PERIODIC ORBITS AND BIRKHOFF SECTIONS OF STABLE HAMILTONIAN STRUCTURES

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Abstract. Stable Hamiltonian structures (SHS) generalize contact forms and define a vector field, known as the Reeb vector field of a stable Hamiltonian structure. As in the contact case, these vector fields preserve a transverse plane field and a volume form. We study two aspects of Reeb vector fields of SHS in 3-manifolds: on one hand we classify all the examples with finitely many periodic orbits; on the other, we give sufficient conditions for the existence of a supporting broken book decomposition and a rational open book decomposition (or a Birkhoff section).

1. Introduction

The aim of this paper is to extend recent results concerning Reeb vector fields defined by contact forms to Reeb vector fields defined by stable Hamiltonian structures (SHS), a larger set of volume-preserving vector fields on closed 3-manifolds. These results concern the number of periodic orbits and establishing a strong relationship between the dynamics of these classes of vector fields and the dynamics of surface diffeomorphisms or homeomorphisms, via the existence of Birkhoff sections.

We deal with several types of transverse surfaces. We consider a non-singular vector field X on a closed 3-manifold M. A *section* or *global section* of X is an embedded closed surface that is everywhere transverse to X and intersects all the orbits of X. If the ambient manifold M has boundary, a *section* or *global section* is an embedded surface with boundary, whose boundary is mapped to the boundary of M and satisfies the previous conditions. Observe that in these two cases, the manifold M fibers over $\mathbb{S}^1$ and the dynamics of the flow of X is captured by the first return map to the section. This situation is very unusual, hence we consider more general sections of flows. A surface is a *transverse surface* if it is immersed in M, its interior is embedded and transverse to X, while its boundary is a collection of periodic orbits of X. A transverse surface is a *Birkhoff section* if it intersects all the orbits of X in bounded time. The later surfaces are also known in the literature as *global surfaces of section (GSS)*, but we reserve this notation for Birkhoff section whose boundary is embedded. There is a well-defined

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first return map in the interior of a Birkhoff section, that allows to transfer results from surface dynamics to dynamics of three-dimensional flows and backwards.

A stable Hamiltonian structure is a pair of a differential 1-form λ and a closed 2-form ω, such that $\lambda \wedge \omega \neq 0$ and $\ker \omega \subset \ker d\lambda$. Observe that the second condition implies that the two kernels coincide except at the points where $d\lambda = 0$. Also, $\ker \omega$ is a 1-dimensional distribution, hence the first condition implies that the kernel of λ is transverse to the kernel of ω. The Reeb vector field is defined as the vector field spanning the kernel of ω and such that $\lambda(X) = 1$. These were introduced by Hofer and Zehnder [28], and arise as restriction of a Hamiltonian system to a certain regular energy level sets that generalize contact type hypersurfaces. From a topological perspective, stable Hamiltonian structures were deeply studied in 3-dimensions in seminal works by Cieliebak and Volkov [9, 10], and appear as well in symplectic field theory [20, 3, 8], and other works in symplectic topology [21, 45, 30, 36, 32, 34]. Their dynamical properties recently attracted interest [6, 7, 33, 11, 4], especially since the proof of the Weinstein conjecture in this context [29, 38]. In closed 3-manifolds, Reeb vector fields defined by SHS are also known as volume-preserving geodesible vector fields [38].

Consider a stable Hamiltonian structure (λ, ω). If $d\lambda$ is identically zero, the Reeb vector field admits a global section and the ambient manifold fibers over $\mathbb{S}^1$, by Tischler's theorem [42]. If $d\lambda$ is never zero, then λ is a contact form and X is its Reeb vector field. Since we will extend results from [12, 13], we are concerned with the case in which the set $\{d\lambda = 0\}$ is a non-empty proper subset of the ambient manifold. In this case, Cieliebak and Volkov's Structure Theorem [10] (Theorem 2.2), establishes a way to cut the manifold along invariant tori. The closure of the connected components are manifolds with boundary where the Reeb vector field is either integrable, or admits a section, or is the Reeb vector field of a contact form. The part where it is integrable is called the *integrable region* U, and its connected components are of the form $T^2 \times I$, with $I = [0, 1]$.

Colin, Dehornoy, Hryniewicz, and the second author in [13] and Contreras and Mazzuchelli in [14], established independently that the set of Reeb vector field of contact structures admitting a Birkhoff section contains an open and dense set in C^∞-topology. In analogy, our main result is

Theorem 1.1. *On any closed 3-manifold, the set of stable Hamiltonian structures whose Reeb vector field admits a Birkhoff section contains a C^1-dense, C^2-open set.*

Observe that contrary to the results for Reeb vector fields of contact structures, our theorem is not in the C^∞-topology. This is due to the structure of SHS, where C^1 is the right category to perturb a vector field as studied in [10] and explained in Section 2.1. The result is not perturbative in nature, i.e. we give sufficient conditions for a stable Hamiltonian structure to admit a Birkhoff section (confer

Corollary 5.14). We then check that this condition is dense and that Birkhoff sections exist in a C^2-neighborhood of these SHS.

Consider now the periodic orbits of a non-singular vector field and the linearised Poincaré map of each of them. If the flow preserves a volume, the determinant of this map is equal to 1. The vector field is non-degenerate if the eigenvalues are never equal to 1 (even when one considers the iterations of the map). Hence the linearised Poincaré map of a periodic orbit of a non-degenerate vector field is either an irrational rotation or has two real eigenvalues. In general, this is a C^∞-generic condition among non-singular vector fields, but not among Reeb vector fields defined by SHS. Instead we consider *contact non-degenerate* Reeb vector fields of SHS (Definition 2.3), a weaker condition than the Morse-Bott non-degeneracy considered in [10] and which is C^1-dense.

Assume that a vector field on a closed 3-manifold M admits a Birkhoff section, and let K be the boundary of the Birkhoff section. Then $M \setminus K$ fibers over $\mathbb{S}^1$ and hence we obtain an open book decomposition of M with binding K. Broken book decompositions, introduced in [12], are a more general structure than an open book decomposition. They consist of a link K and a foliation $\mathcal{F}$ of $M \setminus K$. The foliation is non-trivial, but each leaf is proper and its boundary is contained in the binding K (see Definition 5.7). A vector field is supported by a broken book decomposition if it is tangent to K and transverse to the interior of the leaves of $\mathcal{F}$. In this direction we prove

Theorem 1.2. *Let (λ, ω) be a contact non-degenerate stable Hamiltonian structure in a closed 3-manifold M whose Reeb vector field is X. Assume that the slope of $\ker \omega$ is non-constant in each connected component of U. Then there is a C^2-neighborhood of (λ, ω) in the set of SHS, such that every Reeb vector field in this neighborhood is supported by a broken book decomposition.*

We note that the hypothesis on (λ, ω) in the previous theorem is C^1-dense in the set of SHS (confer Section 5.3). Our theorem deals essentially with the case of a non-constant function $f = \frac{d\lambda}{\omega}$ since otherwise U is empty and the result is already known: the Reeb vector field of (λ, ω) is either a suspension flow or a (non-degenerate) Reeb vector field defined by a contact form. In the first case, the Reeb vector field admits a global section, which is a particular instance of a broken book decomposition adapted to the field. In the latter case, by [12] the Reeb vector field is supported by a broken book decomposition.

If we assume that f is non-constant, in the contact parts of M (confer Section 2.1), the results in [12] and [14, 13] imply the existence of a broken book decomposition or a Birkhoff section (both adapted to have boundary) respectively. Theorems 1.1 and 1.2 are proved by pasting either enough leaves of a broken book decomposition or a Birkhoff section of the contact parts of M with the sections that the flow has in the suspension parts. To be able to paste these two families of transverse surfaces, one needs to add some binding components in U. The slope hypothesis in Theorem 1.2 is to assure that U contains periodic orbits and hence candidates for a binding component.

Reeb vector fields of SHS might not have periodic orbits, but it is known that there is always one if the ambient manifolds is not a torus bundle over the circle [29, 38]. We describe all the possible examples without periodic orbits:

Theorem 1.3. *Let X be an aperiodic Reeb vector field of a stable Hamiltonian structure (λ, ω) on a 3-manifold M. Then one of the following holds:*

(1) *$M = T^3$ or a positive parabolic torus bundle over $\mathbb{S}^1$, the Reeb vector field admits a torus global section and it is conjugated to the suspension of an aperiodic symplectomorphism of the torus,*
(2) *M is a hyperbolic torus bundle over $\mathbb{S}^1$, the Reeb vector field does not admit a global section, and after cutting along an invariant torus, it is conjugated to the suspension of an irrational pseudorotation of the annulus with quadratic irrational rotation number.*

If (λ, ω) is assumed to be analytic, then only the first case occurs.

As we will see, it is possible to construct an aperiodic Reeb vector field of a SHS in each of the cases described in Theorem 1.3. The Weinstein conjecture [44] asserts that every Reeb vector field of a contact structure on a closed manifold (in any dimension) has a periodic orbit. The conjecture holds for contact 3-manifolds [40], where it was later proved that there are always at least two periodic orbits [16]. We deduce the following sharp refinement of the Weinstein conjecture for stable Hamiltonian structures.

Corollary 1.4. *Let M be a closed, oriented 3-manifold. The Weinstein conjecture holds for stable Hamiltonian structures on M if and only if M is not the 3-torus, a hyperbolic torus bundle, or a positive parabolic torus bundle.*

Under the contact non-degeneracy assumption, we can characterize as well those SHS whose Reeb vector field has only finitely many periodic orbits. The following theorem is an analogue of the "two or infinitely many periodic orbits" theorem for non-degenerate contact Reeb vector fields proved in [12], with previous results established in [27, 17].

Theorem 1.5. *Let (λ, ω) be a contact non-degenerate SHS on a closed 3-manifold M with at least one periodic orbit. Then there is C^2-neighborhood of (λ, ω) in the set of SHS, such that every Reeb vector field admits infinitely many periodic orbits except in the following cases:*

- *The Reeb vector field is conjugated to the suspension of a symplectomorphism of a surface Σ_g with finitely many periodic points.*
- *The ambient manifold M is the 3-sphere or a lens space, there are exactly two closed Reeb orbits and they are core circles of a genus one Heegaard splitting of M.*

We point out that the proof of Theorem 1.5 involves proving that a diffeomorphism of a surface with boundary, without periodic points along the boundary, has infinitely many periodic points (confer Theorem 4.1) except in the disk and the annulus. This proof is inspired by the ideas in [31]. It is known that symplectomorphisms with finitely many periodic points (and at least one) only exist in finite order isotopy classes of diffeomorphisms of a surface [31, Theorems 1.2 and 1.3] (see also [1]). Those isotopy classes characterize those surface bundles where contact non-degenerate SHS with finitely many periodic orbits do exist.

The paper is organized as follows. We start presenting known facts about stable Hamiltonian structures and surface dynamics in Section 2, in particular we state the Structure Theorem 2.2 and Definition 2.3 of contact non-degenerate SHS. We then study aperiodic SHS in Section 3 and the contact non-degenerate examples with finitely many periodic orbits in Section 4, proving Theorems 1.3 and 1.5 respectively. In Section 5 we give proofs for Theorems 1.1 and 1.2, starting with the construction in Section 5.1 that establishes a very general result for Birkhoff sections in T^2-invariant flows. The main tool are *helix boxes*, which are used to build broken book decompositions or Birkhoff sections. In the final Section 5.3, we establish Theorem 1.1 on the generic existence of Birkhoff sections.

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2. Preliminaries

In this section, we recall some necessary definitions and results that will be used throughout this work.

2.1. Stable Hamiltonian structures.

As mentioned in the introduction, stable Hamiltonian structures generalize contact forms and are defined on closed odd-dimensional manifolds. In this work, we will only consider the 3-dimensional case.

Definition 2.1. A stable Hamiltonian structure (SHS) on an oriented 3-manifold M is a pair (λ, ω) where $\lambda \in \Omega^1(M)$ and $\omega \in \Omega^2(M)$ such that

- $\lambda \wedge \omega > 0$,
- $d\omega = 0$
- $\ker \omega \subset \ker d\lambda$.

It uniquely determines a Reeb vector field X by the equations

$$\begin{cases} \lambda(X) = 1, \\ \iota_X \omega = 0. \end{cases}$$

Observe that X preserves the volume form $\lambda \wedge \omega$ and the plane field $\xi = \ker \lambda$ that is transverse to X. Only in dimension three, Reeb vector fields of SHS can be equivalently characterized as geodesible volume-preserving vector fields normalized to be of unit length [38]. If the 2-form $d\lambda$ is never zero, λ is a contact form and X is its Reeb vector field, while if $d\lambda \equiv 0$ then M fibers over the circle and each fiber is a section of X. This implies that X is conjugated to the suspension of an area-preserving diffeomorphism of a closed surface.

Given a SHS (λ, ω) a 1-form $\widetilde{\lambda}$ is a stabilizing form if $(\widetilde{\lambda}, \omega)$ is again a SHS. In this situation the Reeb vector field of (λ, ω) and the Reeb vector field of $(\widetilde{\lambda}, \omega)$ are parallel (that is one is a multiple of the other by a non-zero function). The following theorem [10, Theorem 4.1] gives a good description of a SHS on a closed 3-manifold.

Theorem 2.2 (SHS structure theorem)**.** *Let (λ, ω) be a stable Hamiltonian structure on a closed 3-manifold M and set $f := d\lambda/\omega$. Then there exists a compact 3-dimensional submanifold N (possibly with boundary, possibly disconnected) of M, invariant under the Reeb flow; a disjoint union $U = U_1 \sqcup ... \sqcup U_k$ of compact integrable regions and a stabilizing 1-form $\widetilde{\lambda}$ such that:*

- $\operatorname{int} U \cup N = U \cup \operatorname{int} N = M$;
- $\widetilde{f} = d\widetilde{\lambda}/\omega$ *is constant in each connected component of* N;
- *on each* $U_i \cong T^2 \times I$ *the stable Hamiltonian structure* (λ, ω) *is* T^2*-invariant, in particular* $f(r, z) = \alpha_i r + \beta_i$ *for some constants* $\alpha_i > 0, \beta_i \in \mathbb{R}$;
- $\widetilde{\lambda}$ *is* C^1*-close to* λ.

We point out a slight difference between this statement and the original one: we consider the U and N so that they intersect only along their boundary. This can be easily achieved.

The domain N splits in three disjoint domains N_+, N_-, N_0 where f (or $\widetilde{f}$ equivalently) is respectively positive, negative or zero. In N_+ and N_-, the one-form λ is of contact type and the Reeb vector field of the SHS is the Reeb vector field of the contact form λ. In N_0 we have $d\widetilde{\lambda} = 0$, implying that N_0 fibers over $\mathbb{S}^1$ and the fibers are transverse to the Reeb vector field. By convention, if f is constant, then $M = N_0$, $M = N_+$ or $M = N_-$ depending on the value of f.

In the SHS structure theorem, the region U can be chosen (but it will always have the same number of connected components). Indeed, since each connected component of U is of the form $T^2 \times I$, one can decide on the *thickness* of U. The Reeb vector field is tangent to the tori in each U_i, so if one asks it to be non-degenerate everywhere (e.g. as in [34]) the vector field has to be constant of irrational slope on each U_i, that is a very strong condition. To apply results from [12], we need that in the contact region $N_+ \cup N_-$ the Reeb vector field is non-degenerate, but we do not need the vector field to be non-degenerate everywhere. We hence define

Definition 2.3. A stable Hamiltonian structure (λ, ω) is contact non-degenerate if λ restricts to a non-degenerate contact form in N_+ and N_-, for some choice of N.

Recall that a contact form is non-degenerate if its Reeb vector field is non-degenerate, and this property is satisfied by a dense set of contact forms in the C^∞-topology. In [10, p 377], Cieliebak and Volkov introduce another condition of non-degeneracy for a Reeb vector field of a SHS, that is analogous to the Morse-Bott Reeb vector fields of contact structures. Their definition is stronger than Definition 2.3, meaning that it implies that the SHS is contact non-degenerate. Both conditions are satisfied by a dense set of SHS in the C^1-topology.

If (λ, ω) is contact non-degenerate, the periodic orbits in $N_+ \cup N_-$ are either elliptic or hyperbolic. If γ is a hyperbolic periodic orbit in $N_+ \cup N_-$, it has a stable and an unstable manifold denoted respectively by $W^u(\gamma)$ and $W^s(\gamma)$. If (λ, ω) is contact non-degenerate, we say that it is *contact strongly non-degenerate*, if the intersections between stable and unstable manifolds of the hyperbolic periodic orbits in $N_+ \cup N_-$ are transverse. Each connected component of $N_+ \cup N_-$ is invariant and, when (λ, ω) is contact non-degenerate, there are no periodic orbits near the boundary. Hence the intersections between stable and unstable manifolds of hyperbolic periodic orbits are in the interior of $N_+ \cup N_-$. The arguments in [10] can be adapted to show that contact strongly non-degenerate SHS are C^1-dense as well, confer Section 5.3.

2.2. **Torus bundles over $\mathbb{S}^1$.** The classification of torus bundles over $\mathbb{S}^1$ will play an important role in the proof of Theorem 1.3. Let us recall the main properties that we will need. An (orientable) torus bundle over $\mathbb{S}^1$ is obtained by considering the mapping torus of an orientation-preserving diffeomorphism $\varphi : T^2 \longrightarrow T^2$ of the torus. The isotopy class of φ in the space of orientation preserving diffeomorphism of the torus $\mathrm{Diff}(T^2)$ is determined by the action on the first homology group of T^2, which is given by an element $A \in SL(2, \mathbb{Z})$. The conjugacy class of A in $SL(2, \mathbb{Z})$ defines the torus bundle up to homeomorphism. These classes can be characterised by the trace of A:

- If $|\operatorname{tr}(A)| < 2$, there are two conjugacy classes for each possible value $-1, 0$ and 1. The torus bundles obtained via these matrices are called *elliptic* torus bundles. In this case, the matrix is not diagonalizable.
- If $|\operatorname{tr}(A)| = 2$, there are two $\mathbb{Z}$-families of conjugacy classes given by matrices of the form

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}, \text{ with } n \in \mathbb{Z}.$$

  We call the torus bundles obtained using these matrices *positive parabolic* and *negative parabolic* torus bundles respectively. In this case, both eigenvectors have rational slope.

- If $|\operatorname{tr}(A)| > 2$, we will only need to know that the matrix always has eigenvectors with irrational slope. We call the torus bundles obtained using a matrix in this class a *hyperbolic* torus bundles.

2.3. **Surface dynamics.** Let us recall a few definitions that we will need concerning homeomorphisms of compact surfaces. For details, we refer to [2, 31].

Let $f : \mathbb{A} \longrightarrow \mathbb{A}$ be a homeomorphism of the closed annulus $\mathbb{A} = \mathbb{S}^1 \times I$ isotopic to the identity. Let $\widetilde{\mathbb{A}} = \mathbb{R} \times I$ be the universal cover of $\mathbb{A}$ and $\widetilde{f} : \widetilde{\mathbb{A}} \longrightarrow \widetilde{\mathbb{A}}$ be a lift of f. Denote by $\pi_1 : \widetilde{\mathbb{A}} \longrightarrow \mathbb{R}$ the projection into the first factor. Given a f-invariant compactly supported Borel probability measure μ, we define the rotation number of μ as

$$\operatorname{rot}_{\widetilde{f}}(\mu) = \int_D (\pi_1 \circ \widetilde{f} - \pi_1) d\mu,$$

where D is any fundamental domain of the covering. The rotation set $\operatorname{rot}(\widetilde{f}) \subset \mathbb{R}$ is the set of rotation numbers for all invariant measures, and defines a compact interval. For two different lifts, the rotation set only differs by an integer number.

A special class of maps of the annulus is that of irrational pseudorotations, which can be characterized by their rotation set.

Definition 2.4. An irrational pseudorotation of the annulus is a homeomorphism of $\mathbb{A}$ isotopic to the identity and whose rotation set reduces to a single irrational number.

One can define as well an irrational pseudorotation of the closed disk. In both cases, we will only consider area-preserving homeomorphisms. In our context, an irrational pseudorotation of the annulus (respectively, the disk) can as well be defined as an area-preserving homeomorphism without periodic points (respectively with exactly one fixed point in the disk). A classical theorem of Franks [25] shows that an area-preserving diffeomorphism of the disk or the annulus with finitely many periodic points is necessarily an irrational pseudorotation.

We will use the following result concerning the conjugacy class of an irrational pseudorotation of the annulus.

Theorem 2.5 ([2])**.** *An irrational pseudorotation of the annulus is conjugated to a homeomorphism of the annulus that is arbitrarily C^0-close to an irrational rigid rotation R_α.*

Another class of homeomorphisms that we will use are the so-called Dehn twist maps, defined on closed surfaces of genus at least 2.

Definition 2.6. Let Σ be a closed surface of genus $g \geq 2$. A Dehn twist map is an orientation preserving homeomorphism $h : \Sigma \longrightarrow \Sigma$ such that:

- there is a finite family of pairwise disjoint invariant essential closed annuli $(A_i)_{i=1,\dots,k}$,
- no connected component of $\Sigma \setminus \bigcup_{i=1}^k A_i$ is an annulus,

- h fixes every point in $\Sigma \setminus \bigcup_{i=1}^k A_i$,
- the map $h|_{A_i}$ us conjugated to τ^{n_i}, where $n_i \neq 0$ and τ is a homeomorphism of $\mathbb{S}^1 \times [0,1]$ that lifts to $\widetilde{\tau}(x,y) = (x+y,y)$.

3. Characterization of aperiodic SHS

Given a stable Hamiltonian structure (λ, ω), it is immediate that if $f = \frac{d\lambda}{\omega}$ never vanishes, then λ is a contact form and so X cannot be aperiodic by Taubes' theorem [40]. Hence, aperiodic examples only occur when f vanishes somewhere and we start by analyzing the cases in which f is not identically zero. In this section, we provide a complete dynamical and topological characterization of counterexamples to the Weinstein conjecture for SHS, proving Theorem 1.3. As an intermediate step, we study contact Reeb dynamics in 3-manifolds with boundary. This will be useful in Section 4 as well.

In order to simplify the statements in what follows, we say that a vector field is conjugated to a suspension if its flow is orbit equivalent to a suspension flow. Orbit equivalence means that there is a homeomorphism (or diffeomorphism) that maps orbits of one flow to orbits of the other flow, it is a weak conjugacy, since conjugacy of flows imposes restrictions on the parametrizations. In our context, the conjugacy is a topological conjugacy.

3.1. **Reeb vector fields on manifolds with T^2-invariant boundary.** Denote by I the interval $[0,1]$ and by $\mathbb{A} = \mathbb{S}^1 \times I$ the closed annulus. We say that a vector field on a region of the form $T^2 \times I$, for I a closed interval, is T^2 invariant if its restriction to each torus $T^2 \times \{t\}$ is a linear vector field, i.e. the slope only depends on the coordinate t. In order to study the cases where f is not identically zero, we start by

Lemma 3.1. *Let α be a contact form defining a non-degenerate Reeb vector field on a compact oriented manifold with boundary M, that is T^2 invariant near the boundary and has finitely many periodic orbits. Then either $M \cong T^2 \times I$ and X is conjugated to the suspension of an irrational pseudorotation of the annulus, or $M \cong \mathbb{S}^1 \times D^2$ and X is conjugated to an irrational pseudorotation of the disk.*

Proof. Following [29, Section 5.2], the Reeb vector field along a neighborhood $U \cong T^2 \times [0, \delta]$ of a boundary component of M with coordinates (x, y, t) is of the form

$$R = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y}, \tag{1}$$

for a_1, a_2 functions of t and the contact form can be assumed to be

$$\lambda = \frac{1}{2}\rho(t)^2(a_2 dx - a_1 dy) + \alpha_2 dy, \tag{2}$$

where $\rho^2(t)$ is a smooth function on $[0, \delta]$ and $\rho(t)$ is smooth and strictly increasing on $(0, \delta]$. We can collapse the boundary torus to a circle by identifying those points that have the same x coordinate along the torus $t = 0$ and (ρ, x) become polar coordinates on the disk. Doing this along each component of the boundary we obtain a closed 3-manifold M_1, and λ becomes a smooth contact form λ_1 on M_1 just by using the expression (2) along each solid torus U_1 obtained from U. The Reeb vector field R_1 of λ_1 in $M_1 \setminus \{\gamma_i\}$, where γ_i are the circles that we constructed by collapsing the boundary tori, coincides with R in the interior of M.

The Reeb vector field R_1 is non-degenerate and by hypothesis, it admits finitely many periodic orbits. By [12, Theorem 1.2] the manifold M_1 is a sphere or a lens space and R_1 has exactly two periodic orbits that we denote by γ_1 and γ_2. By [29, Theorem 1.2], the periodic orbits are core circles of solid tori of a genus one Heegaard splitting of M_1. This implies that either $M \cong T^2 \times I$ or $M \cong \mathbb{S}^1 \times D^2$. Finally, by [15, Corollary 1.10], each closed Reeb orbit of R_1 bounds a disk-like Birkhoff section (one could alternatively find an annulus-like Birkhoff section induced by the projection of holomorphic cylinders, see [29, Section 4.6]). Take the disk-like section bounded by γ_2, and denote the first-return map by h which is smooth since near the boundary there is an invariant foliation by circles and h is conjugated to a rigid rotation in each circle. The diffeomorphism h is conjugated to an irrational pseudorotation of the disk. We know that R_1 is tangent to a foliation by tori close to each periodic orbit obtained by collapsing a boundary component of M. If $M \cong \mathbb{S}^1 \times D^2$, then R_1 is tangent to a foliation by invariant tori near, say, the boundary that corresponds to the circle γ_2. We can restrict h to a disk D' whose boundary is an invariant circle close enough to γ_2, and hence R_1 is conjugated to the suspension of $h|_{D'}$, an irrational pseudorotation of the disk. Hence R is conjugated to the suspension of an irrational pseudorotation and we conclude in this case.

In the other case, the original manifold is $M \cong T^2 \times I$, which means that near the boundary γ_2 and near the fixed point of h there is a foliation by invariant circles of h. Along those circles, the diffeomorphism is conjugated to a rigid rotation. By restricting h to some annulus $\mathbb{A}$ bounded by two invariant circles, one close enough to the fixed point and one close enough to γ_2, we obtain an area-preserving diffeomorphism $h|_{\mathbb{A}}$ of the closed annulus. It follows that R_1 is conjugated to the suspension of h, and the same holds for R in M. Hence $M \cong T^2 \times I$ and R is conjugated to the suspension of an irrational pseudorotation of the annulus. □

3.2. **A global section up to cutting along a torus.** In this section, we prove that aperiodic SHS always admit a global section if one allows to cut open the ambient manifold along an invariant torus. A key fact is that the suspension of an irrational pseudorotation of $\mathbb{A}$ admits global sections inducing any homology class on the boundary.

Proposition 3.2. *Let X be a vector field whose flow is the suspension of an irrational pseudorotation $h : \mathbb{A} \longrightarrow \mathbb{A}$ of the annulus. For any integer homology class $\sigma_a \in H_1(T^2; \mathbb{Z})$, there is an annulus-like global section Σ of X such that $\Sigma \cong \mathbb{A}$ that induces a circle with homology class σ_a on each boundary component of $T^2 \times I$.*

Proof. By Theorem 2.5, the conjugacy class of $h : \mathbb{A} \longrightarrow \mathbb{A}$ contains the rigid rotation R_α in its C^0-closure. In other words, given any $\varepsilon > 0$ we can find some homeomorphism of the annulus $\widetilde{h}$ conjugated to h satisfying

$$||\widetilde{h} - R_\alpha||_{C^0} < \varepsilon, \tag{3}$$

for some α. The homeomorphism $\phi_0 : \mathbb{A} \longrightarrow \mathbb{A}$ satisfying $h = \phi_0^{-1} \circ \widetilde{h} \circ \phi_0$ is isotopic to the identity, see [2, Proposition 5.1]. Let ϕ be a C^∞-diffeomorphism of the annulus such that

$$||\phi - \phi_0||_{C^0} < \varepsilon, \tag{4}$$

which exists because any homeomorphism of a compact surface with or without boundary can be arbitrarily well approximated by diffeomorphisms in the C^0-topology [35]. Since the space of homeomorphism is locally contractible [5], for ε small enough ϕ is also isotopic to the identity. Then for any $\delta > 0$ there exists $\varepsilon > 0$ verifying the inequalities (3) and (4), such that $\widehat{h} = \phi \circ h \circ \phi^{-1}$ satisfies

$$||\widehat{h} - R_\alpha||_{C^0} < \delta.$$

Denote by X_α the vector field on $T^2 \times I$ obtained by suspension of R_α, and by $X_{\widehat{h}}$ the suspension of $\widehat{h}$. Since $\widehat{h}$ and h are (smoothly) conjugated, the flows of the vector fields X_h and $X_{\widehat{h}}$ are smoothly orbit equivalent by some diffeomorphism

$$\varphi : T^2 \times I \longrightarrow T^2 \times I,$$

which is isotopic to the identity because so was ϕ. The vector field X_α is a rotation of constant irrational slope in each torus fiber of $T^2 \times I$. It follows that X_α admits a smooth global section Σ_a such that $\Sigma_a \cap \partial(T^2 \times I)$ is a circle representing a given non-trivial homology class $\sigma_a \in H_1(T^2; \mathbb{Z})$. Taking δ perhaps even smaller, the vector field $X_{\widehat{h}}$ can be taken to be arbitrarily C^0-close to X_α, hence Σ_a is also a (smooth) global section of $X_{\widehat{h}}$. We deduce that the surface $\Sigma'_a = \varphi^{-1}(\Sigma_a)$ is a global section of X_h. The map φ^{-1} is isotopic to the identity, so the annulus Σ'_a induce circles in the boundary whose homology class is σ_a. □

The previous proposition can be used to obtain a global section, up to cutting open M along an invariant torus, for any aperiodic SHS Reeb vector field.

Theorem 3.3. *Let X be the Reeb vector field of a stable Hamiltonian structure (λ, ω) without periodic orbits on a 3-manifold M. If f is non-constant, then given any invariant torus $T \subset M$ of f, cutting along T yields a manifold with boundary $\widetilde{M} \cong T^2 \times I$ in which the Reeb vector field admits an annulus-like section and is conjugated to the suspension of an irrational pseudorotation.*

Proof. Apply the structure Theorem 2.2 to decompose our manifold into $U = U_1 \sqcup ... \sqcup U_k$ and $N = N_0 \sqcup N_+ \sqcup N_-$. This decomposition can be taken so that any two connected domains in N are disjoint, and the boundary of each component of N is also a boundary component of U. Choose a torus T to be the boundary of any connected component of N. Denote by $\overline{M}$ the closure of the manifold obtained by cutting open M along T, whose boundary is given by two tori. Each U_i is an integrable region $T^2 \times I$ where (λ, ω) can be assumed to be T^2-invariant. The Reeb vector field has constant irrational slope in each integrable region U_i, hence it is conjugated to the suspension of a rigid irrational rotation. In each connected component of N_+ or N_-, the form λ is of contact type and X is the contact Reeb vector field of λ. There are no periodic orbits, so by Lemma 3.1 the Reeb flow is conjugated to the suspension of an irrational pseudorotation of the annulus. Similarly, each component of N_0 is also $T^2 \times I$ and the Reeb flow is a suspension of an irrational pseudorotation of the annulus, see [38, page 20] or [29, Section 5.3]. We have shown that $\overline{M}$ is obtained by gluing along their boundary components a finite number of connected domains $V_i \cong T^2 \times I$ where X is conjugated to the suspension of an irrational pseudorotation or a rotation.

Consider the decomposition above and take one of the connected domains $V_1 \cong T^2 \times I$ whose boundary contains $T \subset \overline{M}$. By Proposition 3.2, we can choose any non-trivial homology class $\sigma_1 \in H_1(\partial V_1; \mathbb{Z})$ and find an annulus-like section Σ_1 of X inducing the homology class σ_1 in each boundary component. One boundary component of V_1 is glued to a boundary component of another connected domain, say $V_2 \cong T^2 \times I$. Denote this boundary torus by $T_2 \subset \partial V_2$. The surface Σ_1 induces on T_2 a circle with homology class σ_2 (the class σ_1 understood in $H_1(\partial V_2; \mathbb{Z})$). Applying Proposition 3.2, there is an annulus-like section Σ_2 of X in V_2 inducing circles with homology class σ_2 in each boundary component.

Denote by γ_1 and γ_2 the circles $\Sigma_1 \cap T_2$ and $\Sigma_2 \cap T_2$. Up to a small C^∞ perturbation of Σ_2, we might assume that γ_1 and γ_2 intersect transversely. Each curve comes equipped with an orientation, induced respectively by the orientations of Σ_1 and Σ_2 inherited by the positive direction of the vector field X. Since $[\gamma_1] = [\gamma_2]$ in $H_1(T_2; \mathbb{Z})$, there is some 2-chain C such that $\partial C = \gamma_1 - \gamma_2$.

Assume that $\gamma_1 \cap \gamma_2 \neq \emptyset$. The fact that γ_1 and γ_2 intersect transversely implies that the interior of C is given by a finite collection of disjoint open disks. The boundary of the closure of each one of those disks is given by a segment in γ_1, and segment in γ_2 and two points in $\gamma_1 \cap \gamma_2$. We can assign to each disk a sign depending on whether the orientation induced by the disk on the boundary of its closure coincides or not with the orientation induced on γ_1 and γ_2 at those points which are not in set $\gamma_1 \cap \gamma_2$. Let D be one of the disks of negative sign, and consider a small neighborhood V of this disk. Take a chart

$$\phi : (U, (x, y)) \longrightarrow V$$

with $(x, y) \in (-\delta, 1 + \delta)^2$ of this neighborhood such that $\phi^{-1}(\gamma_1 \cap V) = \{y = 0\}$ and $\phi^{-1}(\gamma_2) = \{y = f(x)\}$ where f is a smooth function such that $f \pitchfork 0$ and $f \cap 0 = \{(0, 0), (1, 0)\} = U \cap \gamma_1 \cap \gamma_2$. The integral curves of X are transverse to both curves (and are dense in T^2), so they come inside D through one of the curves and leave D through the other curve since otherwise there would be a fixed point of X in D. Assume that these integral curves are oriented by X from $\{y = 0\}$ to $\{y = f(x)\}$. Denote by φ_t the flow of X at time t. We can now construct a non-negative function $g(x)$, compactly supported in $x \in (-\varepsilon, 1 + \varepsilon)$ with $\varepsilon < \delta$, such that

$$\gamma_1' = \varphi_{g(x)}(\{y=0\})$$

is a curve whose y coordinate at any point $(x,y)\in\gamma_1'$ is greater than $f(x)$, and which coincides with $\{y=0\}$ close to $x=-\delta$ and $x=+\delta$. Since we have done this isotopy using the flow of X, the curve γ_1' is still transverse to X globally. Doing this at every disk of negative sign, we find a curve γ_1' that is transverse to X and disjoint from γ_2. Since we constructed them via an isotopy, we can find a family of curves κ_s in T_2 with $s\in[0,1]$ such that

$$\begin{cases} \kappa_0 = \gamma_1 \\ \kappa_s = \gamma_1' \text{ for } s\in[1-\varepsilon,1] \\ \kappa_s \pitchfork X \text{ for all } s\in[0,1]. \end{cases}$$

Recall that a neighborhood of T_2 lies inside an integrable region U_j, so it is foliated by tori invariant by the Reeb flow. Let $T_2^s, s\in[0,1]$ be a family of such invariant tori, all lying in V_1, such that $T_2^1=T_2$. We can assume that $\Sigma_1\cap\mathbb{T}_2^s=\gamma_1$, and we can isotope Σ_1 to another global surface of section Σ_1' in V_1 such that $\Sigma_1'\cap T_2^s=\kappa_s$. In particular, we have $\Sigma_1'\cap T_2^s=\gamma_1'$ for all $s\in[1-\varepsilon,1]$.

Up to renaming γ_1' as γ_1 and Σ_1' as Σ_1, we proved that we can ensure that γ_1 and γ_2 are disjoint. Since they are still in the same homology class, they bound a cylinder. Arguing as before, we can find smooth positive function $g:\gamma_1\longrightarrow\mathbb{R}$ such $\varphi_{g(p)}(\gamma_1)=\gamma_2$: the flowlines of X enter the cylinder bounded by γ_1 and γ_2 enter the cylinder through one of the curves and come out through the other curve. Let r be a coordinate of a small T^2-invariant neighborhood $W=T^2\times[-\delta,\delta]$ of T_2, where X is just an irrational vector field of constant slope and such that $V_1\cap W=\{r\leq 0\}$ and $V_2\cap W=\{r\geq 0\}$. We might assume that $\Sigma_1\cap\{r=-\delta\}=\gamma_1$ and $\Sigma_2\cap\{r=\delta\}=\gamma_2$ (where we have abused notation by taking letting γ_i be the translated curve in any of the tori). Now define surface Σ_3 which is equal to Σ_1 in $V_1\setminus W$, equal to Σ_2 in $V_2\setminus W$ and such that in W it is given by $(\varphi_{h(r,p)}(\gamma_1),r)$ for a function $h(r)$ which is equal to 0 near $r=-\delta$ and equal to $g(p)$ for near $r=\delta$. Then Σ_3 is an annulus-like surface of section of the Reeb vector field in $V_1\cup V_2$. We can apply this argument iteratively by gluing each domain V_i, showing that the Reeb vector field admits an annulus-like surface of section in all $\overline{M}\cong T^2\times I$ and hence that it is conjugated to the suspension of an irrational pseudorotation of the annulus. □

3.3. **Admissible torus bundles.**

In this section, we analyze aperiodic Reeb vector fields of SHS for which $f=\frac{d\lambda}{\omega}$ is not constant and deduce obstructions on the topology of the ambient T^2-bundle by applying Theorem 3.3.

Proposition 3.4. *Let (λ,ω) be a SHS defining an aperiodic Reeb vector field such that $f=\frac{d\lambda}{\omega}$ is not constantly equal to zero. Then $M\cong T^3$ or M is a hyperbolic torus bundle over $\mathbb{S}^1$.*

Proof. Since f vanishes somewhere (and not everywhere), applying Theorem 2.2 we find an integrable domain $U \cong T^2 \times I$ where (λ, ω) is T^2-invariant. By cutting open the manifold M along one of the torus fibers, we obtain a manifold with boundary $\overline{M} \cong T^2 \times I$ and coordinates (x, y, t) such that (λ, ω) is T^2-invariant near the boundary. The Reeb vector field X defined by (λ, ω) is conjugated to the suspension of an irrational pseudorotation of the annulus by Theorem 3.3, denote by α its irrational rotation number. Since the vector field X corresponds to an irrational flow on each invariant torus near the boundary, it has irrational slope equal to α on each torus. Write near $t = 0$ the form λ in a T^2-invariant form $\lambda = h_1(t)dx + h_2(t)dy$. Similarly, write the Reeb vector field X as

$$X = a_1(t)\frac{\partial}{\partial x} + a_2(t)\frac{\partial}{\partial y},$$

and we know that $\frac{a_2(t)}{a_1(t)} \equiv \alpha$.

Let us first assume that the slope $\frac{h_1(t)}{h_2(t)}$ is non-constant. Then, modulo changing the cutting torus, we can assume that $\frac{h_1(t)}{h_2(t)}$ is irrational and different from α. Denote by $\varphi : T^2 \to T^2$ the gluing diffeomorphism such that M is obtained by gluing $t = 1$ with $t = 0$ via φ. Such diffeomorphism must preserve the irrational foliation by curves of X, and the foliation given by the kernel of λ restricted to $t = 1$ must be sent to the foliation spanned by the kernel of λ restricted to $t = 0$. Let δ be the slope of the kernel of λ at $t = 1$, and β be the slope of the kernel of λ along $t = 0$. Write the map ϕ induced by φ on the universal cover $\mathbb{R}^2$ as

$$\phi(x, y) = (g_1(x, y), g_2(x, y)),$$

where g_1 and g_2 can be expressed in a unique way as

$$g_1(x, y) = l_1 + p_1$$
$$g_2(x, y) = l_2 + p_2$$

where $l_1, l_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are linear functions and $p_1, p_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are periodic functions. The fact that the kernel of λ is preserved implies that there is some function $G : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$g_2(x, y) - \beta g_1(x, y) = G(y - \delta x).$$

We can now argue as in [38, Lemma 4.4]: the function $p_2 - \beta p_1$ is constant on the straight line of irrational slope $y = \delta x$ which implies that it is constant everywhere. This shows that $G(z) = az + b$ for some constants a, b and so

$$g_2(x, y) - \beta g_1(x, y) = a(y - \delta x) + b.$$

The same argument applied to the foliation of slope α shows that

$$g_2(x, y) - \alpha g_1(x, y) = c(y - \alpha x) + d,$$

for some constants c, d. Then

$$c(y - \alpha x) + d + \alpha g_1(x, y) = a(y - \delta x) + b + \beta g_1(x, y),$$

and since $\beta \neq \alpha$, we obtain that g_1 and g_2 are linear functions. Up to a translation, the diffeomorphism is given by a matrix $A \in SL(2,\mathbb{Z})$ which necessarily admits an eigenvector with irrational slope α. This proves that A is the identity or a hyperbolic matrix of $SL(2,\mathbb{Z})$.

Otherwise, the slope $\frac{h_2(t)}{h_1(t)}$ is constant. We can assume that we chose the invariant domain U such that $f \neq 0$ there. This implies that $d\lambda \neq 0$ on U, and it has the form

$$d\lambda = h_1'(t)dt \wedge dx + h_2'(t)dt \wedge dy.$$

Since $\iota_X d\lambda = 0$, it follows that $\frac{h_2'(t)}{h_1'(t)} \equiv -\frac{1}{\alpha}$. Using that the slope of λ is constant we deduce that $\frac{h_2(t)}{h_1(t)} = \frac{h_2'(t)}{h_1'(t)} = -\frac{1}{\alpha}$. In other words, we have shown that the slope of the foliation induced by $\ker\lambda$ can be assumed to be irrational on the boundary of $\widetilde{M}$. We are now in the same situation as in the previous case, finishing the proof. □

3.4. **Proof of Theorem 1.3.** Let M be a closed 3-dimensional manifold equipped with a stable Hamiltonian structure (λ, ω) whose Reeb vector field X is aperiodic. This implies that M is a torus bundle over $\mathbb{S}^1$. Denote by f the function $\frac{d\lambda}{\omega}$. This function necessarily vanishes somewhere since otherwise, the one-form λ is a positive or negative contact form and the Reeb vector field X admits a periodic orbit by the Weinstein conjecture [40].

First case: $f \equiv 0$. Assume first that the function f is constant and equal to 0. In this case, the one form λ is closed, and by Tischler's theorem [42] there is a surface fiber bundle over $\mathbb{S}^1$ such that each fiber is a global section of X. Then the fiber is necessarily a torus. Hence X is conjugated to the suspension of a symplectomorphism of the torus $\varphi : T^2 \to T^2$, whose conjugacy class corresponds to a matrix $A \in SL(2,\mathbb{R})$. The Lefschetz number of φ is given by

$$\Gamma_\varphi = \sum_{i=0}^{2} (-1)^i \operatorname{tr}(\varphi_* H_i(X,\mathbb{Q})) = 2 - \operatorname{tr}(A).$$

By hypothesis φ admits no periodic points, which implies that $\Gamma_\varphi = 0$ and hence $\operatorname{tr}(A) = 2$. We deduce that M is either T^3 or a positive parabolic torus bundle, and X admits a global section.

Second case: $f \not\equiv 0$ and vanishes somewhere. In this case Proposition 3.4 implies that M is either the three torus or a hyperbolic torus bundle. Theorem 3.3 shows that there is some embedded invariant torus T such that the closed manifold $\overline{M}$ obtained by cutting open M along T is diffeomorphic to $T^2 \times I$ where X admits some annulus-like global section Σ and X is conjugated to an irrational pseudorotation of the annulus. If $M = T^3$, then Σ defines in M an immersed surface with boundary, embedded in the interior, and whose boundary is given by two circles defining the same homology class in $H_1(T^2;\mathbb{Z})$. We might argue exactly as in the proof of Theorem 3.3 to deform a bit Σ so that it glues smoothly along T and yields a global section of X diffeomorphic to a torus.

If M is a hyperbolic torus bundle, the missing part is to prove that X does not admit a global section. Assume that there is such surface Σ, that we assume connected. Then after cutting along an invariant torus T, we obtain a global section $\overline{\Sigma}$ of X in $\overline{M}\cong T^2\times I$. Since X in $\overline{M}$ admits an annulus-like global section, $\overline{\Sigma}$ has to be an annulus and it induces in the boundary of $\overline{M}$ circles with non-trivial homology class. The fact that $\overline{\Sigma}$ defines a smooth closed surface in M after the identification of the two boundary components of $\overline{M}$ implies that such identification preserves some integer homology class. In other words, the matrix $A\in SL(2,\mathbb{Z})$ of the mapping class of φ admits an eigenvector with integer coordinates, leading to a contradiction with the fact that A is hyperbolic. We deduce that X does not admit a global section, even if after cutting along T it is conjugated to the suspension of an irrational pseudorotation of the annulus. Observe that the rotation number of the annulus defines the slope of X in appropriate coordinates. Arguing as in the proof of Proposition 3.4, we know that the gluing diffeomorphism between the two components of ∂M is induced by the matrix A. Once we have our well-chosen generators of $H_1(T;\mathbb{Z})$, the irrational slope of X is necessarily the same as the slope of some eigenvector of A. Writing the matrix A in the form

$$A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

it follows that the slope α of an eigenvector satisfies $b\alpha^2+(a-d)\alpha-c=0$, a quadratic equation.

When (λ,ω) is assumed to be analytic, it was shown in [38, Section 4] that M is either T^3 or a parabolic T^2-bundle over $\mathbb{S}^1$. Combining it with our previous discussion, we deduce that $M\cong T^3$ and only case (1) occurs.This finishes the proof of Theorem 1.3.

Remark 3.5. The fact that the rotation number is quadratic implies that it is not of Liouville type. In this case, no example is known of a smooth irrational pseudorotation that is not conjugated to a rigid irrational rotation. The existence of an aperiodic SHS, on some hyperbolic torus bundle, which is not tangent to a foliation by invariant tori (where the flow is conjugated to a linear flow) would then imply the existence of a non-trivial irrational pseudorotation. It is an open question whether non-trivial irrational pseudorotations with Diophantine (i.e. non Liouville) rotation number exist [23].

3.5. **Aperiodic examples.** To deduce Corollary 1.4, we only need to show that in each case of Theorem 1.3 there is at least an example of an aperiodic stable Hamiltonian Reeb vector field. In T^3, an example is given by the suspension of the time one map of any linear vector field with irrational slope. In a positive parabolic bundle, an example is given by the suspension of the map

$$\begin{aligned} \phi : T^2 &\longrightarrow T^2 \\ (x, y) &\longmapsto (x + ny, y + \alpha), \end{aligned}$$

where α is an irrational number and $n \neq 0$ is an integer that determines the homeomorphism type of the resulting (positive) parabolic torus bundle. The following proposition shows that hyperbolic torus bundles also admit an aperiodic example.

Proposition 3.6. *Any hyperbolic torus bundle admits an aperiodic stable Hamiltonian Reeb vector field.*

Note that by Theorem 1.3, this Reeb vector field cannot have a global section.

Proof. Let A be an hyperbolic element (i.e. with trace greater than 2) of $SL(2, \mathbb{Z})$.

$$A = \begin{pmatrix} p & q \\ n & m \end{pmatrix}$$

that has an eigenvector of the form $(1, \alpha)$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and eigenvalue $\mu \neq 0$.

Let R_α be a rigid rotation of the annulus $\mathbb{A}$ of angle $2\pi\alpha$. Its suspension yields a vector field X_α on $M_1 = T^2 \times I$ such that each torus $T^2 \times \{t\}$ is invariant and X_α is a linear vector field of the form

$$X_\alpha = \alpha \frac{\partial}{\partial y} + \frac{\partial}{\partial x},$$

where x, y are coordinates in the T^2 component of $T^2 \times I$. Consider a reparametrization of X_α given by

$$X = g(t) \left(\alpha \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \right),$$

where $g(t)$ is such that $g(t) = 1$ close to $t = 0$ and $g(t) = \frac{1}{\mu}$ close to $t = 1$. We identify $\{t = 0\}$ with $\{t = 1\}$ via the linear Anosov diffeomorphism $f : T^2 \longrightarrow T^2$ given by $\mathbf{x} = A.\mathbf{x}$. Since X_α is an eigenvector of A with eigenvalue μ, it follows that X_α is well-defined in the quotient torus bundle. Let us denote again by X_α the vector field obtained in the closed manifold M, which is tangent to each fiber of the torus bundle.

Lemma 3.7. *Up to reparametrization, X is the Reeb vector field of a SHS.*

Proof. Consider a pair (λ, ω) where

$$\omega = u(t)(-\alpha dt \wedge dx + dt \wedge dy),$$

whit $u(t)$ a strictly positive function such that $u = 1$ near $t = 0$ and $u = \mu$ near $t = 1$, and the 1-form is

$$\lambda = h(t)(dy - \alpha dx) + dy + C dx,$$

where $C = \frac{-\alpha(p-1)-n}{(m-1)+\alpha q}$ and $h(t)$ such that $h=0$ near $t=0$ and $h=D=(p-1)+qC$ near $t=1$. Since $g>0$, we might assume that $\lambda(X)=g(t)(\alpha+C)>0$ (if $\alpha+C$ is negative, just change either λ or X of sign). Furthermore

$$d\lambda = h'(t)dt\wedge(dy-\alpha dx),$$

and it follows that for any choice of h we have $\iota_X d\lambda=0$. Near $t=0$ we have $\lambda|_{t=0}=dy+Cdx$, and near $t=1$ we have $\lambda|_{t=1}=(1+D)dy+(C-\alpha D)dx$. To conclude, observe that

$$\begin{pmatrix} p & q \\ n & m \end{pmatrix}\begin{pmatrix} 1 \\ C \end{pmatrix} = \begin{pmatrix} 1+D \\ C-\alpha D \end{pmatrix},$$

and so λ is well defined in the quotient. The two-form ω is also well defined in the quotient, and we end up with a pair (λ,ω) satisfying the following equations.

$$\begin{cases} \lambda(X)>0 \\ \iota_X d\lambda = 0 \\ \iota_X\omega = 0 \\ d\omega = 0 \end{cases}$$

Furthermore ω is non-degenerate and since $\lambda(X)>0$ where X spans the kernel of ω, we deduce that $\lambda\wedge\omega\neq 0$. By reparametrizing X to $\overline{X}=\frac{1}{\lambda(X)}X$, it follows that $\overline{X}$ is the Reeb vector field of the SHS (λ,ω) defined on M. ☐

This concludes the proof of Proposition 3.6. ☐

4. SHS with finitely many periodic orbits

In Section 3 we have completely understood aperiodic stable Hamiltonian Reeb vector fields. In this section, we are interested in understanding when a contact non-degenerate SHS admits, or not, infinitely many periodic orbits.

4.1. Symplectomorphism of surfaces aperiodic at the boundary.

Let Σ be a compact surface. A diffeomorphism $\phi:\Sigma\longrightarrow\Sigma$ is called a symplectomorphism if there exists an area form ω in Σ such that $\phi^*\omega=\omega$. To simplify the statement of the following theorem, we require that ϕ is aperiodic along the boundary. However, the same theorem holds if we require only that there is a set of connected components of the boundary that is ϕ-invariant and where ϕ has no periodic points.

Theorem 4.1. *Let Σ be a surface with boundary. Let $\phi:\Sigma\longrightarrow\Sigma$ be a symplectomorphism without periodic points along the boundary. Then ϕ admits periodic points of arbitrarily large period in the interior of Σ unless either $\Sigma\cong D^2$ and ϕ is an irrational pseudorotation of the disk, or $\Sigma\cong\mathbb{A}$ and ϕ is an irrational pseudorotation of the annulus.*

Proof. Let Σ_g^b be a surface of genus g with b boundary components. Up to considering an iterate of ϕ, we can assume that each boundary components is preserved. Then ϕ restricts to each boundary component as a diffeomorphism of the circle that is conjugated to an irrational rotation. We will only need that this is true in one of the boundary components. We can blow-down the boundary components, obtaining from ϕ a symplectomorphism of a closed surface

$$\varphi : \Sigma_g \longrightarrow \Sigma_g,$$

that has at least b fixed points with irrational rotation number. Choose one of them $z \in \Sigma_g$. For the proof we consider three cases: $g = 0$, $g = 1$ and $g \geq 2$.

If $g = 0$, then a theorem of Franks [25] shows that either φ has infinitely many periodic points of arbitrarily large periods, or there are exactly two fixed points, *i.e.* ϕ is an irrational pseudorotation of the two-sphere. We deduce that ϕ has infinitely many periodic points of arbitrarily large period or Σ_g^b is either a disk D^2 or an annulus $\mathbb{A}$, and in both cases ϕ is an irrational pseudorotation. Observe that if a positive iterate of a diffeomorphism is an irrational pseudorotation, the same holds for the diffeomorphism itself, so there is no issue if we considered a positive iterate of ϕ.

Assume now that the surface is a torus $\Sigma_g = \Sigma_1 = T^2$. Let $\widetilde{\Sigma}_1 \cong \mathbb{R}^2$ be the universal covering space of T^2, and take a lift of φ that we denote by

$$\widetilde{\varphi} : \widetilde{\Sigma}_1 \longrightarrow \widetilde{\Sigma}_1,$$

and satisfies that some lift $\widetilde{z}$ of z is fixed by $\widetilde{\varphi}$. The homeomorphism $\widetilde{\varphi}$ is area-preserving. We identify $\mathbb{R}^2$ with the open unit disk D^2 via the map $\rho(x,y) = \frac{(x,y)}{||(x,y)||+1}$. Decompose $\widetilde{\varphi}$ as $\widetilde{\varphi}(x,y) = A(x,y) + g(x,y)$, where A is a linear map and g is $\mathbb{Z}$-periodic in both coordinates. The matrix A is an element of $SL(2,\mathbb{Z})$, which determines the mapping class of φ. We can now easily construct a continuous extension of $\widehat{\varphi} = \rho \circ \widetilde{\varphi} \circ \rho^{-1}$ to $\overline{D^2}$ as follows. Any sequence of points of the form (x_n, y_n) such that $(x_n, y_n) \to (x, y) \in \partial D_2$ satisfies

$$\lim_{n\to\infty} \rho \circ \widetilde{\varphi} \circ \rho^{-1}(x_n, y_n) = \frac{A(x,y)}{||A(x,y)||}.$$

So we can extend the map $\widehat{\varphi} : \overline{D^2} \longrightarrow \overline{D^2}$ as above. If A is hyperbolic, the map φ necessarily admits infinitely many periodic points of arbitrarily large period. If A is of finite order, i.e. an elliptic element of $SL(2,\mathbb{Z})$, then $\widehat{\varphi}$ corresponds to a rotation along ∂D^2. If A is parabolic, we can find an eigenvector of A which induces a fixed point of $\widehat{\varphi}$ along the boundary. If A is the identity, then $\widehat{\varphi}$ is just the identity on the boundary ∂D^2. In the three cases, observe that $\widehat{\varphi}$ admits either a fixed point or a periodic point along the boundary of the disk. This implies that the rotation number of $\widehat{\varphi}$ along the boundary is rational.

We blow up $\tilde{z}$, obtaining an area-preserving homeomorphism of the closed annulus $\mathbb{A}$. In the boundary component obtained by blowing up $\tilde{z}$, the rotation number is irrational. On the other boundary component, the rotation number is rational. Hence, these numbers are different and we deduce that the rotation set of the homeomorphism of the annulus contains some open interval $I \subset \mathbb{R}$. Take a subinterval $J \subset I$ which does not contain the rotation numbers at the boundary, now [24, Corollary 2.4] shows that there is a compact subset K of D^2 with periodic points of arbitrarily large period. The compactness of K implies that each point in Σ_g has only finitely many preimage points in K by the projection from the universal cover, so we deduce that the projection of K in Σ_g contains periodic points of φ of arbitrarily large period. It follows that $\phi : \Sigma_1^b \longrightarrow \Sigma_1^b$ also admits infinitely many periodic points which are further of arbitrarily large period.

It only remains to analyze the case where the surface is Σ_g with $g \geq 2$. By Nielsen-Thurston's decomposition [41, 22], we know that there is a q such that one of the following holds:

- φ^q is isotopic to the identity,
- φ^q is isotopic to a Dehn twist,
- the decomposition of φ has at least one pseudo-Anosov component.

If φ has a pseudo-Anosov component, it is well-known that φ admits infinitely many periodic points of arbitrarily large periods [26]. Otherwise, denote φ^q by f. The universal cover of Σ_g is Poincaré disk $\widetilde{\Sigma}_g \cong D^2$ and consider a lift of f

$$\widetilde{f} : D^2 \longrightarrow D^2$$

such that $\widetilde{f}(\widetilde{z}) = \widetilde{z}$, where $\widetilde{z}$ is one lift of the fixed point z obtained by blowing down one boundary component. Such homeomorphism admits an extension to the boundary of the disk $S_\infty = \partial D^2$. We claim that $\widetilde{f}|_{S_\infty}$ has rational rotation number.

Let f' be a diffeomorphism isotopic to f, that is either the identity or a Dehn twist. We can lift the isotopy, obtaining a homemorphism $\widetilde{f'}$ isotopic to $\widetilde{f}$. Abusing notation, we denote also by $\widetilde{f'}$ and $\widetilde{f}$ the extended homeomorphisms to the closed disk. If f' is the identity then $\widetilde{f'}$ coincides with a deck transformation of D^2, which is given by a hyperbolic translation. This readily implies that $\widetilde{f'}$ admits two fixed points (the endpoints of the translation axis) along S_∞. Since $\widetilde{f'}|_{S_\infty} = \widetilde{f}|_{S_\infty}$, we deduce that $\widetilde{f}$ also admits two fixed points along the boundary of the disk. This proves our claim in the first case.

Otherwise, f' is a Dehn twist and we can find some closed curve γ of non-trivial homology that is preserved by f'. The lift of γ to the universal cover is an infinite family of disjoint open segments, and the closure of each intersects S_∞ at two points. Fix one of such lifts $\widetilde{\gamma}_0$, with boundary points $p_1, p_2 \in S_\infty$. Since γ is preserved by f', $\widetilde{f'}(\widetilde{\gamma}_0)$ is another segment $\widetilde{\gamma}_1$ with boundary points $q_1, q_2 \in S_\infty$ (that projects into γ). Both segments $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ (with their boundary points included) are either equal or disjoint. If $\widetilde{\gamma}_0 = \widetilde{\gamma}_1$, then $(\widetilde{f'})^2(p_1) = (\widetilde{f})^2(p_1) = p_1$ and the claim is proved. If $\widetilde{\gamma}_0 \neq \widetilde{\gamma}_1$, assume without loss of generality that $\widetilde{f'}(p_1) = q_1$ and $\widetilde{f'}(p_2) = q_2$.

Let us show that $\widetilde{f'}$ admits a periodic point at the boundary. Let I be the closed interval in S_∞ whose boundary is p_1 and p_2 and such that q_1, q_2 are not in the interior of I. This is possible because the curves $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ are disjoint. Denote by J the interval $\widetilde{f'}(I)$ which satisfies $\partial J = \{q_1, q_2\}$. Two cases can occur. In the first case, we have $I \subset J$, and we can apply Brouwer's fixed point theorem to $(\widetilde{f'})^{-1}$, deducing that $\widetilde{f}'$ admits a fixed point. In the second case, the interior of I and J are disjoint. Consider the interval $H=(\widetilde{f}')^2(I)$ and we claim either $H\subseteq I$ or $I\subset H$. To see this, note first that H is the image of J by $\widetilde{f}'$ and hence lies in the complementary of the interior of J, because $J=\widetilde{f}'(I)$ and we are assuming that I and J are disjoint along their interior. Furthermore, the boundary points of H lie in the boundary of some curve $(\widetilde{f}')^2(\widetilde{\gamma})$ which is another lift of γ and so is either equal to or disjoint with $\widetilde{\gamma}$. This shows that the boundary points of H are either both contained in I or both in $S_\infty\setminus I$, since otherwise the curves $\widetilde{\gamma}$ and $(\widetilde{f}')^2(\widetilde{\gamma})$ would be different but with non-trivial intersection.

We conclude that if the boundary points of H lie in I then $H\subseteq I$, and if they lie outside of I then $I\subset H$. We can apply Brouwer's fixed point theorem to $(\widetilde{f}')^2$ or to $(\widetilde{f}')^{-2}$ to conclude that $\widetilde{f}'$ admits a periodic point of period two. Since $\widetilde{f}'|_{S_\infty}=\widetilde{f}|_{S_\infty}$, we deduce that $\widetilde{f}$ always admits a periodic point at the boundary.

We can now finish the argument as in the case of the torus. The lift $\widetilde{f}$ fixes a point $\widetilde{z}$ with irrational rotation number, and since it admits a periodic point along S_∞, it has rational rotation number along S_∞. We can blow up the fixed point $\widetilde{z}$ to find an area-preserving homeomorphism of the closed annulus with different rotation numbers at each boundary component, and conclude as before that φ, and hence ϕ, admits periodic points of arbitrarily large period. □

Remark 4.2. The previous proof can be shortened at the expense of calling in recent strong results by Le Calvez [31] instead of direct arguments.

A corollary is the following.

Corollary 4.3. *Let (λ,ω) be a stable Hamiltonian structure (eventually degenerate). If any component of N_0 is not a solid torus or a thickened torus ($T^2\times I$), then the Reeb vector field admits infinitely many periodic orbits.*

Proof. We might assume that in every invariant torus in U, the linear vector field has irrational slope. Then we can perturb $\widetilde{\lambda}$ in Theorem 2.2 so that it defines a multiple of an integer cohomology class, and so the Reeb vector field in N_0 admits a global section. The Reeb flow in N_0 is conjugated to the suspension of a symplectomorphism of a surface with boundary, which by hypothesis is neither a disk nor a torus. It follows from Theorem 4.1 that the Reeb vector field admits infinitely many periodic orbits. □

4.2. **Proof of Theorem 1.5.** Let (λ,ω) be a contact non-degenerate SHS on a closed 3-manifold M, and we assume that the Reeb vector field X is not aperiodic. If the function $f=\frac{d\lambda}{\omega}$ is constant and non vanishing, then the Reeb vector field X is a non-degenerate Reeb vector field of the contact form λ. The Reeb vector field of any C^2-close SHS $(\widetilde{\lambda},\widetilde{\omega})$ will be the Reeb vector field of the contact form $\tilde{\lambda}$, which is C^2-close to λ. It follows from [12, Theorem 1.2] that the Reeb vector field of $(\widetilde{\lambda},\widetilde{\omega})$ always admits infinitely many periodic orbits unless: either M is the sphere or M is a lens space and there are exactly two periodic orbits that are core circles of a genus one Heegaard splitting of M. If f is constantly equal to zero, the Reeb vector field of any SHS C^2-close to (λ, ω) is conjugated to the suspension of a symplectomorphism, arguing as in the first case of the proof of Theorem 1.3. We deduce that X is conjugated to the suspension of a surface symplectomorphism with finitely many periodic points.

The last case to analyze is when f is non-constant. Let N and U be as in the Structure Theorem 2.2, and denote by N_+, N_0, N_- the disjoint regions of N where f is respectively positive, zero or negative. If the Reeb vector field has no periodic orbits, then in each invariant torus (given by a regular level set of f) the vector field is conjugated to a linear flow with irrational slope. If f never vanishes, then the Reeb vector field is also the Reeb vector field defined by the contact form λ, and is everywhere non-degenerate so we conclude as before. Otherwise N_0 is nonempty, and in each connected component of N_0, we can argue as in the first case of the proof of Theorem 1.3 to deduce that the Reeb vector field X is conjugated to a suspension of symplectomorphism of a surface with boundary Σ_g^b. In the boundary of N_0 the Reeb vector field is conjugated to an irrational linear flow, so in the boundary Σ_g^b this symplectomorphism has no periodic points. By Theorem 4.1 we deduce that either X admits infinitely many periodic points in the interior of N_0, or each component of N_0 is diffeomorphic to a solid torus or a thick torus and X is conjugated to the suspension of an irrational pseudorotation. In each connected component of N_+, N_-, the vector field X is a non-degenerate Reeb vector field of some contact form that is T^2-invariant near the boundary. This follows from the fact that there is a neighborhood of the boundary of N_+ and N_- that is foliated by regular level set of f, where X is T^2-invariant. By Lemma 3.1, each connected component of N_+ or N_- is diffeomorphic to a solid torus or a thick torus, and X is conjugated to a suspension of an irrational pseudorotation there.

The connected components of N are glued along their boundary to boundary components of U, each diffeomorphic to $T^2 \times I$, and where X is conjugated to the suspension of an irrational rotation. In conclusion, the whole manifold is obtained by gluing a finite number of copies of $\mathbb{S}^1 \times D^2$ and $T^2 \times I$ along their boundaries. Since M is assumed to be connected, we deduce that there are either two or zero copies of $\mathbb{S}^1 \times D^2$. In the first case, the vector field X does not admit any periodic orbit, since in each domain $T^2 \times I$ it is conjugated to the suspension of an irrational pseudorotation of the annulus. Since we assumed that X has at least one periodic orbit, we deduce that exactly two connected components in N are diffeomorphic to $\mathbb{S}^1 \times D^2$. After gluing together all the components that are diffeomorphic to $T^2 \times I$ iteratively to one of the solid tori, we obtain a decomposition of M into two solid tori V_1, V_2 that share a common boundary. This shows that M is either a sphere or a lens space, and the periodic orbits of X are core circles of V_1 and V_2 which define a genus one Heegaard splitting of M. This proves the statement for contact non-degenerate SHS.

It remains to show that there exists an open C^2-neighborhood (in the set of SHS) of (λ, ω), where the conclusions of Theorem 1.5 hold.

Let U denote the integrable region (according to the structure theorem) of (λ, ω). By [10, Theorem 3.7], given a stable Hamiltonian structure $(\widetilde{\lambda}, \widetilde{\omega})$ sufficiently C^2-close to (λ, ω), it will have T^2-invariant integrable regions K_i inside each component U_i of U of almost full measure in U. We assume that the slope of the Reeb vector field of $(\widetilde{\lambda}, \widetilde{\omega})$ has constant irrational slope, since otherwise there infinitely many periodic orbits. The complement of this integrable region $K = \sqcup K_i$ is diffeomorphic to $N_0 \sqcup N_c$. In N_0, the Reeb vector field still admits a global section, since there it is C^1-close to the Reeb vector field of (λ, ω), which is conjugated to a suspension flow. In the contact region N_c, the Reeb vector field of $(\widetilde{\lambda}, \widetilde{\omega})$ is the Reeb vector field of the contact form $\widetilde{\lambda}$, which is C^2-close to λ. The simple observation is that the proof of Lemma 3.1, applies to (contact) Reeb vector fields in a C^1-small neighborhood of non-degenerate ones, since so does [12, Theorem 1.2]. Then our proof above applies as well to the Reeb vector field of $(\widetilde{\lambda}, \widetilde{\omega})$. This concludes the proof of Theorem 1.5.

Remark 4.4. Notice that if [12, Theorem 1.2] and hence Lemma 3.1 can be proven without the non-degeneracy hypothesis, then our proof applies and Theorem 1.5 holds without the contact non-degeneracy hypothesis.

5. Broken books and Birkhoff sections for SHS

The aim of this section is to generalize the main results of [12] and [13, 14], from Reeb vector fields of a contact form to Reeb vector fields of a SHS. Broken book decompositions, introduced in [12] (consult Definition 5.7), provide a strong tool for studying the dynamics of a vector field. In particular, they are the starting point to prove the existence of Birkhoff sections under generic hypotheses (see [13] and [14]). We will study the existence of this structure for contact non-degenerate Reeb vector fields of a SHS (see Definition 2.3) and prove that every contact non-degenerate Reeb vector field of a SHS, satisfying that the slope in each integrable region $U_i \subset U$ is non-constant, is supported by some broken book decomposition. Proving Theorem 1.2.

The rough idea is the following. First, we assume that the Reeb vector field of a SHS is neither a suspension, nor a Reeb vector field of a contact form. In N_0 we have global sections whose boundary is contained in ∂N_0, while in N_+ and N_- we have a broken book decomposition whose pages might have boundary components contained in ∂N_+ and ∂N_- respectively. We want to paste these two types of transverses surfaces along their boundary, these is done with *helix boxes* of the form $T^2 \times I$ contained in the region U, as explained in Section 5.1.

A Reeb vector field of an SHS is contact strongly non-degenerate if it is contact non-degenerate and the stable and unstable manifolds of the hyperbolic periodic orbits in the contact part $N_+ \cup N_-$ intersect transversally. To deduce Theorem 1.1, we use that contact strongly non-degenerate Reeb vector fields of SHS are C^1-dense in their class, and having non-constant slope in U is a C^∞-dense condition. Combined with the fact that in the contact region Birkhoff sections exist on a C^∞-dense and open subset [14, 13], we can construct the Birkhoff sections for a C^1-dense and C^2-open set of Reeb vector fields of SHS.

We start with the construction of the helix boxes and the construction of Birkhoff sections for T^2-invariant flows in Section 5.1. Then in Section 5.2 we prove Theorem 1.2 and in Section 5.3 we prove Theorem 1.1.

5.1. **Birkhoff sections in $T^2 \times I$.** Let us recall that for a vector field X on a 3-manifold, a transverse surface is a surface with boundary whose interior is embedded and transverse to X and whose boundary is immersed and composed of periodic orbits. A Birkhoff section is a transverse surface for which there exists some $T > 0$ such that for all $x \in M$, we have $\{\phi_X^t(x) \mid t \in [0,T]\} \cap S \neq \emptyset$, where ϕ_X^t denotes the flow defined by X.

Proposition 5.1. *Let X be a vector field on $T^2 \times I$ that is T^2-invariant, with a periodic orbit ν in $T^2 \times \{t^*\}$ for some $t^* \in (0,1)$. Let γ_0, γ_1 be two connected closed curves respectively in $T^2 \times \{0\}$ and $T^2 \times \{1\}$. Assume that:*

1. *$\gamma_0 \times I$ and $\gamma_1 \times I$ are transverse to X respectively in $T^2 \times [0,t^*]$ and $T^2 \times [t^*,1]$;*
2. *the homology classes $[\gamma_0]$ and $[\nu]$ generate $H_1(T^2;\mathbb{Z})$.*

Write $[\gamma_1] = p[\gamma_0] + q[\nu]$. Then X admits a Birkhoff section Σ with binding ν and such that $\Sigma \cap \{t = 1\} = \gamma_1$ and $\Sigma \cap \{t = 0\}$ is given by p parallel copies of γ_0.

Proof. In a small neighborhood $V = T^2 \times [t^* - \varepsilon, t^* + \varepsilon]$ of $T^2 \times \{t^*\}$, the vector field X has slope close to the one of ν. Hence the surfaces $\gamma_0 \times I$ and $\gamma_1 \times I$ are transverse to X in V. We construct a Birkhoff section in V that extends to $T^2 \times I$ trivially with the surfaces $\gamma_0 \times I$ and $\gamma_1 \times I$: the Birkhoff section intersects $T^2 \times \{t^* - \varepsilon\}$ in p copies of the curve $(\gamma_0 \times [0,t^*]) \cap \{t^* - \varepsilon\}$ and intersects $T^2 \times \{t^* + \varepsilon\}$ in the same curve as the surface $\gamma_1 \times [t^*, 1]$. In V the vector field is almost constant, thus we simplify the situation by considering: X to be a constant vector field parallel to the vertical direction (understood as the direction of ν), γ_0 to be a horizontal curve and we identify V with I^3 a cube. In this cube, assume further that ν is the curve $\{x = 1/2, t = 1/2\}$.

Consider the cube I^3 with coordinates (x,y,t) under the identifications $(0,y,t) \sim (1,y,t)$ and $(x,0,t) \sim (x,1,t)$, so that

$$I^3 /_\sim \cong T^2 \times I.$$

Let X denote the vector field $\frac{\partial}{\partial y}$ and γ_0 the curve $\{y = 0\}$ and $\{t = 0\}$. The closed curve γ_1 in $T^2 \times \{1\}$ can be assumed to be a linear constant slope curve up to isotopy and still be transverse to X. Recall that $[\gamma_1] = p[\gamma_0] + q[\nu]$, for some integers $p, q \in \mathbb{Z}$. To avoid trivial cases, we will assume that $p, q \neq 0$.

Lemma 5.2. *Assume that $[\gamma_1]$ is a primitive homology class in $H_1(T^2;\mathbb{Z})$. With the notation above, the vector field X admits a Birkhoff section Σ in $T^2 \times I$ whose boundary is the periodic orbit $\{x = 1/2, t = 1/2\}$ and such that $[\Sigma \cap \{t = 0\}] = p[\gamma_0]$ and $\Sigma \cap \{t = 1\} = \gamma_1$.*

The idea of the proof is to construct suitable curves in the boundary of I^3 and then use them to span a surface whose boundary is ν and the curves in ∂I^3, see Figure 2.

Proof. Let us assume that $q > 0$. Up to a translation, we might assume that the curve γ_1 intersects the point $(0, 0, 1)$. Let $H = \{y = 0, t = 1\}$ be the horizontal bottom side of the square $I^2 \times \{t = 1\}$, and $V = \{x = 1, t = 1\}$ be the vertical right side of the same square. The curve γ_1 is represented in $I^2 \times \{t = 1\}$ by $p + q - 1$ disjoint segments.

We will now construct q piecewise linear curves λ_i in the boundary of ∂I^3, that is $i = 1, 2, \ldots, q$. We proceed iteratively as follows. Start at $(1/2, 0, 0)$, follow the boundary of I^3 along $y = 0$ towards $(0, 0, 0)$ and up to $(0, 0, 1)$. Add the segment of γ_1 starting at $(0, 0, 1)$ to λ_i. We have to consider several cases. If the other endpoint of this segment intersects H, the first segment λ_1 is done, otherwise, we intersected V along a point $(1, \widetilde{y}, 1)$. If $\widetilde{y} = 1$, we add to λ_1 the segments obtained by considering segments from $(1, 1, 1)$ to $(1, 1, 0)$ and from there to $(1/2, 1, 0)$, finishing the construcion of λ_1. Otherwise $\widetilde{y} \neq 1$ and we add the curve obtained considering line segments connecting the points $(1, \widetilde{y}, 1)$, $(1, \widetilde{y}, 0)$, $(0, \tilde{y}, 0)$ and $(0, \widetilde{y}, 1)$. Repeat the process with the segment of γ_1 starting at $(0, \widetilde{y}, 1)$.

When the first segment is done, we consider the point $(\widetilde{x}, 0, 1)$ in γ_1 immediately at the right of $(0, 0, 1)$ along H, and apply the same recipe as before. We take the segment of γ_1 starting at $(\widetilde{x}, 0, 1)$ and continue accordingly if the endpoint of the segment intersects H or V. Two examples of the curves λ_i are pictured in Figure 1.

Let (r, θ) be polar coordinates of the square $\{y = 0\}$ centered at $\{x = 1/2, t = 1/2\}$. The key feature of each path λ_i is that it can be parametrized as

$$\lambda_i(s) = (y(s), r(s), \theta(s)), \quad s \in [0, 1]$$

in a way that $\theta'(s) > 0$ for all s. Construct for each λ_i a surface Σ_i in I^3 with boundary parametrized by $\phi_i : I^2 \longrightarrow I^3$ with

$$\phi_i(s, \rho) = ((1 - \rho)s + \rho y(s), \rho r(s), \theta(s)). \tag{5}$$

Choosing well the parameter s of each λ_i, we can achieve that the surfaces Σ_i are pairwise disjoint except along the boundary segment $\{x = 1/2, t = 1/2\}$. This can be achieved by choosing a parameter s that varies a small quantity when the curve $\lambda_i(s)$ moves along a plane of fixed y coordinate, and s varies approximately as the y coordinate along each segment of γ_1.

The surfaces Σ_i are well defined in the quotient space of $T^2 \times I$, and we obtain a continuous surface Σ in $T^2 \times I$ whose boundary $\partial\Sigma$ is given by $\gamma_1 \subset \{t = 1\}$, p curves in $\{t = 0\}$ parallel to γ_0, and the central orbit of the flow given by

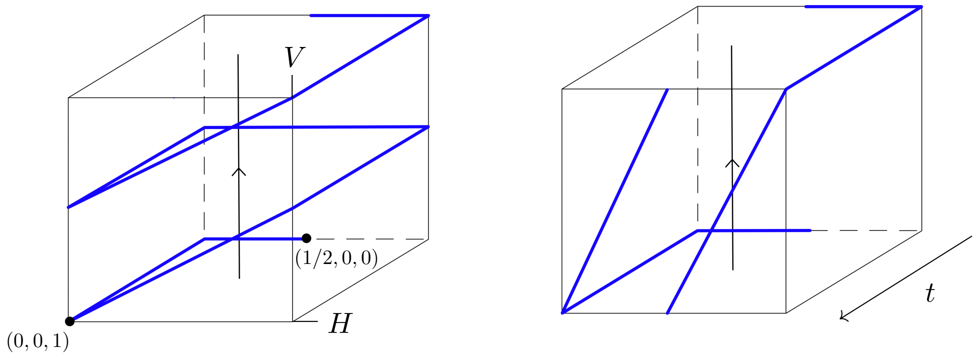

FIGURE 1. The curves λ_i for $i = 1, 2, \ldots, q$, in the cases for $p = 2, q = 1$ on the left hand side and $p = 1, q = 2$ on the right hand side.

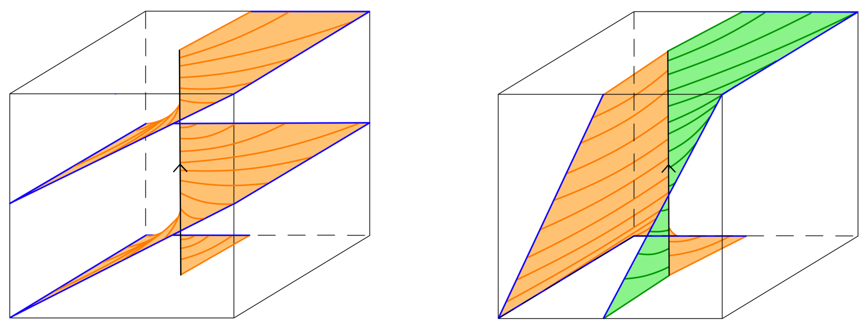

FIGURE 2. Surfaces for $p = 2, q = 1$ and $p = 1, q = 2$

$\{x = 1/2, t = 1/2\}$. We can easily construct surfaces Σ_i that induce a C^∞-immersed surface Σ. It is embedded except maybe along $\{x = 1/2, t = 1/2\}$ where it is embedded only if $q = 1$. The parametrization ϕ_i satisfies

$$d\phi_i\left(\frac{\partial}{\partial \rho}\right) = (y(s) - s)\frac{\partial}{\partial y} + r(s)\frac{\partial}{\partial r}$$
$$d\phi_i\left(\frac{\partial}{\partial s}\right) = (1 - \rho + \rho y'(s))\frac{\partial}{\partial y} + \rho r'(s)\frac{\partial}{\partial r} + \theta'(s)\frac{\partial}{\partial \theta},$$

and since $r(s)$ and $\theta'(s)$ do not vanish in the complement of $\{x = 1/2, t = 1/2\}$, we deduce that for $\rho \neq 0$, the image of $d\phi$ never contains $\frac{\partial}{\partial y}$.

Note that if $q < 0$, the only difference is that we need to adapt the construction with curves λ_i that satisfy $\theta'(s) < 0$. $\square$

Back to the proof of Proposition 5.1, we only need to consider the constructed Birkhoff section inside $U = T^2 \times [t^* - \varepsilon, t^* + \varepsilon]$ and extend this surface trivially to $T^2 \times I$. □

Remark 5.3. If γ_1 is a finite number of parallel copies of a closed curve of primitive homology class, we just need to take parallel copies of the surface Σ constructed in Lemma 5.2 and Proposition 5.1 also holds.

Remark 5.4. Similar surfaces were first considered in [43] in the context of open book decompositions for contact structures. They were later used in dynamical systems by Dehornoy [18, 19] to classify global surfaces of section of the geodesic flow on the flat torus. His construction uses several bindings instead of one, and yields embedded surfaces (instead of surfaces that are only immersed along the boundary). Those are constructed by piling up vertically and horizontally copies of the surface obtained in Lemma 5.2 when choosing a linear curve γ with $p = 1, q = 1$. Using this approach, one obtains an embedded Birkhoff section, at the cost of needing several parallel binding components along the torus $t = \{1/2\}$. Similarly, Proposition 5.1 yields an embedded Birkhoff section if there are sufficient parallel periodic orbits along $T^2 \times \{t^*\}$.

In [18, 19], the name *helix box* refers to the construction in Lemma 5.2 for a curve γ such that $p = 1, q = 1$. Since it is a natural generalization, the construction in Lemma 5.2 will be called a helix box as well, independently of the curve γ. We proceed to the proof of the main result of this section, obtained by concatenating suitable helix boxes.

Proposition 5.5. *Let X be a non-singular vector field in $T^2 \times I$ that is T^2-invariant and whose slope is non-constant. Let Γ_0, Γ_1 be two families of embedded closed curves such that $\Gamma_0 \subset T^2 \times \{0\}$, $\Gamma_1 \subset T^2 \times \{1\}$, and such that $X|_{t=0}$ and $X|_{t=1}$ is respectively transverse to Γ_0 and Γ_1. Then there exists a Birkhoff section Σ of X such that $\Sigma \cap \{t = 1\} = \Gamma_1$ and $\Sigma \cap \{t = 0\} = \Gamma_0$.*

Proof. We might assume that Γ_0 and Γ_1 are just parallel copies of linear closed curves γ_0, γ_1 that have primitive integer homology classes in $H_1(T^2; \mathbb{Z})$, respectively. In homology, we have $[\Gamma_0] = r.[\gamma_0]$ and $[\Gamma_1] = \ell.[\gamma_1]$. As before, we can consider these curves to be linear constant slope curves in a torus.

Let $k(t)$ denote the slope of the vector field X on $T^2 \times \{t\}$, with respect to some coordinate system. Let $\varepsilon > 0$ be such that the slope $k(t)$ is not constant, varies only by a small value and satisfies $k'(t) \geq 0$ for $0 \leq t \leq \varepsilon$ (if $k'(t) \leq 0$, an analogous argument works). In particular, we can assume that $\Gamma_0 \times [0, \varepsilon]$ is transverse to X. Choose two primitive integer homology classes b_1, b_2 of T^2 represented by two positively oriented periodic orbits of the flow on two tori $\{t = s_1\}$ and $\{t = s_2\}$ where X has rational slope and such that $0 < s_1 < s_2 < \varepsilon$. We might further impose that b_1, b_2 are generators of $H_1(T^2; \mathbb{Z})$ and that $[\gamma_0], b_1$ are also generators of $H_1(T^2; \mathbb{Z})$. We can choose coordinates x, y in T^2 such that $\frac{\partial}{\partial x}$ is parallel to γ_0 and $\frac{\partial}{\partial y}$ is parallel to b_1. Since the slope of the Reeb vector field is of positive derivative for $t \in [0, \varepsilon]$, it follows that b_2 can be chosen such that $b_2 = Nb_1 + [\gamma_0]$ for some $N >> 0$.

Set $t_0 = 0$ and partition the interval $[\varepsilon, 1]$ into disjoint subintervals

$$[t_1, t_2], ..., [t_{n-1}, t_n],$$

where $t_1 = \varepsilon$ and $t_n = 1$, so that in each interval the slope is non-constant and varies only by some amount smaller in absolute value than $\theta_0 << 1$. We can assume that $\Gamma_1 \times I$ is transverse to X for $t \in [t_{n-1}, 1]$. Construct a family of closed curves σ_i with $i = 1, ..., n-1$ in $T^2 \times \{t_i\}$ such that:

- $[\sigma_1] = p' b_1 + q'[\gamma_0]$ for some coprime integers such that $p' < 0$ and $q' \geq r$, and $[\sigma_1], b_2$ are generators of $H_1(T^2; \mathbb{Z})$,
- $\sigma_i \times I$ is transverse to X for $t \in [t_{i-1}, t_{i+1}]$ for each $i = 1, ..., n-1$.

This is possible because the slope of the vector field does not vary more than θ_0 in each interval, so we might iteratively choose σ_i for $i = 2, ..., n-1$ to be a curve whose slope is approximately minus the inverse of the slope of the vector field in $T^2 \times \{t_i\}$ (with respect to some fixed coordinates in $T^2 \times I$). For $i = 1$ there are many choices for σ_1.

Having the curves σ_i, we choose, in each domain $T^2 \times [t_i, t_{i+1}]$, a periodic orbit of the flow whose homology class together with $[\sigma_i]$ gives a base of $H_1(T^2; \mathbb{Z})$.

We apply Proposition 5.1 first in $T^2 \times [t_{n-1}, t_n]$ where the curve in $T^2 \times \{t_n\}$ is Γ_1 (by Remark 5.3, Proposition 5.1 applies even if Γ_1 is not connected). We find a Birkhoff section whose boundary along $t = 1$ is Γ_1 and along $t = t_{n-1}$ is a finite collection of parallel copies of σ_{n-1}. Iteratively, we apply Proposition 5.1 to each domain $T^2 \times [t_i, t_{i+1}]$ with $i = n-2, ..., 1$, choosing as curve in $T^2 \times \{t_{i+1}\}$ the boundary of the Birkhoff section constructed in $[t_{i+1}, t_{i+2}]$, which is given by a finite number of copies of σ_{i+1}. Again by Remark 5.3, there is no issue in considering multiple copies of a curve. After reaching $i = 1$, we constructed a Birkhoff section Σ in $T^2 \times [\varepsilon, 1]$ that intersects $\{t = 1\}$ along Γ_1, and intersects $\{t = \varepsilon\}$ along some finite number of parallel copies of σ_1. To simplify the notation, denote γ_ε the curve σ_1, and Γ_ε the finite collection of parallel copies of γ_ε given by the intersection of the Birkhoff section with $\{t = \varepsilon\}$.

By construction $[\gamma_\varepsilon] = p' b_1 + q'[\gamma_0]$, for some coprime integers p', q' satisfying $p' < 0$ and $q' \geq r$. Since $[\Gamma_\varepsilon] = s[\gamma_\varepsilon] = pb_1 + q[\gamma_0]$ for some positive integer s, we have $p < 0$ and $q \geq r$. We can decompose the homology class of Γ_ε as

$$[\Gamma_\varepsilon] = [\Gamma_0] + (p - (q-r)N)b_1 + (q-r)b_2,$$

using that b_1, b_2 are generators. In vectorial notation, we have $[\gamma_0] = (0, 1)$, $b_1 = (1, 0)$, and the equality above is tantamount to

$$(p, q) = (0, r) + (p - (q-r)N)(1, 0) + (q-r)(N, 1).$$

Choose some $\varepsilon' \in (s_1, s_2)$ and a set of closed (linear) curves Γ'_ε in $T^2 \times \{\varepsilon'\}$ such that $[\Gamma_{\varepsilon'}] = [\Gamma_0] + (p - (q-r)N)b_1$. We would like to apply Proposition 5.1 in

$T^2 \times [0, \varepsilon']$ and in $T^2 \times [\varepsilon', \varepsilon]$, choosing as curves $\Gamma_0, \Gamma_{\varepsilon'}$ and with binding in $T^2 \times \{s_1\}$, and $\Gamma_{\varepsilon'}, \Gamma_\varepsilon$ with binding in $T^2 \times \{s_2\}$. The curves and bindings are schematically depicted in Figure 3.

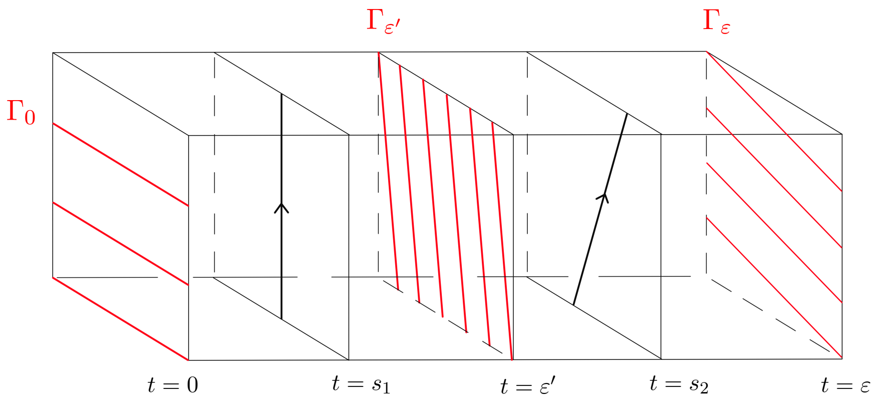

FIGURE 3. Last step in $T^2 \times [0, \varepsilon]$

To be able to do it, we need to check that $\Gamma_{\varepsilon'} \times I$ is transverse to the flow for $t \in [s_1, s_2]$.

We claim that the condition $p < 0, q > r$ ensures that $\Gamma_{\varepsilon'} \times I$ is transverse to the vector field for $t \in [s_1, \varepsilon]$. Recall that s_1 is such that in $T^2 \times \{s_1\}$ the orbits of X have homology b_1. The transversality follows from the fact that the vector field "rotates" in the direction that increases the angle between X and $\Gamma_{\varepsilon'} \times [s_1, \varepsilon]$ as t increases. This is represented in Figure 3: the vector field rotates clockwise as t increases, becoming increasingly transverse to $\Gamma_{\varepsilon'} \times [s_1, \varepsilon]$.

Formally, the vector field along any torus $T^2 \times \{t^*\}$ with $t^* \in [s_1, \varepsilon]$ is of the form $X_{t^*} = m\frac{\partial}{\partial x} + n\frac{\partial}{\partial y}$ where n, m are real numbers such that $n >> m > 0$. On the other hand, the section is given by an integral curve of the vector field $Y = r\frac{\partial}{\partial x} + \big(p - (q-r)N\big)\frac{\partial}{\partial y}$. The determinant of the matrix whose columns are the coefficients of these vector fields is

$$\det \begin{pmatrix} r & m \\ p-(q-r)N & n \end{pmatrix} = rn - m\big(p - (q-r)N\big).$$

It follows from the fact that $p < 0$ and that $q \geq r$ that this determinant is always positive, as claimed.

We are now able to apply Proposition 5.1 in $T^2 \times [0, \varepsilon']$, and construct a Birkhoff section of the flow such that along $t = 0$ it defines the curves Γ_0 and at $t = \varepsilon'$ it defines $\Gamma_{\varepsilon'}$, and whose binding is a periodic orbit of the flow along $t = s_1$. The fact in this step is that the homology of $\Gamma_{\varepsilon'}$ expressed in the base γ_0, b_1 has a coefficient r in γ_0. This ensures that along $t = 0$ the section can be chosen to coincide with Γ_0, i.e. with exactly r copies of γ_0.

Finally, we have

$$[\Gamma_\varepsilon] = [\Gamma_{\varepsilon'}] + (q-r)b_2,$$

and the key fact is again that in the base $[\gamma_{\varepsilon'}], b_2$, both sets of curves have the same coefficient in $[\gamma_{\varepsilon'}]$. We apply again Proposition 5.1 to construct a Birkhoff section of the vector field in $T^2 \times [\varepsilon', \varepsilon]$ with a binding component which is a closed curve of the flow in $\{t = s_2\}$. This Birkhoff section coincides with $\Gamma_{\varepsilon'}$ and Γ_ε when intersected with $t = \varepsilon'$ and $t = \varepsilon$ respectively, finishing the proof Proposition 5.5. □

Even for flows in $T^2 \times I$ that are not necessarily T^2 invariant, Proposition 5.1 and Proposition 5.5 hold under more general conditions.

Remark 5.6. Proposition 5.5 holds as well if we want a global surface of section instead of a Birkhoff section (i.e. the surface is embedded even along the boundary). This can be done by using several bindings, see Remark 5.4.

5.2. **Broken books for contact non-degenerate SHS.** Let us first recall the definition of broken book decomposition, introduced in [12].

Definition 5.7. A broken book decomposition of a closed 3-manifold M is a pair $(K, \mathcal{F})$ such that:

- K is a link, called the binding,
- $\mathcal{F}$ is a smooth cooriented foliation of $M \setminus K$ such that each leaf S of $\mathcal{F}$ is properly embedded in $M \setminus K$ and admits a compactification $\overline{S}$ in M which is a compact surface, that we call a page. The boundary of S is contained in K.

The link K is composed of the radial part K_r and the broken part K_b. A connected component k_r of K is radial if $\mathcal{F}$ foliates a tubular neighborhood of k_r by annuli all having exactly one boundary component on k_r. A tubular neighborhood V of broken component of $k_b \in K$ is foliated by $\mathcal{F}$ and $\mathcal{F}|_V$ has two types of leaves: radial ones that are annuli with a boundary component in k_r and the other boundary component in ∂V; and hyperbolic ones that are annuli with both boundary components in ∂V. Then ∂V is separated into radial and hyperbolic sectors, accordingly. In a broken book decomposition, ∂V has either two or four hyperbolic open sectors, separated by two or four radial closed sectors.

Remark 5.8. Observe that a radial sector in ∂V, where V is a tubular neighborhood of a broken component of K, might contain just one leaf of $\mathcal{F}|_V$. Also, the definitions of radial and hyperbolic leaves are only local.

There is a finite number of pages that do not belong to the interior of one-parameter families of homeomorphic pages, which are called *rigid pages*. The boundary of a rigid page must contain a broken binding component and there are finitely many rigid pages. Each connected component of the complement of the rigid pages fibers over $\mathbb{R}$ and the fibers can be taken to be leaves of $\mathcal{F}$.

Given a vector field X on a 3-manifold equipped with a broken book decomposition $(K, \mathcal{F})$, we say that the broken book carries (or supports) the vector field X if the binding K is composed of periodic orbits of X, while the other orbits are transverse to the interior of the leaves of $\mathcal{F}$. If the broken book decomposition has no broken components, it is a rational open book decomposition. In this case, the foliation $\mathcal{F}$ is a fibration over $\mathbb{S}^1$ in $M \setminus K$.

Remark 5.9. In our construction, a periodic orbit of a vector field X carried by a broken book decomposition that belongs to K_b has to be non-degenerate and hyperbolic. In this case, it can be either positive or negative hyperbolic according to the sign of the eigenvalues of the linearised Poincaré map.

This will be easily achieved since K_b will be contained in the contact part of the Reeb vector field of a SHS, and hence the conclusion will follow from the construction in [12].

We cite two definitions from [12] for a smooth non-singular vector field X on a closed 3-manifold M. Given a periodic orbit γ of X we denote by Σ_γ the unit normal bundle $(TM_\gamma/T\gamma)/\mathbb{R}_+$ to γ and by M_γ the normal blow-up of M along γ, that is the manifold $(M \setminus \gamma) \cup \Sigma_\gamma$. The vector field X extends to a vector field X_γ on the torus Σ_γ and hence to a vector field on M_γ and tangent to the boundary. Observe that X_γ restricted to the interior of M_γ coincides with X in $M \setminus \gamma$. We abuse notation and still denote this extension by X. If S is a transverse surface in M with $\gamma \in \partial S$, we denote by $\partial_\gamma S$ its extension to Σ_γ.

Definition 5.10. Let S be a (not necessarily connected) transverse surface with boundary.

- An orbit γ of X is *asymptotically linking* S if for every $T \in \mathbb{R}$ the arcs $\gamma([T, +\infty))$ and $\gamma((-\infty, T])$ intersect S.
- If γ is a non-degenerate periodic orbit in ∂S, consider its unit normal bundle Σ_γ. The *self-linking* of γ with S is the rotation number of the extension of X to Σ_γ, with respect to the 0-slope given by the curve in Σ_γ defined by the intersection with the extension of S.

Definition 5.11. A transverse surface S is *∂-strong* if, for every boundary orbit γ of S, the extension $\partial_\gamma S$ is a collection of embedded curves in Σ_γ that are transverse to the extended vector field X. If S is a Birkhoff section, S is *∂-strong* if moreover $\partial_\gamma S$ defines a section for X restricted Σ_γ.

We proceed with the proof of the main theorem of this section, Theorem 1.2.

Proof of Theorem 1.2. The cases in which f is constant are known. We assume that f is non-constant and start by recalling the ingredients of the construction of a broken book decomposition supporting a vector field X. These correspond to the properties enumerated in [12, Lemma 3.6] and used in the construction of a broken book. We refer to [12] for the proof.

Proposition 5.12. *Let X be a non-singular vector field on closed a 3-manifold M. Assume there is a finite collection of transverse surfaces with boundary $S_1, S_2, \dots S_\ell$ such that:*

1. *the transverse surfaces have two by two disjoint interiors;*
2. *every orbit intersects $\cup_i S_i$;*
3. *$\cup_i \partial S_i = K$;*
4. *$M \setminus (\cup_i S_i)$ fibers over $\mathbb{R}$;*
5. *if an orbit of X is not asymptotically linking $\cup_i S_i$, it converges to one of their boundary components which is a non-degenerate hyperbolic periodic orbit γ with tubular neighborhood V. In this case, each one of the quadrants transversally delimited by the stable and unstable manifolds of γ is intersected by at least one S_i such that a connected component of $S_i \cap V$ contained in this quadrant has γ as a boundary component.*

Then there is a broken book decomposition $(K, \mathcal{F})$ supporting X and having the surfaces S_i as pages.

The orbits in Proposition 5.12 (5) are the orbits in K_b, that is the broken components of the binding of the broken book obtained. Given a vector field carried by a broken book decomposition, one can choose a finite collection of pages satisfying Proposition 5.12: the set of rigid pages is one possibility, but also a set of nearby regular pages.

Let (λ, ω) be a contact non-degenerate SHS, and denote by X its Reeb vector field. By Theorem 2.2, we can decompose our manifold into $U_1, ..., U_k, N_+, N_0$ and N_-. In the connected components of the "contact region" $N_c = N_+ \cup N_-$, the Reeb vector field is non-degenerate and T^2-invariant near the boundary.

Let W denote a connected component of N_c. Near ∂W, the Reeb vector field is parallel to a linear vector field of constant irrational slope: arguing as in the proof of Lemma 3.1 we can compactify W into a closed 3-manifold $\overline{W}$, and find a contact form α in $\overline{W}$. The invariant boundary components of W yield closed non-degenerate elliptic orbits of the Reeb vector field R_α that are surrounded by a foliation of invariant tori T_ρ for ρ a real parameter. Let Γ be the collection of these periodic orbits of R_α and let V be a disjoint union of tubular invariant open neighborhoods of the orbits in Γ, bounded by invariant tori. By construction X restricted to the interior of N_c coincides with R_α in $\overline{W} \setminus \Gamma$.

Applying Theorem 1.1 in [12], we know that R_α is carried by a broken book decomposition $(K, \mathcal{F})$. Consider $\gamma \in \Gamma$, that is an elliptic periodic orbit, hence, it cannot be a broken component of K. It follows that either γ is a radial component of K or γ is everywhere transverse to $\mathcal{F}$. In both cases, the $\mathcal{F}$ induces in each invariant torus T_ρ a foliation by closed curves transverse to R_α, which is a linear irrational vector field in each torus. We deduce that in each connected component of N_+ or N_- the Reeb vector field X is carried by a broken book decomposition, whose pages are transverse to the boundary tori and such that K is in the interior of N_c.

Let us still denote by $(K, \mathcal{F})$ the broken book decomposition of N_c: that is $(K, \mathcal{F})$ restricts to a broken book decomposition in each component of N_c.

On the other hand, in each connected component of N_0, the Reeb vector field is transverse to the fibers of a surface bundle over the circle as argued in Section 3.4. This means that in the boundary of any integrable region U, which connects two components in N, there are two different induced homology classes of curves: a torus in ∂U is also a boundary torus of a connected component of N. Thus, the curves we consider are the intersection of any page of $(K, \mathcal{F})$ with the torus in the contact case or the intersection of any fiber of the surface bundle with the torus in the suspension case.

Choose a finite collection of non-rigid pages $P = P_1 \cup \dots \cup P_k$ of $\mathcal{F}$ in N_c satisfying Proposition 5.12. Observe that since N_c has boundary, the surfaces P_i might have boundary components in ∂N_c.

Take an integrable region $U_i \cong T^2 \times I$ of U and set $\partial U_i = T_0 \cup T_1$ where $T_k \cong T^2 \times \{k\}$ for $k = 0, 1$. If $T_k \in \partial N_c$ let Γ_k be the collection of closed curves $P \cap T_k$. If $T_k \in \partial N_0$ let Γ_k be the intersection of a fiber with T_k. We obtain two families of closed curves Γ_0 and Γ_1 transverse to the Reeb vector field along T_0 and T_1 respectively. Applying Proposition 5.5, we construct a Birkhoff section in U_i that glues together the surfaces defined on each component of N.

We can do this at each integrable region U_i, obtaining a finite collection $S = S_1 \cup \dots \cup S_\ell$ of transverse surfaces with boundary. We claim that S satisfies Proposition 5.12. By construction, the surfaces have disjoint interiors and intersect all the orbits of X. Observe that in the closure of $M \setminus N_c$ the restriction of S is a Birkhoff section, hence the orbits that are not asymptotically linking S are contained in N_c and by the choice of P they satisfy (5) of Proposition 5.12. The same argument implies that the complement of S fibers over $\mathbb{R}$.

The existence of a broken book decomposition carrying X follows from Proposition 5.12.

It remains to show that there exists an open C^2-neighborhood (in the set of SHS) of (λ, ω), where each Reeb vector field is also carried by a broken book decomposition. By [10, Theorem 3.7], a SHS $(\widetilde{\lambda}, \widetilde{\omega})$ sufficiently C^2-close to (λ, ω) will have T^2-invariant integrable regions K_i inside each component U_i of U of almost full measure in U. Since the slope of the Reeb vector field of (λ, ω) on U was non-constant, the slope of the Reeb vector field of $(\widetilde{\lambda}, \widetilde{\omega})$ has non-constant slope if the SHS is enough C^2-close to (λ, ω). The complement of $K = \sqcup K_i$ is diffeomorphic to $N_0 \sqcup N_c$. In N_0, the Reeb vector field still admits a global section, since it is C^1-close to a suspension flow. In the contact region N_c, the Reeb vector field is C^1-close to the Reeb vector field of (λ, ω), and the contact form $\widetilde{\lambda}$ is C^2-close to λ. The broken book decomposition that carries de Reeb vector field of (λ, ω) is such that in the contact region, the binding components are non-degenerate periodic orbits and the flow is δ-strong carried by the broken book (see Remark 5.15). It was shown in [12, Section 4.7] that the Reeb vector field defined by a contact form that is C^2-close to a non-degenerate contact form is also be supported by a broken book decomposition. We deduce that even if $(\widetilde{\lambda}, \widetilde{\omega})$ is a priori not contact non-degenerate, the Reeb vector field is supported by a broken book decomposition in the contact region (since it is supported by a broken book decomposition in $\overline{W}$). The slope in the integrable region is non-constant in each connected component, so we can still construct a broken book decomposition supporting the Reeb vector field of $(\widetilde{\lambda}, \widetilde{\omega})$ arguing step by step as before. □

Remark 5.13. We only required the contact non-degeneracy to deduce that the Reeb vector field in the contact regions is carried by a broken book decomposition. Additionally, by construction, if the broken book decomposition $(K, \mathcal{F})$ given in the contact region N_c is a rational open book decomposition, the previous theorem implies that the Reeb vector field of (λ, ω) is carried by a rational open book decomposition. Indeed, the only broken binding components arise in the original broken book of the contact region, and nowhere else.

Corollary 5.14. *Let (λ, ω) be a contact strongly non-degenerate stable Hamiltonian structure in a closed 3-manifold M with respect to some partition $M = N \cup U$. Assume that the slope of $\ker \omega$ is non-constant in each connected component of U. Then the Reeb vector field of (λ, ω) admits a Birkhoff section.*

Proof. Let us keep the notation of the proof of Theorem 1.2. If the Reeb vector field is strongly non-degenerate in the contact region N_c, it is also strongly non-degenerate in the closed 3-manifold $\overline{W}$ obtained by blowing down the boundary components of N_c. By [14, Theorem A], the broken open book decomposition adapted to the Reeb vector field obtained by [12, Theorem 1.1] can be modified into a proper rational open book decomposition adapted to the Reeb vector field. Applying Theorem 1.2 and Remark 5.13, we deduce that the Reeb vector field of (λ, ω) is supported by a rational open book decomposition. One of its pages defines a Birkhoff section of the Reeb vector field. □

Remark 5.15. Both in Theorem 1.2 and in Corollary 5.14, in the contact region N_c, the binding components are non-degenerate Reeb orbits and the flow is δ-strong carried by the broken book decomposition or the Birkhoff section there. This follows from the broken book constructed in $\overline{W}$, that we obtain by applying [12, Theorem 1.1]. In [14, Theorem A], the Birkhoff section is constructed by referring to the proof of [12, Theorem 1.1], so the same holds.

5.3. **Density of SHS admitting a Birkhoff section.** We prove Theorem 1.1 by applying Corollary 5.14. Hence we need to show that strongly contact non-degenerate SHS whose slope is non-constant in each connected component of the integrable region U are dense in some topology in the set of SHS.

Lemma 5.16. *Let (λ, ω) be a stable Hamiltonian structure. There exist an arbitrarily C^∞-small perturbation of (λ, ω), compactly supported in the integrable region U and T^2-invariant, that yields a stable Hamiltonian structure such that* *the slope of* $\ker\omega$ *is non-constant in each connected component of* U *(for some choice of* N, U*).*

Proof. In any connected component of U, we take coordinates (x, y, t) in $T^2 \times I$. We know that λ is of the form

$$\lambda = g_1(t)dx + g_2(t)dy + g_3(t)dt,$$

and ω is of the form

$$\omega = h_1(t)dt \wedge dx + h_2(t)dt \wedge dy.$$

The fact (λ, ω) defines a stable Hamiltonian structure implies that

$$g_1{}'h_2 - g_2{}'h_1 = 0 \qquad \text{and} \qquad h_1g_2 - h_2g_1 > 0.$$

If $\ker\omega$ has constant slope, then $\frac{h_2}{h_1}$ is constant. We might simply perturb the function h_2 to $\widetilde{h}_2(t)$ in a way that $h_2(t) = \widetilde{h}_2(t)$ for $|t - \frac{1}{2}| \geq \varepsilon$, the slope is no longer constant and $\widetilde{h}_2$ is arbitrarily C^∞-close to h_2. This defines an arbitrarily small C^∞ perturbation of ω, that we denote by

$$\widetilde{\omega} = h_1(t)dt \wedge dx + \widetilde{h}_2(t)dt \wedge dy.$$

We can now perturb λ by a C^∞-small perturbation to

$$\widetilde{\lambda} = g_1(t)dx + \widetilde{g}_2(t)dy + g_3(t)dt,$$

where $\widetilde{g}_2(t)$ is determined by the equations

$$\begin{cases} \widetilde{g}_2{}' = \frac{g_1{}'\widetilde{h}_2}{h_1}, \\ \widetilde{g}_2(0) = g_2(0). \end{cases}$$

The function $\widetilde{g}_2$ coincides with g_2 for $t \leq \frac{1}{2} - \varepsilon$. It coincides with g_2 for $t > \frac{1}{2} + \varepsilon$ if and only if

$$\int_0^1 \frac{g_1{}'\widetilde{h}_2}{h_1} dt = g_2(1) - g_2(0).$$

It is clear that we can choose some $\widetilde{h}_2$ satisfying this condition (see also [10, Lemma 3.15] for more general statements on this type of perturbations, considered in the C^1-topology). It follows that $(\widetilde{\lambda}, \widetilde{\omega})$ is a stable Hamiltonian structure that is C^∞-close to (λ, ω), coincides with it outside of $V = T^2 \times (1/2 - \varepsilon, 1/2 + \varepsilon) \subset T^2 \times I$ and whose Reeb vector field has a non-constant slope in V. The perturbation can hence be applied in each connected component of U. □

In the following proof, we show that Lemma 5.16 and the techniques in [10, Theorem 4.6] allow us to prove C^1-density of either contact non-degenerate or strongly contact non-degenerate SHS with non-constant slope in the integrable region.

Proof of Theorem 1.1. Let (λ, ω) be a SHS on a closed 3-manifold M. By [10, Theorem 4.6], there exists an arbitrarily C^1-close stable Hamiltonian structure (λ', ω') such that in the contact regions we have $d\lambda' = c\omega'$ for some constant

$c \in \mathbb{R}$, and which is further contact non-degenerate. We can perturb the SHS by a perturbation of the form

$$(\lambda + \eta, c(\omega + d\eta)),$$

where η is a one-form that is compactly supported in the interior of the contact region that makes the stable Hamiltonian structure contact strongly non-degenerate [14, 39, 37]. This is possible because close to the boundary of each connected component of N the flow is just integrable of constant irrational slope. By Lemma 5.16, we can make another arbitrarily C^∞-small perturbation, compactly supported in the integrable region, such that the slope of the Reeb vector field in each connected component of U is non-constant. This shows that stable Hamiltonian structures in the hypotheses of Corollary 5.14 are C^1-dense in the space of stable Hamiltonian structures.

Finally, we want to show that given any SHS (λ, ω) satisfying the hypotheses of Corollary 5.14, there is C^2-open neighborhood of stable Hamiltonian structures around it that also admit a Birkhoff section. We argue exactly as in the last step of the proof of Theorem 1.2. The Birkhoff section obtained in Corollary 5.14 is such that in the contact regions it is δ-strong and the binding components are non-degenerate periodic orbits by Remark 5.15. Then [13, Proposition 5.1] shows that any Reeb vector field sufficiently C^1-close to the Reeb vector field of a strongly non-degenerate contact form also admits a Birkhoff section. Hence the Reeb vector field of $(\widetilde{\lambda}, \widetilde{\omega})$ admits a Birkhoff section in the contact region. The slope in the integrable region is non-constant in each connected component, so by Remark 5.13 and Theorem 1.2 we can still construct a Birkhoff section for the Reeb vector field of $(\widetilde{\lambda}, \widetilde{\omega})$. □

Remark 5.17. In [9], it was shown that any SHS is exact stable homotopic (through an homotopy that is not small in any C^k-topology) to one which is supported by an open book decomposition, which is equivalent to the Reeb vector field admitting a global surface of section. Given a SHS (λ, ω), the C^1-perturbation in the proof above can be done through an exact stable homotopy (as per Lemma 5.16 and [10, Theorem 4.6]). Hence, there is C^1-small exact stable homotopy $(\lambda_t, \omega + d\mu_t)$ such that $\lambda_0 = 0$, $\mu_0 = 0$ and such that the Reeb vector field of $(\lambda_1, \omega + d\mu_1)$ admits a Birkhoff section. Equivalently, this means that $(\lambda_1, \omega + d\mu_1)$ is supported by a rational open book decomposition.

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