# A striking correspondence between the dynamics generated by the vector fields and by the scalar parabolic equations

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*À la mémoire de Michelle Schatzman, notre chère et regrettée collègue et amie.*

The results of this paper have been presented at the Conference in honour of the sixthieth birthday of Michelle Schatzman. The second author would like to express her gratitude towards Michelle, who guided her first steps in the research in mathematics and was a true friend.

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**Abstract**

The purpose of this paper is to enhance a correspondence between the dynamics of the differential equations $\dot{y}(t) = g(y(t))$ on $\mathbb{R}^d$ and those of the parabolic equations $\dot{u} = \Delta u + f(x, u, \nabla u)$ on a bounded domain $\Omega$. We give details on the similarities of these dynamics in the cases $d = 1$, $d = 2$ and $d \geq 3$ and in the corresponding cases $\Omega = (0, 1)$, $\Omega = \mathbb{T}^1$ and $\dim(\Omega) \geq 2$ respectively. In addition to the beauty of such a correspondence, this could serve as a guideline for future research on the dynamics of parabolic equations.
**Keywords:** finite- and infinite-dimensional dynamical systems, vector fields, scalar parabolic equation, Kupka-Smale property, genericity.
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# 1 Introduction

In this paper, we want to point out the similarities between the dynamics of vector fields in $\mathbb{R}^d$ and those of reaction-diffusion equations on bounded domains. More precisely, we consider the following classes of equations.

### Class of vector fields

Let $d \geq 1$ and $r \geq 1$ and let $g \in \mathcal{C}^r(\mathbb{R}^d, \mathbb{R}^d)$ be a given vector field. We consider the ordinary differential equation

$$\begin{cases} \dot{y}(t) = g(y(t)) \quad t > 0 \\ y(0) = y_0 \in \mathbb{R}^d \end{cases} \tag{1.1}$$

where $\dot{y}(t)$ denotes the time-derivative of $y(t)$.
The equation (1.1) defines a local dynamical system $T_g(t)$ on $\mathbb{R}^d$ by setting $T_g(t)y_0 = y(t)$. We assume that there exists $M > 0$ large enough such that

$$\forall y \in \mathbb{R}^d, \ \|y\| \geq M \ \Rightarrow \ \langle y | g(y) \rangle < 0 \ .$$

This condition ensures that $T_g(t)$ is a global dynamical system. Moreover, the ball $B(0, M)$ attracts the bounded sets of $\mathbb{R}^d$. Therefore, $T_g(t)$ admits a compact global attractor[1] $\mathcal{A}_g$. The attractor $\mathcal{A}_g$ contains the most interesting trajectories such as periodic, homoclinic

[1] To make the reading of this article easier for the reader, who is not familiar with dynamical systems theory or with the study of PDEs, we add a short glossary at the end of the paper.

and heteroclinic orbits[1] and any $\alpha-$ or $\omega-$limit set[1]. Therefore, if one neglects the transient dynamics, the dynamics on $\mathcal{A}_g$ is a good representation of the whole dynamics of $T_g(t)$.

**Class of scalar parabolic equations**

Let $d' \geq 1$ and let $\Omega$ be either a regular bounded domain of $\mathbb{R}^{d'}$, or the torus $\mathbb{T}^{d'}$. We choose $p > d'$ and $\alpha \in ((p+d')/2p, 1)$. We denote $X^\alpha \equiv D((-\Delta_N)^\alpha)$ the fractional power space[1] associated with the Laplacian operator $\Delta_N$ on $\mathbb{L}^p(\Omega)$ with homogeneous Neumann boundary conditions. It is well-known[1] that $X^\alpha$ is continuously embedded in the Sobolev space $W^{2\alpha,p}(\Omega)$ and thus it is compactly embedded in $\mathcal{C}^1(\overline{\Omega})$. Let $r \geq 1$ and $f \in \mathcal{C}^r(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{d'}, \mathbb{R})$. We consider the parabolic equation

$$\begin{cases} \dot{u}(x,t) = \Delta u(x,t) + f(x, u(x,t), \nabla u(x,t)) & (x,t) \in \Omega \times (0, +\infty) \\ \frac{\partial u}{\partial \nu}(x,t) = 0 & (x,t) \in \partial\Omega \times (0, +\infty) \\ u(x,0) = u_0(x) \in X^\alpha \end{cases} \tag{1.2}$$

where $\dot{u}(t)$ is the time-derivative of $u(t)$.
Eq. (1.2) defines a local dynamical system $S_f(t)$ on $X^\alpha$ (see [41]) by setting $S_f(t)u_0 = u(t)$. We assume moreover that there exist $c \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$, $\varepsilon > 0$ and $\kappa > 0$ such that $f$ satisfies

$$\forall R > 0,\ \forall \xi \in \mathbb{R}^{d'}, \qquad \sup_{(x,z) \in \overline{\Omega} \times [-R,R]} |f(x,z,\xi)| \leq c(R)(1 + |\xi|^{2-\varepsilon})$$

$$\text{and } \forall z \in \mathbb{R},\ \forall x \in \overline{\Omega}, \qquad |z| \geq \kappa \ \Rightarrow\ z f(x,z,0) < 0 \ .$$

Then, Eq. (1.2) defines a global dynamical system in $X^\alpha$ which admits a compact global attractor $\mathcal{A}$ (see [73]).

The reader, which is not familiar with partial differential equations, may neglect all the technicalities about $X^\alpha$, the Sobolev spaces and the parabolic equations in a first reading. The most important point is that $S_f(t)$ is a dynamical system defined on an *infinite-dimensional function space*. Compared with the finite-dimensional case, new difficulties arise. For example, the existence of a compact global attractor requires compactness properties, coming here from the smoothing effect of (1.2). We also mention that, even if the backward uniqueness property holds, backward trajectories do not exist in general for (1.2). The reader interested in the dynamics of (1.2) may consult [25], [41], [35], [73] or [79].

The purpose of this paper is to emphasize the different relationships between the dynamics of (1.1) and (1.2). The correspondence is surprisingly perfect. It can be summarized by Table 1. This correspondence has already been noticed for some of the properties of the table. We complete here the correspondence for all the known properties of the dynamics of the parabolic equation. Table 1 will be discussed in more details in Section 2 and, for cooperative systems, in Section 4. Some of the properties presented in the table concerning finite-dimensional dynamical systems are trivial, other ones are now well-known.

| ODE | | PDE |
|---|---|---|
| $d = 1$<br><br>(Or more generally tridiagonal cooperative system of ODEs) | • Gradient dynamics<br>• Convergence to an equilibrium point<br>• Automatic transversality of stable and unstable manifolds<br>• Genericity of Morse-Smale property<br>• Knowledge of the equilibrium points implies knowledge of the whole dynamics<br>• Dimension of the attractor equal to the largest dimension of the unstable manifolds<br>• Realisation of the ODE in the PDE | $\Omega = (0, 1)$ |
| $d = 2$<br>General case<br><br>(Or more generally cyclic tridiagonal cooperative system of ODEs) | • Poincaré-Bendixson property<br>• Automatic transversality of stable and unstable manifolds of two orbits if one of them is a hyperbolic periodic orbit or if both are equilibrium points with different Morse indices.<br>• Non-existence of homoclinic orbits for periodic orbits<br>• Genericity of Morse-Smale property<br>• Realisation of the ODE in the PDE | $\Omega = \mathbb{T}^1$<br>General case |
| $d = 2$<br>$g$ radially symmetric | • Automatic transversality of stable and unstable manifolds of equilibrium points and periodic orbits<br>• No homoclinic orbit<br>• Knowledge of the equilibrium points and of the periodic orbits implies knowledge of the whole dynamics<br>• Genericity of the Morse-Smale property.<br>• Dimension of the attractor equal to the largest dimension of the unstable manifolds<br>• Realisation of the ODE in the PDE | $\Omega = \mathbb{T}^1$<br>$f(x, u, \nabla u) \equiv f(u, \nabla u)$ |
| $d \geq 3$ | • Existence of persistent chaotic dynamics<br>• Genericity of Kupka-Smale property (ODE)<br>• Generic transversality of homoclinic and heteroclinic orbits (PDE)<br>• Realisation of the ODE in the PDE | $\dim(\Omega) \geq 2$ |
| Any $d$<br>$g \equiv \nabla G$ | • Gradient dynamics<br>• Genericity of the Morse-Smale property<br>• Realisation of a generic ODE in the PDE | Any $\Omega$<br>$f(x, u, \nabla u) \equiv f(x, u)$ |

Table 1: the correspondence between the dynamics of vector fields and the ones of parabolic equations.

However, the corresponding results for the parabolic equation are more involved and some of them are very recent. These properties are mainly based on Sturm-Liouville arguments and unique continuation properties for the parabolic equations as explained in Section 3. The study of the dynamics generated by vector fields in dimension $d \geq 3$ is still a subject of research. Taking into account the correspondence presented in Table 1 should give a guideline for research on the dynamics of the parabolic equations. Some examples of open questions are given in Section 5.

We underline that we only consider the dynamics on the compact global attractors. Hence, we deal with dynamical systems on compact sets. It is important to be aware of the fact that, even if the dimension of the compact global attractor $\mathcal{A}$ of the parabolic equation (1.2) is finite, it can be made as large as wanted by choosing a suitable function $f$. This is true even if $\Omega$ is one-dimensional. Therefore, all the possible properties of the dynamics of (1.2) do not come from the low dimension of $\mathcal{A}$ but from properties, which are very particular to the flow of the parabolic equations.

Finally, we remark that most of the results described here also hold in more general frames than (1.1) and (1.2). For example, $\mathbb{R}^d$ could be replaced by a compact orientable manifold without boundary. We could also choose for (1.2) more general boundary conditions than Neumann ones, or less restrictive growing conditions for $f$. The domain $\Omega$ may be replaced by a bounded smooth manifold. Finally, notice that the case $\Omega = \mathbb{T}^{d'}$ can be seen as $\Omega = (0,1)^{d'}$ with periodic boundary conditions.

# 2 Details and comments about the correspondence table

We expect the reader to be familiar with the basic notions of the theory of dynamical systems and flows. Some definitions are briefly recalled in the glossary at the end of this paper. For more precisions, we refer for example to [50], [57], [62], [81] or [85] for finite-dimensional dynamics and to [37], [41], [82] or [36] for the infinite-dimensional ones.

We first would like to give short comments and motivations concerning the properties appearing in Table 1. Notice that we do not deal in this section with the cooperative systems of ODEs. The properties of these systems are discussed in Section 4.

A *generic property of the dynamics* is a property satisfied by a countable intersection of open dense subsets of the considered class of dynamical systems. Generic dynamics represent the typical behaviour of a class of dynamical systems. For finite-dimensional flows, we mainly consider classes of the form $(T_g(t))_{g\in\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d)}$. The parameter is the vector field $g$, which belongs to the space $\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d)$ endowed with either the classical $\mathcal{C}^1$ or the $\mathcal{C}^1$ Whitney topology. Notice that the question wether or not a property is generic for $g \in \mathcal{C}^r(\mathbb{R}^d,\mathbb{R}^d)$ for some $r \geq 2$ may be much more difficult than $\mathcal{C}^1$ genericity. We will not discuss this problem here. In some cases, we restrict the class of vector fields to subspaces of $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$ such as radially symmetric, gradient vector fields or cooperative systems. In a similar way, for infinite-dimensional dynamics, we consider families of the type $(S_f(t))_{f\in\mathcal{C}^1(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^{d'},\mathbb{R})}$, where $\mathcal{C}^1(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^{d'},\mathbb{R})$ is endowed with either the classical $\mathcal{C}^1$ or the $\mathcal{C}^1$ Whitney topology. For some results, we restrict the class of nonlinearities $f$ to homogeneous ones or to ones, which are independent of the last variable $\xi$.

*Poincaré-Bendixson property* and *the convergence to an equilibrium or a periodic orbit* are properties related to the following question: how simple are the $\alpha-$ and $\omega-$limit sets of the trajectories ? For vector fields, the restriction of the complexity of the limit sets may come from the restriction of freedom due to the low dimension of the flow. As said above, there is no restriction on the dimension of the global attractor for the parabolic equations. The possible restrictions of the complexity of the limit sets come from particular properties of the parabolic equations, see Section 3.

*Hyperbolicity of equilibria and periodic orbits, tranversality of stable and unstable manifolds, Kupka-Smale and Morse-Smale properties* are properties related to the question of stability of the local and global dynamics respectively. Morse-Smale property is the strongest one. It implies the structural stability of the global dynamics: if the dynamical system $T_g(t)$ satisfies the Morse-Smale property, then for $\tilde{g}$ close enough to $g$, the dynamics of $T_{\tilde{g}}(t)$, restricted to its attractor $\mathcal{A}_{\tilde{g}}$, are qualitatively the same as the ones of $T_g(t)$ on $\mathcal{A}_g$, see [61], [63] and [62]. The same structural stability result holds for parabolic equations satisfying the Morse-Smale property, see [37], [36] and [60]. It is natural to wonder if almost all the dynamics satisfy these properties, that is if these properties are generic.

The fact that *the knowledge of the equilibria and the periodic orbits implies the knowledge of the whole dynamics* may be studied at different levels. Two equilibria or periodic orbits being given, can we know if they are connected or not by a heteroclinic orbit ? Are two dynamics with the same equilibria and periodic orbits equivalent ? Is there a simple algorithm to determine the global dynamics from the position of the equilibria and the periodic orbits ? These questions are among the rare dynamical questions coming from the study of partial differential equations and not from the study of vector fields. Indeed, for finite-dimensional dynamical systems, the answers, either positive or negative, are too simple. In contrast, such kinds of results are probably among the most amazing ones for the dynamics of the parabolic equations.

*The persistent chaotic dynamics* and the fact that *the dimension of the attractor is equal to the largest dimension of the unstable manifolds*, are related to the following question: how complicated may be the dynamics ? In general, the dimension of the attractor of a dynamical system may be larger than the largest dimension of the unstable manifolds. The classes of systems, where these dimensions automatically coincide, are strongly constrained, which in some sense implies a simple behaviour. On the contrary, chaotic dynamics have very complicated behaviour. Chaotic dynamics may occur through several phenomena, and the notion of chaotic behaviour depends on the authors. In this paper, "persistent chaotic dynamics" refers to the presence of a tranversal homoclinic orbit generating a Smale horseshoe (see [89]). The persistent chaotic dynamics provide complicated dynamics, which cannot be removed by small perturbations of the system. Such an open set of chaotic dynamics is a counter-example to the genericity of the Morse-Smale systems.

The question of *the realization of vector fields in the parabolic equations* is as follows: a vector field $g \in \mathcal{C}^r(\mathbb{R}^d, \mathbb{R}^d)$ being given, can we find a function $f$ and an invariant manifold $M \subset L^p(\Omega)$ such that the dynamics of the parabolic equation (1.2) restricted to $M$ is equivalent to the dynamics generated by the vector field $g$ ? A positive answer to this question implies that the dynamics of the considered class of parabolic equations is at least as complicated as the dynamics of the considered class of vector fields. Such a realization result is very interesting since, on the opposite, the other properties stated in Table 1 roughly say that the dynamics of the parabolic equation (1.2) cannot be much more complicated than the ones of the corresponding class of finite-dimensional flows. One has to keep in mind that the manifold $M$, on which the finite-dimensional dynamics are realized, is not necessarily stable with respect to the dynamics of the parabolic equation. Typically, $M$ cannot be stable if the finite-dimensional system contains a stable periodic orbit, since all periodic orbits of (1.2) are unstable (see for example [45]).

Now, we give short comments and references for the correspondences stated in Table 1.

• $d = 1$ **and** $\Omega = (0, 1)$

The dynamics generated by a one-dimensional vector field is very simple. Its attractor consists in equilibrium points and heteroclinic orbits connecting two of them. The existence of these heteroclinic orbits is easily deduced from the positions of the equilibrium points. Moreover, these heteroclinic connections are trivially transversal. Finally, (1.1) is clearly a gradient system with associated Lyapounov functional $F(y) = -\int_0^y g(s)ds$. As a consequence, the Morse-Smale property is equivalent to the hyperbolicity of all the equilibrium points, which holds for a generic one-dimensional vector field.

The dynamics $S_f(t)$ generated by (1.2) for $\Omega = (0, 1)$ is richer since its attractor may have a very large dimension. However, these dynamics satisfy similar properties. These similarities are mainly due to the constraints coming from the non-increase of the number of zeros of solutions of the linear parabolic equation (see Theorem 3.1). Zelenyak has proved in [97] that $S_f(t)$ admits an explicit Lyapounov function and thus that it is gradient. He also showed that the $\omega-$limit sets of the trajectories consist in single equilibrium points. In Proposition 3.2, we give a short proof of this result, due to Matano. The fact that the stable and unstable manifolds of equilibrium points always intersect transversally comes from Theorem 3.1 and the standard Sturm-Liouville theory. This property has been first proved by Henry in [40] and later by Angenent [2] in the weaker case of hyperbolic equilibria. As a consequence of the previous results, the Morse-Smale property is equivalent to the hyperbolicity of the equilibrium points and is satisfied by the parabolic equation on $(0, 1)$ generically with respect to $f$. The most surprising result concerning (1.2) on $\Omega = (0, 1)$ is the following one. Assuming that every equilibrium point is hyperbolic and that the equilibrium points $e_1$,...,$e_p$ are known, one can say if two given equilibria $e_i$ and $e_j$ are connected or not by a heteroclinic orbit. This property has been proved by Brunovský and Fiedler in [12] for $f = f(u)$ and by Fiedler and Rocha in [21] in the general case. The description of the heteroclinic connections is obtained from the Sturm permutation which is a permutation generated by the respective positions of the values $e_i(0)$ and $e_i(1)$ of the equilibrium points at the endpoints of $\Omega = (0, 1)$. The importance of Sturm permutation has been first underlined by Fusco and Rocha in [28]. We also refer to the work of Wolfrum [96] , which presents a very nice formalism for this property. Fiedler and Rocha showed in [23] that the Sturm permutation characterizes the global dynamics of (1.2) on $(0, 1)$. They proved in [22] that it is possible to give the exact list of all the permutations which are Sturm permutations for some nonlinearity $f$ and thus to give the list of all the possible dynamics of the parabolic equation on $(0, 1)$. The fact that the dimension of the attractor is equal to the largest dimension of the unstable manifolds has been shown by Rocha in [83]. The previous works of Jolly [46] and Brunovský [11] deal with the particular case $f \equiv f(u)$, but show a stronger result: the attractor can be embedded in a $\mathcal{C}^1$ invariant graph of dimension equal to the largest dimension of the unstable manifolds. Finally, let us mention that it is easy to realize any one-dimensional flow in an invariant manifold of the semi-flow generated by the one-dimensional parabolic equation. For example, in the simplest case of Neumann boundary conditions as in (1.2), one can realize the flow of any vector field $g$ as the restriction of the dynamics of the equation $\dot{u} = \Delta u + g(u)$ to the subspace of spatially constant functions.

• **$d = 2$ and $\Omega = \mathbb{T}^1$, general case**

Even if they are richer than in the one-dimensional case, the flows generated by vector fields on $\mathbb{R}^2$ are constrained by the Poincaré-Bendixson property (see the original works of Poincaré [67] and Bendixson [7] or any textbook on ordinary differential equations). In particular, this constraint precludes the existence of non-trivial non-wandering points in Kupka-Smale dynamics. Due to the low dimension of the dynamics, the stable and unstable manifolds of hyperbolic equilibria or periodic orbits always intersect transversally if either one of the manifold corresponds to a periodic orbit or if the invariant manifolds correspond to two equilibrium points with different Morse indices. Moreover, there is no homoclinic trajectory for periodic orbits. Using these particular properties, Peixoto proved in [64] that the Morse-Smale property holds for a generic two-dimensional vector field.

The first correspondence between two-dimensional flows and the dynamics of the parabolic equation (1.2) on the circle $\Omega = \mathbb{T}^1$ has been obtained by Fiedler and Mallet-Paret in [20]. They proved that the Poincaré-Bendixson property holds for (1.2) on $\mathbb{T}^1$, by using the properties of the zero number (see Theorem 3.1). The realization of any two-dimensional flow in a two-dimensional invariant manifold of the parabolic equation on the circle has been proved by Sandstede and Fiedler in [86]. Very recently, Czaja and Rocha have shown in [18] that the stable and unstable manifolds of two hyperbolic periodic orbits always intersect transversally and that there is no homoclinic connection for a periodic orbit. The other automatic transversality results and the proof of the genericity of the Morse-Smale property have been completed by the authors in [48] and [49].

• **$d = 2$ and $\Omega = \mathbb{T}^1$, radial symmetry and $\mathbb{T}^1-$equivariance**
When the vector field $g$ satisfies a radial symmetry, the dynamics of the two-dimensional flow generated by (1.1) becomes roughly one-dimensional. The closed orbits consist in 0, circles of equilibrium points and periodic orbits being circles described with a constant rotating speed. The dynamics are so constrained that the closed orbits being given, it is possible to describe all the heteroclinic connections. Notice that no homoclinic connection is possible. We also underline that the Morse-Smale property is generic in the class of radially symmetric vector fields.

If the two-dimensional radial vector fields are too simple to attract much attention, the corresponding case for the parabolic equation (1.2) on $\Omega = \mathbb{T}^1$ with homogeneous nonlinearity $f(x, u, \partial_x u) \equiv f(u, \partial_x u)$ has been extensively studied. Since Theorem 3.1 holds for (1.2) with any one-dimensional domain $\Omega$, it is natural to expect results for (1.2) on $\Omega = \mathbb{T}^1$ similar to the ones obtained for (1.2) on $\Omega = (0, 1)$. In particular, one may wonder if it is possible to describe the global dynamics of (1.2) knowing the equilibria and the periodic orbits only. However, this property is still open for general non-linearities $f(x, u, \partial_x u)$ in the case $\Omega = \mathbb{T}^1$. Moreover, if one believes in the correspondence stated in this paper, one can claim that it is in fact false for a general nonlinearity $f(x, u, \partial_x u)$. Therefore, it was natural to first study the simpler case of homogeneous nonlinearities $f \equiv f(u, \partial_x u)$. Indeed, the dynamics in this case are much simpler, in particular the closed orbits are either homogeneous equilibrium points $e(x) \equiv e \in \mathbb{R}$, or circles of non-homogeneous equilibrium points, or periodic orbits consisting in rotating waves $u(x, t) = u(x - ct)$ (notice the correspondence with the closed orbits of a radially symmetric two-dimensional flow). This property is a consequence of the zero number property of Theorem 3.1 and has been proved in [4] by Angenent and Fiedler. The works of Matano and Nakamura [55] and of Fiedler, Rocha and Wolfrum [24] show that the unstable and stable manifolds of the equilibria and the periodic orbits always intersect transversally and that no homoclinic orbit can occur. Moreover, in [24], the authors give an algorithm for determining the global dynamics of the parabolic equation (1.2) on $\Omega = \mathbb{T}^1$ with homogeneous nonlinearity $f \equiv f(u, \partial_x u)$. This algorithm uses the knowledge of the equilibria and the periodic orbits only. In [84], Rocha also characterized all the dynamics, which may occur. Due to the automatic transverse intersection of the stable and unstable manifolds and due to the possibility of transforming any circle of equilibrium points into a rotating periodic orbit (see [24]), one can show that the Morse-Smale property holds for the parabolic equation on $\mathbb{T}^1$ for a generic homogeneous nonlinearity $f(u, \partial_x u)$ (see [48]). Finally, the realization of any radially symmetric two-dimensional flow in the dynamics of (1.2) on $\mathbb{T}^1$ for some $f \equiv f(u, \partial_x u)$ and the fact that the dimension of the attractor is equal to the largest dimension of the unstable manifolds are shown in [36].

• **$d \geq 3$ and dim($\Omega$)$\geq 2$**
The genericity of the Kupka-Smale property for vector fields in $\mathbb{R}^d$, $d \geq 3$, has been proved

independently by Kupka in [51] and by Smale in [88]. Their proofs have been simplified by Peixoto in [65] (see [1] and [62]). The strong difference with the lower dimensional vector fields is that, when $d \geq 3$, (1.1) may admit transversal homoclinic orbits consisting in the transversal intersection of the stable and unstable manifolds of a hyperbolic periodic orbit. The existence of such an intersection is stable under small perturbations and yields a Smale horseshoe containing an infinite number of periodic orbits and chaotic dynamics equivalent to the dynamics of the shift operator, see [89]. Therefore, the Morse-Smale property cannot be dense in vector fields on $\mathbb{R}^d$ with $d \geq 3$. Even worse, the set of vector fields, whose dynamics are structurally stable under small perturbations, is not dense (notice that this set contains the vector fields satisfying the Morse-Smale property). Indeed, as shown in [33], there exists an open set $\mathcal{U}$ of vector fields of $\mathbb{R}^3$ and a foliation of $\mathcal{U}$ by $2-$codimensional leaves $(\mathcal{U}_\lambda)_{\lambda\in\mathbb{R}^2}$ such that each $g \in \mathcal{U}$ admits a Lorenz attractor $\mathcal{A}_g$ and such that the dynamics of two attractors $\mathcal{A}_g$ and $\mathcal{A}_{\tilde{g}}$ are qualitatively equivalent if and only if $g$ and $\tilde{g}$ belong to the same leave $\mathcal{U}_\lambda$. The possible presence of other chaotic dynamics such as Anosov systems or wild dynamics is also noteworthy, see [5], [56] and [9]. For the interested reader, we refer to [81] or [90].

In [13], Brunovský and both authors proved that the stable and unstable manifolds of hyperbolic equilibria or periodic orbits of the parabolic equation (1.2) are generically transversal. To obtain the genericity of Kupka-Smale property, it remains to obtain the generic hyperbolicity of periodic orbits. Even if this problem is still open, we may strongly believe that the Kupka-Smale property is generic for the parabolic equation (1.2). There exist several results concerning the embedding of the finite-dimensional flows into the parabolic equations. Poláčik has shown in [69] that any ordinary differential equation on $\mathbb{R}^d$ can be embedded into the flow of (1.2) for some $f$ and for some domain $\Omega \subset \mathbb{R}^d$. The constraint that the dimension of $\Omega$ is equal to the dimension of the imbedded flow is removed in [70], however the result concerns a dense set of flows only. A similar result has been obtained by Dancer and Poláčik in [19] for homogeneous nonlinearities $f(u, \nabla u)$ (see also [74]). These realization results imply the possible existence of persistent chaotic dynamics in the flow of the parabolic equation (1.2) as soon as $\Omega$ has a dimension larger than one: transversal homoclinic orbits, Anosov flows on invariant manifolds of any dimension, Lorenz attractors etc...

• **Gradient case**

When $g$ is a gradient vector field $\nabla G$ with $G \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$, then $-G$ is a strict Lyapounov function and (1.1) is a gradient system. In this case, the Kupka-Smale property is equivalent to the Morse-Smale property. The genericity of the Morse-Smale property for gradient vector fields has been obtained by Smale in [87].

In the case where the nonlinearity $f \in \mathcal{C}^r(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ (that is, $f \equiv f(x, u)$ does not depend on $\nabla u$), the parabolic equation (1.2) admits a strict Lyapounov function given by $E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u(x)|^2 - F(x, u(x))\right) dx$, where $F(x, u) = \int_0^u f(x, s)ds$ is a primitive of $f$, and hence generates a gradient system. Brunovský and Poláčik have shown in [14] that the Morse-Smale property holds for the parabolic equation, generically with respect to $f(x, u)$. It is noteworthy that the Morse-Smale property is no longer generic if one restricts the nonlinearities to the class of homogeneous functions $f \equiv f(u)$ (see [72]). Poláčik has shown in [71] that any generic gradient vector field of $\mathbb{R}^d$ can be realized in the flow of the parabolic equation (1.2) on a bounded domain of $\mathbb{R}^2$ with an appropriate nonlinearity $f(x, u)$. The paper [71] also contains the realization of particular dynamics such as non-transversal intersections of stable and unstable manifolds.

• **Caveat: general ODEs or cooperative systems?**
In Table 1, we have given the striking correspondence between the flow generated by Eq. (1.1) and the semiflow generated by the parabolic equation (1.2). In addition, we have pointed out that some classes of cooperative systems are also involved in this correspondence. In fact, the reader should be aware that the dynamics of (1.2) is much closer to the ones of a cooperative system than to the ones of the general vector field (1.1). Indeed, the semiflow $S_f(t)$ generated by the parabolic equation (1.2) belongs to the class of strongly monotone semiflows, which means that this semiflow has more constraints than the flow $T_g(t)$ generated by a general vector field $g$ (see Section 4). That is why, it could be more relevant to write Table 1 in terms of cooperative systems only (for example, by replacing the case of the general ODE with $d \geq 3$ by the case of a cooperative system of ODEs in dimension $d \geq 4$). However, we have chosen to mainly write Table 1 in terms of general ODEs for several reasons:
- as far as the properties stated in Table 1 are concerned, there is no difference between the dynamics of a general ODE and the ones of a parabolic PDE,
- the dynamics of general ODEs are common knowledge, whereas speaking in terms of cooperative systems may not give a good insight of the dynamics of (1.2),
- not all the properties stated in Table 1 are known for the class of cooperative systems (for example the genericity of Kupka-Smale property is not yet known for $d \geq 4$).

# 3 Zero number and unique continuation properties for the scalar parabolic equation

The results presented in Table 1 and in Section 2 strongly rely on properties specific to the parabolic equations. The purpose of this section is to give a first insight of these particular properties and of their use, to the reader.

Dynamical systems generated by vectors fields are flows on $\mathbb{R}^d$, whereas the phase-space of the parabolic equation is an infinite-dimensional space $X^\alpha$. It is important to be aware of the fact that the parabolic equations generate only a small part of all possible dynamical systems on the Banach space $X^\alpha$. On one hand, this implies less freedom in perturbing the dynamics and hence in obtaining density results. In particular, whereas one can easily construct perturbations of a vector field $g$ which are localized in the phase space $\mathbb{R}^d$, the perturbations of the nonlinearity $f$ act in a non local way on $X^\alpha$ (many different functions $u$ can have the same values of $u$ and $\nabla u$ at a given point $x$). Therefore, it is important to obtain unique continuation results in order to find values $(x, u, \nabla u)$, which are reached only once by a given periodic, heteroclinic or homoclinic orbit. On the other hand, the small class of dynamics generated by the parabolic equations admits special properties. These properties may in particular yield the constraints, which make the dynamics similar to the ones of low-dimensional vector fields.

The scalar parabolic equation in space dimension one ($\Omega = (0,1)$ or $\mathbb{T}^1$) satisfies a very strong property: the number of zeros of the solutions of the linearized equation is nonincreasing in time. This property is often called Sturm property since its idea goes back to Sturm [95] in 1836. There are different versions of this result, which have been proved by Nickel [58], Matano [53, 54], Angenent and Fiedler [3, 4] and Chen [16] (see also [30] for a survey). By similar technics, a geometrical result on braids formed by solutions of the one-dimensional parabolic equation is obtained in [32].

**Theorem 3.1.** *Let $\Omega = (0,1)$ with Neumann boundary conditions or $\Omega = \mathbb{T}^1$. Let $T > 0$, $a \in W^{1,\infty}(\overline{\Omega} \times [0,T], \mathbb{R})$ and $b \in \mathbb{L}^\infty(\overline{\Omega} \times [0,T], \mathbb{R})$. Let $v : \overline{\Omega} \times (0,T) \to \mathbb{R}$ be a bounded non-trivial classical solution of*

$$\partial_t v = \partial^2_{xx} v + a(x,t)\partial_x v + b(x,t)v \quad , \quad (x,t) \in \Omega \times (0,T) \ .$$

*Then, for any $t \in (0,T)$, the number of zeros of the function $x \in \overline{\Omega} \mapsto v(x,t)$ is finite and non-increasing in time. Moreover, it strictly decreases at $t = t_0$ if and only if $x \mapsto v(x,t_0)$ has a multiple zero.*

Theorem 3.1 is the fundamental ingredient of almost all the results given in Table 1 in the cases $\Omega = (0,1)$ and $\Omega = \mathbb{T}^1$. It can be used either as a strong comparison principle or as a strong unique continuation property, as shown in the following examples of applications. General surveys can be found in [25], [35] and [36].

In the first application presented here, Theorem 3.1 is used as a strong maximum principle. In some sense, it yields an order on the phase space which is preserved by the flow. This illustrates how Theorem 3.1 may imply constraints similar to the ones of low-dimensional vector fields. The following result was first proved in [97] and the proof given here comes from [53] (see also [25]).

**Proposition 3.2.** *Let $\Omega = (0,1)$, let $u_0 \in X^\alpha$ and let $u(x,t)$ be the corresponding solution of the parabolic equation* (1.2) *with homogeneous Neumann boundary conditions. The $\omega-$limit set of $u_0$ consists of a single equilibrium point.*

***Proof:*** We first notice that $v(x,t) = \partial_t u(x,t)$ satisfies the equation

$$\partial_t v(x,t) = \partial^2_{xx} v(x,t) + f'_u(x, u(x,t), \partial_x u(x,t))v(x,t) + f'_{\partial_x u}(x, u(x,t), \partial_x u(x,t))\partial_x v(x,t) \ .$$

Due to the Neumann boundary conditions, we have $\partial_x u(0,t) = \partial^2_{xt} u(0,t) = \partial_x v(0,t) = 0$ for all $t > 0$. In particular, as soon as $v(0,t) = 0$, $v(t)$ has a double zero at $x = 0$. Due to Theorem 3.1, either $v$ is a trivial solution, that is $v \equiv 0$ for all $t$, and $u$ is an equilibrium point, or $v(0,t)$ vanishes at most a finite number of times since $v(t)$ can have a multiple zero only a finite number of times. Assume that $u$ is not an equilibrium, then $u(0,t)$ must be monotone for large times and thus converges to $a \in \mathbb{R}$. Any trajectory $w$ in the $\omega-$limit set of $u_0$ must hence satisfy $w(0,t) = a$ for all $t$. Therefore, $\partial_t w(0,t) = 0$ for all $t$ and $\partial_t w(0,t)$ has a multiple zero at $x = 0$ for all times. Using Theorem 3.1, we deduce as above that $w$ is an equilibrium point of (1.2). But there exists at most one equilibrium $w$ satisfying $w(0) = a$ and the Neumann boundary condition $\partial_x w(0) = 0$. Therefore, the $\omega-$limit set of $u_0$ is a single equilibrium point $w$. □

The second application comes from [48]. It shows how Theorem 3.1 can be used as a unique continuation property. This kind of property roughly says that, if two solutions coincide too much near a point $(x_0, t_0)$, then they must be equal everywhere. The motivation beyond this example of application is the following. We consider a time-periodic solution of (1.2) on $\Omega = \mathbb{T}^1$. The problem is to find a perturbation of the nonlinearity $f$, which makes this periodic orbit hyperbolic. As enhanced above, such a perturbation is nonlocal in the phase space of (1.2). To be able to perform perturbation arguments, it is important to show that one can find a perturbation of $f$ which acts only locally on the periodic orbit. To this end, one proves the following result.

**Proposition 3.3.** *Let $p(x,t)$ be a periodic orbit of (1.2) on $\Omega = \mathbb{T}^1$. Let $T > 0$ be its minimal period. Then, the map*

$$(x,t) \in \mathbb{T}^1 \times [0,T) \mapsto (x, p(x,t), \partial_x p(x,t))$$

*is one to one.*

***Proof:*** Assume that this map is not injective. Then there exist $x_0$, $t_0 \in [0,T)$ and $t_1 \in [0,T)$, $t_0 \neq t_1$ such that

$$p(x_0,t_0) = p(x_0,t_1) \quad \text{and} \quad \partial_x p(x_0,t_0) = \partial_x p(x_0,t_1) \; .$$

The function $v(x,t) = p(x, t + t_1 - t_0) - p(x,t)$ is a solution of the equation

$$\partial_t v(x,t) = \partial^2_{xx} v(x,t) + a(x,t) v(x,t) + b(x,t) \partial_x v(x,t) \; ,$$

where $a(x,t) = \int_0^1 f'_u(x, p(x,t) + s(p(x,t+t_1-t_0) - p(x,t)), \partial_x p(x,t+t_1-t_0)) ds$ and $b(x,t) = \int_0^1 f'_{u_x}(x, p(x,t), \partial_x(p(x,t) + s(p(x,t+t_1-t_0) - p(x,t))) ds$. Moreover, the function $v(x,t)$ satisfies $v(x_0,t_0) = 0$ and $\partial_x v(x_0,t_0) = 0$ and does not vanish everywhere since $|t_1 - t_0| < T$. Due to Theorem 3.1, the number of zeros of $v(t)$ drops strictly at $t = t_0$ and never increases. However, $v(t)$ is a periodic function of period $T$, and thus, its number of zeros is periodic. This leads to a contradiction and proves the proposition. $\square$

In a domain $\Omega$ of dimension $d' \geq 2$, there is no known counterpart for Theorem 3.1 as shown in [29]. In particular, Proposition 3.3 does no longer hold. However, to be able to construct relevant perturbations of periodic orbits, one needs a result similar to Proposition 3.3, even if weaker. The following result can be found in [13]. Its proof is based on a generalization of the arguments of [38] and on unique continuations properties of the parabolic equations.

**Theorem 3.4.** *Let $p(x,t)$ be a periodic orbit of* (1.2) *with minimal period $T > 0$. There exists a generic set of points $(x_0, t_0) \in \Omega \times [0,T)$ such that if $t \in [0,T)$ satisfies $p(x_0,t) = p(x_0,t_0)$ and $\nabla p(x_0,t) = \nabla p(x_0,t_0)$, then $t = t_0$.*

# 4 Cooperative systems of ODEs

We consider a system of differential equations

$$\dot{y}(t) = g(y(t)) \ , \quad y(0) = y_0 \in \mathbb{R}^N \ , \tag{4.1}$$

where $g = (g_i)_{i=1\ldots N}$ is a $\mathcal{C}^1$ vector field.
Due to the analogy with biological models, the following definitions are natural. We say that (4.1) is a *cooperative* (resp. *competitive*) system if for any $y \in \mathbb{R}^N$ and $i \neq j$, $\frac{\partial g_i}{\partial y_j}(y)$ is non-negative (resp. non-positive) and the matrix $(\frac{\partial g_i}{\partial y_j})(y)$ is irreducible i.e. it is not a block diagonal matrix (the simpler assumption that all the coefficients $\frac{\partial g_i}{\partial y_j}(y)$ are positive is sometimes made instead of the irreducibility). We say that (4.1) is a *tridiagonal* system if $\frac{\partial g_i}{\partial y_j} = 0$ for $|i-j| \geq 2$ and a *cyclic tridiagonal* system if the indices $i$ and $j$ are considered modulo $N$, i.e. if, in addition, we allow $\frac{\partial g_1}{\partial y_N}$ and $\frac{\partial g_N}{\partial y_1}$ to be non-zero. For the reader interested in cooperative systems, we refer to [94].

In this section, we only consider cooperative systems. However, notice that, by changing $t$ into $-t$ or $y_i$ into $-y_i$, we obtain similar results for competitive systems and for systems with different sign conditions.

The dynamics of cooperative systems may be as complicated as the dynamics of general vector fields. Indeed, Smale has shown in [91] that any vector field in $\mathbb{R}^{N-1}$ can be realized in a invariant manifold of a cooperative system in $\mathbb{R}^N$. Notice that this realization result implies that any one-dimensional vector field can be imbedded in a tridiagonal cooperative system and any two-dimensional vector field can be imbedded in a cyclic tridiagonal cooperative system. This explains why we present the tridiagonal cooperative systems in Table 1 as generalization of one- and two-dimensional vector fields.

However, the dynamics of a cooperative system (4.1) is really different from the ones of the general ODE (1.1) since a cooperative system generates a strongly monotone flow, that is, a flow which preserves a partial order. It is noteworthy that the semiflow $S_f(t)$ generated by the parabolic equation (1.2) also belongs to the class of strongly monotone semiflows (it preserves the order of $X^\alpha$ induced by the classical order of $\mathcal{C}^0(\Omega)$). Therefore, the semiflow of (1.2) is much closer to the one of the cooperative system (4.1), both admitting more constraints than the flow $T_g(t)$ generated by a general vector field $g$. In [42] and [45] for example, Hirsch has shown that almost all bounded trajectories of a strongly monotone semiflow are quasiconvergent, that is, their $\omega$-limit sets consist only of equilibria. More precisely, all initial data, which have bounded nonquasiconvergent trajectories, form a meager subset (that is, the complement of a generic subset) of the phase space. Later, in [68], Poláčik has proved that the set of all initial data $u_0 \in X^\alpha$, which have bounded nonconvergent trajectories in the semiflow of the parabolic equation (1.2), is meager in $X^\alpha$.

Moreover, since the works of Hirsch and Smillie, it is known that the dynamics of cooperative systems, which are in addition tridiagonal, are very constrained in any dimension $N$. Indeed, in [43], [44] and [92], strong properties of the limit sets of cooperative systems are proved. In particular, any three-dimensional cooperative system satisfies the Poincaré-Bendixson property and the trajectory of any tridiagonal cooperative system converges to a single equilibrium point. Inspired by the articles of Henry and Angenent about the parabolic equation on $(0,1)$, Fusco and Oliva (see [26]) showed a theorem similar to Theorem 3.1 (see [93] for a more general statement).

**Theorem 4.1.** *Let $\mathcal{N}$ be the set of vector $y \in \mathbb{R}^N$ such that, for all $i = 1 \dots N$, either $y_i \neq 0$ or $y_i = 0$ and $y_{i-1}y_{i+1} < 0$ (where $y_0 = y_{N+1} = 0$). For every $y \in \mathcal{N}$, we set $N(y)$ to be the number of sign changes for $y_i$, when $i$ goes from 1 to $N$. Let $y(t) \neq 0$ be a solution of*

$$\dot{y}(t) = A(t)y(t) \ , \tag{4.2}$$

*where $A \in \mathcal{C}^0(\mathbb{R}, \mathcal{M}_N(\mathbb{R}))$ satisfies $A_{ij}(t) > 0$ for all $t \in \mathbb{R}$ and all $i \neq j$.*
*Then, the times $t$ where $y(t) \notin \mathcal{N}$ are isolated and, if $y(t_0) \notin \mathcal{N}$, then, for every $\varepsilon > 0$ small enough, $N(y(t+\varepsilon)) < N(y(t-\varepsilon))$.*

In other words, the number of sign changes of the solutions of the linear equation (4.2) is non-increasing in time and strictly drops at $t_0$ if and only if $y(t_0)$ has in some sense a multiple zero. The parallel with Theorem 3.1 is of course striking. Using Theorem 4.1, Fusco and Oliva have shown that the stable and unstable manifolds of equilibrium points of a tridiagonal cooperative system always intersect transversally. As a consequence, the Morse-Smale property is generic in the class of tridiagonal cooperative systems.

Theorem 4.1 also holds for cyclic tridiagonal cooperative systems, see [27] and [93]. Using this fundamental property, Fusco and Oliva have shown in [27] that the stable and unstable manifolds of periodic orbits of cyclic tridiagonal cooperative systems always intersect transversally. In addition, Mallet-Paret and Smith have shown in [52] that cyclic tridiagonal cooperative systems satisfy the Poincaré-Bendixson property. Notice that, following [48] and [49], one should be able to prove the genericity of the Morse-Smale property for cyclic tridiagonal cooperative systems. This has been proved very recently by Percie du Sert (see [66])

Considering all these results, it is not surprising that there exists a parallel between tridiagonal cooperative systems and the parabolic equation on $(0, 1)$. Indeed, consider a solution $v$ of the linear one-dimensional parabolic equation

$$\dot{v}(x,t) = \partial^2_{xx} v(x,t) + a(x,t)\partial_x v(x,t) + b(x,t)v(x,t) \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{4.3}$$

We discretize the segment $(0, 1)$ by a sequence of points $x_k = (k-1)/(N-1)$ with $k = 1 \ldots N$. The natural approximation of $v$ is given by $y_k \approx v(x_k)$ solution of

$$\dot{y}_k(t) = \frac{y_{k+1}(t) - 2y_k(t) + y_{k-1}(t)}{h^2} + a_k(t)\frac{y_{k+1} - y_k}{h} + b_k(t)y_k(t) \tag{4.4}$$

where $a_k(t) = a(x_k, t)$, $b_k(t) = b(x_k, t)$ and $h = 1/(N - 1)$. If $h$ is small enough, (4.4) is a tridiagonal cooperative system. The relation between Theorems 3.1 and 4.1 is obvious in this framework.

# 5 Kupka-Smale property and other open problems

One of the main goals of the study of dynamical systems is to understand the behaviour of a generic dynamical system. The most recent results concerning the parabolic equations are related to the genericity of Kupka-Smale property. In [13], the generic transversality of homoclinic and heteroclinic orbits of hyperbolic equilibria or periodic orbits of (1.2) is proved. However, the generic hyperbolicity of periodic orbits is still an open problem (see the discussion in [13]). Its resolution would complete the whole correspondance of Table 1 and is one of the most important open problem concerning the qualitative dynamics of the parabolic equation (1.2).

However, even if the generic Kupka-Smale property is obtained, it cannot give a good insight of the complex and chaotic dynamics that may be generated by homoclinic connections. For this reason, the study of finite-dimensional flows has been pursued further the Kupka-Smale property and is still in progress. The corresponding results should serve as a guideline for the study of the flow generated by the parabolic equation (1.2). For vector fields, one of the main steps beyond Kupka-Smale property is Pugh's closing lemma: if $p$ is a non-wandering point of the dynamical system $T_g(t)$ generated by (1.1), then there exists a $\mathcal{C}^1-$perturbation $\tilde{g}$ of $g$ such that $p$ is a periodic point of $T_{\tilde{g}}(t)$ (the case of a $\mathcal{C}^r-$perturbation with $r \geq 2$ is still open). The proof of Pugh in [75] concerns discrete dynamical systems. It has been adapted to the case of flows by Pugh and Robinson in [77] (see also [80] for an introduction to the proof). A direct consequence of Pugh closing lemma is the general density theorem: for a generic finite-dimensional flow, the non-wandering points are is the closure of the periodic points (see [76] and [81]). Other connecting lemmas have been proved by Hayashi [39] and Bonatti and Crovisier [8]. They enable a better understanding of generic dynamics. For example, the class of finite-dimensional dynamical systems which either satisfy the Morse-Smale property or admit a transversal homoclinic connection is generic (see [78], [10] and [17] for discrete dynamical systems in dimensions $d = 2$, $d = 3$ and $d \geq 4$ respectively, and see [6] for three-dimensional flows). Obtaining similar results for the flow of the parabolic equation should be a very interesting and difficult challenge.

Other interesting open problems concern the realization of finite-dimensional dynamics in the semiflow of parabolic equations. Indeed, we only know that one can realize the dynamics of a dense set of general ODEs in the flow of a parabolic equation (1.2) on a two-dimensional domain. One may wonder if it is possible to realize the dynamics of all ODEs. Since the parallel between parabolic equations and cooperative systems is stronger, the following strong realization conjecture may be more plausible: *any* flow of a cooperative system of ODEs can be realized in an invariant manifold of the flow of a parabolic equation (1.2) on a two-dimensional domain.

Finally, the genericity of the Morse- and Kupka-Smale properties is also an interesting problem for other classes of partial differential equations. The genericity of the Morse-Smale property is known for the wave equations $\ddot{u} + \gamma\dot{u} = \Delta u + f(x, u)$ with constant damping $\gamma > 0$ (see [15]) and with variable damping $\gamma(x) \geq 0$ in space dimension one (see [47]). We recall that, in both cases, the associated dynamical system is gradient. Nothing is known for other classes of PDEs, in particular for the equations of fluids dynamics and for systems of parabolic equations $\dot{U} = \Delta U + f(x, U)$, with $U(x, t) \in \mathbb{R}^N$. In all these cases, the main problem consists in understanding how the perturbations act on the phase plane of the PDE. Either one proves unique continuation results similar to Theorem 3.4 in order to be able to use local perturbations of the flow (as in [48], [49] and [13]), or one uses particular non-local perturbations in a very careful way (as in [14], [15] and [47]).

# Glossary

In this section, $S(t)$ denotes a general continuous dynamical system on a Banach space $X$. An orbit of $S(t)$ is denoted by $x(t) = S(t)x_0$ with $t \in I$, where $I = [0, +\infty)$, $I = (-\infty, 0]$ or $I = (-\infty, +\infty)$ in the case of a positive, negative or global trajectory respectively.
**Compact global attractor:** if it exists, the compact global attractor $\mathcal{A}$ of $S(t)$ is a compact invariant set which attracts all the bounded sets of $X$. Notice that $\mathcal{A}$ is then the set of all the bounded global trajectories. See [34].
$\alpha-$ **and** $\omega-$**limit sets:** let $x_0 \in X$. The $\alpha-$limit set $\alpha(x_0)$ and the $\omega-$limit set $\omega(x_0)$ of $x_0$ are the sets of accumulation points of the negative and positive orbits coming from $x_0$

respectively. More precisely,

$$\begin{aligned}\alpha(x_0) = \{x \in X \;/\; \exists (t_n)_{n\in\mathbb{N}},\ t_n \xrightarrow[n\to\infty]{} -\infty \text{ and a negative trajectory } x(t) \\ \text{such that } x(0) = x_0 \text{ and } x(t_n) \xrightarrow[n\to\infty]{} x\} \\ \omega(x_0) = \{x \in X \;/\; \exists (t_n)_{n\in\mathbb{N}},\ t_n \xrightarrow[n\to\infty]{} +\infty \text{ such that } S(t_n)x_0 \xrightarrow[n\to\infty]{} x\}\end{aligned}$$

The limit sets $\alpha(x_0)$ and $\omega(x_0)$ are non-empty connected compact sets.
**Homoclinic or heteroclinic orbit:** let $x(t) = S(t)x_0$ be a global trajectory of $S(t)$. Assume that the $\alpha-$ and $\omega-$limit sets of $x_0$ exactly consists in one orbit, denoted $x_-(t)$ and $x_+(t)$ respectively, this orbit being either an equilibrium point or a periodic orbit. The trajectory $x(t)$ is said to be a homoclinic orbit if $x_-(t) = x_+(t)$ and a heteroclinic orbit if $x_-(t) \neq x_+(t)$.
**Backward uniqueness property:** $S(t)$ satisfies the backward uniqueness property if for any time $t_0 > 0$ and any trajectories $x_1(t)$ and $x_2(t)$, $x_1(t_0) = x_2(t_0)$ implies $x_1(t) = x_2(t)$ for all $t \in [0, t_0]$. Notice that this does not mean that $S(t)$ admits negative trajectories.
**Hyperbolic equilibrium points or periodic orbits:** an equilibrium point $e$ of $S(t)$ is hyperbolic if the linearized operator $x \mapsto D_eS(1)x$ has no spectrum on the unit circle. Let $p(t)$ be a periodic solution of $S(t)$ with minimal period $T$. For each $x \in X$, we denote $t \mapsto \Pi(t, 0)x$ the corresponding trajectory of the linearization of $S(t)$ along the periodic solution $p(t)$. Then, $p(t)$ is said hyperbolic if the linear map $x \mapsto \Pi(T, 0)x$ has no spectrum on the unit circle except the eigenvalue 1 which is simple. Remark that then, for any integer $k \neq 0$, the linear map $x \mapsto \Pi(kT, 0)x$ has no spectrum on the unit circle except the eigenvalue 1 (which is simple).
**Stable and unstable manifolds:** let $e$ be a hyperbolic equilibrium point of $S(t)$. There exists a neighbourhood $\mathcal{N}$ of $e$ such that the set

$$W^u_{loc}(e) = \{x_0 \in X \ ,\ \exists \text{ a negative trajectory } x(t) \text{ with } x(0) = x_0 \\ \text{and, } \forall t \leq 0,\ x(t) \in \mathcal{N}\}$$

is a submanifold of $X$, in which all negative trajectories converge to $e$ when $t$ goes to $-\infty$. The manifold $W^u_{loc}(e)$ is called the local unstable manifold of $e$. Pushing $W^u_{loc}(e)$ by the flow $S(t)$, one can define the (global) unstable set $W^u(e) = \cup_{t\geq 0} S(t)W^u_{loc}(e)$, which consists in all the negative trajectories converging to $e$ when $t$ goes to $-\infty$. This unstable set $W^u(e)$ is an immersed submanifold under suitable properties. For instance, backward uniqueness properties are needed to extend the manifold structure.
In the same way, one defines the local stable manifold

$$\begin{aligned}W^s_{loc}(e) &= \{x_0 \in X \ ,\ \forall t \geq 0,\ S(t)x_0 \in \mathcal{N}\} \\ &= \{x_0 \in X \ ,\ \forall t \geq 0,\ S(t)x_0 \in \mathcal{N} \text{ and } S(t)x_0 \xrightarrow[t\to+\infty]{} e\} \ .\end{aligned}$$

General partial differential equations (and parabolic equations in particular) do not admit negative trajectories. Therefore, it is less easy to extend the local stable manifold to a global stable manifold. However, one can define the stable set $W^s(e)$ of $e$ as follows

$$W^s(e) = \{x_0 \in X \ , \ S(t)x_0 \xrightarrow[t\to+\infty]{} e\} \ .$$

Under suitable additional properties (which are satisfied by the parabolic equation (1.2)), one can show that $W^s(e)$ is also an immersed submanifold. For instance, backward uniqueness properties of the adjoint dynamical system $S^*(t)$ on $X^*$ and finite-codimensionality of $W^s_{loc}(e)$ are needed (see [41] for more details).
If $p(t)$ is a hyperbolic periodic orbit, one defines its unstable and local stable manifolds in the same way. See for example [62] for more details.
**Non-wandering set:** a point $x_0 \in X$ is non-wandering if for any neighbourhood $\mathcal{N} \ni x_0$ and any time $t_0 > 0$, there exists $t \geq t_0$ such that $S(t)\mathcal{N} \cap \mathcal{N} \neq \emptyset$.
**The Kupka-Smale and Morse-Smale properties:** $S(t)$ satisfies the Kukpa-Smale property if all its equilibrium points and periodic orbits are hyperbolic and if their stable and unstable manifolds intersect transversally. It satisfies the Morse-Smale property if in addition its non-wandering set consists only in a finite number of equilibrium points and periodic orbits. We refer to [62] for more precisions on these notions.
**Gradient dynamical systems:** $S(t)$ is gradient if it admits a Lyapounov functional, that is a function $\Phi \in \mathcal{C}^0(X,\mathbb{R})$ such that, for all $x_0 \in X$, $t \mapsto \Phi(S(t)x_0)$ is non-increasing and is constant if and only if $x_0$ is an equilibrium point. We recall that a gradient dynamical system does not admit periodic or homoclinic orbits.
**Cooperative system of ODEs:** see Section 4.
**Generic set and Baire space:** a generic subset of a topological space $Y$ is a set which contains a countable intersection of dense open subsets of $Y$. A property is generic in $Y$ if it is satisfied for a generic set of $Y$. The space $Y$ is called a Baire space if any generic set is dense in $Y$. In particular a complete metric space is a Baire space.
**Whitney topology:** let $k \geq 0$ and let $M$ be a Banach manifold. The Whitney topology on $\mathcal{C}^k(M,\mathbb{R})$ is the topology generated by the neighbourhoods

$$\{g \in \mathcal{C}^k(M,\mathbb{R}) \ , \ |D^i f(x) - D^i g(x)| \leq \delta(x), \ \forall i \in \{0,1,\ldots,k\}, \ \forall x \in M\} \ ,$$

where $f$ is any function in $\mathcal{C}^k(M,\mathbb{R})$ and $\delta$ is any positive function in $\mathcal{C}^k(M,(0,+\infty))$. Notice that $\mathcal{C}^k(M,\mathbb{R})$ endowed with the Whitney topology is a Baire space even if it is not a metric space when $M$ is not compact. We refer for instance to [31].
**The fractional power space $X^\alpha$:** let $A$ be a positive self-adjoint operator with compact inverse on $\mathbb{L}^2(\Omega)$. Let $(\lambda_n)$ be the sequence of its eigenvalues, which are positive, and let $(\varphi_n)$ be the corresponding sequence of eigenfunctions, which is an orthonormal basis of $\mathbb{L}^2(\Omega)$. For each $\alpha \in \mathbb{R}$, we define the fractional power of $A$ by $A(\sum_n c_n\varphi_n) = \sum_n c_n\lambda_n^\alpha\varphi_n$. In particular, $A^0 = Id$ and $A^1 = A$. The space $X^\alpha$ is the domain of $A^\alpha$ that is

$$X^\alpha = \{\varphi \in \mathbb{L}^2(\Omega) \ , \ \varphi = \sum_n c_n\varphi_n \text{ such that } (c_n\lambda_n^\alpha) \in \ell^2(\mathbb{N})\} \ .$$

It is possible to define fractional powers of more general operators, called sectorial operators, see [41].
**The Sobolev space $W^{s,p}(\Omega)$:** if $s$ is a positive integer, $W^{s,p}(\Omega)$ is the space of (classes of) functions $f \in \mathbb{L}^p(\Omega)$, which are $s$ times differentiable in the sense of distributions and whose derivatives up to order $s$ belong to $\mathbb{L}^p(\Omega)$. It is possible to extend this notion to positive numbers $s$ which are not integers by using interpolation theory.
**Unique continuation properties:** let us consider a partial differential equation on $\Omega$ and let $u(x,t)$ be any solution of it. A unique continuation property for this PDE is a result stating that if $u(x,t)$ vanishes on a subset of $\Omega \times \mathbb{R}_+$ which is too large in some sense, then $u(x,t)$ must vanish for all $(x,t)$ in $\Omega \times \mathbb{R}_+$.

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# References

[1] R. Abraham and J. Robbin, *Transversal mappings and flows.* W. A. Benjamin, Inc., New York-Amsterdam 1967.

[2] S.B. Angenent, The Morse-Smale property for a semilinear parabolic equation, *Journal of Differential Equations* **62** (1986), 427-442.

[3] S.B. Angenent, The zero set of a solution of a parabolic equation, *Journal für die Reine und Angewandte Mathematik*, **390** (1988), 79-96.

[4] S.B. Angenent and B. Fiedler, The dynamics of rotating waves in scalar reaction diffusion equations, *Transactions of the American Mathematical Society* **307** (1988), 545-568.

[5] D.V. Anosov, Geodesic flows on closed Riemannian manifolds of negative curvature, *Trudy Matematicheskogo Instituta Imeni V. A. Steklova* **90** (1967).

[6] A. Arroyo and F. Rodriguez Hertz, Homoclinic bifurcations and uniform hyperbolicity for three-dimensional flows, *Annales de l'Institut Henri Poincaré, Analyse Non Linéaire* **20** (2003), 805-841.

[7] I. Bendixson, Sur les courbes définies par des équations différentielles, *Acta Mathematica* **24** (1901), 1-88.

[8] C. Bonatti and S. Crovisier, Récurrence et généricité, *Inventiones Mathematicae* **158** (2004), 33-104.

[9] C. Bonatti and L. Díaz, Connexions hétéroclines et généricité d'une infinité de puits et de sources, *Annales Scientifiques de l'École Normale Supérieure* **32** (1999), 135-150.

[10] C. Bonatti, S. Gan and L. Wen, *On the existence of non-trivial homoclinic classes, Ergodic Theory and Dynamical Systems* **27** (2007), 1473-1508.

[11] P. Brunovský, The attractor of the scalar reaction diffusion equation is a smooth graph, *Journal of Dynamics and Differential Equations* **2** (1990), 293-323.

[12] P. Brunovský and B. Fiedler, Connecting orbits in scalar reaction diffusion equations. II. The complete solution, *Journal of Differential Equations* **81** (1989), 106-135.

[13] P. Brunovský, R. Joly and G. Raugel, Generic transversality of heteroclinic and homoclinic orbits for scalar parabolic equations, manuscript.

[14] P. Brunovský and P. Poláčik, The Morse-Smale structure of a generic reaction-diffusion equation in higher space dimension, *Journal of Differential Equation* **135** (1997), 129-181.

[15] P. Brunovský and G. Raugel, Genericity of the Morse-Smale property for damped wave equations, *Journal of Dynamics and Differential Equations* **15** (2003), 571-658.

[16] X.-Y. Chen, A strong unique continuation theorem for parabolic equations, *Mathematische Annalen* **311** (1998), 603-630.

[17] S. Crovisier, Birth of homoclinic intersections: a model for the central dynamics of partially hyperbolic systems, *Annals of Mathematics* **172** (2010), 1641-1677.

[18] R. Czaja and C. Rocha, Transversality in scalar reaction-diffusion equations on a circle, *Journal of Differential Equations* **245** (2008), 692-721.

[19] E.N. Dancer and P. Poláčik, *Realization of vector fields and dynamics of spatially homogeneous parabolic equations*, Memoirs of the American Mathematical Society **140** (1999), n°668.

[20] B. Fiedler and J. Mallet-Paret, A Poincaré-Bendixson theorem for scalar reaction-diffusion equations, *Archive for Rational Mechanics and Analysis* **107** (1989), 325-345.

[21] B. Fiedler and C. Rocha, Heteroclinic orbits of semilinear parabolic equations, *Journal of Differential Equations* **125** (1996), 239-281.

[22] B. Fiedler and C. Rocha, Realization of meander permutations by boundary value problems, *Journal of Differential Equations* **156** (1999), 282-308.

[23] B. Fiedler and C. Rocha, Orbit equivalence of global attractors of semilinear parabolic differential equations, *Transactions of the American Mathematical Society* **352** (2000), 257-284.

[24] B. Fiedler, C. Rocha and M. Wolfrum, Heteroclinic orbits between rotating waves of semilinear parabolic equations on the circle, *Journal of Differential Equations* **201** (2004), 99-138.

[25] B. Fiedler and A. Scheel, Spatio-temporal dynamics of reaction-diffusion patterns, in *Trends in nonlinear analysis*, (Springer Verlag, Berlin, 2003), pp. 23-152.

[26] G. Fusco and W.M. Oliva, Jacobi matrices and transversality, *Proceedings of the Royal Society of Edinburgh Sect. A* **109** (1988), 231-243.

[27] G. Fusco and W.M. Oliva, Transversality between invariant manifolds of periodic orbits for a class of monotone dynamical systems, *Journal of Dynamics and Differential Equations* **2** (1990), 1-17.

[28] G. Fusco and C. Rocha, A permutation related to the dynamics of a scalar parabolic PDE, *Journal of Differential Equations* **91** (1991), 111-137.

[29] G. Fusco and S.M. Verduyn Lunel, Order structures and the heat equation, *Journal of Differential Equations* **139** (1997), 104-145.

[30] V.A. Galaktianov and P. J. Harwin, Sturm's theorems on zero sets in nonlinear parabolic equations, in *Sturm Liouville Theory: Past and Present*, W. O. Amrein, A. M. Hinz and D. B. Pearson, Editors (Birkhäuser Verlag, 2005).

[31] M. Golubitsky and V. Guillemin, *Stable mapping and their singularities*, Graduate Texts in Mathematics Vol. 14, (Springer Verlag, New York-Heidelberg, 1973).

[32] R.W. Ghrist and R.C. Vandervorst, Scalar parabolic PDEs and braids, *Transactions of the American Mathematical Society* **361** (2009), 2755-2788.

[33] J. Guckenheimer and R.F. Williams, Structural stability of Lorenz attractors, *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* **50** (1979), 59-72.

[34] J.K. Hale, *Asymptotic behavior of dissipative systems*, Mathematical Surveys and Monographs, Vol. 25 (American Mathematical Society, Providence, RI, 1988).

[35] J.K. Hale, Dynamics of a scalar parabolic equation, *Canadian Applied Mathematics quartely* **5** (1997), 209-305.

[36] J.K. Hale, R. Joly and G. Raugel, book in preparation.

[37] J.K. Hale, L. Magalhães and W. Oliva, *An introduction to infinite dimensional dynamical systems*, Applied Mathematical Sciences Vol. 47 (Springer-Verlag, 1984). Second edition, *Dynamics in infinite dimensions*, (Springer Verlag, 2002).

[38] R. Hardt and L. Simon, Nodal sets for solutions of elliptic equations, *Journal of Differential Geometry* **30** (1989), 505-522.

[39] S. Hayashi, Connecting invariant manifolds and the solution of the $C^1$ stability and $\Omega$-stability conjectures for flows, *Annals of Mathematics* **145** (1997), 81-137.

[40] D.B. Henry, Some infinite-dimensional Morse-Smale systems defined by parabolic partial differential equations, *Journal of Differential Equations* **59** (1985), 165-205.

[41] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, Vol. 840 (Springer-Verlag, Berlin-New York, 1981).

[42] M.W. Hirsch, Differential equations and convergence almost everywhere in strongly monotone semiflows, *Contemp. Math.* **17** (1983), 267-285.

[43] M.W. Hirsch, Systems of differential equations which are competitive or cooperative. I. Limit sets, *SIAM Journal on Mathematical Analysis* **13** (1982), 167-179.

[44] M.W. Hirsch, Systems of differential equations that are competitive or cooperative. II. Convergence almost everywhere, *SIAM Journal on Mathematical Analysis* **16** (1985), 423-439.

[45] M.W. Hirsch, Stability and convergence on strongly monotone dynamical systems, *Journal für die Reine und Angewandte Mathematik* **383** (1988), 1-53.

[46] M. Jolly, Explicit construction of an inertial manifold for a reaction diffusion equation, *J. Differential Equations* **78** (1989), 220-261.

[47] R. Joly, Generic transversality property for a class of wave equations with variable damping, *Journal de Mathématiques Pures et Appliquées* **84** (2005), 1015-1066.

[48] R. Joly and G. Raugel, Generic hyperbolicity of equilibria and periodic orbits of the parabolic equation on the circle, *Transactions of the American Mathematical Society* **362** (2010), 5189-5211.

[49] R. Joly and G. Raugel, Generic Morse-Smale property for the parabolic equation on the circle, *Annales de l'Institut Henri Poincaré, Analyse Non Linéaire* **27** (2010), 1397-1440.

[50] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems.*, with a supplementary chapter by Katok and Leonardo Mendoza in *Encyclopedia of Mathematics and its Applications*, Vol. 54 (Cambridge University Press, Cambridge, 1995).

[51] I. Kupka, Contribution à la théorie des champs génériques,*Contributions to Differential Equations* **2** (1963),457-484. Addendum and corrections, ibid. **3** (1964), 411-420.

[52] J. Mallet-Paret and H.L. Smith, The Poincaré-Bendixson theorem for monotone cyclic feedback systems, *Journal of Dynamics and Differential Equations* **2** (1990), 367-421.

[53] H. Matano, Convergence of solutions of one-dimensional semilinear parabolic equations, *Journal of Mathematics of Kyoto University* **18** (1978), 221-227.

[54] H. Matano, Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **29** (1982), 401-441.

[55] H. Matano and K.-I. Nakamura, The global attractor of semilinear parabolic equations on $\mathbb{T}^1$, *Discrete and Continuous Dynamical Systems* **3** (1997), 1-24.

[56] S.E. Newhouse, Diffeomorphisms with infinitely many sinks, *Topology* **13** (1974), 9-18.

[57] S.E. Newhouse, Lectures on dynamical systems, in *J. Guckenheimer, J. Moser and S.E. Newhouse Dynamical Systems: C.I.M.E. Lectures, Bressanone, Italy, June 1978*, Progress in Mathematic Vol. 8 (Birkhäuser, 1980).

[58] K. Nickel, Gestaltaussagen über Lösungen parabolischer Differentialgleichungen, *Journal für die Reine und Angewandte Mathematik* **211** (1962), 78-94.

[59] Z. Nitecki, *Differentiable dynamics*, (MIT Press Cambridge, Mass., 1971).

[60] W. M. Oliva, Morse-Smale semiflows. Openess and A-stability, in *Differential equations and dynamical systems (Lisbon, 2000)*, Fields Inst. Commun., Vol 31 (Amer. Math. Soc., Providence, RI, 2002), pp. 285–307,

[61] J. Palis, On Morse-Smale dynamical systems,*Topology* **8** (1968) , pp. 385-404.

[62] J. Palis and W. de Melo, *Geometric theory of dynamical systems. An introduction*, Translated from the Portuguese by A. K. Manning (Springer-Verlag, New York-Berlin, 1982).

[63] J. Palis and S. Smale, Structural stability theorems, in *Global Analysis*, Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968 (American Mathematical Society, Providence, R.I., 1970), pp. 223–231.

[64] M.M. Peixoto, Structural stability on two-dimensional manifolds, *Topology* **1** (1962), 101-120.

[65] M.M. Peixoto, On an approximation theorem of Kupka and Smale, *Journal of Differential Equations* **3** (1967), 214-227.

[66] M. Percie du Sert, in preparation.

[67] H. Poincaré, Sur les courbes définies par une équation différentielle, *Comptes-Rendus de l'Académie des Sciences, Paris*, **90** (1880), 673-675 (see also *Œuvres*, Gauthier-Villars, Paris, Vol. 1, (1928)).

[68] P. Poláčik, Convergence in smooth strongly monotone flows defined by semilinear parabolic equations, *Journal of Differential Equations* **79** (1989), 89-110.

[69] P. Poláčik, Imbedding of any vector field in a scalar semilinear parabolic equation, *Proceedings of the American Mathematical Society* **115** (1992), 1001-1008.

[70] P. Poláčik, High-dimensional $\omega$-limit sets and chaos in scalar parabolic equations, *Journal of Differential Equations* **119** (1995), 24-53.

[71] P. Poláčik, Reaction-diffusion equations and realization of gradient vector fields, in *International Conference on Differential Equations (Lisboa, 1995)*, (World Sci. Publ., River Edge, NJ, 1998), pp 197–206.

[72] P. Poláčik, Persistent saddle connections in a class of reaction-diffusion equations, *Journal of Differential Equations* **56** (1999), 182-210.

[73] P. Poláčik, Parabolic equations: asymptotic behavior and dynamics on invariant manifolds, in *Handbook of dynamical systems*, Vol 2 (North-Holland, Amsterdam, 2002), pp. 835–883.

[74] M. Prizzi and K. Rybakowski, Complicated dynamics of parabolic equations with simple gradient dependence, *Transactions of the American Mathematical Society* **350** (1998), 3119-3130.

[75] C.C. Pugh, The closing lemma, *American Journal of Mathematics* **89** (1967), 956-1009.

[76] C.C. Pugh, An improved closing lemma and a general density theorem, *American Journal of Mathematics* **89** (1967), 1010-1021.

[77] C.C. Pugh and C. Robinson, The $C^1$ closing lemma, including Hamiltonians, *Ergodic Theory and Dynamical Systems* **3** (1983), 261-313.

[78] E.R. Pujals and M. Sambarino, Homoclinic tangencies and hyperbolicity for surface diffeomorphisms, *Annals of Mathematics* **151** (2000), 961-1023.

[79] G. Raugel, Global attractors in partial differential equations, in *Handbook of dynamical systems*, Vol 2 (North-Holland, Amsterdam, 2002), pp 885–982.

[80] C. Robinson, Introduction to the closing lemma, in *The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977)*, Lecture Notes in Mathematics, Vol 668, (Springer, Berlin, 1978), pp 225–230.

[81] C. Robinson, *Dynamical systems. Stability, symbolic dynamics, and chaos.* Second edition, Studies in Advanced Mathematics (CRC Press, Boca Raton, FL, 1999).

[82] J.C. Robinson, *Infinite-dimensional dynamical systems. An introduction to dissipative parabolic PDEs and the theory of global attractors*, Cambridge Texts in Applied Mathematics (Cambridge University Press, Cambridge, 2001).

[83] C. Rocha, Properties of the attractor of a scalar parabolic PDE, *Journal of Dynamics and Equations* **3** (1991), 575–591.

[84] C. Rocha, Realization of period maps of planar Hamiltonian systems, *Journal of Dynamics and Differential Equations* **19** (2007), 571-591.

[85] D. Ruelle, *Elements of differentiable dynamics and bifurcation theory*, (Academic Press, Inc., Boston, MA, 1989).

[86] B. Sandstede and B. Fiedler, Dynamics of periodically forced parabolic equations on the circle, *Ergodic Theory and Dynamical Systems* **2** (1992), 559-571.

[87] S. Smale, On gradient dynamical systems, *Annals of Mathematics* **74** (1961), 199-206.

[88] S. Smale, Stable manifolds for differential equations and diffeomorphisms, *Annali della Scuola Normale Superiore di Pisa* **17** (1963), 97-116.

[89] S. Smale, Diffeomorphisms with many periodic points, in *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, (Princeton Univ. Press, Princeton, N.J., 1965), pp 63–80.

[90] S. Smale, Differentiable dynamical systems, *Bulletin of the American Mathemathical Society* **73** (1967), 747-817.

[91] S. Smale, On the differential equations of species in competition, *Journal of Mathematical Biology* **3** (1976), 5-7.

[92] J. Smillie, Competitive and cooperative tridiagonal systems of differential equations, *SIAM Journal on Mathematical Analysis* **15** (1984), 530-534.

[93] H.L. Smith, A discrete Lyapunov function for a class of linear differential equations, *Pacific Journal of Mathematics* **144** (1990), 345-360.

[94] H.L. Smith, *Monotone dynamical systems. An introduction to the theory of competitive and cooperative systems*, Mathematical Surveys and Monographs, Vol 41 (American Mathematical Society, Providence, RI, 1995).

[95] C. Sturm, Sur une classe d'équations à différences partielles, *Journal de Mathématiques Pures et Appliquées* **1** (1836), 373-444.

[96] M. Wolfrum, A sequence of order relations: encoding heteroclinic connections inscalar parabolic PDE, *Journal of Differential Equations* **183** (2002), 56-78.

[97] T.I. Zelenyak, Stabilization of solutions of boundary value problems for a second-order parabolic equation with one space variable, *Differencialnye Uravnenija* **4** (1968), 34-45. Translated in *Differential equations* **4**, 17-22.