ is not metrizable and the classical perturbation theorems are usually proved in Banach spaces, we will "replace" $\mathfrak{G}$ by a Banach space in the following way.
Since $H^s(S^1)$ is continuously embedded in $C^1(S^1)$, there exists a positive integer $k_0 > 1$ such that $\|v\|_{C^1} \leq k_0\|v\|_{H^s}$, for any $v \in H^s(S^1)$. Now, for any M_0, we introduce the restriction operator $R(M_0) : g \in \mathfrak{G} \mapsto Rg \in C^2(S^1 \times [-k_0(M_0+2), k_0(M_0+2)] \times [-k_0(M_0+$

$2), k_0(M_0+2)], \mathbb{R})$ defined by

$$R(M_0)g = g_{|S^1\times[-k_0(M_0+2),k_0(M_0+2)]\times[-k_0(M_0+2),k_0(M_0+2)]} \ .$$

The map $R(M_0)$ is continuous, open and surjective from $\mathfrak{G}$ into $R(M_0)\mathfrak{G}$.

In what follows, we need the following two auxiliary lemmas. The first lemma allows to construct appropriate neighborhoods of the equilibria $e_0^{\pm}$ in $H^s(S^1)$, when f is close to f_0 in a small enough neighborhood of f_0 in C^2. This lemma is classical and is proved as [8, Lemma 4.c.2] (see also [9, Lemma 4.10]). Its proof mainly uses the continuous dependence of the equilibria and local unstable or stable manifolds with respect to the non-linearity f.

Lemma 5.1. *Let M_0 be a given positive constant and $f_0 \in \mathfrak{G}$ be given such that all its equilibrium points are hyperbolic. Then, f_0 has a finite number of equilibria e_j, $1 \le j \le N_0$ such that $\|e_j\|_{H^s} \le M_0$. There exist $r_0 > 0$, $R_0 > 0$, $R_1 > 0$, with $r_0 < R_0 < R_1$, and a small neighborhood $\mathcal{V}(f_0) \equiv \mathcal{V}(f_0, M_0)$ of f_0 in $R(M_0)\mathfrak{G}$, depending only on f_0 and M_0, such that the following properties hold:*

1. *For any $f \in \mathcal{V}(f_0, M_0)$ and any j, $1 \le j \le N_0$, there exists an equilibrium point $e_j(f)$ of $S_f(t)$ in $B_{H^s}(e_j(f_0), r_0)$. The equilibrium $e_j(f)$ is unique in the closed ball $B_{H^s}(e_i(f_0), R_1))$ and has the same Morse index as $e_j(f_0)$.*

2. *R_1 can be chosen so that $B_{H^s}(e_i(f_0), R_1) \cap B_{H^s}(e_j(f_0), R_1) = \emptyset$, if $i \neq j$.*

3. *There exist small neighborhoods $\mathcal{N}_j(f)$ of $e_j(f)$, (with $B_{H^s}(e_j(f_0), R_0) \subset \mathcal{N}_j(f) \subset B_{H^s}(e_j(f_0), R_1)$), which converge to $\mathcal{N}_j(f_0)$ in $H^s(S^1)$ as f converges to f_0 in $R(M_0)\mathfrak{G}$ and satisfy the following property:*
   the local stable set $W^s_{loc,f}(e_j(f), \mathcal{N}_j(f))$ and the local unstable set $W^u_{loc,f}(e_j(f), \mathcal{N}_j(f))$ are C^1-manifolds of codimension $i(e(f_0))$ and dimension $i(e(f_0))$ respectively. Moreover, $W^u_{loc,f}(e_j(f), \mathcal{N}_j(f)) \cap W^s_{loc,f}(e_j(f), \mathcal{N}_j(f)) = \{e_j(f)\}$.

4. *If $u(t) = S_f(t)u^*$ is a solution of (1.1) such that $u(t)$ belongs to $B_{H^s}(e_j(f_0), R_0)$ for all $t \ge t_0$ (respectively, to $B_{H^s}(e_j(f_0), R_0)$ for all $t \le t_1$), then $u(t)$, $t \ge t_0$ belongs to the local stable manifold $W^s_{loc,f}(e_j(f), \mathcal{N}_j(f))$ (respectively $u(t)$, $t \le t_1$ belongs to the local unstable manifold $W^u_{loc,f}(e_j(f), \mathcal{N}_j(f))$).*

5. *Moreover, there exists a positive constant c_0 such that, if u_1, u_2 are two solutions of (1.1) with $f = f_1$ and $f = f_2$, where f_i, $i = 1, 2$ belong to $\mathcal{V}(f_0, M_0)$, and if $u_1(t)$, $u_2(t)$ belong to $B_{H^s}(e_j(f_0), R_0)$ for all $t \in J$, where $J = (-\infty, t_0]$ or $J = [t_0, +\infty)$ for some $t_0 \in \mathbb{R}$, then,*

$$\sup_{t\in J} \|u_1(t) - u_2(t)\|_{H^s} \le c_0\big(\|f_1 - f_2\|_{C^1} + \|u_1(t_0) - u_2(t_0)\|_{H^s}\big) \qquad (5.1)$$

We also need the following auxiliary lemma about convergence of connecting orbits. Its proof is the same as the proofs of [8, Lemma 4.c.3] and [9, Lemma 4.11]. See the proof of Lemma 6.2 for similar arguments.

Lemma 5.2. *Let f_0, M_0 and $r_0 < R_0$ be as in Lemma 5.1. Let $e^-(f_0)$ and $e^+(f_0)$ be two (hyperbolic) equilibria of f_0 satisfying the conditions of Lemma 5.1. Let $\rho_0 > 0$ be any positive number such that $r_0 < \rho_0 < R_0$.*

Let $f_\nu \in \mathcal{V}(f_0)$ be a sequence converging in $R(M_0)\mathfrak{G}$ to some function $f_\infty \in \mathcal{V}(f_0)$. Assume that, for $\nu = 1, 2, \ldots$, u_ν is a solution of (1.1) for $f = f_\nu$ such that,

$$\begin{aligned} &u_\nu(t) \in B_{H^s}(0, M_0) \ , \quad \forall t \in \mathbb{R} \\ &u_\nu(t) \in B_{H^s}(e^-(f_0), \rho_0) \ , \quad \forall t \in (-\infty, -t_0] \\ &u_\nu(t) \in B_{H^s}(e^+(f_0), \rho_0) \ , \quad \forall t \in [t_0, \infty) \ , \end{aligned} \tag{5.2}$$

where t_0 is a positive time. If $e^-(f_0) = e^+(f_0) = e(f_0)$, we assume in addition that there exists a sequence of times $t_\nu \in (-t_0, t_0)$ such that $u_\nu(t_\nu) \notin B_{H^s}(e(f_0), R_0)$.
Then, u_ν admits a subsequence u_{ν_j} that converges in $C^0_b(\mathbb{R}, H^s(S^1))$ to a non trivial connecting orbit u_∞ of (1.1) for $f = f_\infty$, connecting the equilibria $e^-(f_\infty)$ and $e^+(f_\infty)$.

Remark: If, in the case where $e^-(f_0) = e^+(f_0) = e(f_0)$, we do not require that there exists a sequence of times $t_\nu \in (-t_0, t_0)$ such that $u_\nu(t_\nu) \notin B_{H^s}(e(f_0), R_0)$, then the subsequence u_{ν_j} could converge in $C^0_b(\mathbb{R}, H^s(S^1))$ to an equilibrium point $e(f_\infty)$ of $S_{f_\infty}(t)$.

In [8], in order to give a functional characterization of the transversality of the connecting orbits $\mathcal{C}(e^-, e^+)$, Brunovský and Poláčik have introduced a functional defined on a subspace $\mathcal{E}$ of the continuous bounded mappings from $\mathbb{R}$ into L^2. In our situation, following the path of [8], we could introduce the spaces,

$$\mathcal{E} = C^{1,\delta}(\mathbb{R}, L^2(S^1)) \cap C^{0,\delta}(\mathbb{R}, H^2(S^1)) \ , \quad \delta > 0 \ , \quad \mathcal{Z} = C^{0,\delta}(\mathbb{R}, L^2(S^1)) \ .$$

Then, we would fix a non-linearity f_0, the equilibrium points of which are all hyperbolic, and fix two such equilibria e_0^- and e_0^+. By Lemma 5.1, if a solution $u(t) = S_{f_0}(t)u_0$ belongs to the ball $B_{H^s}(e_0^+(f_0), R_0)$ for every time $t \geq t_0$ and to the ball $B_{H^s}(e_0^-(f_0), R_0)$ for every time $t \leq -t_0$, where $t_0 > 0$, then $u(t)$ is a connecting orbit from e_0^- to e_0^+. This leads us to introduce the open subset $\mathcal{U}_0 \subset \mathcal{E}$,

$$\begin{aligned} \mathcal{U}_0 \equiv \mathcal{U}_0(e_0^-, e_0^+) = \{w(t) \in \mathcal{E} \mid\, & w(t) \in B_{H^s}(e_0^+(f_0), R_0) \text{ for every } t \geq t_0 \ , \\ & w(t) \in B_{H^s}(e_0^-(f_0), R_0) \text{ for every } t \leq -t_0\} \ . \end{aligned} \tag{5.3}$$

We could finally define the functional $\Phi(w, f) : (w, f) \in \mathcal{U}_0 \times \mathcal{V}(f_0) \mapsto \Phi(w, f) \in \mathcal{Z}$ by

$$\Phi(w, f) = w_t(x, t) - w_{xx}(x, t) - f(x, w(x, t), w_x(x, t)) \ . \tag{5.4}$$

As in [8, Lemma 4.b.5 and Corollary 4.b.6], we could show that, if (u, f) belongs to $\Phi^{-1}(0) \cap (\mathcal{U}_0 \times \mathcal{V}(f_0))$, then the linearized operator $D_u\Phi(u, f)$ is a Fredholm operator of index $i(e_0^-) - i(e_0^+)$. Moreover, we could show that, if 0 is a regular value of the map $u \in \mathcal{U}_0 \mapsto \Phi(u, f)$, then all the connecting orbits $\tilde{u}(t)$ such that $(\tilde{u}, f) \in \mathcal{U}_0 \times \mathcal{V}(f_0)$ are transverse. This is a consequence of a functional characterization of the transversality similar to the one of Appendix B. Then, we would apply the Sard-Smale theorem (Theorem A.1) to the function Φ to deduce that, generically with respect to f, 0 is a regular value of the map $u \in \mathcal{U}_0 \mapsto \Phi(u, f)$.

However, we have seen, in [9], that it is more convenient to use a *discretized* version of the functional Φ, that is, to work with bounded sequences $(w(n\tau))_{n\in\mathbb{N}}$ rather than with bounded continuous mappings $w(t)$. In the next section, as in [9], we shall discretize the time variable and replace the functional $\Phi(\cdot)$ defined on bounded functions on $\mathbb{R}$ by a discrete analog, defined on bounded sequences.

5.2 Proof of Theorem 1.5

As already explained, the proof of Theorem 1.5 essentially consists in using the (discrete) functional characterization of the transversality given in Appendix B and in applying the Sard-Smale theorem to an appropriate discretization of the functional (5.4). The application of the Sard-Smale theorem involves some technical difficulties. The way to overcome them is now well understood (see [8] and [9]). The verification of the surjectivity of the functional Φ is the crucial point in the application of the Sard-Smale theorem.

The proof of Theorem 1.5 can be decomposed into several steps.

Step 1: Choice of particular neighborhoods and reduction to a simpler problem

We introduce the sequence of bounded open sets, $m \in \mathbb{N}$, given by

$$B_m = B_{H^s}(0, m) = \{v \in H^s(S^1) \,|\, \|v\|_{H^s} < m\} .$$

Since $H^s(S^1) = \cup_m \overline{B}_m$, Theorem 1.5 will be proved if we show that, for each m, there exists a generic set in $\mathfrak{G}$, such that, for any f in this generic set, any orbit $\tilde{u}(t)$ of (1.1), connecting two (hyperbolic) equilibria and satisfying $\tilde{u}(t) \in \overline{B}_m$, $t \in \mathbb{R}$, is transverse. We recall that, since $H^s(S^1)$ is continuously embedded in $C^1(S^1)$, there exists a positive integer k_0 such that $\|v\|_{C^1} \leq k_0\|v\|_{H^s}$, for any $v \in H^s(S^1)$. In [31, Proposition 3.2], we have shown that the set

$$\mathcal{O}^h_m = \{f \in \mathfrak{G} \,|\, \text{ any equilibrium } e \text{ of (1.1) with } \|e\|_{C^1(S^1)} \leq k_0(m+1) \text{ is hyperbolic}\}$$

is open and dense in $\mathfrak{G}$.
As in Section 5.1, we want to work in subspaces of $C^2(S^1 \times [-k_0(m+2), k_0(m+2)] \times [-k_0(m+2), k_0(m+2)], \mathbb{R})$ and hence, we use the restriction operator $R(m)$ that we simply denote

R. We set $\mathcal{RO}_m = R(\mathcal{O}_m^h)$ endowed with the topology of $C^2(S^1 \times [-k_0(m+2), k_0(m+2)] \times [-k_0(m+2), k_0(m+2)], \mathbb{R})$, which is a separable Banach space. The set $\mathcal{RO}_m$ is an open subset of $C^2(S^1 \times [-k_0(m+2), k_0(m+2)] \times [-k_0(m+2), k_0(m+2)], \mathbb{R})$ and the map R is continuous, open and surjective.

As already remarked in [8, Proof of Theorem 4.c.1, p. 165] (and also in [9, Proposition 4.12]), Theorem 1.5 will be proved by using the following proposition.

Proposition 5.3. *Assume that, for any $m \in \mathbb{N}$ and any $f_0 \in \mathcal{RO}_m$, there exist a small neighborhood $\mathcal{V}_{f_0}$ of f_0 in $\mathcal{RO}_m$ (or simply in $C^2(S^1 \times [-k_0(m+2), k_0(m+2)] \times [-k_0(m+2), k_0(m+2)], \mathbb{R})$) and a generic set $\mathcal{G}_{f_0,m}$ in $\mathcal{V}_{f_0}$ such that, for any $f \in \mathcal{G}_{f_0,m}$, any solution $\tilde{u}(t)$ of (1.1), connecting two (hyperbolic) equilibria and satisfying $\|\tilde{u}(t)\|_{H^s} \leq m$, for any $t \in \mathbb{R}$, is transverse. Then Theorem 1.5 holds.*

Proof: Let m be given. Since $\mathcal{RO}_m$ is separable, there exists a countable set of functions f_i, $i \in \mathbb{N}$, such that the family of corresponding neighborhoods $(\mathcal{V}_{f_i})$, $i \in \mathbb{N}$, covers $\mathcal{RO}_m$. Let $(\mathcal{G}_{f_i,m})$, $i \in \mathbb{N}$, be the corresponding generic sets and let $\tilde{\mathcal{G}}_{f_i,m} = \mathcal{G}_{f_i,m} \cup (\mathcal{RO}_m \setminus \mathcal{V}_{f_i})$, which is a generic subset of $\mathcal{RO}_m$. The set $\mathcal{G}_m = \cap_{i\in\mathbb{N}} \tilde{\mathcal{G}}_{f_i,m}$ is generic in $\mathcal{RO}_m$. Moreover, for any $f \in \mathcal{G}_m$, any solution $\tilde{u}(t)$ of (1.1), connecting two (hyperbolic) equilibria and satisfying $\|\tilde{u}(t)\|_{H^s} \leq m$, $t \in \mathbb{R}$, is transverse. Since the map R is continuous, open and surjective, $R^{-1}(\mathcal{G}_m)$ and $R^{-1}(\mathcal{G}_m) \cap \mathcal{O}_h$ (where $\mathcal{O}_h$ is the generic set introduced in Theorem 1.3) are generic subsets of $\mathfrak{G}$. Finally, we notice that $\mathcal{O}_M = \cap_{m\in\mathbb{N}} R^{-1}(\mathcal{G}_m) \cap \mathcal{O}_h$ is the generic set given in Theorem 1.5. □

The interest of Proposition 5.3 is that we can now work in a small neighborhood $\mathcal{V}_{f_0}$ of f_0 in $\mathcal{RO}_m$ instead of working in $\mathcal{RO}_m$. This neighborhood can be chosen as small as is needed.

From now on, we fix $m \in \mathbb{N}$ and $f_0 \in \mathcal{RO}_m$. We apply Lemma 5.1 with $M_0 = m$. Hence, there exists a small neighborhood $\mathcal{V}(f_0)$ of f_0 in $\mathcal{RO}_m$ such that all the properties described in Lemma 5.1 are satisfied. In particular, let $e_j(f_0)$, $1 \leq j \leq N_0$ be the (hyperbolic) equilibrium points of $S_{f_0}(t)$ such that $\|e_j(f_0)\|_{H^s} \leq m$. Since $S_{f_0}(t)$ has also a finite number of (hyperbolic) equilibrium points in the closed ball $B_{H^s}(0, m+1)$, there exist two real numbers k and k_1 such that $1 < k_1 < k < k_0(m+2)/(m+1)$ and that $S_{f_0}(t)$ has only N_0 equilibrium points in the closed ball $B_{H^s}(0, km)$. Moreover, we may choose the neighborhood $\mathcal{V}(f_0)$ of f_0 small enough so that, for any $f \in \mathcal{V}(f_0)$, $S_f(t)$ has only N_0 equilibrium points in the closed balls $B_{H^s}(0, km)$ and has no equilibrium points in $\{v \in H^s(S^1) \,|\, k_1 m \leq \|v\|_{H^s} \leq km\}$. Let r_0 and R_0 be chosen as in Lemma 5.1. We notice that r_0 and R_0 can be chosen small enough so that, for any $1 \leq j \leq N_0$, $B_{H^s}(e_j(f_0), R_0) \subset B_{H^s}(0, km)$. For later use, we also fix ρ_0 and ρ_1 such that $r_0 < \rho_0 < \rho_1 < R_0$.

For any (even) integer d, we introduce the set $\mathcal{E}_d$ of equilibria of Eq. (1.1) for $f = f_0$ in $B_{H^s}(0, m)$, the Morse indices of which are equal to d and we set

$$\mathcal{D}_d = \cup_{e_j(f_0)\in\mathcal{E}_d} B_{H^s}(e_j(f_0), r_0) \ .$$

For any integer ℓ and (even) integer d, we denote by $\mathcal{G}_m^{\ell,d}$ the set of all $f \in \mathcal{V}(f_0)$ that have the following property: every solution u of (1.1) satisfying

$$\begin{aligned} u(t) &\in \overline{B}_m\ , \quad \forall t \in \mathbb{R} \\ u(t) &\in \mathcal{D}_d\ , \quad \forall t \in (-\infty, -\ell] \cup [\ell, +\infty)\ , \end{aligned} \tag{5.5}$$

is transverse. We notice that, due to Lemma 5.1, any non-trivial orbit satisfying the conditions (5.5) is a connecting orbit, connecting two equilibria contained in $\mathcal{D}_d$. Due to the choice of $\mathcal{V}(f_0)$, the set

$$\mathcal{G}_m \;=\; \cap_{\ell,d} \mathcal{G}_m^{\ell,d}$$

satisfies the transversality conditions stated in Proposition 5.3. It is therefore sufficient to prove that each $\mathcal{G}_m^{\ell,d}$ is open and dense in $\mathcal{V}(f_0)$.

Step 2: Proof of the openness of $\mathcal{G}_m^{\ell,d}$

Assume that (f_ν) is a sequence in $\mathcal{V}(f_0) \setminus \mathcal{G}_m^{\ell,d}$ which converges to $f_\infty \in \mathcal{V}(f_0)$. We want to show that f_∞ does not belong to $\mathcal{G}_m^{\ell,d}$. Since $f_\nu \in \mathcal{V}(f_0) \setminus \mathcal{G}_m^{\ell,d}$, there exists a solution of (1.1) for $f = f_\nu$, distinct from any equilibrium and connecting two equilibria in $\mathcal{D}_d$. Since there is only a finite number of sets $B_{H^s}(e_j(f_0), r_0)$ in $\mathcal{D}_d$, passing to a subsequence, we may suppose that there exist indices i_1, i_2, with $1 \le i_1 \le N_0$, $1 \le i_2 \le N_0$ such that

$$\begin{aligned} u_\nu(t) &\in B_{H^s}(e^{i_1}(f_0), r_0)\ , \quad \forall t \in (-\infty, -\ell], \\ u_\nu(t) &\in B_{H^s}(e^{i_2}(f_0), r_0)\ , \quad \forall t \in [\ell, +\infty)\ . \end{aligned}$$

Moreover, if $i_1 = i_2 \equiv i$, since the solution $u_\nu(t)$ is not an equilibrium point, there exists a time τ_ν such that $u_\nu(\tau_\nu) \notin B_{H^s}(e^i(f_0), R_0)$. By Lemma 5.2, there exists a subsequence u_{ν_j} that converges in $C_b^0(\mathbb{R}, H^s(S^1))$ to a non trivial connecting orbit u_∞ of (1.1) for $f = f_\infty$, connecting the equilibria $e^{i_1}(f_\infty)$ and $e^{i_2}(f_\infty)$. Since this non-trivial orbit connects two equilibrium points with same Morse index, it cannot be a transverse orbit, which proves the openness of $\mathcal{G}_m^{\ell,d}$.

It remains to show that $\mathcal{G}_m^{\ell,d}$ is dense in $\mathcal{V}(f_0)$. This will be done in the next (and remaining) steps of the proof, by introducing a discrete version of the functional Φ described in Section 5.1 and applying the Sard-Smale theorem to it. To this end, we first need to discretize the semi-flow $S_f(t)$.

Step 3: Discretization of the semi-flow $S_f(t)$

For any $u_0 \in H^s(S^1)$ and f close to f_0, we consider the image by the time τ-map $S_f(\tau)u_0 \in H^s(S^1)$ of u_0,

$$G(u_0) \;:=\; G_f(u_0) \;=\; e^{A\tau}u_0 + \int_0^\tau e^{A(\tau-\sigma)} f(x, S_f(\sigma)u_0, \partial_x(S_f(\sigma)u_0))\, d\sigma\ , \tag{5.6}$$

where $A = \partial_{xx}$. As we do not assume global existence of solutions in this section, $G(u_0)$ may not be defined if $S_f(t)u_0$ blows up in a time shorter than τ. To overcome this difficulty, for any given $m > 0$, we choose a time $\tau_m > 0$ such that G is well defined for any $u_0 \in B_{H^s}(0,m)$ and any f in a neighborhood of f_0. Then, since $u \mapsto f(.,u,u_x)$ belongs to $\mathcal{C}^r(H^s(S^1), L^2(S^1))$, $r \geq 1$, the mapping G belongs to $\mathcal{C}^r(B_{H^s}(0,m), H^s(S^1))$ and, for any $v_0 \in H^s(S^1)$,

$$
\begin{aligned}
DG(u_0)v_0 \;=\; & e^{A\tau_m} v_0 + \int_0^{\tau_m} e^{A(\tau_m-\sigma)} \Big(D_u f(x, S_f(\tilde{s})u_0, \partial_x(S_f(\sigma)u_0))((DS_f(\sigma)u_0)v_0) \\
& + D_{u_x} f(x, S_f(\sigma)u_0, \partial_x(S_f(\sigma)u_0))((DS_f(\sigma)u_0)v_0)_x \Big) \, d\sigma \;, \qquad (5.7)
\end{aligned}
$$

that is, $DG(u_0)v_0$ is the image at the time $t = \tau_m$ of the (classical) solution of the linearized equation,

$$
\begin{aligned}
\partial_t v \;&=\; Av + D_u f(x, S_f(t)u_0, \partial_x(S_f(t)u_0))v + D_{u_x} f(x, S_f(t)u_0, \partial_x(S_f(t)u_0))v_x \;, t > 0, \\
v(0) \;&=\; v_0 \;.
\end{aligned}
$$

In other words, if $\tilde{u}(t)$ is a bounded orbit of (1.1) with $\sup_{t\in\mathbb{R}} \|\tilde{u}(t)\|_X \leq m$, then, for any $n \in \mathbb{Z}$,

$$
G(\tilde{u}(n\tau_m)) \;=\; \tilde{u}((n+1)\tau_m) \;, \quad DG(\tilde{u}(n\tau_m))v_0 \;=\; T_{\tilde{u}}((n+1)\tau_m, n\tau_m)v_0 \;, \qquad (5.8)
$$

where $T_{\tilde{u}}(t,s)$ is the evolution operator (on $L^2(S^1)$) defined by the linearized equation along the bounded orbit $\tilde{u}(t)$ (see Eq. (B.11) in Appendix B.3).

In the next step, we shall introduce the discretized version of the functional Φ defined in (5.4) and the "discretized" open set corresponding to $\mathcal{U}_0$. We require several smallness conditions on the time step τ_m. We assume that $\tau \equiv \tau_m \equiv \tau_{m,k,k_1}$ and $\mathcal{V}(f_0)$ are small enough such that:
(i) if $\|u_0\|_{H^s} \leq m$, then $S_{f_0}(t)u_0 \in B_{H^s}(0, k_1 m)$, for $0 \leq t \leq \tau_m$,
(ii) if $\|u_0\|_{H^s} \leq k_1 m$, then $S_{f_0}(t)u_0 \in B_{H^s}(0, km)$, for $0 \leq t \leq \tau_m$,
(iii) if $u_0 \in B_{H^s}(e_j(f_0), \rho_0)$, $1 \leq j \leq N_0$ and $f \in \mathcal{V}(f_0)$, then $S_f(t)u_0 \in B_{H^s}(e_j(f_0), \rho_1)$ for $0 \leq t \leq \tau_0$.
With these conditions, we control the behavior to the continuous solution $S_f(t)u_0$ between two time steps $n\tau$ and $(n+1)\tau$. For example, (iii) ensures that if $u(n\tau)$ belongs to $B_{H^s}(e_j(f_0), \rho_0)$ for any large enough n, then $u(t)$ belongs to $B_{H^s}(e_j(f_0), R_0)$ for t large enough and thus $u(t)$ belongs to the local stable manifold of $e_j(f_0)$.

Step 4: A functional characterization of the transversality

We are now ready to define the discretized version of the functional Φ introduced in Section 5.1. This follows the lines of [9].

We recall that the integer m, the function f_0 and the neighborhood $\mathcal{V}(f_0)$ are fixed. Since the Sard-Smale theorem requires that Φ is defined on open sets and the set $\overline{B}_m$ used in the definition of $\mathcal{G}_m^{\ell,d}$ is closed, we need to introduce the following set

$$
B_m^* \;=\; \{v \in H^s(S^1) \mid \|v\|_{H^s} < k_1 m\} \;.
$$

Let $\mathcal{E}_d$ be the set of equilibrium points of $S_{f_0}(t)$ in $\overline{B}_m$ of Morse index d. We set

$$\mathcal{D}_d^* = \cup_{e_j(f_0)\in\mathcal{E}_d} B_{H^s}(e_j(f_0), \rho_0) .$$

For any integer ℓ and any (even) integer d, we finally introduce the following subspace of $\ell^\infty(\mathbb{Z}, H^s(S^1))$,

$$\mathcal{X} \equiv \mathcal{X}_{m,\ell,d} = \{w(\cdot\tau) \in \ell^\infty(\mathbb{Z}, H^s(S^1)) \,|\, \forall |n| \geq \ell,\ w(n\tau) \in \mathcal{D}_d^* \text{ and } \forall n \in \mathbb{Z},\ w(n\tau) \in B_m^*\} . \tag{5.9}$$

We notice that $\mathcal{X}$ is open in $\ell^\infty(\mathbb{Z}, H^s(S^1))$ and contains the discretizations of all connecting orbits of $S_f(t)$ satisfying (5.5). We next define the discretized map $\Phi \equiv \Phi_{m,\ell,d} : \mathcal{X}_{m,\ell,d} \times \mathcal{V}(f_0) \to l^\infty(\mathbb{Z}, H^s(S^1))$ by

$$\Phi(w, f)(n) \equiv \Phi_{m,\ell,d}(w, f)(n) = w((n+1)\tau) - G_f(w(n\tau)) , \quad \forall n \in \mathbb{Z} , \tag{5.10}$$

where G_f has been defined in (5.6).

Arguing as in [9, Section 4.2], we obtain the following characterization of the transversality. Its proof is based on the abstract formulation of transversality given in Appendix B. Without loss of generality, we may replace $\mathcal{V}(f_0)$ by a smaller neighborhood and thus assume that $\mathcal{V}(f_0)$ is actually a convex neighborhood.

Theorem 5.4. *The above map $\Phi : \mathcal{X}_{m,\ell,d} \times \mathcal{V}(f_0) \to l^\infty(\mathbb{Z}, H^s(S^1))$ is of class $\mathcal{C}^1$. A pair (u, f) belongs to $\Phi^{-1}(0)$ if and only if u is the discretization of a connecting orbit $\tilde{u}(t)$ (or an equilibrium point) of $S_f(t)$ contained in $B_{H^s}(0, km)$ whose discretization belongs to B_m^*. Moreover, for any $(u, f) \in \Phi^-1(0)$ the mapping $D_u\Phi(u, f)$ is a Fredholm operator of index 0.*
If 0 is a regular value of the map $u \in \mathcal{X}_{m,\ell,d} \mapsto \Phi(u, f)$, then all the connecting orbits the discretizations of which are contained in $\mathcal{X}_{m,\ell,d}$ are transverse, i.e. $f \in \mathcal{G}_m^{\ell,d}$.

Proof: The description of the zeros (u, f) of Φ is obvious. Indeed, $G_f(u(n\tau)) = S_f(\tau)u(n\tau)$ and thus u is the discretization of a trajectory $\tilde{u}(t)$ of $S_f(t)$. Moreover, due to the definition of $\mathcal{X} \equiv \mathcal{X}_{m,\ell,d}$ and to Lemma 5.1, it is a connection between two equilibrium points $e^-(f)$ and $e^+(f)$ of same index d and is a constant sequence if and only if it coincides with an equilibrium $e^-(f) = e^+(f)$.

It is straightforward to prove that the mapping $\Phi : \mathcal{X} \times \mathcal{V} \mapsto l^\infty(\mathbb{Z}, H^s(S^1))$ is of class C^1 (see [8, Lemma 4.c.4] or [9, Lemma 4.13]). Moreover, the first derivative $D\Phi(u, f)$, for $(u, f) \in \mathcal{X} \times \mathcal{V}(f_0)$, is given by

$$\begin{aligned} D\Phi(u,f)(Y,h)(n) &= Y((n+1)\tau) - D_uG_f(u(n\tau)))Y(n\tau) - D_fG_f(u(n\tau)) \cdot h \\ &= Y((n+1)\tau) - T_{u,f}((n+1)\tau, n\tau)Y(n\tau) - D_fG_f(u(n\tau)) \cdot h \\ &\equiv (\mathcal{L}_{u,f}Y)(n\tau) - D_fG_f(u(n\tau)) \cdot h , \end{aligned} \tag{5.11}$$

where (Y, h) is any element of $l^\infty(\mathbb{Z}, H^s(S^1)) \times \mathfrak{G}$ and where $T_{u,f}$, $t \geq s$, is the evolution operator defined by the linearized equation (B.11) and $\tilde{u}$ is the solution of (1.1), the discretization of which is given by u.

The expression (5.11) of the derivative $D\Phi$ shows that u is a regular zero of the mapping $u \in \mathcal{X}_{m,\ell,d} \mapsto \Phi(u,f)$ if and only if the mapping $\mathcal{L}_{u,f}$ is surjective. Corollary B.14 implies that the map $\mathcal{L}_{u,f}$ is surjective if and only if $\tilde{u}$ is transverse. Corollary B.14 also tells that $\mathcal{L}_{u,f}$ is a Fredholm operator of index equal to $i(e^-(f)) - i(e^+(f)) = 0$.
As noticed above, if $\tilde{u} = \{\ldots, e, e, \ldots,\}$ is a constant sequence, then, by the construction of the various neighborhoods made in Lemma 5.1, e is a hyperbolic equilibrium point of (1.1), which implies that $e = e^-(f) = e^+(f)$. Again, by Theorem B.7, the surjectivity of the map $\mathcal{L}_{u,f}$ is then equivalent to the hyperbolicity of $e^-(f)$. □

Step 5: Surjectivity of $D\Phi$
As already explained, we will apply the Sard-Smale theorem to the functional Φ introduced in Step 4 and consider the set $\Phi^{-1}(0)$ in particular. One of the main hypotheses of the Sard-Smale theorem is the fact that 0 is a regular value of the map $(w,f) \in \mathcal{X} \times \mathcal{V}(f_0) \mapsto \Phi(w,f)$. This property will be shown in the next theorem as consequence of Corollary B.14 and the one-to-one property of homoindexed orbits proved in Proposition 3.6.

Theorem 5.5. *Assume that $\mathcal{X}$ and Φ are given as in Step 4.*
1) The pair $(\tilde{u}, f)$ is a regular zero of the map $(w,f) \in \mathcal{X} \times \mathcal{V}(f_0) \mapsto \Phi(w,f)$, if and only if, for any nontrivial bounded solution $\psi(t) \in C^0_b(\mathbb{R}, L^2(S^1))$ of the adjoint equation (B.12), *there exists $\tilde{g} \in R\mathfrak{G}$ such that*

$$\int_{-\infty}^{+\infty} \langle \psi(t), \tilde{g}(\tilde{u}(t)) \rangle_{L^2(S^1)} \, \mathrm{d}t \neq 0 ,$$

where $\tilde{u}(t) = S_f(t)\tilde{u}(0)$ is the (continuous) trajectory corresponding to the sequence $(\tilde{u}(n\tau))$.

2) As a consequence of the first statement and of Proposition 3.6, 0 *is a regular value of the map Φ.*

Proof: We first prove the second statement of the theorem, which is a direct consequence of Proposition 3.6. If $\Phi(\tilde{u}, f) = 0$ and $\tilde{u}$ is the discretization of an (hyperbolic) equilibrium point, then as explained in the proof of Theorem 5.4, the map $\mathcal{L}_{\tilde{u},f}$ is surjective and thus $D\Phi(\tilde{u}, f)$ is also surjective. Thus, it remains to consider the case where $(\tilde{u}, f) \in \Phi^{-1}(0)$ and $\tilde{u}$ is not the discretization of an equilibrium point. By the first statement, the operator $D\Phi(\tilde{u}, f) \in L(\mathcal{X} \times \mathcal{V}, l^\infty(\mathbb{Z}, H^s(S^1)))$ is surjective if and only if, for any nontrivial bounded solution $\psi(t) \in C^0_b(\mathbb{R}, L^2(S^1))$ of the adjoint equation (B.12), there exists $\tilde{g} \in R\mathfrak{G}$ such that

$$\int_{S^1} \int_{-\infty}^{+\infty} \psi(x,t) \tilde{g}(x, \tilde{u}(x,t), \tilde{u}_x(x,t)) dx dt \neq 0 . \tag{5.12}$$

Since $\psi(t)$ is a non trivial solution of the adjoint equation, there exist $x_0 \in S^1$ and t_0 such that $\psi(x_0, t_0) \neq 0$. Due to the injectivity property of Proposition 3.6, for x_0 fixed, there exists no other time t_1 such that $\tilde{u}(x_0,t_1)=\tilde{u}(x_0,t_0)$ and $\tilde{u}_x(x_0,t_1)=\tilde{u}_x(x_0,t_0)$. Moreover, Proposition 3.6 also implies that $(\tilde{u}(x_0,t),\tilde{u}_x(x_0,t))$ stays outside a small neighborhood of $(\tilde{u}(x_0,t_0),\tilde{u}_x(x_0,t_0))$ for t close to $\pm\infty$. Therefore, one easily constructs a regular bump function $\tilde{g}$ which vanishes outside a small neighborhood of $(x_0,\tilde{u}(x_0,t_0),\tilde{u}_x(x_0,t_0))$ and is positive in this neighborhood; so that the function $(x,s)\mapsto\tilde{g}(x,\tilde{u}(x,s),\tilde{u}_x(x,s))$ is a regular bump function concentrated around (x_0,t_0). For such a choice of $\tilde{g}$, the condition (5.12) is thus satisfied.

We now prove the first statement of the theorem. This proof is nothing else as the proof of [9, Theorem 4.7]. For the reader's convenience, we reproduce it here.

As already explained, if $(\tilde{u},f)$ belongs to $\Phi^{-1}(0)$, then $\tilde{u}$ is a discretization of a trajectory $\tilde{u}(t)$, $t\in\mathbb{R}$, of (1.1), connecting two equilibria $e^-(f)$ and $e^+(f)$. Without loss of generality, we may assume that $\tilde{u}$ is a nonconstant sequence. Indeed, if $\tilde{u}=\{\ldots,e,e,\ldots,\}$, then $e=e^-(f)=e^+(f)$ is a hyperbolic equilibrium point and so $\tilde{u}$ is a regular zero of Φ. On the other hand, the adjoint equation (B.12) has no nontrivial bounded solution.

Thus, we assume that $\tilde{u}$ is not a constant sequence. We recall that, by (5.11), for any $(Y,\tilde{g})\in l^\infty(\mathbb{Z},H^s(S^1))\times R\mathfrak{G}$

$$(D\Phi(\tilde{u},f)\cdot(Y,\tilde{g}))(n\tau)\;=\;(\mathcal{L}_{\tilde{u},f}Y)(n\tau)-D_fG_f(\tilde{u}(n\tau))\cdot\tilde{g}\;.$$

We notice that $\mathcal{L}_{\tilde{u},f}$ corresponds to the operator $\mathcal{L}$ defined in (B.2). A sequence $H\in\ell^\infty(\mathbb{Z},H^s(S^1))$ is in the range of $D\Phi$ if and only if one can choose $\tilde{g}\in R\mathfrak{G}$ such that $H+D_fG_f(\tilde{u}).\tilde{g}$ is in the range of $\mathcal{L}_{\tilde{u}}$. According to Corollary B.14, this is equivalent to finding $\tilde{g}$ such that

$$\sum_{n=-\infty}^{+\infty}\langle\psi((n+1)\tau),D_fG_f(\tilde{u}(n\tau))\cdot\tilde{g}\rangle_{L^2(S^1)}\;=\;-\sum_{n=-\infty}^{+\infty}\langle\psi((n+1)\tau,H(n\tau)\rangle_{L^2(S^1)}\;,\quad(5.13)$$

for every nontrivial sequence $\psi(n\tau)=T^*(n\tau,0)\psi_0$, $\psi_0\in L^2(S^1)$, which is bounded in $L^2(S^1)$. We can choose such a $\tilde{g}\in R\mathfrak{G}$ if, given a basis $\psi_1,\psi_2,\ldots,\psi_q$ of the (necessarily) finite-dimensional vector space of bounded sequences $\psi(n)=T^*(n\tau,0)\psi_0$, the mapping

$$\tilde{g}\in\mathfrak{G}\mapsto(\sum_{-\infty}^{+\infty}\langle\psi_j((n+1)\tau),D_fG_f(\tilde{u}(n\tau))\cdot\tilde{g}\rangle_{L^2(S^1)})_{1\le j\le q}\in\mathbb{R}^q\qquad(5.14)$$

is surjective. If the range of the mapping (5.14) is not the whole vector space $\mathbb{R}^q$, there exists a vector $(\alpha_1,\ldots,\alpha_q)$ orthogonal to the range, that is, there exists a bounded sequence $\psi=\sum\alpha_j\psi_j\neq0$ such that, for any $\tilde{g}\in R\mathfrak{G}$,

$$\sum_{n=-\infty}^{+\infty}\langle\psi((n+1)\tau),D_fG_f(\tilde{u}(n\tau))\cdot\tilde{g}\rangle_{L^2(S^1)}\;=\;0\;.$$

Thus, $D\Phi$ is surjective if and only if, for any bounded sequence $\psi(n\tau) = T^*(n\tau, 0)\psi_0$, there exists $\tilde{g} \in R\mathfrak{G}$ such that,

$$\sum_{n=-\infty}^{+\infty} \langle \psi((n+1)\tau, D_f G_f(\tilde{u}(n\tau)) \cdot \tilde{g}\rangle_{L^2(S^1)} \neq 0 \ . \tag{5.15}$$

Since the solution of (1.1) is differentiable with respect to f, we can differentiate (1.1) formally with respect to f to deduce that, for any $\tilde{g}$,

$$D_f G_f(\tilde{u}(n\tau)) \cdot \tilde{g} = \int_{n\tau}^{(n+1)\tau} T_{\tilde{u}}((n+1)\tau, \sigma)\tilde{g}(\tilde{u}(\sigma))\, d\sigma \ . \tag{5.16}$$

Using the expression (5.16) in the condition (5.15) yields

$$\begin{aligned}
&\sum_{n=-\infty}^{+\infty} \langle \psi(n+1), D_f G_f(\tilde{u}(n\tau)) \cdot \tilde{g}\rangle_{L^2(S^1)} \\
&\qquad = \sum_{n=-\infty}^{+\infty} \int_{n\tau}^{(n+1)\tau} \langle \psi((n+1)\tau), T_{\tilde{u}}((n+1)\tau, \sigma)\tilde{g}(\tilde{u}(\sigma))d\sigma\rangle_{L^2(S^1)} \\
&\qquad = \sum_{n=-\infty}^{+\infty} \int_{n\tau}^{(n+1)\tau} \langle T^*_{\tilde{u}}(\sigma, (n+1)\tau)\psi((n+1)\tau), \tilde{g}(\tilde{u}(\sigma))d\sigma\rangle_{L^2(S^1)} \\
&\qquad = \int_{-\infty}^{+\infty} \langle \psi(\sigma), \tilde{g}(\tilde{u}(\sigma))d\sigma\rangle_{L^2(S^1)} \ ,
\end{aligned} \tag{5.17}$$

where, for any $\sigma \in \mathbb{R}$, $\psi(\sigma) = T^*_{\tilde{u}}(\sigma, (n+1)\tau)\psi((n+1)\tau) = T^*_{\tilde{u}}(\sigma, 0)\psi_0$. We remark that $\psi(\cdot)$ belongs to $\mathcal{C}^0_b(\mathbb{R}, L^2(S^1))$ and is a bounded solution of (B.11). Theorem 5.5 is thus proved. □

Step 6: Application of the Sard-Smale theorem and density of $\mathcal{G}^{\ell,d}_m$

After all the preliminaries given in the previous steps, we are now ready to apply the Sard-Smale theorem (in the form recalled in Appendix A). This follows the lines of [8] and [9].

We recall that the subspace $\mathcal{X}$ of $\ell^\infty(\mathbb{Z}, H^s(S^1))$ has been defined in (5.9). We introduce the open subset $\mathcal{Y} = \mathcal{V}(f_0)$ of $\mathcal{RO}_m$ and the Banach space $\mathcal{Z} = \ell^\infty(\mathbb{Z}, H^s(S^1))$.
We recall that the mapping $\Phi \equiv \Phi_{m,\ell,d} : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ has been given in (5.10) as follows:

$$\Phi(w, f)(n) = w((n+1)\tau) - G_f(w(n\tau)) \ , \quad \forall n \in \mathbb{Z} \ ,$$

where G_f has been defined in (5.6).
We now check that all the hypotheses of Theorem A.1 are satisfied with $\xi = 0$.

By Theorem 5.4, Φ is a C^1-mapping from $\mathcal{X} \times \mathcal{Y}$ into $\mathcal{Z}$ and it is also a Fredholm operator of index 0 for any $(w, f) \in \Phi^{-1}(0)$. Thus Hypothesis 1 of Theorem A.1 holds.

By Theorem 5.5, $D\Phi(w, f) : T_w\mathcal{X} \times T_f\mathcal{Y} \to T_0\mathcal{Z}$ is surjective, for $(w, f) \in \Phi^{-1}(0)$. Thus Hypothesis 2 of Theorem A.1 also holds.

Taking into account Lemma 5.2 and the remark following this lemma, we can prove Hypothesis 3(b) by following the lines of the proof of the corresponding property in [9] (see [9, Step 3 of the proof of Proposition 4.12]).

Thus, the functional Φ satisfies all the assumptions of the Sard-Smale theorem. Due to the relation between Φ and the transversality of connecting orbits (Theorem 5.4), this shows the genericity of $\mathcal{G}_m^{\ell,d}$ in $\mathcal{V}(f_0)$ and concludes the proof of Theorem 1.5.

6 The non-wandering set

To prove Theorem 1.6, it remains to show that generically there does not exist non-wandering elements which are not critical elements and that generically the number of critical elements is finite. We emphasize that the dynamics of (1.1) may have non-trivial non-wandering elements. Indeed, as shown in [57], every two-dimensional flow can be realized in the dynamics of (1.1). Thus, one can for example create a sequence of periodic orbits, which piles up on a homoclinic orbit. This orbit is then a non-wandering orbit which is not critical. However, non-trivial non-wandering orbits are generically precluded.

Proposition 6.1. *Assume that f is a non-linearity such that the dynamics of* (1.1) *satisfy the following properties:*
- there exists a compact global attractor for (1.1).
- All the equilibria and periodic orbits of (1.1) *are hyperbolic.*
- There is no homoclinic orbit and all the heteroclinic orbits are transversal.
Then, the set of non-wandering elements consists in a finite number of equilibrium points and periodic orbits.

As usual in this article, the property stated in Proposition 6.1 has its equivalent for two-dimensional dynamical systems. It mainly relies on Poincaré-Bendixson property, proved in [14] for (1.1). Proposition 6.1 is the key point to deduce the genericity of Morse-Smale property from the genericity of Kupka-Smale property. The genericity of Morse-Smale property for dynamical systems of orientable surfaces shown in [48] also relies on a similar property, see [43].

We enhance that, if f is such that (1.1) admits a compact global attractor and that any equilibrium point and any periodic orbit are hyperbolic, then there is at most a finite number of equilibrium points. However, as we explained above, there could exist an infinite number of hyperbolic periodic orbits: think of a sequence of hyperbolic periodic orbits piling up to a homoclinic orbit. One can only ensure that there is a finite number of hyperbolic periodic orbits with a period less than a given number.

We begin the proof of Proposition 6.1 by several lemmas. We assume in the whole section that f has been chosen so that the hypotheses of Proposition 6.1 are satisfied.

Let $\mathcal{C}$ be a hyperbolic equilibrium point or periodic orbit of (1.1). We recall that there exists an open neighborhood B of $\mathcal{C}$ in $H^s(S^1)$ such that each global solution $u(t)$ of (1.1), satisfying $u(t) \in \overline{B}$ for all $t \leq 0$, belongs to the local unstable manifold $W^u_{loc}(\mathcal{C})$. We refer for example to [18], [51].

Lemma 6.2. *Let $\mathcal{C}$ be a hyperbolic equilibrium point or periodic orbit of* (1.1) *and let B be the neighborhood of $\mathcal{C}$ as described above. Let $(u_n(t))_{n\in\mathbb{N}}$ be a sequence of solutions of* (1.1) *such that, for each $n \in \mathbb{N}$, there exist three times $\sigma_n < t_n < \tau_n$ such that the following properties hold. For all $t \in (\sigma_n, \tau_n)$, $u_n(t) \in B$, $u_n(\sigma_n) \in \partial B$, $u_n(\tau_n) \in \partial B$ and,*

$$d\left(u_n(t_n), \mathcal{C}\right) := \inf_{c\in\mathcal{C}} \|u_n(t_n) - c\|_{H^s(S^1)} \xrightarrow[n\longrightarrow+\infty]{} 0\ .$$

Then, there exist an extraction φ and a globally defined and bounded solution $u_\infty(t)$ of (1.1) *such that $u_\infty(t) \in W^u_{loc}(\mathcal{C})$, $t \leq 0$, and*

$$\forall\ T > 0, \quad \sup_{t\in[-T,T]} \left\|u_{\varphi(n)}(\tau_{\varphi(n)} + t) - u_\infty(t)\right\|_{H^s(S^1)} \xrightarrow[n\longrightarrow+\infty]{} 0\ . \tag{6.1}$$

Proof: First, we claim that $\tau_n - t_n \longrightarrow +\infty$ when $n \to +\infty$. Indeed, if this is not true, since $\mathcal{C}$ is compact, we can extract a subsequence $u_{\psi(n)}(t_{\psi(n)})$ converging to some $c \in \mathcal{C}$ and such that $\tau_{\psi(n)} - t_{\psi(n)}$ converges to some $t \geq 0$. Then, by continuity of the Cauchy problem related to (1.1), $u_{\psi(n)}(\tau_{\psi(n)})$ converges to a point of $\mathcal{C}$, which contradicts the fact that $u_{\psi(n)}(\tau_{\psi(n)}) \in \partial B$.
We set $T = 1$. Since (1.1) admits a compact global attractor and that (τ_n) converges to $+\infty$, the sequence $u_n(\tau_n - T)$ is precompact in $H^s(S^1)$ and there is an extraction φ_1 such that $u_{\varphi_1(n)}(\tau_{\varphi_1(n)} - T)$ converges to some $u_\infty(-T) \in H^s(S^1)$. Let $u_\infty(t) = S(t+T)u_\infty(-T)$, $t \geq -T$, be the solution of (1.1) associated to $u_\infty(-T)$. By continuity of the Cauchy problem related to (1.1), $u_{\varphi_1(n)}(\tau_{\varphi_1(n)} + t)$ converges to $u_\infty(t)$ uniformly with respect to $t \in [-T, T]$. To achieve the proof of uniform convergence of $u_n(t)$ to $u_\infty(t)$ on any compact set of time, it is sufficient to repeat the argument for all $T \in \mathbb{N}$ and to use the diagonal extraction $\varphi(n) = \varphi_n \circ ... \circ \varphi_1(n)$.
Finally, let us notice that $u_\infty(t)$ belongs to $W^u_{loc}(\mathcal{C})$. Indeed, since $u_n(t) \in B$ for all $t \in [t_n, \tau_n)$ and that $\tau_n - t_n \longrightarrow +\infty$, $u_\infty(t) \in \overline{B}$ for all $t \leq 0$. Due to the choice of B, this implies that $u_\infty(t) \in W^u_{loc}(\mathcal{C})$. □

Let $M \in \mathbb{R} \cup \{+\infty\}$. We use the notation $[\![1, M+1]\!] = \{k \in \mathbb{N}, 1 \leq k \leq M+1\}$. We say that a sequence of critical elements $(\mathcal{C}_k)_{k\in[\![1,M+1]\!]}$ is connected if for any $k \in [\![1, M]\!]$, there exists a heteroclinic orbit $u_k(t)$ such that the $\alpha-$limit set of $u_k(t)$ is $\mathcal{C}_k$ and its $\omega-$limit set is $\mathcal{C}_{k+1}$. We recall that a chain of heteroclinic orbits denotes the sequence of heteroclinic orbits corresponding to a connected sequence of critical elements $(\mathcal{C}_k)_{k\in[\![1,p+1]\!]}$ with $\mathcal{C}_{p+1} = \mathcal{C}_1$.

Lemma 6.3. *Assume that f is as in Proposition 6.1. Then, there is no connected sequence of critical elements with infinite length. As a consequence, there is no chain of heteroclinic orbits and, every $\omega-$limit set and every non-empty $\alpha-$limit set of trajectories of* (1.1) *consist exactly of one critical element*

Proof: Let $M \in \mathbb{N}\cup\{+\infty\}$ and let $(\mathcal{C}_k)_{k\in[\![1,M+1]\!]}$ be a connected sequence of closed orbits with heteroclinic connections $(u_k(t))_{k\in[\![1,M]\!]}$. We consider the Morse indices $i(\mathcal{C}_k)$ of the closed orbits. We have several cases:
- if $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$ are both periodic orbits, then Theorem 1.2 shows that $i(\mathcal{C}_k) > i(\mathcal{C}_{k+1})$.
- if $\mathcal{C}_k$ is an equilibrium point and if $\mathcal{C}_{k+1}$ is an equilibrium point or a periodic orbit, then $\dim(W^u(\mathcal{C}_k))$=i$(\mathcal{C}_k)$ and codim$(W^s(\mathcal{C}_{k+1}))$=i$(\mathcal{C}_{k+1})$. Thus, since the intersection of $W^u(\mathcal{C}_k)$ and $W^s(\mathcal{C}_{k+1})$ is non-empty and transversal, one must have $i(\mathcal{C}_k) > i(\mathcal{C}_{k+1})$.
- if $\mathcal{C}_k$ is a periodic orbit and $\mathcal{C}_{k+1}$ is an equilibrium, then $\dim(W^u(\mathcal{C}_k))$=i$(\mathcal{C}_k)$+1 and codim$(W^s(\mathcal{C}_{k+1}))$=i$(\mathcal{C}_{k+1})$. Therefore, $i(\mathcal{C}_k) \geq i(\mathcal{C}_{k+1})$.
Hence, the Morse index of $\mathcal{C}_k$ is non-increasing and decreases except if u_k goes from a periodic orbit to an equilibrium point. However, a sequence cannot consist only of connections from a periodic orbit to an equilibrium and the Morse index must decrease at least every two steps. Therefore, since the Morse indices are non-negative, M is bounded by $2i(\mathcal{C}_0)$.

The non-existence of connected sequence of critical elements of infinite length precludes the existence of chains of heteroclinic orbits since every chain $(\mathcal{C}_k)_{k\in[\![1,p+1]\!]}$ with $\mathcal{C}_{p+1} = \mathcal{C}_1$ can be extended to a periodic connected sequence of critical elements and thus to a connected sequence of infinite length.

Let $u_0 \in H^s(S^1)$ be chosen such that its $\omega-$limit set $\omega(u_0)$ is not a unique periodic orbit. Then, the Poincaré-Bendixson property stated in Theorem 1.1 shows that $\omega(u_0)$ consists of equilibrium points and homoclinic or heteroclinic orbits connecting them. We know that homoclinic orbits are precluded. Moreover, since there is no connected sequence of equilibria of infinite length, there exists an equilibrium point e where no connected sequence can be extended, that is, such that $W^u(e) \cap \omega(u_0) = \{e\}$. Let B_e be a small neighborhood of e in $H^s(S^1)$ such that any solution $u(t)$ of (1.1) satisfying $u(t) \in \overline{B}_e$ for any $t \leq 0$, belongs to the unstable manifold $W^u(e)$. If $\omega(u_0) \neq \{e\}$, then one easily constructs three sequences of times (σ_n), (t_n) and (τ_n) going to $+\infty$ and satisfying the hypotheses of Lemma 6.2 with $\mathcal{C} = \{e\}$ and $u_n(t) = S(t)u_0$. But then Lemma 6.2 implies that there exists a solution $u_\infty(t)$ of (1.1) belonging to the unstable manifold $W^u(e)$ and with $u(\tau_{\varphi(n)})$ converging $u_\infty(0)$. Therefore $u_\infty(0) \in \omega(u_0) \cap \partial B_e$ and $W^u(e) \cap \omega(u_0) \neq \{e\}$, which leads to a contradiction. Therefore, $\omega(u_0) = \{e\}$. □

Proof of Proposition 6.1: Let $\tilde{u}_0 \in H^s(S^1)$ be a non-wandering element and let $\tilde{u}(t) = S(t)\tilde{u}_0$. Using the definition of a non-wandering element, we easily construct a sequence of trajectories $u_n(t)$ such that $u_n(0)$ converges to $\tilde{u}_0$ and, for any sequence (t_n), there exists a sequence (t'_n) such that $t'_n > t_n$ and $u_n(t'_n)$ converges to $\tilde{u}_0$. By Lemma 6.3, there exists a hyperbolic critical element $\mathcal{C}_1$ such that $\omega(\tilde{u}_0) = \{\mathcal{C}_1\}$. Let B_1 be a neighborhood of $\mathcal{C}_1$ as in Lemma 6.2. Assume that $\tilde{u}_0 \notin \mathcal{C}_1$. Replacing B_1 by a smaller neighborhood if needed, we may assume that $\tilde{u}_0 \notin \overline{B}_1$. There is a sequence of times (t_n) and a point $c \in \mathcal{C}_1$ such that $\tilde{u}(t_n) \to c$. By continuity of the Cauchy problem, we may assume without loss of generality that $u_n(t_n) \to c$. As we can find a sequence of times t'_n such that $t'_n > t_n$ and $u_n(t'_n) \to \tilde{u}_0 \notin \overline{B}_1$, there exists a sequence of times τ_n^1 such that $u_n(t) \in B_1$ for $t \in [t_n, \tau_n^1)$ and $u_n(\tau_n^1) \in \partial B_1$. By Lemma 6.2, we may assume without loss of generality that $u_n(\tau_n^1 + t)$ converges to some $\tilde{u}_1(t) \in W^u(\mathcal{C}_1)$. Now, we can repeat the arguments: there exist a critical element $\mathcal{C}_2$ such that $\omega(\tilde{u}_1(t)) = \{\mathcal{C}_2\}$ and a sequence of times τ_n^2 such that, up to an extraction, $u_n(\tau_n^2 + t)$ converges to some solution $\tilde{u}_2(t) \in W^u(\mathcal{C}_2)$ and so on... Thus, we are constructing a connected sequence of critical elements of infinite length, which is precluded by Lemma 6.3. This means that $\tilde{u}_0$ belongs to $\mathcal{C}_1$, that is, that our non-wandering element either is an equilibrium point or belongs to a periodic orbit.

To finish the proof of Proposition 6.1, it suffices to show that the number of critical elements is finite. First, as noticed earlier, due to the compactness of the global attractor and the hyperbolicity of the equilibrium points and the periodic orbits, the number of equilibrium points and periodic orbits of smallest period bounded by a given number is finite. Thus we only need to show that there does not exist an infinite sequence of periodic orbits $\gamma_n(x,t)$ of smallest period p_n, where p_n tends to infinity when n goes to infinity. If we had such a sequence, we would be able to repeat the arguments of the first part of the proof and, by using lemmas 6.2 and 6.3, construct a connected sequence of critical elements of infinite length, which leads to a contradiction. $\square$

Appendices

A Fredholm operator and Sard-Smale Theorem

As we have explained in the introduction, an ingredient of the proof of the generic non-existence of homoindexed orbits is the Sard-Smale theorem. We recall the precise statement of the version of the Sard-Smale theorem, that we are applying in Section 5.

Let $\mathcal{X}$, $\mathcal{Z}$ be two differentiable Banach manifolds and $\Phi : \mathcal{X} \to \mathcal{Z}$ be a C^1 map. A point z is a *regular value* of Φ if, for any $x \in \Phi^{-1}(z)$ the derivative $D\Phi(x)$ is surjective and its kernel splits, i.e. has a closed complement in $T_x\mathcal{X}$ (sometimes this property is denoted by $\Phi \pitchfork z$). A point $z \in \mathcal{Z}$ which is not regular is called a *critical value* of Φ. A subset in a topological space is *generic* or *residual* if it is a countable intersection of open dense sets.

We recall that a continuous linear map $L : X \to Z$ between two Banach spaces X and Z, is a Fredholm map if its range $R(L)$ is closed and if both $\dim \ker(L)$ and $\operatorname{codim} R(L)$ are finite. The *index* $\operatorname{ind}(L)$ is the integer $\operatorname{ind}(L) = \dim \ker(L) - \operatorname{codim} R(L)$.

The version of the Sard-Smale theorem given here has been proved in [25] (for weaker versions, we also refer to [52] and [54]). The next theorem has been widely used in the genericity proofs in [8] and [9].

Theorem A.1. Sard-Smale Theorem

Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be three smooth Banach manifolds, $\Phi : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a mapping of class C^r, $r \geq 1$ and ξ an element of $\mathcal{Z}$. Assume that the following hypotheses hold:

1. *for each $(x, y) \in \Phi^{-1}(\xi)$, $D_x\Phi(x, y)$ is a Fredholm operator of index strictly less than r;*
2. *for each $(x, y) \in \Phi^{-1}(\xi)$, $D\Phi(x, y) : T_x\mathcal{X} \times T_y\mathcal{Y} \to T_\xi\mathcal{Z}$ is surjective;*
3. *one of the following properties is satisfied:*
   (a) $\mathcal{X}$ and $\mathcal{Y}$ are separable metric spaces;
   (b) the map $(x, y) \in \Phi^{-1}(\xi) \mapsto y \in \mathcal{Y}$ is σ-proper, that is, there is a countable system of subsets $V_n \subset \Phi^{-1}(\xi)$ such that $\cup_n V_n = \Phi^{-1}(\xi)$ and for each n the map $(x, y) \in V_n \mapsto y \in \mathcal{Y}$ is proper (i.e. any sequence $(x_k, y_k) \in V_n$ such that y_k is convergent in $\mathcal{Y}$ has a convergent subsequence in V_n).

Then, the set $\{y \in \mathcal{Y} \,|\, \xi$ is a regular value of $\Phi(., y)\}$ contains a countable intersection of open dense sets (and hence is dense) in $\mathcal{Y}$.

B Exponential dichotomies, shifted exponential dichotomies and applications to transversality

In the proof of the generic non-existence of homoindexed connections between equilibria, we will use a functional characterization of the transversality property. A main tool in this proof is the notion of dichotomy. Also, when studying the asymptotics of the solutions of linearized equations along connecting orbits connecting a hyperbolic periodic orbit to another critical element, we will need to consider iterates of maps and thus, in particular, discrete shifted dichotomies. We begin this appendix by recalling the definition and the basic properties of the exponential dichotomies and shifted exponential dichotomies. Then, we give some applications to the scalar parabolic equation on S^1.
The results, that we recall here, are all contained in [22], [36], [19], [24], [10], [8], [9], [45] and [46].

B.1 Generalities

Let X be a Banach space and J be an interval in $\mathbb{Z}$. Let $\{T_n \in L(X,X) \,|\, n \in J\}$ be a family of continuous maps from X into X. We define the family of evolution operators

$$T(m,m) \,=\, I \;, \quad T(n,m) \,=\, T_{n-1} \circ \ldots \circ T_m \;, \forall\, n > m \;\text{ in } J \;,$$

where I is the identity in X.

Definition B.1. *We say that the family of linear operators T_n, $n \in J$, or the family of evolution operators $\{T(n,m) \,|\, n \geq m$ in $J\}$, admits an* exponential dichotomy *(or* discrete dichotomy*) on the interval J with* exponent $\beta > 0$ *(or* constant $e^{-\beta}$*),* bound $M > 0$ *and* projections $P(n)$ *if there is a family of continuous projections $P(n)$, $n \in J$, such that the following properties hold for any n in J:*

(i) *$T(n,m)P(m) = P(n)T(n,m)$ for $n \geq m$ in J,*

(ii) *the restriction $T(n,m)|_{R(P(m))}$ is an isomorphism of $R(P(m))$ onto $R(P(n))$, for $n \geq m$ in J, and $T(m,n)$ is defined as the inverse from $R(P(n))$ onto $R(P(m))$,*

(iii) *$\|T(n,m)(I - P(m))\|_{L(X)} \leq Me^{-\beta(n-m)}$ for $n \geq m$ in J,*

(iv) *$\|T(n,m)P(m)\|_{L(X)} \leq Me^{-\beta(m-n)}$ for $n \leq m$.*

If $\dim R(P(n)) = k$ is finite for some $n \in J$, the equality holds for all $n \in J$ by (ii) and we say that the dichotomy has finite *rank* k.

Remarks:

1) We could also have defined (continuous) exponential dichotomies on an interval $J \subset \mathbb{R}$ (see [22] for instance). However, here we restrict our discussion to discrete dichotomies on time intervals $J \subset \mathbb{Z}$ for two reasons. First, as it was already pointed out by Henry in [24], the theory of dichotomies is much simpler with discrete time and there is little loss in restricting to this case. Moreover, in our applications to the asymptotics near periodic orbits, we really need to work with maps and not with continuous evolution operators. Finally, we remark that constructing discrete dichotomies from continuous ones or conversely is an easy task (see Theorem 1.3 of [24] for example).

2) Let A be a sectorial operator on a Banach space Y. For any $J \subset \mathbb{Z}$ and $n \geq m$ in J, we define $T_n = e^A$ (independent of n) and $T(n,m) = e^{A(n-m)}$ on the Banach space $X = Y^\alpha$, $\alpha \in [0,1]$. Then $\{T(n,m) \,|\, n \geq m$ in $J\}$ is a family of evolution operators on X. If the spectrum $\sigma(A)$ satisfies $\sigma(A) \cap \{\mu \,|\, \operatorname{Re}\mu = 0\} = \emptyset$, then, for any $t_0 > 0$, we can define the projection P by

$$P \,=\, I - \frac{1}{2i\pi} \int_{|z|=1} (zI - e^{At_0})^{-1} \,\mathrm{d}z \;. \tag{B.1}$$

And $T(n,m)$ has an exponential dichotomy with projection P. If the spectrum $\sigma(A)$ satisfies $\sigma(A) \cap \{\mu \,|\, -\beta \leq \operatorname{Re}\mu \leq \beta\} = \emptyset$ for some $\beta > 0$, then there exists a positive constant M such that $T(n,m)$ has an exponential dichotomy with projection P, exponent β and constant M. If the essential spectrum of e^{At} is strictly inside the unit circle, the dichotomy has finite rank. This is the case for the linear parabolic equation.

Let again $\{T_n \in L(X,X) \,|\, n \in J\}$ be a family of continuous maps from X into X. We define the family of operators on X^*, given by

$$T^*(m,n) = (T(n,m))^* \,.$$

Definition B.2. *We say that the family of maps T_n^*, $n \in J$, or the family of evolution operators $\{T^*(m,n) \,|\, n \geq m \text{ in } J\}$, admits a* reverse exponential dichotomy *on the interval J with* exponent $\beta > 0$, bound $M > 0$ *and projections $P^*(t)$ if there is a family of continuous projections $P^*(n)$, $n \in J$, such that the following properties hold, for any n in J:*

(i) *$T^*(m,n)P^*(n) = P^*(m)T^*(m,n)$ for $n \geq m$ in J,*

(ii) *the restriction $T^*(m,n)|_{R(P^*(n))}$ is an isomorphism of $R(P^*(n))$ onto $R(P^*(m))$, for $n \geq m$ in J, and $T^*(n,m)$ is defined as the inverse from $R(P^*(m))$ onto $R(P^*(n))$,*

(iii) *$\|T^*(m,n)(I - P^*(n))\|_X \leq Me^{-\beta(n-m)}$ for $n \geq m$ in J,*

(iv) *$\|T^*(m,n)P^*(n)\|_X \leq Me^{-\beta(m-n)}$ for $n \leq m$.*

The following natural property is proved in [8] for example.

Lemma B.3. *If the family of evolution operators $T(n,m)$, $n,m \in J$ on the Banach space X admits an exponential dichotomy on the interval J, with projections $P(n)$, exponent β and bound M, then $T^*(m,n) = (T(n,m))^*$ admits reverse exponential dichotomy on J with the same exponent and bound and with the projections $P^*(n) = (P(n))^*$.*

In our applications to the asymptotics near a periodic orbit, we cannot directly use the concept of exponential dichotomy since 1 always belongs to the spectrum of the period map $\Pi(p,0)$ associated to a periodic orbit $\gamma(t)$ of (1.1) of minimal period p. For this reason, we also recall the notion of *shifted exponential dichotomy*, which is a generalization of the notion of *exponential dichotomy*. Calling these dichotomies *shifted*, we follow the terminology of [19]; alternatively it is sometimes called *pseudodichotomy*. If $\lambda_1 < 1 < \lambda_2$, it reduces to the usual exponential dichotomy.

Definition B.4. *We say that the family of linear operators T_n, $n \in J$, or the family of linear evolution operators $\{T(n,m) \,|\, n \geq m \text{ in } J\}$, admits a* shifted exponential dichotomy *on the interval J with* gap $[\lambda_1, \lambda_2]$, bound $K > 0$ *and projections $P(n)$, $Q(n) = I - P(n)$ if there is a family of continuous projections $P(n)$, $n \in J$, such that the following properties hold for any n in J:*

i) *$T(n,m)P(m) = P(n)T(n,m)$ for $n \geq m$ in J,*

ii) *the restriction $T(n,m)|_{R(P(m))}$ is an isomorphism of $R(P(m))$ onto $R(P(n))$, for $n \geq m$ in J, and on $R(P(n))$, $T(m,n)$ is defined as the inverse from $R(P(n))$ onto $R(P(m))$,*

iii) *$\|T(n,m)(I - P(m))\|_X \leq K\lambda_1^{n-m}$ for $n \geq m$ in J,*

iv) *$\|T(n,m)P(m)\|_X \leq K\lambda_2^{n-m}$ for $n \leq m$.*

We also say that the family of operators T_n, $n \in J$, or the family of evolution operators $\{\Phi(n,m) \,|\, n \leq m \text{ in } J\}$, admits a reverse shifted exponential dichotomy *on the interval J with* gap $[\lambda_1, \lambda_2]$, bound $K > 0$ *and projections $P(n)$, $Q(n) = I - P(n)$ if the above properties hold with $n \geq m$ in J (resp. $n \leq m$ in J) replaced by $n \leq m$ in J (resp. $n \geq m$ in J).*

Before describing the applications of these abstract notions to our problem, we recall two properties, which are very useful in proving that linearized equations along connecting orbits of (1.1) admit exponential dichotomies. The next roughness property is given in [24, Corollary 1.9] and is a consequence of [24, Theorem 1.5 and Lemma 1.6] (see also [22, Theorem 7.6.7] as well as Proposition C.1 below). The extension of the result below to trichotomies is stated in [19].

Theorem B.5. Perturbation of exponential dichotomies

Let $n_0 > 0$ (resp. $n_0 < 0$) be a given integer and let $T(n+1,n)$, $n \in \mathbb{Z}^+$, $n \geq n_0$, (respectively $n \in \mathbb{Z}^-$, $n \leq n_0$), be a discrete family of evolution operators on a Banach space X admitting a discrete dichotomy on $[n_0, +\infty)$ (respectively on $(-\infty, n_0]$), with exponent β, constant M and projections $P^T(n)$. Let $M_1 > M$, $0 < \beta_1 < \beta$ and $0 < \varepsilon \leq (\frac{1}{M} - \frac{1}{M_1})\frac{e^{-\beta_1} - e^{-\beta}}{1 + e^{-(\beta+\beta_1)}}$. If $S(n+1,n)$, $n \in \mathbb{Z}^+$, $n \geq n_0$ (respectively $n \in \mathbb{Z}^-$, $n \leq n_0$), is a discrete family of evolution operators on X with $\|S(n+1,n) - T(n+1,n)\|_{L(X,X)} \leq \varepsilon$, for all $n \geq n_0$ in $\mathbb{Z}^+$ (respectively for all $n \leq n_0$ in $\mathbb{Z}^-$), then $S(n+1,n)$ admits a discrete dichotomy on $[n_0, +\infty)$ (respectively on $(-\infty, n_0]$) with exponent β_1, constant M_1 and projections $P^S(n)$. Moreover, the projections $P^S(n)$ satisfy $\sup_n \|P^T(n) - P^S(n)\|_{L(X,X)} = O(\sup_n \|T(n+1,n) - S(n+1,n)\|_{L(X,X)})$ as $\sup_n \|T(n+1,n) - S(n+1,n)\|_{L(X,X)}$ tends to 0. Furthermore, there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, if $T(n+1,n)$ has a dichotomy of finite rank m, then the dichotomy of $S(n+1,n)$ is also of finite rank m.

The next result, which is proved in [24, Theorem 1.14], allows to extend dichotomies from smaller to larger "time intervals". The continuous version of it is proved in [36] and the extension to trichotomies is stated in [19].

Theorem B.6. Extension of exponential dichotomies

Let $T(n+1,n)$, $n \in \mathbb{Z}^-$, $n < n_1$, be a discrete family of evolution operators on a Banach space X, and suppose that, for $n < n_0$, with $n_0 < n_1$, $T(n+1,n)$ admits a discrete

dichotomy with finite rank m, exponent β, constant M and projections $P(n)$, $n \le n_0$. Assume moreover that $T(n_1, n_0)\big|_{R(P(n_0))}$ is injective. Then, $T(n+1, n)$, for $n < n_1$, admits a discrete dichotomy with the same rank m, same exponent and projections $\tilde{P}(n)$, $n \le n_1$, such that $\|P(n) - \tilde{P}(n)\|_{L(X)} \to 0$ exponentially as n goes to $-\infty$. The constant M has to be replaced by a larger one.
Let $T(n+1, n)$, $n \in \mathbb{Z}^+$, $n \ge n_0$, be a discrete family of evolution operators on a Banach space X. Suppose that, for $n \ge n_1$, with $n_0 < n_1$, $T(n+1, n)$ admits a discrete dichotomy with finite rank m, exponent β, constant M and projections $P(n)$, $n \ge n_1$. Assume moreover that the adjoint operator $T^*(n_0, n_1)\big|_{R(P^*(n_1))}$ is injective, then $T(n+1, n)$, for $n \ge n_0$, admits a discrete dichotomy with the same rank m, same exponent β and projections $\tilde{P}(n)$, $n \ge n_0$, such that $\|P(n) - \tilde{P}(n)\|_{L(X)} \to 0$ exponentially as n goes to $+\infty$. The constant M has to be replaced by a larger one.
In both cases, the convergence of $\|P(n) - \tilde{P}(n)\|_{L(X)}$ is of order $O(e^{-2\beta|n|})$.

B.2 Dichotomies and Fredholm property

In Section 5, in proving the generic non-existence of homoindexed connecting orbits, we use a functional characterization of the transversality, which is based on dichotomies and a Fredholm property. In this appendix, we quickly recall the tools and basic facts, which lead to this functional characterization.

Let X be a Banach space. We introduce the following spaces

$$\mathcal{Z} = \ell^\infty(\mathbb{Z}, X) \ , \ (\text{respectively } \mathcal{Z}^\pm = \ell^\infty(\mathbb{Z}^\pm, X)) \ .$$

Given a family $T(n, m)$ of evolution operators on X for $n, m \in \mathbf{Z}$, we define the mapping $\mathcal{L}$ from $X^{\mathbf{Z}}$ into $X^{\mathbf{Z}}$ (respectively the mapping $\mathcal{L}^\pm$ from $X^{\mathbf{Z}^\pm}$ into $X^{\mathbf{Z}^\pm}$) by

$$(\mathcal{L}Y)(n) = Y(n+1) - T(n+1, n)Y(n) \ , \quad \forall n \in \mathbf{Z} \ , \tag{B.2}$$

(respectively $(\mathcal{L}^\pm Y)(n) = Y(n+1) - T(n+1, n)Y(n)$, $\forall n \in \mathbf{Z}^\pm$).
We say that $Y = \{Y(n)\}_{n \in \mathbf{Z}}$ belongs to the domain $D(\mathcal{L})$ if $\sup_{n \in \mathbf{Z}} \|Y(n+1) - T(n+1, n)Y(n)\|_X < \infty$ (likewise, we define $D(\mathcal{L}^\pm)$). This allows to define the operator $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{Z} \to \mathcal{Z}$ by (B.2) (likewise, we may define the operator $\mathcal{L}^\pm : D(\mathcal{L}^\pm) \subset \mathcal{Z}^\pm \to \mathcal{Z}^\pm$).

In [22, Theorem 7.6.5], Henry has given the following characterization of the existence of a discrete dichotomy for $T(n+1, n)$ (see also [24]; for a finite-dimensional version, see [45], [46]). The family of evolution operators $T(n+1, n)$ has a discrete dichotomy if and only if, for every bounded sequence $F \in \mathcal{Z}$, there is a unique bounded sequence $Y \in \mathcal{Z}$ with $(\mathcal{L}Y)(n) := Y(n+1) - T(n+1, n)Y(n) = F(n)$, for any $n \in \mathbf{Z}$. Moreover, the unique bounded solution is given by

$$Y(n) = \sum_{k=-\infty}^{+\infty} \mathcal{G}(n, k+1)F(k) \ , \tag{B.3}$$

where $\mathcal{G}(n,m) = T(n,m)(I - P(m))$ for $n \geq m$, $\mathcal{G}(n,m) = -T(n,m)P(m)$ for $n < m$, is called the Green function.

Henry has also proved in Theorem 1.13 of [24] that any discrete family of evolution operators $T(n+1,n)$ admits a discrete dichotomy on $\mathbf{Z}$ if and only if the restrictions to both $\mathbf{Z}^+$ and $\mathbf{Z}^-$ have dichotomies and also $X = S_0 \oplus U_0$ where

$$\begin{aligned} U_0 &= \{x_0 \,|\, \exists \{x_n\}_{n \leq 0} \in \mathcal{Z}^- \text{ with } x_{n+1} = T(n+1,n)x_n \,,\, n < 0\} \\ S_0 &= \{x_0 \,|\, \exists \{x_n\}_{n \geq 0} \in \mathcal{Z}^+ \text{ with } x_{n+1} = T(n+1,n)x_n \,,\, n \geq 0\} \,. \end{aligned} \tag{B.4}$$

When the dichotomies in $\mathbf{Z}^+$ and $\mathbf{Z}^-$ have finite rank, the equality $X = S_0 \oplus U_0$ means that they have the same rank.

The previous existence result of a dichotomy on $\mathbf{Z}$ is actually a particular case of the following more general result, which is also proved in Theorem 1.15 of [24] (see also [55], [45], [19] and [8, Theorem 4.a.4] in the case of ordinary differential, functional differential and parabolic equations).

We recall that $\langle \cdot, \cdot \rangle_{X,X^*}$ denotes the duality pairing between X and X^*.

Theorem B.7. Fredholm alternative

Let $T(n+1,n)$ be a discrete family of evolution operators on a Banach space X, admitting discrete dichotomies of finite rank on both $\mathbb{Z}^+$ and $\mathbb{Z}^-$, with projections $P^+(n)$ and $P^-(n)$. Then the operator $\mathcal{L}$, defined by (B.2)*, belongs to $L(D(\mathcal{L}), \mathcal{Z})$ and is a Fredholm operator with index* $\text{ind}(\mathcal{L})$ *given by*

$$\text{ind}(\mathcal{L}) \;=\; \dim(R(P^-(0))) - \dim(R(P^+(0))) \;. \tag{B.5}$$

The codimension $\text{codim}R(\mathcal{L})$ *of $R(\mathcal{L})$ is given by* $\text{codim}R(\mathcal{L}) = \dim[R(I - P^{-*}(0)) \cap R(P^{+*}(0))]$.
A sequence $F \in \mathcal{Z}$ belongs to $R(\mathcal{L})$ if and only if

$$\sum_{n=-\infty}^{+\infty} \langle F(n), \Psi(n+1) \rangle_{X,X^*} \;=\; 0 \;, \tag{B.6}$$

for every sequence $\Psi(n) = T^*(n,0)\Psi_0$, $\Psi_0 \in X^*$, which is bounded.

The proof of Theorem B.7 uses the following two auxiliary lemmas. First, recall that, for any operator $Q \in L(X)$, one has

$$\ker(Q^*) \;=\; (R(Q))^{\perp} \;, \tag{B.7}$$

where, for any subspace $X_0 \in X$, $X_0^{\perp} = \{\psi \in X^* \,|\, \langle x, \psi \rangle = 0 \,, \forall x \in X_0\}$.
If $Q \in L(X)$ is a projection, we have, in addition,

$$R(I - Q^*) \;=\; \ker(Q^*) \;=\; (R(Q))^{\perp} \;. \tag{B.8}$$

Lemma B.8. *Let $T(n,m)$ be an evolution operator admitting discrete dichotomies of finite rank on both $\mathbb{Z}^+$ and $\mathbb{Z}^-$. Then, any element $\Psi_0 \in X^*$ belongs to $(R(P^-(0)))^\perp \cap (R(I - P^+(0)))^\perp$ if and only if the sequence*

$$\Psi(m) \,=\, T^*(m,0)\Psi_0 \ , \quad m \in \mathbb{Z} \ ,$$

(which is defined for all m due to the property (ii) of the reverse dichotomy) is bounded (that is belongs to $\ell^\infty(\mathbb{Z}, X^*)$). In this case, $\Psi(m)$ belongs to $R(I - P^{-*}(m))$ for $m \leq 0$ and to $R(P^{+*}(m))$ for $m \geq 0$.

The next lemma emphasizes the formulas given in (B.3).

Lemma B.9. *We assume that the hypotheses of Theorem B.7 hold. Then,*
(i) if $F \in \mathcal{Z}^-$, there exists $Y \in \mathcal{Z}^-$ such that $F = \mathcal{L}^- Y$ if and only if, for any $n \in \mathbb{Z}^-$,

$$\begin{aligned} Y(n) \,=\, T(n,0)P^-(0)Y(0) - \sum_{k=n}^{-1} T(n,k+1)P^-(k+1)F(k) \\ + \sum_{k=-\infty}^{n-1} T(n,k+1)(I - P^-(k+1))F(k) \ ; \end{aligned} \tag{B.9}$$

(ii) similarly, if $F \in \mathcal{Z}^+$, there exists $Y \in \mathcal{Z}^+$ such that $F = \mathcal{L}^+ Y$ if and only if, for any $n \in \mathbf{Z}^+$,

$$\begin{aligned} Y(n) = T(n,0)(I - P^+(0))Y(0) + \sum_{k=0}^{n-1} T(n,k+1)(I - P^+(k+1))F(k) \\ - \sum_{k=n}^{+\infty} T(n,k+1)P^+(k+1)F(k) \ . \end{aligned} \tag{B.10}$$

We remark that these "variation of constants formulas" generalize the formula (B.3). They have already been given in [46] under this discrete form in the finite dimensional context (see [46, Formula (13) of Lemma 2.7]) and they are contained in Theorem 1.15 of [24]. In the continuous case for parabolic equations, they are well-known and can be found in [22] and in [8].

B.3 Application to the parabolic equation on S^1

In this section, we apply the previous abstract results to the homoclinic and heteroclinic orbits between equilibrium points of the scalar parabolic equation (1.1) and we give some equivalent formulations of transversality.

We assume in this section that $\tilde{u}(t) \in \mathcal{C}^0_b(\mathbb{R}, H^s(S^1))$ is a bounded trajectory of $S(t) = S_f(t)$ satisfying $\lim_{t\to\pm} \tilde{u}(t) = e^\pm$, where $e^\pm$ are hyperbolic equilibria of finite Morse index

$i(e^{\pm})$. We recall that $\tilde{u}$ belongs to $C^0_b(\mathbb{R}, H^2(S^1)) \cap C^\theta(\mathbb{R}, H^s(S^1))$, where $0 < \theta \leq 1$. We consider the linearized equation along $\tilde{u}$, that is, the linear equation for $t \geq s$,

$$v_t(t) = v_{xx}(t) + D_u f(x, \tilde{u}, \tilde{u}_x)v(t) + D_{u_x} f(x, \tilde{u}, \tilde{u}_x)v_x(t) \equiv C_{\tilde{u}}(t)v(t) \ , \ t > \sigma \ , \ v(\sigma) = v_0 \ . \tag{B.11}$$

We recall that, for any $v_0 \in L^2(S^1)$, for any $\sigma \in \mathbb{R}$, there exists a unique classical solution $v(t) \in \mathcal{C}^0([\sigma, +\infty), L^2(S^1)) \cap \mathcal{C}^0((\sigma, +\infty), H^s(S^1))$ of (B.11) such that $v(\sigma) = v_0$. We set $T(t, \sigma)v_0 = T_{\tilde{u}}(t, \sigma)v_0 = v(t)$.

We next introduce the adjoint linearized equation to (B.11), that is, the linear equation for $\sigma \leq t$,

$$\partial_t \psi(\sigma) = -C^*_{\tilde{u}}(\sigma)\psi(\sigma) \ , \quad \sigma \leq t \ , \quad \psi(t) = \psi_0 \ . \tag{B.12}$$

Since $\tilde{u}$ belongs to $C^\theta(\mathbb{R}, H^s(S^1))$ for any $\theta \leq 1$ and thus that $(C_{\tilde{u}}(t) - \partial_{xx})^*$ is locally Hölder continuous with exponent $\gamma > s/2$ as a mapping from $\mathbb{R}$ into $L(H^s(S^1), L^2(S^1))$, (B.12) has a unique classical solution $\psi(\sigma) = \psi(\sigma, t; \psi_0)$ on $(-\infty, t)$, for any $\psi_0 \in L^2(S^1)$ (see [22, Theorem 7.3.1] for example). We denote this solution $\psi(s) := \psi(s, t; \psi_0)$.

With $T_{\tilde{u}}(t, s)$, we associate the adjoint evolution operator on $L^2(S^1)$, given by

$$T^*_{\tilde{u}}(\sigma, t) = (T_{\tilde{u}}(t, \sigma))^* \ , \quad t \geq \sigma \ . \tag{B.13}$$

It is well-known (see [22, Theorem 7.3.1]) that, for any $\psi_0 \in L^2(S^1)$,

$$T^*_{\tilde{u}}(\sigma, t_0)\psi_0 = \psi(\sigma, t_0; \psi_0) \ , \quad \forall \sigma \leq t_0 \ . \tag{B.14}$$

We also remark that the adjoint operator $(T_{\tilde{u}}(t, \sigma))^*$ is injective and that its range is dense in $H^s(S^1)$.

From now on, we discretize the evolution operators. We fix a time step $\tau > 0$ and consider the discretizations $S(n\tau)$ and $T_{\tilde{u}}(n\tau, m\tau)$, with $n, m \in \mathbb{Z}$. The hyperbolic equilibria $e^{\pm}$ of $S(t)$, their stable and unstable sets coincide with those of the discretization $S(n\tau)$; the discretization of the trajectory $\tilde{u}(t)$ connecting e^- to e^+ is a heteroclinic or homoclinic orbit connecting these equilibria for the discretized semi-flow. Let $\beta^{\pm} > 0$ be chosen such that

$$\sigma(e^{L_{e^{\pm}}}) \cap \{z \,|\, e^{-\beta^{\pm}} \leq |z| \leq e^{\beta^{\pm}}\} = \emptyset \ ,$$

where the linearized operator $L_{e^{\pm}}$ has been defined in (2.4). As explained in the remarks of Section B.1, $e^{L_{e^{\pm}}\tau}$ admits an exponential dichotomy with projection $P^{\pm}$ (see (B.1)), exponent $\beta^{\pm}$ and constant M in $H^{2\alpha}(S^1)$ for any $\alpha \in [0, 1)$. Thus, we will be able to deduce from Theorems B.5 and B.6 that $T_{\tilde{u}}(n\tau, (n-1)\tau)$ admits exponential dichotomies on $\mathbb{Z}^-$ and on $\mathbb{Z}^+$ of respective index $i(e^-)$ and $i(e^+)$ in $H^{2\alpha}(S^1)$, for any $\alpha \in [0, 1)$. We will only give a sketch of the proof. For a more detailed proof in the case of ordinary differential equations (resp. functional differential equations, resp. parabolic equations, resp. in the case damped wave equations), we refer the reader to [45], [46] (resp. to [36], [8] and [9]).

Theorem B.10. *For any $\beta_1^\pm \in (0, \beta^\pm)$, the discrete family of evolution operators $T(n,m) = T_{\tilde{u}}(n\tau, m\tau)$ admits exponential dichotomies on $\mathbb{Z}^\pm$ in $L^2(S^1)$ (resp. $H^s(S^1)$) of finite rank equal to the index $i(e^\pm)$ of $e^\pm$, with exponent $\beta_1^\pm$, constant $M^\pm$ and projections $\tilde{P}_{\tilde{u}}^\pm(n)$ (resp. $P_{\tilde{u}}^\pm(n)$), satisfying*

$$\lim_{n\to\pm\infty} \|\tilde{P}_{\tilde{u}}^\pm(n) - P^\pm\|_{L(L^2,L^2)} = 0 \ , \ \textit{(resp.} \ \lim_{n\to\pm\infty} \|P_{\tilde{u}}^\pm(n) - P^\pm\|_{L(H^s,H^s)} = 0 \ \textit{)} \ . \tag{B.15}$$

Moreover, $P_{\tilde{u}}^\pm$ is the restriction of $\tilde{P}_{\tilde{u}}^\pm$ to $H^s(S^1)$, that is,

$$\tilde{P}_{\tilde{u}}^\pm(n)_{|_{H^s(S^1)}} = P_{\tilde{u}}^\pm(n) \ . \tag{B.16}$$

Proof: In the proof of Corollary C.7 below, (see (C.25)), we show that, for $n \in \mathbb{Z}^\pm$, with $|n|$ large enough, we have the exponential asymptotic convergence

$$\|T_{\tilde{u}}((n+1)\tau, n\tau) - e^{L_{e^\pm}}\|_{L(H^s,H^s)} \le Ce^{-C|n|} \ , \tag{B.17}$$

where C is a positive constant. Thus we may apply Theorem B.5, which implies that there exists $n_0 \in \mathbb{Z}^+$ such that $T_{\tilde{u}}((n+1)\tau, n\tau)$ admits an exponential dichotomy on $\mathbb{Z}^+$ (resp. $\mathbb{Z}^-$) in $H^s(S^1)$, for $n \ge n_0$ (resp. $n \le -n_0$) of finite rank $i(e^+)$ (resp. $i(e^-)$) and projections P_T^+ (resp. P_T^-) satisfying the properties of Theorem B.5. Applying Theorem B.6 with $X = H^s(S^1)$, we next extend these dichotomies in $H^s(S^1)$ to $\mathbb{Z}^\pm$ and thus prove that $T_{\tilde{u}}((n+1)\tau, n\tau)$ admits exponential dichotomies on $\mathbb{Z}^\pm$ in $H^s(S^1)$ of finite rank equal to $i(e^\pm)$, with exponent $\beta_1^\pm$, constant $M^\pm$ and projections $P_{\tilde{u}}^\pm(n)$, satisfying the property (B.15). Using next [22, Lemma 7.6.2 and Exercise 5 of Chapter 7], we may extend these projections $P_{\tilde{u}}^\pm(n)$ in $H^s(S^1)$ to projections $P_{\tilde{u}}^\pm$ in $L^2(S^1)$ satisfying the properties (B.15) and (B.16).

An alternative proof consists in showing first that, for any $n \in \mathbb{Z}^\pm$, with $|n|$ large enough, we have the exponential asymptotic convergence

$$\|T_{\tilde{u}}((n+1)\tau, n\tau) - e^{L_{e^\pm}}\|_{L(L^2,H^s)} \le Ce^{-C|n|} \ , \tag{B.18}$$

where C is a positive constant. Then we may apply Theorem B.5 in the space $L^2(S^1)$, which implies that there exists $n_0 \in \mathbb{Z}^+$ such that $T_{\tilde{u}}((n+1)\tau, n\tau)$ admits an exponential dichotomy on $\mathbb{Z}^+$ (resp. $\mathbb{Z}^-$) in $L^2(S^1)$, for $n \ge n_0$ (resp. $n \le -n_0$) of finite rank $i(e^+)$ (resp. $i(e^-)$) and projections $\tilde{P}_T^+$ (resp. $\tilde{P}_T^-$) satisfying the properties of Theorem B.5. Applying Theorem B.6 with $X = L^2(S^1)$, we next extend these dichotomies in $L^2(S^1)$ to $\mathbb{Z}^\pm$ and thus prove that $T_{\tilde{u}}((n+1)\tau, n\tau)$ admits exponential dichotomies on $\mathbb{Z}^\pm$ in $L^2(S^1)$ of finite rank equal to $i(e^\pm)$, with exponent $\beta_1^\pm$, constant $M^\pm$ and projections $\tilde{P}_{\tilde{u}}^\pm(n)$, satisfying the property (B.15). We remark that, by the property (ii) of the definition B.1 and by the property (B.18), the image of $\tilde{P}_{\tilde{u}}^\pm(n)$ belongs to $H^s(S^1)$, which implies, together with [22, Lemma 7.6.2 of Chapter 7], that the restrictions $P_{\tilde{u}}^\pm$ of the projections $\tilde{P}_{\tilde{u}}^\pm(n)$ to $H^s(S^1)$ define an exponential dichotomy of $T_{\tilde{u}}((n+1)\tau, n\tau)$ on $\mathbb{Z}^\pm$ in $H^s(S^1)$ of finite rank equal to $i(e^{\pm})$, with exponent $\beta_1^{\pm}$, constant $M^{\pm}$. $\square$

We notice that Theorem B.10 and Lemma B.3 imply that $T^*(n, n{+}1) = T_{\tilde{u}}((n{+}1)\tau, n\tau)^*$ admits a reverse exponential dichotomy on $\mathbb{Z}^{\pm}$ in $L^2(S^1)$ (resp. $H^{-s}(S^1)$) with rank $i(e^{\pm})$, exponent $\beta_1^{\pm}$ and projections $(\tilde{P}_{\tilde{u}}^{\pm}(n))^*$ (resp. $(P_{\tilde{u}}^{\pm}(n))^*$).

Lemma 4.2 (on page 376) and Appendix C of [10] yield the important characterization of the range of $P_{\tilde{u}}^{\pm}(n)$ given in the next proposition.

Proposition B.11. *We have the following equalities in $H^s(S^1)$,*

$$\begin{aligned} R(P_{\tilde{u}}^-(n)) &= T_{\tilde{u}(n)}W^u(e^-) \ , \quad \forall n \in \mathbb{Z}^- \\ R(I - P_{\tilde{u}}^+(n)) &= T_{\tilde{u}(n)}W^s(e^+) \ , \quad \forall n \in \mathbb{Z}^+ \ . \end{aligned} \tag{B.19}$$

Let $\tilde{u}(t)$ belongs to $W^u(e^-) \cap W^s(e^+)$. We say that the bounded orbit $\tilde{u}$ is *transverse* at $\tilde{u}(0)$ if

$$W^u(e^-) \pitchfork_{\tilde{u}(0)} W^s_{loc}(e^+) \ ,$$

which means that $T_{\tilde{u}(0)}W^u(e^-)$ contains a closed complement of $T_{\tilde{u}(0)}W^s(e^+)$ in $H^s(S^1)$ (notice that, as $W^u(e^-) \cap W^s(e^+)$ are immersed manifolds in $H^s(S^1)$, this notion is well-defined, see [34, Page 23]). It is easily seen that, since the linearized operator $T(t, \sigma)$ is injective and has dense range in $H^s(S^1)$, the above condition implies that, for any $t \in \mathbb{R}$,

$$W^u(e^-) \pitchfork_{\tilde{u}(t)} W^s_{loc}(e^+) \ ,$$

which allows to simply say that the orbit $\tilde{u}$ is a *transverse* connecting orbit.

From the previous proposition and the equalities (B.7) and (B.8) as well as the property that the range of $\tilde{P}_{\tilde{u}}^{\pm}$ is contained in $H^s(S^1)$, we at once deduce the following equivalences.

Proposition B.12. *(i) The trajectory $\tilde{u}(t)$ is transverse in $H^s(S^1)$ if and only if*

$$R(P_{\tilde{u}}^-(0)) + R(I - P_{\tilde{u}}^+(0)) = H^s(S^1) \ , \tag{B.20}$$

or equivalently, since $R(P_{\tilde{u}}^-(0))$ is finite-dimensional and, thus, this sum is closed,

$$[R(P_{\tilde{u}}^-(0))]^{\perp} \cap [R(I - P_{\tilde{u}}^+(0))]^{\perp} = \{0\} \ , \tag{B.21}$$

or also

$$R(I - P_{\tilde{u}}^-(0)^*) \cap R(P_{\tilde{u}}^+(0)^*) = \{0\} \ . \tag{B.22}$$

(ii) Moreover, the trajectory $\tilde{u}(t)$ is transverse in $H^s(S^1)$ if and only if

$$R(\tilde{P}_{\tilde{u}}^-(0)) + R(I - \tilde{P}_{\tilde{u}}^+(0)) = L^2(S^1) \ , \tag{B.23}$$

or equivalently,

$$[R(\tilde{P}_{\tilde{u}}^-(0))]^{\perp} \cap [R(I - \tilde{P}_{\tilde{u}}^+(0))]^{\perp} = \{0\} \ , \tag{B.24}$$

or also

$$R(I - \tilde{P}_{\tilde{u}}^-(0)^*) \cap R(\tilde{P}_{\tilde{u}}^+(0)^*) = \{0\} \ . \tag{B.25}$$

Applying Lemma B.8 and Proposition B.12, we obtain the next characterization of transversality.

Corollary B.13. *The trajectory $\tilde{u}(t)$ is transverse if and only if there does not exist any element $\psi_0 \in (H^s(S^1))^*$, $\psi_0 \neq 0$, such that the sequence $(T^*_{\tilde{u}}(n,0)\psi_0)_{n\in\mathbb{Z}}$ is bounded in $(H^s(S^1))^*$ or equivalently, if and only if there does not exist any element $\psi_1 \in L^2(S^1)$, $\psi_1 \neq 0$, such that the sequence $(T^*_{\tilde{u}}(n,0)\psi_1)_{n\in\mathbb{Z}}$ is bounded in $L^2(S^1)$.*

Finally we show how to apply Theorem B.7 to obtain a corollary, which plays a crucial role in the proof of the genericity of the non-existence of homoindexed orbits between equilibrium points. We introduce the operator $\mathcal{L}_{\tilde{u}} \equiv \mathcal{L}$ defined in (B.2) with $T(n,m) = T_{\tilde{u}}(n\tau, m\tau)$, $X = H^s(S^1)$ and $\mathcal{Z} = \ell^\infty(\mathbb{Z}, X)$. Likewise, we introduce the extension $\tilde{\mathcal{L}}_{\tilde{u}}$ of $\mathcal{L}_{\tilde{u}}$ to $\tilde{\mathcal{Z}} = \ell^\infty(\mathbb{Z}, L^2(S^1))$. We notice that, due to the smoothing properties of $T_{\tilde{u}}(n\tau, m\tau)$, a sequence $F \in \mathcal{Z}$ belongs to $R(\mathcal{L}_{\tilde{u}})$ if and only if $F \in \mathcal{Z}$ belongs to $R(\tilde{\mathcal{L}}_{\tilde{u}})$. This remark, Theorem B.7 and Corollary B.13 imply the following results.

Corollary B.14. Functional characterization of transversality
The operators $\mathcal{L}_{\tilde{u}} : D(\mathcal{L}_{\tilde{u}}) \to \mathcal{Z}$ and $\tilde{\mathcal{L}}_{\tilde{u}} : D(\tilde{\mathcal{L}}_{\tilde{u}}) \to \tilde{\mathcal{Z}}$ defined above are Fredholm operators of index $i(e^-) - i(e^+)$. In particular, the codimension of $R(\mathcal{L}_{\tilde{u}})$ in $H^s(S^1)$ and of $R(\tilde{\mathcal{L}}_{\tilde{u}})$ in $L^2(S^1)$ is equal to codim $[R(\tilde{P}^-_{\tilde{u}}(0)) + R(I - \tilde{P}^+_{\tilde{u}}(0))]$.
Moreover, a sequence $F \in \mathcal{Z}$ belongs to $R(\mathcal{L}_{\tilde{u}})$ if and only if

$$\sum_{n=-\infty}^{+\infty} \langle F(n), \Psi(n+1)\rangle_{L^2(S^1)} = 0 \ ,$$

for every bounded sequence $\Psi(n) = T^*_{\tilde{u}}(n\tau, 0)\Psi_0$, $\Psi_0 \in L^2(S^1)$. Finally, the connecting orbit $\tilde{u}(t)$ is transverse if and only if $\mathcal{L}_{\tilde{u}}$ is surjective or also if and only if $\tilde{\mathcal{L}}_{\tilde{u}}$ is surjective.

C Asymptotics of solutions of perturbations of linear autonomous equations

In this section, X denotes a Banach space and J an interval of $\mathbb{Z}$.

In several proofs of the previous sections, we need to know the asymptotics of the bounded solutions (as $t \to \pm\infty$) of the linearized equations along orbits, connecting hyperbolic equilibrium points or periodic orbits of (1.1). These asymptotics will be described in the second and third sections of this appendix.

C.1 Abstract results

Here, we are going to describe these asymptotics for a general linearized equation or for iterates of a general linearized mapping. Thus, in the first place, we are interested in the asymptotic behaviour of the bounded sequences $u(n)$, $n \in \mathbb{Z}^+$ (resp. $n \in \mathbb{Z}^-$), defined by

$$u(n+1) = Tu(n) + \Sigma(n)u(n) \equiv L(n)u(n) \ , \tag{C.1}$$

where $T \in L(X)$ and $\Sigma(n) \in L(X)$, for any $n \in \mathbb{Z}^+$ (resp. $n \in \mathbb{Z}^-$).

All the statements given in this appendix are already known and are mainly results or generalizations of results of [23], [24], [10] and [9]. The main theorems C.5 and C.6 are a refinement of Theorem B.5 of [10] and have been proved in Appendix B of [9]. Here, we closely follow Appendix B of [9]. We want to point out that all the statements contained in this appendix are more or less common knowledge, at least in finite dimensions. For additional references, see [12], [45], [46], [47], [19] and [55] for example.

For $n \geq m$ in J, we define the evolution operator $\Phi(n,m) = L(n-1) \circ \ldots \circ L(m)$.

For any linear operator $T : X \to X$, we denote by $\mathcal{R}(T)$ the set of all nonnegative numbers ρ for which

$$\sigma(T) \cap \{z \in \mathbb{C} \,|\, |z| = \rho\} \neq \emptyset \ .$$

For $\rho \notin \mathcal{R}(T)$, we denote by P_ρ, Q_ρ the spectral projections associated with the partition of the spectrum $\sigma(T)$ into its subsets $\sigma(T) \cap \{|z| > \rho\}$ and $\sigma(T) \cap \{|z| < \rho\}$ respectively. The following proposition gives a sufficient condition for $L(t)$ defined by (C.1) to admit a shifted dichotomy.

Proposition C.1. *Let $L(n) = T + \Sigma(n)$ where T, $\Sigma(n)$, $n \in \mathbb{Z}^+$, belong to $L(X)$ and $\Sigma(n)$ satisfies the asymptotic condition*

$$\|\Sigma(n)\|_{L(X)} \ = \ O(r^n) \ , \ \textit{for some } r < 1, \ \textit{when } n \to +\infty \ . \tag{C.2}$$

Assume that $0 < \rho_1^* < \rho_1 \leq \rho_2 < \rho_2^*$ are such that $\sigma(T) \cap \{\rho_1^* \leq |z| \leq \rho_2^*\} = \emptyset$, which implies that $Q_{\rho_2^*} - Q_{\rho_1^*} = 0$. Suppose also that T admits shifted exponential dichotomy with gap $[\rho_1^*, \rho_2^*]$ and with projections $P_{\rho_1^*}$, $Q_{\rho_1^*}$. Then the family $L(\cdot)$ admits shifted dichotomy on $J = \mathbb{Z}^+$ with gap $[\rho_1, \rho_2]$ and projections $P(n)$, $Q(n)$. Moreover, when $n \in \mathbb{Z}^+$ is sufficiently large,

$$\|Q(n) - Q_{\rho_1}\|_{L(X)} \ = \ O(r^n) \ , \ (\textit{ resp. } \|P(n) - P_{\rho_2}\|_{L(X)} \ = \ O(r^n)) \ , \tag{C.3}$$

and $\tilde{Q}(n) := Q(n)\big|_{R(Q_{\rho_1})} : R(Q_{\rho_1}) \to R(Q(n))$ (resp. $\tilde{Q}_1(n) = Q_{\rho_1}\big|_{R(Q(n)} : R(Q(n)) \to R(Q_{\rho_1})$) is an isomorphism satisfying

$$\begin{aligned} \max(\|\tilde{Q}(n) - I\|_{L(X)}, \|\tilde{Q}_1(n) - I\|_{L(X)}) \ &\leq \ \|Q(n) - Q_{\rho_1}\|_{L(X)} = O(r^n) \ , \\ \max(\|\tilde{Q}^{-1}(n) - I\|_{L(X)}, \|\tilde{Q}_1^{-1}(n) - I\|_{L(X)}) \ &\leq \ \frac{\|Q(n) - Q_{\rho_1}\|_{L(X)}}{1 - \|Q(n) - Q_{\rho_1}\|_{L(X)}} = O(r^n) \ . \end{aligned} \tag{C.4}$$

The same statement holds on $J = \mathbb{Z}^-$ if the condition "$n \in \mathbb{Z}^+$" is replaced by "$n \in \mathbb{Z}^-$" and $r < 1$ by $r > 1$.

We next recall three results about the asymptotic behaviour of $u(n)$ for n large enough, where $u(n)$ is given by (C.1). The first theorem has been proved by D. Henry (see Theorem 2 in [23]) ; here we state it under the form given by Chen, Chen and Hale in [10, Theorem B.2].

Theorem C.2. *Let $T \in L(X, X)$ and let $0 < \rho_1 \leq \rho_2$ be such that $[\rho_1, \rho_2] \cap \mathcal{R}(T) = \emptyset$. Let $u(n) \neq 0$, $n \geq n_0$ in $\mathbb{Z}^+$, be a sequence in X such that*

$$\lim_{n \to +\infty} \frac{\|u(n+1) - Tu(n)\|_X}{\|u(n)\|_X} = 0 \ . \tag{C.5}$$

Then, either

$$(i) \lim_{n \to +\infty} \frac{\|Pu(n)\|_X}{\|Qu(n)\|_X} = +\infty \ , \ \text{and} \quad \liminf_{n \to +\infty} \|u(n)\|_X^{1/n} \geq \rho_2 \ ;$$

or

$$(ii) \lim_{n \to +\infty} \frac{\|Pu(n)\|_X}{\|Qu(n)\|_X} = 0 \ , \ \text{and} \quad \limsup_{n \to +\infty} \|u(n)\|_X^{1/n} \leq \rho_1 \ ;$$

where $P = P_{\rho_1} = P_{\rho_2}$ and $Q = Q_{\rho_1} = Q_{\rho_2}$. The same property holds if, in the above statements, $n \geq n_0$ in $\mathbb{Z}^+$ and $n \to +\infty$ are replaced by $n \leq -n_0$ in $\mathbb{Z}^-$ and $n \to -\infty$ respectively.

Theorem C.2 gives a close relation between the spectrum of T and the growth rate of $u(n)$. As direct corollary, Chen, Chen and Hale (see [10, Corollary B.3]) have proved the following property.

Theorem C.3. *Let T be a continuous linear operator from X into X such that $\mathcal{R}(T)$ is nowhere dense in $[0, +\infty)$ and let $u(n) \neq 0$, $n \geq n_0$ in $\mathbb{Z}^+$, be a sequence in X satisfying the property* (C.5)*. Then, there exists $\rho \in \mathcal{R}(T)$ such that*

$$\lim_{n \in \mathbb{Z}^+ \to +\infty} \|u(n)\|_X^{1/n} = \rho \ . \tag{C.6}$$

The same property holds when $\mathbb{Z}^+$ is replaced by $\mathbb{Z}^-$.

Remark: For sequences $u(n)$, $n \in \mathbb{Z}^+$, given by the recursion formula (C.1) with $\Sigma(n)$ satisfying the hypothesis (C.2), the condition (C.5) obviously holds. We thus deduce from Theorem C.3 that there exists $\rho \in \mathcal{R}(T)$ such that $\lim_{n \in \mathbb{Z}^+ \to +\infty} \|u(n)\|_X^{1/n} = \rho$.

In this paper, we cannot directly apply this corollary since $\mathcal{R}(T)$ has an accumulation point at 0. However, we know that the sequences $u(n)$ that we will consider do not converge faster to zero than an exponential.

For any $\psi \in X$, for any integers m, n, with $n \geq m$, following the notations of [10], we set,

$$u(n,m;\psi) = \Phi(n,m)\psi \ ,$$

where $\Phi(n,m) = L(n-1) \circ \ldots \circ L(m)$ and $L(n)$ is defined by (C.1). We also introduce the quantity

$$r_\infty(m,\psi) = \limsup_{n \to +\infty} \|u(n,m;\psi)\|_X^{1/n} \ .$$

Let T be a continuous linear operator from X into X such that $\mathcal{R}(T)$ is a bounded sequence converging to 0 and that $\rho_{j+1} \leq \rho_j$, for any integer j. We then introduce the spaces

$$E_j^+(m) = \{\psi \in X \,|\, r_\infty(m,\psi) \leq \rho_j\} \ .$$

Then,

$$\ldots E_2^+(m) \subset E_1^+(m) \subset E_0^+(m) \ .$$

If we assume that, for any $\psi \in X$, $\psi \neq 0$ and any integer m, $r_\infty(m,\psi) > 0$ (that is $\Phi(n,m)\psi$ does not decay faster to 0 than an exponential), then

$$\bigcap_{j=0}^{j=\infty} E_j^+(m) = \{0\} \ .$$

Taking into account the above considerations and following the proof of Corollary B.3 of [10], we obtain the following theorem.

Theorem C.4. *Let T be a continuous linear operator from X into X such that $\mathcal{R}(T)$ is a bounded sequence converging to 0 and that $\rho_{j+1} \leq \rho_j$, for any integer j. We also assume that the property* (C.2) *holds. Let $u(n) \equiv u(n,m;\psi)$, be a sequence in X such that $r_\infty(m,\psi) > 0$. Then, there exists $\rho \in \mathcal{R}(T)$ such that*

$$\lim_{n \in \mathbb{Z}^+ \to +\infty} \|u(n)\|_X^{1/n} \ = \ \rho \ . \tag{C.7}$$

The same property holds when $\mathbb{Z}^+$ is replaced by $\mathbb{Z}^-$.
In particular, if for any $\psi \in X$, $\psi \neq 0$ and any integer m, $r_\infty(m,\psi) > 0$, then

$$X = E_0^+(m) = \{\psi \in X \,|\, r_\infty(m,\psi) \leq \rho_1\} \ ,$$

and

$$E_j^+(m) - E_{j+1}^+(m) = \{\psi \in X \,|\, r_\infty(m,\psi) = \rho_{j+1}\} \ .$$

The next theorem is nothing else as Theorem B.6 of [9] and is actually a refinement of Theorem B.5 of [10] about the asymptotics of sequences $u(n)$ given by the recurrence formula (C.1) when n goes to $\pm\infty$.

Theorem C.5. Convergence to a solution of the asymptotic equation
Let $T \in L(X, X)$ *and suppose that there exist positive numbers* δ_1, $\tilde{\delta}_1$, δ, $\tilde{\delta}$, *with* $0 < \delta_1 < \delta$ *and* $0 < \tilde{\delta}_1 < \tilde{\delta}$, *and* $\rho \in \mathcal{R}(T)$ *such that*

$$\emptyset \neq \sigma(T) \cap \{z \in \mathbb{C} \,|\, \rho - \delta \leq |z| \leq \rho + \tilde{\delta}\} \subset \{z \in \mathbb{C} \,|\, \rho - \delta_1 < |z| < \rho + \tilde{\delta}_1\} \ . \tag{C.8}$$

Suppose also that T *admits shifted dichotomy with gap* $[\rho + \tilde{\delta}_1, \rho + \delta^*]$, *for some* $\delta^* > \tilde{\delta}$ *(resp. with gap* $[\rho - \delta^*, \rho - \delta_1]$, *for some* $\delta^* > \delta$*). Let* $u(n)$, $n \in \mathbb{Z}^+$ *(resp.* $n \in \mathbb{Z}^-$*) be a sequence given by the recurrence formula* (C.1), *with* $\Sigma(n)$ *satisfying the hypothesis* (C.2), *where* $(\rho + \tilde{\delta})r < \rho - \delta$ *(resp. where* $(\rho - \delta)r < \rho + \tilde{\delta}$*), such that*

$$\rho - \delta_1 \leq \lim_{n \to +\infty} \|u(n)\|_X^{1/n} \leq \rho + \tilde{\delta}_1 \qquad \textit{(resp. } \rho - \delta_1 \leq \lim_{n \to -\infty} \|u(n)\|_X^{1/n} \leq \rho + \tilde{\delta}_1\textit{)} \ . \tag{C.9}$$

Denote by T_ρ *the operator*

$$T_\rho \,=\, [Q_{\rho+\tilde{\delta}} - Q_{\rho-\delta}]T[Q_{\rho+\tilde{\delta}} - Q_{\rho-\delta}] \ .$$

Then, there exists a non-vanishing sequence $u_{+\infty}(n)$, $n \in \mathbb{Z}^+$, *(resp.* $u_{-\infty}(n)$, $n \in \mathbb{Z}^-$,*) in* $R(Q_{\rho+\tilde{\delta}} - Q_{\rho-\delta})$ *and satisfying*

$$u_{+\infty}(n+1) \,=\, T_\rho u_{+\infty}(n) \qquad \textit{(resp.} u_{-\infty}(n+1) \,=\, T_\rho u_{-\infty}(n)\textit{)} \ , \tag{C.10}$$

and

$$\|u(n) - u_{+\infty}(n)\|_X \,=\, O((\rho - \delta)^n) \ , \qquad \textit{as } n \longrightarrow +\infty \tag{C.11}$$

$$\textit{(resp. } \|u(n) - u_{-\infty}(n)\|_X \,=\, O((\rho + \tilde{\delta})^n) \ , \qquad \textit{as } n \longrightarrow -\infty \textit{).} \tag{C.12}$$

The previous theorem allows to specify the asymptotics of the bounded sequences $u(n)$ given by (C.1). As consequences of Theorem C.5, one obtains corresponding results for solutions of evolutionary partial differential equations. More precisely, let Y be a Banach space and A be the infinitesimal generator of an analytic semigroup on Y. Let $\alpha \in [0, 1)$ be a real number. We introduce the fractional space $X = Y^\alpha$ and consider the equation

$$\partial_t U(t) \,=\, (A + G(t))U(t) \ , \quad t > 0 \ , \quad U(0) \,=\, U_0 \ , \tag{C.13}$$

where $U(t)$, $t \geq 0$, and U_0 belong to X, and $G : t \in \mathbb{R} \mapsto G(t) \in L(Y^\alpha, Y)$ is such that

$$\|G(t)\|_{L(Y^\alpha, Y)} \,=\, O(e^{-rt}) \ , \qquad \text{as } t \in \mathbb{R} \to +\infty \ , \text{ where } r > 0 \ . \tag{C.14}$$

If $\alpha = 0$, it suffices to assume that A is the generator of a C^0- semigroup.

The proof of the next theorem, which is a consequence of Theorem C.5, follows the lines of the proof of Theorem B.8 of [9].

Theorem C.6. *Suppose that there exist positive constants d_1, $\tilde{d}_1$, d, $\tilde{d}$ with $0 < d_1 < d$, $0 < \tilde{d}_1 < \tilde{d}$, and $\mu \in \mathbb{R}$ such that*

$$\emptyset \neq \sigma(A) \cap \{z \in \mathbb{C} \,|\, \mu - d \leq \mathrm{Re} z \leq \mu + \tilde{d}\} \subset \{z \in \mathbb{C} \,|\, \mu - d_1 < \mathrm{Re} z < \mu + \tilde{d}_1\} \ . \quad \text{(C.15)}$$

Suppose also that e^A admits shifted dichotomy with gap $[e^{\mu+\tilde{d}_1}, e^{\mu+d^*}]$, for some $d^* > \tilde{d}$ (resp. with gap $[e^{\mu-d^*}, e^{\mu-d_1}]$, for some $d^* > d$). Let $U(t)$, $t \in \mathbb{R}^+$ (resp. $t \in \mathbb{R}^-$), be a solution of (C.13) *with $G(t)$ satisfying the hypothesis* (C.14) *and with $e^{(\mu+\tilde{d}-r)} < e^{\mu-d}$ (resp. with $e^{(\mu-d-r)} > e^{\mu+\tilde{d}}$), such that*

$$\mu - d_1 \leq \lim_{t \to +\infty} \ln(\|U(t)\|_X^{1/t}) \leq \mu + \tilde{d}_1 \ . \quad \text{(C.16)}$$

Denote

$$A_\mu \ = \ [Q_{\mu+\tilde{d}} - Q_{\mu-d}]A[Q_{\mu+\tilde{d}} - Q_{\mu-d}] \ ,$$

where now $Q_{\mu+\tilde{d}}$ and $Q_{\mu-d}$ denote the spectral projections associated with the parts of the spectrum $\sigma(A) \cap \{\mathrm{Re} z > \mu + \tilde{d}\}$ and $\sigma(A) \cap \{\mathrm{Re} z > \mu - d\}$.
Then, there exists a non-vanishing solution $U_{+\infty}(t)$, $t \in \mathbb{R}^+$, (resp. $U_{-\infty}(t)$, $t \in \mathbb{R}^-$,) in $R(Q_{\mu+\tilde{d}} - Q_{\mu-d})$ and satisfying

$$\partial_t U_{+\infty}(t) \ = \ A_\mu U_{+\infty}(t) \qquad \textit{(resp. } \partial_t U_{-\infty}(t) \ = \ A_\mu U_{-\infty}(t) \ \textit{)} \ , \quad \text{(C.17)}$$

and

$$\|U(t) - U_{+\infty}(t)\|_X \ = \ O(e^{(\mu-d)t}) \ , \qquad \textit{as } t \longrightarrow +\infty \ . \quad \text{(C.18)}$$

$$\textit{(resp. } \|U(t) - U_{-\infty}(t)\|_X \ = \ O(e^{(\mu+\tilde{d})t}) \ , \qquad \textit{as } t \longrightarrow -\infty \ \textit{)} \ . \quad \text{(C.19)}$$

C.2 Applications to the parabolic equation near an equilibrium point

The results of Appendix B and those of the first part of this appendix, together with those of Section 2.2, will be applied here in order to determine the asymptotics of the solutions $u(t)$ of (1.1), which belong to the local unstable or local stable manifolds of hyperbolic equilibria or periodic orbits, as well as the asymptotics of the solutions of the corresponding linearized equations.

Let e be a hyperbolic equilibrium point of (1.1). In accordance with Section 2.2, we denote by L_e the corresponding linearized operator and by λ_i, $i \geq 1$ its eigenvalues, counted with their multiplicity.

Corollary C.7. *Let $u(t)$ be a trajectory of* (1.1) *belonging to the unstable manifold $W^u(e)$ of e (resp. the local stable manifold $W^s_{loc}(e)$) and let $v(t) = u(t) - e$.*
Then, there exists an eigenvalue λ_i of L_e such that $\mathrm{Re}(\lambda_i) > 0$ (resp. $\mathrm{Re}(\lambda_i) < 0$) and

$$\lim_{t \to -\infty} \ln \|v(t)\|_{H^s} = \mathrm{Re}(\lambda_i) \quad \textit{(resp. } \lim_{t \to +\infty} \ln \|v(t)\|_{H^s} = \mathrm{Re}(\lambda_i)\textit{)} \ .$$

More precisely, the asymptotic behavior of v in $H^s(S^1)$ is as follows:
(i) if λ_i is a simple real eigenvalue with eigenfunction φ_i, then there exists $a \in \mathbb{R} - \{0\}$ such that $v(t) = ae^{\lambda_i t}\varphi_i + o(e^{\lambda_i t})$.
(ii) If $\lambda_i = \lambda_{i+1}$ is a double real eigenvalue with two independent eigenfunctions φ_i and φ_{i+1}, then there exist $(a, b) \in \mathbb{R}^2 - \{(0, 0)\}$ such that $v(t) = ae^{\lambda_i t}\varphi_i + be^{\lambda_i t}\varphi_{i+1} + o(e^{\lambda_i t})$.
(iii) If $\lambda_i = \lambda_{i+1}$ is an algebraically double real eigenvalue with eigenfunction φ_i and with generalized eigenfunction φ_{i+1}, then there exist $(a, b) \in \mathbb{R}^2 - \{(0, 0)\}$ such that $v(t) = (a + bt)e^{\lambda_i t}\varphi_i + be^{\lambda_i t}\varphi_{i+1} + o(e^{\lambda_i t})$.
(iv) If $\lambda_{i+1} = \overline{\lambda_i}$ is a (simple) nonreal eigenvalue with eigenfunction $\varphi_{i+1} = \overline{\varphi_i}$, then there exist $(a, b) \in \mathbb{R}^2 - \{(0, 0)\}$ such that $v(t) = e^{\mathrm{Re}(\lambda_i)t}\big[(a\cos(\mathrm{Im}(\lambda_i)t) - b\sin(\mathrm{Im}(\lambda_i)t))\mathrm{Re}(\varphi_i) - (a\sin(\mathrm{Im}(\lambda_i)t) + b\cos(\mathrm{Im}(\lambda_i)t))\mathrm{Im}(\varphi_i)\big] + o(e^{\mathrm{Re}(\lambda_i)t})$.

Let $j \in \mathbb{N} \setminus \{0\}$ be such that λ_i belongs to the pair of eigenvalues $(\lambda_{2j-1}, \lambda_{2j})$, or let $j = 0$ if $\lambda_i = \lambda_0$. As a consequence of the asymptotic behaviour, there exists $t_0 \in \mathbb{R}$ such that, for all $t \le t_0$ (resp. $t \ge t_0$), $v(t)$ has exactly $2j$ zeros which are simple.

Proof: Since the proofs are very similar when t tends to $\pm\infty$, we shall only prove the corollary when $u(t)$ belongs to the local stable manifold $W^s_{loc}(e)$. To prove the corollary, we shall apply Theorem C.6, so we have to check that the hypotheses of Theorem C.6 are satisfied.
Since $u(t)$ belongs to $W^s_{loc}(e)$, there exist two positive constants c_1 and κ such that

$$\|u(t) - e\|_{H^s} \le c_1 e^{-\kappa t} \ , \qquad \text{as } t \to +\infty \ . \tag{C.20}$$

The function $v(t) \equiv u(t) - e$ is a classical solution of the equation

$$v_t = v_{xx} + D_{u_x}f(x, e, e_x)v_x + D_u f(x, e, e_x)v + a(x, t)v_x + b(x, t)v \ , \tag{C.21}$$

where

$$a(x,t) = \int_0^1 \big(f'_{u_x}(x, e + \theta(u - e), e_x + \theta(u_x - e_x)) - f'_{u_x}(x, e, e_x)\big)d\theta$$
$$b(x,t) = \int_0^1 \big(f'_u(x, e + \theta(u - e), e_x + \theta(u_x - e_x)) - f'_u(x, e, e_x)d\theta\big)$$

One at once checks that $\|a(x,t)\|_{C^0} + \|b(x,t)\|_{C^0} \le c_2\|u(t) - e\|_{H^s}$, which implies that

$$\|a(x,t)v_x + b(x,t)v\|_{L^2} \le c_3 e^{-\kappa t}\|v\|_{H^s} \tag{C.22}$$

Thus, v is the solution of an equation of the form (C.13), with $G(t)v = a(x,t)v_x + b(x,t)v$ satisfying the condition (C.14).
We next remark that

$$v(n + 1) = Tv(n) + \Sigma(n)v(n) \ ,$$

where $T = T(1)$, $\Sigma(t)v(t) = \int_0^1 T(1-\sigma)G(t+\sigma)v(t+\sigma)d\sigma$ and $T(t)$ is the linear semigroup associated with the linear equation $v_t = v_{xx} + f'_{u_x}(x, e, e_x)v_x + f'_u(x, e, e_x)v \equiv L_e v$. We next verify that $\Sigma(n)$ satisfies the condition (C.2), for some $r < 1$, when n goes to infinity. First, $\Sigma(n)$ is a continuous linear operator from $H^s(S^1)$ into $H^s(S^1)$. Moreover, since $T(t)$ is an analytic linear semigroup, we obtain the following inequality, for $0 < \tau \leq 1$,

$$\|v(t+\tau)\|_{H^s} \leq Ce^{\alpha\tau}\|v(t)\|_{H^s} + C\int_0^\tau e^{\alpha(\tau-\sigma)}(\tau-\sigma)^{-s/2}\|v(t+\sigma)\|_{H^1} \ , \tag{C.23}$$

where $\alpha > 0$. Using a generalized Gronwall inequality (see [22, Lemma 7.1.1]), we deduce from (C.23) that, for $0 < \tau \leq 1$,

$$\|v(t+\tau)\|_{H^s} \leq Ce^{(\alpha+K)\tau}\|v(t)\|_{H^s} \ , \tag{C.24}$$

where K is a positive constant. The definition of $\Sigma(n)$ and the properties (C.22) and (C.24) imply that, for $n > 0$ large enough,

$$\|\Sigma(n)v(n)\|_{H^s} \leq C\int_0^1 e^{\alpha(1-\sigma)}(1-\sigma)^{-s/2}c_3 e^{-\kappa n}\|v(n+\sigma)\|_{H^1} \leq C^* e^{(\alpha+K)}e^{-\kappa n}\|v(n)\|_{H^s} \ , \tag{C.25}$$

and hence $\Sigma(n)$ satisfies the property (C.2).
Since $\Sigma(n)$ satisfies the property (C.25) and that T admits a shifted exponential dichotomy on $\mathbb{Z}^+$, the family $L(\cdot) = T + \Sigma(\cdot)$ admits a shifted dichotomy on $\mathbb{Z}^+$ by Proposition C.1. As, by a result of Agmon ([1], see also [35]), every non-zero solution of a linear parabolic equation does not go to zero faster than an exponential when t tends to infinity, the hypotheses of Theorem C.4 are satisfied, that is, for any integer m and any $\psi \in H^s(S^1)$, $r_\infty(m, \psi) > 0$. Hence, by Theorem C.4, there exists $\rho_i \in \mathcal{R}(T)$, $\rho_i < 1$, (and thus μ_i belonging to the spectrum of T), such that

$$\lim_{n\to\infty} \|v(n)\|_{H^s}^{1/n} = \rho_i = |\mu_i| \ ,$$

or, in other terms, there exists an eigenvalue λ_i of the linearized operator L_e such that

$$\lim_{t\to\infty} \ln(\|v(t)\|_{H^s}^{1/t}) = |\mathrm{Re}(\lambda_i)| \ .$$

We can now apply Theorem C.6. Let E_i be the generalized (real) eigenspace associated with the eigenvalue λ_i and $L_{e,i}$ be the restriction of the operator L_e to this eigenspace E_i. By Theorem C.6, there exists a nonvanishing solution $\psi_{\infty,i}(t) \in E_i$ of the equation

$$\partial_t \psi_{\infty,i}(t) = L_{e,i}\psi_{\infty,i}(t) \ , \tag{C.26}$$

such that

$$\|v(t) - \psi_{\infty,i}(t)\|_{H^s} = o(e^{\mathrm{Re}(\lambda_i)t}) \ . \tag{C.27}$$

Now the corollary is an elementary consequence of the properties (C.26) and (C.27) and of Proposition 2.2. According to this proposition, λ_i is either a simple real eigenvalue (and the dimension of E_i is one), or a double real eigenvalue or a simple non-real eigenvalue (in which cases, the dimension of E_i is equal to two). $\square$

In the same way, we prove the following corollary.

Corollary C.8. *Let $u(t)$ be a trajectory of* (1.1) *belonging to the unstable manifold $W^u(e)$ of e (resp. the local stable manifold $W^s_{loc}(e)$). Let $v_0 \in T_{u(0)}W^u(e)$ (resp. $v_0 \in T_{u(0)}W^s_{loc}(e)$) and let $v(t)$ be the solution for $t \leq 0$ (resp. $t \geq 0$) of the linearized equation*

$$v_t = v_{xx} + D_u f(x,u,u_x)v + D_{u_x} f(x,u,u_x)v_x \ , \quad v(0) = v_0 \ . \tag{C.28}$$

Then, there exists an eigenvalue λ_i of L_e such that $\mathrm{Re}(\lambda_i) > 0$ (resp. $\mathrm{Re}(\lambda_i) < 0$) and

$$\lim_{t\to-\infty} \ln \|v(t)\|_{H^s} = \mathrm{Re}(\lambda_i) \quad \left(\text{resp. } \lim_{t\to+\infty} \ln \|v(t)\|_{H^s} = \mathrm{Re}(\lambda_i) \right) .$$

Moreover, all the possible asymptotic behaviors are the same as those described in Corollary C.7.
In addition, let $j \in \mathbb{N} \setminus \{0\}$ be such that λ_i belongs to the pair of eigenvalues $(\lambda_{2j-1}, \lambda_{2j})$, or let $j = 0$ if $\lambda_i = \lambda_0$. Then, there exists $t_0 \in \mathbb{R}$ such that, for all $t \leq t_0$ (resp. $t \geq t_0$), $v(t)$ has exactly $2j$ zeros which are simple.

Proof: If $u(t)$ belongs to the local stable manifold $W^s_{loc}(e)$, $u(t)$ satisfies the property (C.20) and Eq. (C.28) can be written in the form (C.21), where the functions $a(x,t)$ and $b(x,t)$ satisfy the properties (C.22). We remark that, by [10, Theorem C2], we already know that $\limsup_{n\to\infty} \|v(n)\|^{1/n}_{H^s} < 1$. We thus obtain the asymptotic behavior of $v(t)$ by following the lines of the proof of Corollary C.7. $\square$

C.3 Application to the parabolic equation near a periodic orbit

Before proving analogous corollaries in the case of periodic orbits, we briefly recall the known properties of the local stable and unstable manifolds of hyperbolic periodic orbits.

Let $\gamma(t)$ be a hyperbolic periodic solution of (1.1) of minimal period p and let $\Gamma = \{\gamma(t),\ t \in \mathbb{R}\}$. As in the introduction and in section 2, we introduce the linearized equation (2.6) along the periodic solution $\gamma(t)$ and introduce the associated evolution operator $\Pi(t,0) : H^s(S^1) \longrightarrow H^s(S^1)$, defined by $\Pi(t,0)\varphi_0 = \varphi(t)$ where $\varphi(t)$ is the solution of the linearized equation (2.6). We recall that the operator $\Pi(p,0) = D_u(S_f(p,0)\gamma(0))$ is called the period map and we denote (μ_i) its eigenvalues (the spectral properties of $\Pi(p,0)$ have been given in Proposition 2.3). Since $\gamma(t)$ is a hyperbolic periodic solution, the intersection of the spectrum of $\Pi(p,0)$ with the unit circle of $\mathbb{C}$ reduces to the eigenvalue 1, which a simple (isolated) eigenvalue. We remark that, if $\gamma(a)$, $a \in [0,p)$, is another point of the periodic orbit, the spectrum of $D_u(S_f(p,0)\gamma(a))$ coincides with the one of $\Pi(p,0)$ whereas the corresponding eigenfunctions depend on the point $\gamma(a)$.

We denote $P_u(a)$ (resp. $P_c(a)$, resp. $P_s(a)$) the projection in $H^s(S^1)$ onto the space generated by the (generalized) eigenfunctions of $D_u(S_f(p,0)\gamma(a))$ corresponding to the eigenvalues with modulus strictly larger than 1 (resp. equal to 1, resp. with modulus strictly smaller than 1).

Since a hyperbolic periodic orbit is a particular case of a normally hyperbolic C^1 manifold, we may apply, for example, the existence results of [27], [28] or [51, Theorem 14.2 and Remark 14.3] (see also [21]) and thus, we may state the following theorem.

Theorem C.9. *Let $\Gamma = \{\gamma(t),\ t \in \mathbb{R}\}$ be a hyperbolic periodic orbit of Eq.* (1.1).
1) There exists a small neighborhood U_Γ of Γ in $H^s(S^1)$ such that the local stable and unstable sets

$$W^s_{loc}(\Gamma) \equiv W^s(\Gamma, U_\Gamma) = \{u_0 \in H^s(S^1) \,|\, S_f(t)u_0 \in U_\Gamma\,,\ \forall t \geq 0\}$$
$$W^u_{loc}(\Gamma) \equiv W^u(\Gamma, U_\Gamma) = \{u_0 \in H^s(S^1) \,|\, S_f(t)u_0 \in U_\Gamma\,,\ \forall t \leq 0\}$$

are (embedded) C^1-submanifolds of $H^s(S^1)$ of codimension $i(\Gamma)$ and dimension $i(\Gamma)+1$ respectively.
2) Moreover, $W^s_{loc}(\Gamma)$ and $W^u_{loc}(\Gamma)$ are fibrated by the local strong stable (resp. unstable) manifolds at each point $\gamma(a) \in \Gamma$, that is,

$$W^s_{loc}(\Gamma) = \cup_{a\in[0,p)} W^{ss}_{loc}(\gamma(a))\ , \quad W^u_{loc}(\Gamma) = \cup_{a\in[0,p)} W^{su}_{loc}(\gamma(a))\ ,$$

where there exist positive constants $\tilde{r}$, κ and κ^* such that

$$\begin{aligned}
W^{ss}_{loc}(\gamma(a)) =& \{u_0 \in H^s(S^1) \,|\, \|S_f(t)u_0 - \gamma(a+t)\|_{H^s} < \tilde{r}\ ,\ \forall t \geq 0\,,\\
& \lim_{t\to\infty} e^{\kappa t}\|S_f(t)u_0 - \gamma(a+t)\|_{H^s} = 0\}\ ,\\
W^{su}_{loc}(\gamma(a)) =& \{u_0 \in H^s(S^1) \,|\, \|S_f(t)u_0 - \gamma(a+t)\|_{H^s} < \tilde{r}\ ,\ \forall t \leq 0\,,\\
& \lim_{t\to-\infty} e^{-\kappa^* t}\|S_f(t)u_0 - \gamma(a+t)\|_{H^s} = 0\}\ .
\end{aligned} \tag{C.29}$$

For any $a \in [0,p)$, $W^{ss}_{loc}(\gamma(a))$ (resp. $W^{su}_{loc}(\gamma(a))$) is a C^1-submanifold of $H^s(S^1)$ tangent at $\gamma(a)$ to $P_s(a)H^s(S^1)$ (resp. $P_u(a)H^s(S^1)$).

In the introduction, we have also defined the global stable and unstable sets as follows

$$W^s(\Gamma) = \{u_0 \in H^s(S^1) \,|\, S_f(t)u_0 \underset{t\to+\infty}{\to} \Gamma\}\ ,$$
$$W^u(\Gamma) = \{u_0 \in H^s(S^1) \,|\, S_f(t)u_0 \text{ is well-defined for } t \leq 0 \text{ and } S_f(t)u_0 \underset{t\to-\infty}{\to} \Gamma\}\ .$$

We recall that $W^s(\Gamma)$ and $W^u(\Gamma)$ are injectively immersed C^1-manifolds of codimension $i(\Gamma)$ and dimension $i(\Gamma)+1$ respectively. Moreover,

$$W^u(\Gamma) = \cup_{t\geq 0} S_f(t) W^u_{loc}(\Gamma) \ ,$$

is a union of embedded C^1-submanifolds of $H^s(S^1)$ of dimension $i(\Gamma)+1$ (see [13] or [18] for example).

We are now ready to prove the following corollary.

Corollary C.10. *Let $\Gamma = \{\gamma(t),\ t \in \mathbb{R}\}$ be a hyperbolic periodic orbit of Eq. (1.1). Let $u(t)$ be a trajectory of (1.1) belonging to the strong unstable manifold $W^{su}(\gamma(a)) \setminus \gamma(a)$ (resp. the local strong stable manifold $W^{ss}_{loc}(\gamma(a)) \setminus \gamma(a)$) and let $v(t) = u(t) - \gamma(t+a)$. Then, there exists an eigenvalue μ_i of $\Pi(p,0)$ such that $|\mu_i| > 1$ (resp. $|\mu_i| < 1$) and*

$$\lim_{n\to -\infty} \|v(np)\|^{1/n}_{H^s} = |\mu_i| \ , \ \textit{(resp.} \ \lim_{n\to\infty} \|v(np)\|^{1/n}_{H^s} = |\mu_i| \ \textit{)} \ .$$

More precisely, the asymptotic behavior of $v(np)$ in $H^s(S^1)$ is given by one of the following possibilities:
(i) if μ_i is a simple real eigenvalue with corresponding real eigenfunction $\varphi_i(a) \in H^s(S^1)$, then there exists $b \in \mathbb{R} - \{0\}$ such that $v(np) = b\mu_i^n \varphi_i(a) + o(|\mu_i|^n)$.
(ii) If $\mu_i = \mu_{i+1}$ is a double real eigenvalue with two independent eigenfunctions $\varphi_i(a)$ and $\varphi_{i+1}(a)$, then there exist $(b,c) \in \mathbb{R}^2 - \{(0,0)\}$ such that $v(np) = b\mu_i^n\varphi_i(a) + c\mu_i^n\varphi_{i+1}(a) + o(|\mu_i|^n)$.
(iii) If $\mu_i = \mu_{i+1}$ is an algebraically double real eigenvalue with eigenfunction $\varphi_i(a)$ and generalized eigenfunction $\varphi_{i+1}(a)$, then there exist $(b,c) \in \mathbb{R}^2 - \{(0,0)\}$ such that $v(np) = (b+cn)\mu_i^n\varphi_i(a) + c\mu_i^n\varphi_{i+1}(a) + o(|\mu_i|^n)$.
(iv) If $\mu_i = |\mu_i|e^{i\theta}$ is a (simple) complex eigenvalue with eigenfunction $\varphi_i(a) = \overline{\varphi_{i+1}}(a)$, then there exist $(b,c) \in \mathbb{R}^2 - \{(0,0)\}$ such that $v(np) = |\mu_i|^n \big[(b\cos(n\theta) - c\sin(n\theta))\mathrm{Re}(\varphi_i(a)) - (b\sin(n\theta) + c\cos(n\theta))\mathrm{Im}(\varphi_i(a))\big] + o(|\mu_i|^n)$.

Let $j \in \mathbb{N} \setminus \{0\}$ be such that μ_i belongs to the pair of eigenvalues (μ_{2j-1}, μ_{2j}), or let $j = 0$ if $\mu_i = \mu_0$. As a consequence of the asymptotic behaviour, there exists $t_0 \in \mathbb{R}$ such that, for all $t \leq t_0$ (resp. $t \geq t_0$), $v(t)$ has exactly $2j$ zeros which all are simple.

Proof: Since $u(t)$ belongs to the local strong stable manifold (or strong unstable) manifold of a point $\gamma(a)$ of the non trivial periodic solution $\gamma(t)$ of (1.1), the proof of this corollary is very similar to the one of Corollary C.7. Thus, we will not repeat the whole proof, but only give the details of the beginning of the proof, in order to point out the differences with the proof of Corollary C.7 and also to emphasize the properties of the strong stable or unstable manifolds of $\gamma(a)$, that we are using here.
Since the proofs are very similar when n tends to $\pm\infty$, we only consider the case where $u(t)$ belongs to the local strong stable manifold $W^{ss}_{loc}(\gamma(a)) \setminus \gamma(a)$. To prove the corollary, we shall apply Theorem C.5, so we have to check that the hypotheses of Theorem C.5 are

satisfied. Also, without loss of generality, we may assume that $a = 0$.
Since $u(t)$ belongs to $W^{ss}_{loc}(\gamma(0)) \setminus \gamma(0)$, there exist two positive constants c_1 and κ such that

$$\|u(t) - \gamma(t)\|_{H^s} \leq c_1 e^{-\kappa t} \ , \qquad \text{as } t \to +\infty \ . \tag{C.30}$$

The function $v(t) \equiv u(t) - \gamma(t)$ is a classical solution of the equation

$$v_t = v_{xx} + D_{u_x} f(x, \gamma(x,t), \gamma_x(x,t)) v_x + D_u f(x, \gamma(x,t), \gamma_x(x,t)) v + a(x,t) v_x + b(x,t) v \ , \tag{C.31}$$

where

$$a(x,t) = \int_0^1 \big(f'_{u_x}(x, \gamma + \theta(u - \gamma), \gamma_x + \theta(u_x - \gamma_x)) - f'_{u_x}(x, \gamma, \gamma_x) \big) d\theta$$
$$b(x,t) = \int_0^1 \big(f'_u(x, \gamma + \theta(u - \gamma), e\gamma_x + \theta(u_x - \gamma_x)) - f'_u(x, \gamma, \gamma_x) d\theta \big)$$

One at once checks that $\|a(x,t)\|_{C^0} + \|b(x,t)\|_{C^0} \leq c_2 \|u(t) - \gamma(t)\|_{H^s}$, which implies that $\|a(x,t)\|_{L^2} + \|b(x,t)\|_{L^2}$ satisfies the inequality (C.22). We next remark that

$$v((n+1)p) = Tv(pn) + \Sigma(n) v(pn) \ ,$$

where $T = \Pi(p, 0)$, and

$$\Sigma(n) v(pn) = \int_0^p \Pi(p, \sigma)(a(x, np + \sigma) v_x(x, np + \sigma) + b(x, np + \sigma) v(x, np + \sigma)) d\sigma \ .$$

Arguing as in the proof of Corollary C.7, one checks that $\Sigma(n)$ satisfies the condition (C.2). As the periodic orbit Γ is hyperbolic, $T = \Pi(p, 0)$ admits a shifted exponential dichotomy on $\mathbb{Z}^+$ (see [19] for example). Thus, by Proposition C.1, the family $L(\cdot) = T + \Sigma(\cdot)$ admits a shifted dichotomy on $\mathbb{Z}^+$. As, by [1], every non-zero solution of a linear parabolic equation does not go to zero faster than an exponential when t tends to infinity, the hypotheses of Theorem C.4 hold. Since the exponential decay property (C.30) holds, it follows from Theorem C.4, that there exists $\rho_i \in \mathcal{R}(T)$, $\rho_i < 1$, (and thus μ_i belonging to the spectrum of $\Pi(p, 0)$), such that

$$\lim_{n \to \infty} \|v(n)\|^{1/n}_{H^s} = |\rho_i| = |\mu_i| \ .$$

Corollary C.10 is now an easy consequence of Theorem C.5 and of Proposition 2.3 (for the details, see the proof of Corollary C.7). □

Corollary C.11. *Let $\Gamma = \{\gamma(t),\ t \in \mathbb{R}\}$ be a hyperbolic periodic orbit of Eq.* (1.1) *and $u(t)$ be a trajectory of* (1.1) *belonging to the strong unstable manifold $W^{su}(\gamma(a)) \setminus \gamma(a)$ (resp. the local strong stable manifold $W^{ss}_{loc}(\gamma(a)) \setminus \gamma(a)$). Let $v_0 \in T_{u(0)} W^{su}(\gamma(a))$ (resp.*

$v_0 \in T_{u(0)}W^{ss}_{loc}(\gamma(a))$) and $v(t)$ be the solution for $t \leq 0$ (resp. $t \geq 0$) of the linearized equation (C.28). *Then, there exists an eigenvalue μ_i of $\Pi(p,0)$ such that $|\mu_i| > 1$ (resp. $|\mu_i| < 1$) and*

$$\lim_{n\to-\infty} \|v(np)\|_{H^s}^{1/n} = |\mu_i| \ , \ \textit{(resp.} \ \lim_{n\to\infty} \|v(np)\|_{H^s}^{1/n} = |\mu_i| \ \textit{)} \ .$$

Moreover, all the possible asymptotic behaviors are the same as those described in Corollary C.10.
In addition, let $j \in \mathbb{N} \setminus \{0\}$ be such that μ_i belongs to the pair of eigenvalues (μ_{2j-1}, μ_{2j}), or let $j = 0$ if $\mu_i = \mu_0$. Then, there exists $t_0 \in \mathbb{R}$ such that, for all $t \leq t_0$ (resp. $t \geq t_0$), $v(t)$ has exactly $2j$ zeros which all are simple.

Proof: If $u(t)$ belongs to the local strong stable manifold $W^{ss}_{loc}(\gamma(a)) \setminus \gamma(a)$, $u(t)$ satisfies the property (C.30) and Eq. (C.28) can be written in the form (C.31), where the functions $a(x,t)$ and $b(x,t)$ satisfy the properties (C.22). We emphasize that, by [10, Theorem C6], we already know that $\limsup_{n\to\infty} \|v(np)\|_{H^s}^{1/n} < 1$. We thus obtain the asymptotic behavior of $v(t)$ by following the lines of the proof of Corollary C.10. □

References

[1] S. Agmon, *Unicité et convexité dans les problèmes différentiels*, Presses de l'Université de Montreal, 1966.

[2] A. Andronov and L. S. Pontrjagin, *Systèmes grossiers*, Dokl. Akad. Nauk., n°14 (1937) pp. 247-250.

[3] S. Angenent, *The Morse-Smale property for a semi-linear parabolic equation*, J. Differential Equations, n°62 (1986), pp. 427-442.

[4] S. Angenent, *The zero set of a solution of a parabolic equation*, Journal für die Reine und Angewandte Mathematik n°390 (1988), pp. 79-96.

[5] S. Angenent and B. Fiedler, *The dynamics of rotating waves in scalar reaction diffusion equations*, Trans. Amer. Math. Soc. n°307 (1988), pp. 545-568.

[6] C. Bardos and L. Tartar, *Sur l'unicité rétrograde des équations paraboliques et quelques questions voisines*, Arch. Rational Mech. Analysis n°50 (1973), pp. 10-25.

[7] P. Brunovský, R. Joly and G. Raugel, *Genericity of the Kupka-Smale property for scalar parabolic equations*, manuscript.

[8] P. Brunovský and P. Poláčik, *The Morse-Smale structure of a generic reaction-diffusion equation in higher space dimension*, J. Differential Equations n°135 (1997), pp. 129-181.

[9] P. Brunovský and G. Raugel, *Genericity of the Morse-Smale property for damped wave equations*, Journal of Dynamics and Differential Equations, vol 15 n°2 (2003), pp. 571-658.

[10] M. Chen, X.-Y. Chen and J. K. Hale, *Structural stability for time periodic one dimensional parabolic equations*, J. Differential Equations n°96 (1992), pp. 355-418.

[11] S.-N. Chow, J.K. Hale and J. Mallet-Paret: *An example of bifurcation to homoclinic orbits*, J. Differential Equations n°37 (1980), pp. 351–373.

[12] W.A. Coppel, *Dichotomies in stability theory*, Lecture Notes In Math. n°629 (1978), Springer-Verlag.

[13] R. Czaja and C. Rocha, *Transversality in scalar reaction-diffusion equations on a circle*, J. Differential Equations n°245 (2008), pp. 692-721.

[14] B. Fiedler and J. Mallet-Paret, *A Poincaré-Bendixson theorem for scalar reaction-diffusion equations*, Arch. Rational Mech. Analysis n°107 (1989), pp. 325-345.

[15] B. Fiedler, C. Rocha and, M. Wolfrum, *Heteroclinic orbits between rotating waves of semilinear parabolic equations on the circle*, J. Differential Equationsn°201 (2004), pp. 99-138.

[16] G. Fusco and W.M. Oliva, *Transversality between invariant manifolds of periodic orbits for a class of monotone dynamical systems*, Journal of Dynamics and Differential Equations n°2 (1990), pp. 1-17.

[17] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics n°14, Springer Verlag, New York-Heidelberg, 1973.

[18] J.K. Hale, R. Joly and G. Raugel, book in preparation.

[19] J.K. Hale and X.-B. Lin, *Heteroclinic orbits for retarded functional differential equations*, J. Differential Equations n°65 (1986), pp. 175-202.

[20] J.K. Hale, L. Magalhães and W. Oliva, *An introduction to infinite dimensional dynamical systems*, Applied Mathematical Sciences n°47. Springer-Verlag, 1984. Second edition, *Dynamics in infinite dimensions*, 2002.

[21] J.K. Hale and G. Raugel, *Behaviour near a non-degenerate periodic orbit*, manuscript (2010).

[22] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes In Math. n°840 (1981), Springer-Verlag.

[23] D. Henry, *Some infinite dimensional Morse–Smale systems defined by parabolic differential equations*, J. Differential Equations n°59 (1985), pp. 165-205.

[24] D. Henry, *Exponential dichotomies, the shadowing lemma and homoclinic orbits in Banach spaces*, Resenhas IME-USP n°1 (1994), pp. 381-401.

[25] D. Henry, *Perturbation of the Boundary for Boundary Value Problems of Partial Differential Operators*, with editorial assistance from Jack Hale and Antonio Luiz Pereira. London Mathematical Society Lecture Note Series n°318. Cambridge University Press, Cambridge UK, 2005.

[26] M.W. Hirsch, *Stability and convergence on strongly monotone dynamical systems*, J. Reine Angew. Math. n°383 (1988), pp. 1-53.

[27] M.W. Hirsch, C.C. Pugh and M. Shub, *Invariant manifolds*, Bull. Amer. Math. Soc. n°76 (1970), pp. 1015-1019.

[28] M.W. Hirsch, C.C. Pugh and M. Shub, *Invariant manifolds*, Lecture Notes in Math. n°583 (1977). Springer-Verlag, Berlin-New York.

[29] R. Joly, *Generic transversality property for a class of wave equations with variable damping*, Journal de Mathématiques Pures et Appliquées n°84 (2005), pp. 1015-1066.

[30] R. Joly, *Adaptation of the generic PDE's results to the notion of prevalence*, Journal of Dynamics and Differential Equations n°19 (2007), pp. 967-983.

[31] R. Joly and G. Raugel, *Generic hyperbolicity of equilibria and periodic orbits of the parabolic equation on the circle*, Trans. Amer. Math. Soc. n°362 (2010), pp. 5189-5211.

[32] R. Joly and G. Raugel, *A striking correspondence between the dynamics generated by the vector fields and by the scalar parabolic equations*, preprint, submitted.

[33] I. Kupka, *Contribution à la théorie des champs génériques*, Contributions to Differential Equations n°2 (1963), p. 457-484. Addendum and corrections, ibid. n°3 (1964), pp. 411-420.

[34] S. Lang, *Introduction to differentiable manifolds*, (1962), John Wiley and Sons, USA.

[35] P. Lax, *A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations*, Communications on Pure and Applied Mathematics, n°IX, (1956), pp. 747-766.

[36] X.B. Lin, *Exponential dichotomies and homoclinic orbits in functional differential equations*, J. Differential Equations n°63, (1986), pp. 227-254.

[37] H. Matano, *Convergence of solutions of one-dimensional semilinear parabolic equations*, Journal of Mathematics of Kyoto University n°18 (1978), pp. 221-227.

[38] H. Matano, *Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation*, J. Fac. Sci. Univ. Tokyo Sec. IA, n°29 (1982), pp. 401-441.

[39] K. Nickel, *Gestaltaussagen über Lösungen parabolischer Differentialgleichungen*, Journal für die Reine und Angewandte Mathematik n°211 (1962), pp. 78-94.

[40] W.M. Oliva, *Morse-Smale semiflows. Openness and A-stability*, Differential equations and dynamical systems (Lisbon, 2000), pp. 285-307, Fields Inst. Commun. n°31, Amer. Math. Soc., Providence, RI, 2002.

[41] W. Ott and J.A. Yorke, *Prevalence*, Bulletin of the American Mathematical Society n°42 (2005), pp. 263-290.

[42] J. Palis, *On Morse-Smale dynamical systems*, Topology n°8 (1969), pp. 385-405.

[43] J. Palis and W. de Melo: *Geometric Theory of Dynamical Systems*, Springer–Verlag, Berlin (1982).

[44] J. Palis and S. Smale, *Structural stability theorems* in Global Analysis, Proc. Symp. Pure Math. 14, pp. 223-231, A.M.S. (1970), Providence, R.I.

[45] K.J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Differential Equations n°55 (1984), pp. 225-256.

[46] K.J. Palmer, *Exponential dichotomies, the shadowing lemma and transversal homoclinic points*, Dynamics Reported n°1, U. Kirchgraber and H.O. Walther editors (1988), Wiley and Sons and B.G. Teubner, pp. 265-306.

[47] K.J. Palmer, *Shadowing in Dynamical Systems. Theory and Applications*, Kluwer; Dordrecht, Boston, London, 2000.

[48] M.M. Peixoto, *Structural stability on two-dimensional manifolds*, Topology n°1 (1962), pp. 101-120.

[49] M.M. Peixoto, *On an approximation theorem of Kupka and Smale*, J. Differential Equations n°3 (1967), pp. 214-227.

[50] P. Poláčik, *Parabolic equations: asymptotic behavior and dynamics on invariant manifolds.* Handbook of dynamical systems, n°2, p. 835-883. North-Holland, Amsterdam, 2002.

[51] D. Ruelle: *Elements of Differentiable Dynamics and Bifurcation Theory*, Academic Press, London, 1989.

[52] F. Quinn, *Transversal approximation on Banach manifolds*, Global Analysis (Proceedings of Symposia in Pure Mathematics n°15, Berkeley, 1968) pp. 213-222, Amer. Math. Soc., Providence, 1970.

[53] J.W. Robbin, *Algebraic Kupka-Smale theory.* Dynamical systems and turbulence, Warwick 1980 (Coventry, 1979/1980), pp. 286-301, Lecture Notes in Mathematics n°898, Springer, Berlin-New York, 1981.

[54] J.-C. Saut and R. Temam, *Generic properties of nonlinear boundary value problems*, Communications in PDE n°4 (1979), pp. 293-319.

[55] R.J. Sacker, *Existence of dichotomies and invariant splitting for linear differential systems IV*, J. Differential Equations n°27 (1978), pp. 106-137.

[56] D. Salamon, *Morse theory, the Conley index and Floer homology*, Bulletin London Mathematical Society n°22 (1990), pp. 113-140.

[57] B. Sandstede and B. Fiedler, *Dynamics of periodically forced parabolic equations on the circle*, Ergodic Theory and Dynamical Systems n°12 (1992), pp. 559-571.

[58] S. Smale, *Morse inequalities for a dynamical system*, Bulletin of the AMS n°66 (1960), pp. 43-49.

[59] S. Smale, *Stable manifolds for differential equations and diffeomorphisms*, Annali della Scuola Normale Superiore di Pisa n°17 (1963), pp. 97-116.

[60] S. Smale, *Diffeomorphisms with many periodic points.* Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), pp. 63-80 Princeton Univ. Press, Princeton, N.J., 1965.

[61] C. Sturm, *Sur une classe d'équations à différences partielles*, Journal de Mathématiques Pures et Appliquées n°1 (1826), pp. 373-444.

[62] T.J. Zelenyak, *Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable*, Diff. Equations n°4 (1968), pp. 17-22.