Plurisubharmonic geodesics in non-Archimedean geometry.

Rémi REBOULET

Institut Fourier, Université Grenoble-Alpes

February 8, 2021

Why study non-Archimedean geometry?

A general heuristic.

Non-Archimedean geometry encodes asymptotic phenomena in complex geometry. Examples:

Why study non-Archimedean geometry?

A general heuristic.

Non-Archimedean geometry encodes asymptotic phenomena in complex geometry. Examples:

- degeneration of complex manifolds $\to$ scheme over $\mathbb{C}((t))$

Why study non-Archimedean geometry?

A general heuristic.

Non-Archimedean geometry encodes asymptotic phenomena in complex geometry. Examples:

- degeneration of complex manifolds $\to$ scheme over $\mathbb{C}((t))$
- psh geodesic ray $\to$ trivially valued non-Archimedean psh metric.

Why study non-Archimedean geometry?

A general heuristic.

Non-Archimedean geometry encodes asymptotic phenomena in complex geometry. Examples:

- degeneration of complex manifolds $\to$ scheme over $\mathbb{C}((t))$
- psh geodesic ray $\to$ trivially valued non-Archimedean psh metric.

Recent applications.

$\to$ can study the limit behaviour of the K-energy along a ray directly on a non-Archimedean space:

Why study non-Archimedean geometry?

A general heuristic.

Non-Archimedean geometry encodes asymptotic phenomena in complex geometry. Examples:

- degeneration of complex manifolds $\to$ scheme over $\mathbb{C}((t))$
- psh geodesic ray $\to$ trivially valued non-Archimedean psh metric.

Recent applications.

$\to$ can study the limit behaviour of the K-energy along a ray directly on a non-Archimedean space:

$\Rightarrow$ variational proof of the YTD conjecture (Berman-Boucksom-Jonsson, C. Li, ...), K-moduli...

Why study non-Archimedean geometry?

A general heuristic.

Non-Archimedean geometry encodes asymptotic phenomena in complex geometry. Examples:

- degeneration of complex manifolds $\rightarrow$ scheme over $\mathbb{C}((t))$
- psh geodesic ray $\rightarrow$ trivially valued non-Archimedean psh metric.

Recent applications.

$\rightarrow$ can study the limit behaviour of the K-energy along a ray directly on a non-Archimedean space:

$\Rightarrow$ variational proof of the YTD conjecture (Berman-Boucksom-Jonsson, C. Li, ...), K-moduli...

$\rightarrow$ can understand how volume forms of Calabi-Yau varieties degenerate via non-Archimedean geometry:

Why study non-Archimedean geometry?

A general heuristic.

Non-Archimedean geometry encodes asymptotic phenomena in complex geometry. Examples:

- degeneration of complex manifolds $\rightarrow$ scheme over $\mathbb{C}((t))$
- psh geodesic ray $\rightarrow$ trivially valued non-Archimedean psh metric.

Recent applications.

$\rightarrow$ can study the limit behaviour of the K-energy along a ray directly on a non-Archimedean space:

$\Rightarrow$ variational proof of the YTD conjecture (Berman-Boucksom-Jonsson, C. Li, ...), K-moduli...

$\rightarrow$ can understand how volume forms of Calabi-Yau varieties degenerate via non-Archimedean geometry:

$\Rightarrow$ applications to mirror symmetry.

Why study non-Archimedean geometry?

A general heuristic.

Non-Archimedean geometry encodes asymptotic phenomena in complex geometry. Examples:

- degeneration of complex manifolds $\to$ scheme over $\mathbb{C}((t))$
- psh geodesic ray $\to$ trivially valued non-Archimedean psh metric.

Recent applications.

$\to$ can study the limit behaviour of the K-energy along a ray directly on a non-Archimedean space:

$\Rightarrow$ variational proof of the YTD conjecture (Berman-Boucksom-Jonsson, C. Li, ...), K-moduli...

$\to$ can understand how volume forms of Calabi-Yau varieties degenerate via non-Archimedean geometry:

$\Rightarrow$ applications to mirror symmetry.

$\to$ Arakelov geometry...

Non-Archimedean fields.

Complete valued field $(\mathbb{K}, |\cdot|)$ whose absolute value satisfies ultrametric inequality

$$|x + y| \leq \max(|x|, |y|)$$

for $x,\ y \in \mathfrak{K}$.

What is non-Archimedean geometry?

Non-Archimedean fields.

Complete valued field $(\mathbb{K}, |\cdot|)$ whose absolute value satisfies ultrametric inequality

$$|x+y| \leq \max(|x|, |y|)$$

for $x,\ y \in \mathfrak{K}$.

Examples.

1. Any field! (with $|\cdot|_0$)

What is non-Archimedean geometry?

Non-Archimedean fields.

Complete valued field $(\mathbb{K}, |\cdot|)$ whose absolute value satisfies ultrametric inequality

$$|x + y| \leq \max(|x|, |y|)$$

for x, $y \in \mathfrak{K}$.

Examples.

1. Any field! (with $|\cdot|_0$)
2. Fields of Laurent series (ex. $\mathbb{C}((t))$, $\mathbb{F}_p((t))$) with t-adic absolute value.

What is non-Archimedean geometry?

Non-Archimedean fields.

Complete valued field $(\mathbb{K}, |\cdot|)$ whose absolute value satisfies ultrametric inequality

$$|x + y| \leq \max(|x|, |y|)$$

for x, $y \in \mathfrak{K}$.

Examples.

1. Any field! (with $|\cdot|_0$)
2. Fields of Laurent series (ex. $\mathbb{C}((t))$, $\mathbb{F}_p((t))$) with t-adic absolute value.
3. $\mathbb{Q}_p$ with the p-adic absolute value, the completion of a closure $\mathbb{C}_p$...

Berkovich spaces.

GAGA principle:
X variety over a n-A field $(\mathbb{K}, |\cdot|) \leftrightarrow$ Berkovich analytic space X^{an}.

Berkovich spaces.

GAGA principle:
X variety over a n-A field $(\mathbb{K}, |\cdot|) \leftrightarrow$ Berkovich analytic space X^{an}.

$\rightarrow$ affine case $X = \operatorname{Spec} \mathcal{A}$, $\mathcal{A}$ a $\mathbb{K}$-algebra: $X^{\mathrm{an}} =$ multiplicative seminorms on $\mathcal{A}$ bounded by $|\cdot|$ on $\mathbb{K}$.

Berkovich spaces.

GAGA principle:
X variety over a n-A field $(\mathbb{K}, |\cdot|) \leftrightarrow$ Berkovich analytic space X^{an}.

$\rightarrow$ affine case $X = \mathrm{Spec}\ \mathcal{A}$, $\mathcal{A}$ a $\mathbb{K}$-algebra: $X^{\mathrm{an}} =$ multiplicative seminorms on $\mathcal{A}$ bounded by $|\cdot|$ on $\mathbb{K}$.

$\rightarrow X$ over $(\mathbb{C}, |\cdot|_0)$: $X^{\mathrm{an}} =$ space of valuations on function fields of subvarieties of X, trivial on $\mathbb{C}$.

What is non-Archimedean geometry?

Berkovich spaces.

GAGA principle:
X variety over a n-A field $(\mathbb{K}, |\cdot|) \leftrightarrow$ Berkovich analytic space X^{an}.

$\rightarrow$ affine case $X = \operatorname{Spec} \mathcal{A}$, $\mathcal{A}$ a $\mathbb{K}$-algebra: X^{an} = multiplicative seminorms on $\mathcal{A}$ bounded by $|\cdot|$ on $\mathbb{K}$.

$\rightarrow$ X over $(\mathbb{C}, |\cdot|_0)$: X^{an} = space of valuations on function fields of subvarieties of X, trivial on $\mathbb{C}$.

$\rightarrow$ preserves compactness, connectedness, Hausdorff-ness.

What is non-Archimedean geometry?

Berkovich spaces.

GAGA principle:
X variety over a n-A field $(\mathbb{K}, |\cdot|) \leftrightarrow$ Berkovich analytic space X^{an}.

$\rightarrow$ affine case $X = \operatorname{Spec} \mathcal{A}$, $\mathcal{A}$ a $\mathbb{K}$-algebra: X^{an} = multiplicative seminorms on $\mathcal{A}$ bounded by $|\cdot|$ on $\mathbb{K}$.

$\rightarrow$ X over $(\mathbb{C}, |\cdot|_0)$: X^{an} = space of valuations on function fields of subvarieties of X, trivial on $\mathbb{C}$.

$\rightarrow$ preserves compactness, connectedness, Hausdorff-ness.

The construction also works over Archimedean complete fields. In the case of $\mathbb{C}$: usual analytification of a complex variety.

The complex setup.

- X complex projective manifold, $L \to X$ ample line bundle.
- Plurisubharmonic metric on L = decreasing limit of smooth metrics on L with positive curvature form.

The complex setup.

- X complex projective manifold, $L \to X$ ample line bundle.
- Plurisubharmonic metric on L = decreasing limit of smooth metrics on L with positive curvature form.

(Not the usual definition... but the one that is easiest for us! Błocki-Kołodziej, Demailly.)

The complex setup.

- X complex projective manifold, $L \to X$ ample line bundle.
- Plurisubharmonic metric on L = decreasing limit of smooth metrics on L with positive curvature form.

(Not the usual definition... but the one that is easiest for us! Błocki-Kołodziej, Demailly.)

The non-Archimedean side?

- X projective variety over a non-Archimedean field $\mathbb{K}$, $L \to X$ ample.

The complex setup.

- X complex projective manifold, $L \to X$ ample line bundle.
- Plurisubharmonic metric on L = decreasing limit of smooth metrics on L with positive curvature form.

(Not the usual definition... but the one that is easiest for us! Błocki-Kołodziej, Demailly.)

The non-Archimedean side?

- X projective variety over a non-Archimedean field $\mathbb{K}$, $L \to X$ ample.
- $L^{\mathrm{an}} \to X^{\mathrm{an}}$ their analytifications.

The complex setup.

- X complex projective manifold, $L \to X$ ample line bundle.
- Plurisubharmonic metric on L = decreasing limit of smooth metrics on L with positive curvature form.

(Not the usual definition... but the one that is easiest for us! Błocki-Kołodziej, Demailly.)

The non-Archimedean side?

- X projective variety over a non-Archimedean field $\mathbb{K}$, $L \to X$ ample.
- $L^{\mathrm{an}} \to X^{\mathrm{an}}$ their analytifications.

... we need to find a "good" class of metrics on L^{an} and take their decreasing limits.

Fubini-Study metrics.

- s (pluri)section of $L \to$ singular metric $\log|s|$ on L^{an}.

Fubini-Study metrics.

- s (pluri)section of $L \to$ singular metric $\log|s|$ on L^{an}.

Trivially valued case: $\log|s|(\nu) = -\nu(s)$.

Fubini-Study metrics.

- s (pluri)section of $L \to$ singular metric $\log|s|$ on L^{an}.

Trivially valued case: $\log|s|(\nu) = -\nu(s)$.

Take now a basepoint-free basis $(s_j)_j$ of sections of mL.

$$\phi = m^{-1} \max_j (\log|s_j| + \lambda_j)$$

is a **Fubini-Study metric** on L^{an}.

Fubini-Study metrics.

- s (pluri)section of $L \to$ singular metric $\log|s|$ on L^{an}.

Trivially valued case: $\log|s|(\nu) = -\nu(s)$.

Take now a basepoint-free basis $(s_j)_j$ of sections of mL.

$$\phi = m^{-1} \max_j (\log|s_j| + \lambda_j)$$

is a **Fubini-Study metric** on L^{an}.

Compare with complex case:

$$\phi = m^{-1} \log \sum_j |s_j|^2 e^{2\lambda_j}.$$

Fubini-Study metrics.

- s (pluri)section of $L \to$ singular metric $\log|s|$ on L^{an}.

Trivially valued case: $\log|s|(\nu) = -\nu(s)$.

Take now a basepoint-free basis $(s_j)_j$ of sections of mL.

$$\phi = m^{-1} \max_j (\log|s_j| + \lambda_j)$$

is a **Fubini-Study metric** on L^{an}.

Compare with complex case:

$$\phi = m^{-1} \log \sum_j |s_j|^2 e^{2\lambda_j}.$$

Plurisubharmonic metric: decreasing limit (of a net) of FS metrics.

Theorem (Demailly ($\mathbb{C}$), Boucksom-Favre-Jonsson (n-A).)

The set of plurisubharmonic metrics is the smallest set containing all Fubini-Study metrics, stable under addition of constants, finite maxima, and decreasing limits.

Theorem (Demailly ($\mathbb{C}$), Boucksom-Favre-Jonsson (n-A).)

The set of plurisubharmonic metrics is the smallest set containing all Fubini-Study metrics, stable under addition of constants, finite maxima, and decreasing limits.

Can define Monge-Ampère measures

$$dd^c\phi_0 \wedge \cdots \wedge dd^c\phi_d$$

associated to non-Archimedean Fubini-Study (or even continuous psh) metrics.

Theorem (Demailly ($\mathbb{C}$), Boucksom-Favre-Jonsson (n-A).)

The set of plurisubharmonic metrics is the smallest set containing all Fubini-Study metrics, stable under addition of constants, finite maxima, and decreasing limits.

Can define Monge-Ampère measures

$$dd^c\phi_0 \wedge \cdots \wedge dd^c\phi_d$$

associated to non-Archimedean Fubini-Study (or even continuous psh) metrics.

$\rightarrow$ Chambert-Loir-Ducros approach (via "local tropicalizations"),
$\rightarrow$ Gubler approach via intersection theory.

The Monge-Ampère energy.

If ϕ_0, ϕ_1 are Fubini-Study metrics (complex or n-A!),

$$E(\phi_0, \phi_1) = \frac{1}{V(d+1)} \sum_{k=0}^{d} \int_X (\phi_0 - \phi_1)\,(dd^c\phi_0)^k \wedge (dd^c\phi_1)^{d-k},$$

$d = \dim X,\ V = \mathrm{vol}(L) = (L^d)$.

The Monge-Ampère energy.

If ϕ_0, ϕ_1 are Fubini-Study metrics (complex or n-A!),

$$E(\phi_0, \phi_1) = \frac{1}{V(d+1)} \sum_{k=0}^{d} \int_X (\phi_0 - \phi_1) (dd^c \phi_0)^k \wedge (dd^c \phi_1)^{d-k},$$

$d = \dim X,\ V = \mathrm{vol}(L) = (L^d)$.

$\rightarrow$ higher-dimensional analogue of the energy in classical complex potential theory;

The Monge-Ampère energy.

If ϕ_0, ϕ_1 are Fubini-Study metrics (complex or n-A!),

$$E(\phi_0, \phi_1) = \frac{1}{V(d+1)} \sum_{k=0}^{d} \int_X (\phi_0 - \phi_1) \, (dd^c \phi_0)^k \wedge (dd^c \phi_1)^{d-k},$$

$d = \dim X, \; V = \mathrm{vol}(L) = (L^d)$.

$\rightarrow$ higher-dimensional analogue of the energy in classical complex potential theory;
$\rightarrow$ primitive of the Monge-Ampère operator (in complex and n-A case).

Finite-energy spaces.

Fixing a metric ϕ_{ref},

$$\phi \mapsto E(\phi) := E(\phi, \phi_{\mathrm{ref}})$$

admits an extension to $\mathrm{PSH}(L)$:

Finite-energy spaces.

Fixing a metric ϕ_{ref},

$$\phi \mapsto E(\phi) := E(\phi, \phi_{\mathrm{ref}})$$

admits an extension to $\mathrm{PSH}(L)$: taking a net of FS metrics $\phi_k \searrow \phi$,

$$E(\phi) = \lim_k E(\phi_k).$$

Can be $-\infty$!

Finite-energy spaces.

Fixing a metric ϕ_{ref},

$$\phi \mapsto E(\phi) := E(\phi, \phi_{\mathrm{ref}})$$

admits an extension to $\mathrm{PSH}(L)$: taking a net of FS metrics $\phi_k \searrow \phi$,

$$E(\phi) = \lim_k E(\phi_k).$$

Can be $-\infty$!
We then define the space of finite-energy metrics

$$\mathcal{E}^1(L) = \{\phi \in \mathrm{PSH}(L),\ E(\phi) > -\infty\}.$$

Finite-energy spaces.

Fixing a metric ϕ_{ref},

$$\phi \mapsto E(\phi) := E(\phi, \phi_{\mathrm{ref}})$$

admits an extension to $\mathrm{PSH}(L)$: taking a net of FS metrics $\phi_k \searrow \phi$,

$$E(\phi) = \lim_k E(\phi_k).$$

Can be $-\infty$!
We then define the space of finite-energy metrics

$$\mathcal{E}^1(L) = \{\phi \in \mathrm{PSH}(L),\ E(\phi) > -\infty\}.$$

Thanks to the cocycle formula

$$E(\phi_0, \phi_1) = E(\phi_0, \phi_2) + E(\phi_2, \phi_1),$$

this space is independent of the reference metric!

Metric structures and geodesics in the complex setting.

• Mabuchi '80s: Riemannian (d_2) structure on the space of smooth psh metrics.

Metric structures and geodesics in the complex setting.

• Mabuchi '80s: Riemannian (d_2) structure on the space of smooth psh metrics.

• Semmes, Donaldson '90s: geodesic equation for $d_2 \leftrightarrow$ complex Monge-Ampère equation.

Metric structures and geodesics in the complex setting.

• Mabuchi '80s: Riemannian (d_2) structure on the space of smooth psh metrics.

• Semmes, Donaldson '90s: geodesic equation for $d_2 \leftrightarrow$ complex Monge-Ampère equation.

• Chen '00: existence of $C^{1,1}$ geodesics.

Metric structures and geodesics in the complex setting.

• Mabuchi '80s: Riemannian (d_2) structure on the space of smooth psh metrics.

• Semmes, Donaldson '90s: geodesic equation for $d_2 \leftrightarrow$ complex Monge-Ampère equation.

• Chen '00: existence of $C^{1,1}$ geodesics.

• Darvas: d_p geodesic metric structures on $\mathcal{E}^p$ spaces.

Metric structures and geodesics in the complex setting.

• Mabuchi '80s: Riemannian (d_2) structure on the space of smooth psh metrics.

• Semmes, Donaldson '90s: geodesic equation for $d_2 \leftrightarrow$ complex Monge-Ampère equation.

• Chen '00: existence of $C^{1,1}$ geodesics.

• Darvas: d_p geodesic metric structures on $\mathcal{E}^p$ spaces.

Here, we will focus on the d_1 case.

Theorem (Darvas)

Given two finite-energy metrics ϕ_0, $\phi_1 \in \mathcal{E}^1(L)$, set

$$P(\phi_0, \phi_1) = \sup\{\phi \in \mathrm{PSH}(L), \phi \leq \min(\phi_0, \phi_1)\}.$$

Theorem (Darvas)

Given two finite-energy metrics ϕ_0, $\phi_1 \in \mathcal{E}^1(L)$, set

$$P(\phi_0, \phi_1) = \sup\{\phi \in \mathrm{PSH}(L), \phi \leq \min(\phi_0, \phi_1)\}.$$

Then

$$d_1(\phi_0, \phi_1) = E(\phi_0, P(\phi_0, \phi_1)) + E(\phi_1, P(\phi_0, \phi_1))$$

is a distance on $\mathcal{E}^1(L)$.

Theorem (Darvas)

Given two finite-energy metrics ϕ_0, $\phi_1 \in \mathcal{E}^1(L)$, set

$$P(\phi_0, \phi_1) = \sup\{\phi \in \mathrm{PSH}(L), \phi \leq \min(\phi_0, \phi_1)\}.$$

Then

$$d_1(\phi_0, \phi_1) = E(\phi_0, P(\phi_0, \phi_1)) + E(\phi_1, P(\phi_0, \phi_1))$$

is a distance on $\mathcal{E}^1(L)$.

Theorem (Boucksom-Jonsson '18, R. '20)

Let L be an ample line bundle over a non-Archimedean variety X. Given two finite-energy metrics ϕ_0, $\phi_1 \in \mathcal{E}^1(L^{\mathrm{an}})$, set

$$P(\phi_0, \phi_1) = \sup\{\phi \in \mathrm{PSH}(L^{\mathrm{an}}), \phi \leq \min(\phi_0, \phi_1)\}.$$

Theorem (Darvas)

Given two finite-energy metrics ϕ_0, $\phi_1 \in \mathcal{E}^1(L)$, *set*

$$P(\phi_0, \phi_1) = \sup\{\phi \in \mathrm{PSH}(L), \phi \le \min(\phi_0, \phi_1)\}.$$

Then

$$d_1(\phi_0, \phi_1) = E(\phi_0, P(\phi_0, \phi_1)) + E(\phi_1, P(\phi_0, \phi_1))$$

is a distance on $\mathcal{E}^1(L)$.

Theorem (Boucksom-Jonsson '18, R. '20)

Let L *be an ample line bundle over a non-Archimedean variety* X. *Given two finite-energy metrics* ϕ_0, $\phi_1 \in \mathcal{E}^1(L^{\mathrm{an}})$, *set*

$$P(\phi_0, \phi_1) = \sup\{\phi \in \mathrm{PSH}(L^{\mathrm{an}}), \phi \le \min(\phi_0, \phi_1)\}.$$

Then

$$d_1(\phi_0, \phi_1) = E(\phi_0, P(\phi_0, \phi_1)) + E(\phi_1, P(\phi_0, \phi_1))$$

is a distance on $\mathcal{E}^1(L^{\mathrm{an}})$.

What about geodesics?

What about geodesics?

Complex case.

In the complex case, the maximal psh segments

$$\phi_t = \sup\{\text{psh segments joining } \phi_0, \phi_1\}$$

are d_1 geodesics.
("psh segment" understood as an $\mathbb{S}^1$-invariant metric psh on product of X with an annulus).

What about geodesics?

Complex case.

In the complex case, the maximal psh segments

$$\phi_t = \sup\{\text{psh segments joining } \phi_0, \phi_1\}$$

are d_1 geodesics.
("psh segment" understood as an $\mathbb{S}^1$-invariant metric psh on product of X with an annulus). If smooth endpoints, $C^{1,1}$ by Chen.

What about geodesics?

Complex case.

In the complex case, the maximal psh segments

$$\phi_t = \sup\{\text{psh segments joining } \phi_0, \phi_1\}$$

are d_1 geodesics.
("psh segment" understood as an $\mathbb{S}^1$-invariant metric psh on product of X with an annulus). If smooth endpoints, $C^{1,1}$ by Chen. Remains in $\mathcal{E}^1(L)$ if the endpoints have finite energy by Darvas.

Our goal: obtain similar results for non-Archimedean finite-energy spaces.

Psh segments.

Define the class of Fubini-Study segments as maps $[0,1] \to \mathrm{PSH}(L^{\mathrm{an}})$ of the form

$$\phi_t = m^{-1}(\max_j \log|s_j| + (1-t)\lambda_j + t\lambda'_j)$$

with $(s_j)_j$ basepoint-free basis of $H^0(mL)$.

Psh segments.

Define the class of Fubini-Study segments as maps $[0,1] \to \mathrm{PSH}(L^{\mathrm{an}})$ of the form

$$\phi_t = m^{-1}(\max_j \log |s_j| + (1-t)\lambda_j + t\lambda'_j)$$

with $(s_j)_j$ basepoint-free basis of $H^0(mL)$.

Psh segments = decreasing limits of (nets of) FS segments.

Psh segments.

Define the class of Fubini-Study segments as maps $[0,1] \to \mathrm{PSH}(L^{\mathrm{an}})$ of the form

$$\phi_t = m^{-1}(\max_j \log |s_j| + (1-t)\lambda_j + t\lambda'_j)$$

with $(s_j)_j$ basepoint-free basis of $H^0(mL)$.

Psh segments = decreasing limits of (nets of) FS segments.

R. '20: it is the smallest class of segments containing all FS segments, stable under finite maxima, addition of constants, and decreasing limits.

Psh segments.

Define the class of Fubini-Study segments as maps $[0,1] \to \mathrm{PSH}(L^{\mathrm{an}})$ of the form

$$\phi_t = m^{-1}(\max_j \log |s_j| + (1-t)\lambda_j + t\lambda_j')$$

with $(s_j)_j$ basepoint-free basis of $H^0(mL)$.

Psh segments = decreasing limits of (nets of) FS segments.

R. '20: it is the smallest class of segments containing all FS segments, stable under finite maxima, addition of constants, and decreasing limits. (Compare with complex case, where this follows from the definition of a psh metric.)

Theorem (R. '20)

- *Given two finite-energy metrics* ϕ_0, $\phi_1 \in \mathcal{E}^1(L^{\mathrm{an}})$, *there exists a maximal segment*

$$\phi_t = \sup\{\textit{(non-Archimedean) psh segments joining } \phi_0, \phi_1\}$$

joining them, which is furthermore a geodesic for the d_1 *distance.*

Theorem (R. '20)

• *Given two finite-energy metrics* ϕ_0, $\phi_1 \in \mathcal{E}^1(L^{\mathrm{an}})$, *there exists a maximal segment*

$$\phi_t = \sup\{\textit{(non-Archimedean) psh segments joining } \phi_0, \phi_1\}$$

joining them, which is furthermore a geodesic for the d_1 *distance.*

• *If furthermore the endpoints are continuous, then* $(t, x) \mapsto \phi_t(x)$ *is continuous in both variables.*

Regarding the continuity statement.

To prove that, with continuous endpoints, the geodesic remains continuous, we use continuity of envelopes.

Conjecture: continuity of envelopes.

If ϕ_0 and ϕ_1 are two continuous psh metrics, then $P(\phi_0, \phi_1)$ is continuous.

Regarding the continuity statement.

To prove that, with continuous endpoints, the geodesic remains continuous, we use continuity of envelopes.

Conjecture: continuity of envelopes.

If ϕ_0 and ϕ_1 are two continuous psh metrics, then $P(\phi_0, \phi_1)$ is continuous.

Conjectured to be true in general for X projective normal over any n-A field, L ample.

Regarding the continuity statement.

To prove that, with continuous endpoints, the geodesic remains continuous, we use continuity of envelopes.

Conjecture: continuity of envelopes.

If ϕ_0 and ϕ_1 are two continuous psh metrics, then $P(\phi_0, \phi_1)$ is continuous.

Conjectured to be true in general for X projective normal over any n-A field, L ample.

Known when X is projective normal and L ample, and:

- $\mathbb{K}$ is $(\mathbb{C}, |\cdot|_0)$ or $\mathbb{C}((t))$ (Berman-Boucksom-Jonsson)

Regarding the continuity statement.

To prove that, with continuous endpoints, the geodesic remains continuous, we use continuity of envelopes.

Conjecture: continuity of envelopes.

If ϕ_0 and ϕ_1 are two continuous psh metrics, then $P(\phi_0, \phi_1)$ is continuous.

Conjectured to be true in general for X projective normal over any n-A field, L ample.

Known when X is projective normal and L ample, and:

- $\mathbb{K}$ is $(\mathbb{C}, |\cdot|_0)$ or $\mathbb{C}((t))$ (Berman-Boucksom-Jonsson) (more generally equicharacteristic 0 and discretely or trivally valued);

Regarding the continuity statement.

To prove that, with continuous endpoints, the geodesic remains continuous, we use continuity of envelopes.

Conjecture: continuity of envelopes.

If ϕ_0 and ϕ_1 are two continuous psh metrics, then $P(\phi_0, \phi_1)$ is continuous.

Conjectured to be true in general for X projective normal over any n-A field, L ample.

Known when X is projective normal and L ample, and:

- $\mathbb{K}$ is $(\mathbb{C}, |\cdot|_0)$ or $\mathbb{C}((t))$ (Berman-Boucksom-Jonsson) (more generally equicharacteristic 0 and discretely or trivally valued);
- X is a curve (Thuillier);

Regarding the continuity statement.

To prove that, with continuous endpoints, the geodesic remains continuous, we use continuity of envelopes.

Conjecture: continuity of envelopes.

If ϕ_0 and ϕ_1 are two continuous psh metrics, then $P(\phi_0, \phi_1)$ is continuous.

Conjectured to be true in general for X projective normal over any n-A field, L ample.

Known when X is projective normal and L ample, and:

- $\mathbb{K}$ is $(\mathbb{C}, |\cdot|_0)$ or $\mathbb{C}((t))$ (Berman-Boucksom-Jonsson) (more generally equicharacteristic 0 and discretely or trivally valued);
- X is a curve (Thuillier);
- some cases in equal characteristic p assuming resolution of singularities (Gubler-Jell-Künnemann-Martin).

The continuity statement follows from realizing this envelope as a Legendre transform of another envelope

$$\hat{\phi}_\tau = P(\phi_0, \phi_1 - \tau)$$

for real τ. (Recall continuity of envelopes!)

The continuity statement follows from realizing this envelope as a Legendre transform of another envelope

$$\hat{\phi}_\tau = P(\phi_0, \phi_1 - \tau)$$

for real τ. (Recall continuity of envelopes!)We then proceed to show that

$$\phi_t = \sup_{\tau \in \mathbb{R}} (t\tau + \hat{\phi}_\tau).$$

(Needs non-Archimedean Kiselman principle!)

The continuity statement follows from realizing this envelope as a Legendre transform of another envelope

$$\hat{\phi}_\tau = P(\phi_0, \phi_1 - \tau)$$

for real τ. (Recall continuity of envelopes!)We then proceed to show that

$$\phi_t = \sup_{\tau \in \mathbb{R}} (t\tau + \hat{\phi}_\tau).$$

(Needs non-Archimedean Kiselman principle!)

Not obvious how to show that

$$d_1(\phi_t, \phi_s) = |t - s| \cdot d_1(\phi_0, \phi_1)$$

from this definition...

The continuity statement follows from realizing this envelope as a Legendre transform of another envelope

$$\hat{\phi}_\tau = P(\phi_0, \phi_1 - \tau)$$

for real τ. (Recall continuity of envelopes!)We then proceed to show that

$$\phi_t = \sup_{\tau \in \mathbb{R}} (t\tau + \hat{\phi}_\tau).$$

(Needs non-Archimedean Kiselman principle!)

Not obvious how to show that

$$d_1(\phi_t, \phi_s) = |t - s| \cdot d_1(\phi_0, \phi_1)$$

from this definition...
We treat the continuous case (finite-energy case follows by approximation), which can be dealt with using "quantization" techniques.

The continuity statement follows from realizing this envelope as a Legendre transform of another envelope

$$\hat{\phi}_\tau = P(\phi_0, \phi_1 - \tau)$$

for real τ. (Recall continuity of envelopes!)We then proceed to show that

$$\phi_t = \sup_{\tau \in \mathbb{R}} (t\tau + \hat{\phi}_\tau).$$

(Needs non-Archimedean Kiselman principle!)

Not obvious how to show that

$$d_1(\phi_t, \phi_s) = |t - s| \cdot d_1(\phi_0, \phi_1)$$

from this definition...
We treat the continuous case (finite-energy case follows by approximation), which can be dealt with using "quantization" techniques. (Compare with Berndtsson, Phong-Sturm, Darvas-Lu-Rubinstein...)

Quantization from below.

$\rightarrow$ for all k positive integers, ϕ_0 and ϕ_1 define (sup)norms $N_k(\phi_0)$, $N_k(\phi_1)$ on the space $H^0(kL)$ via

$$N_k(\phi)(s) = \sup_X |s|_{k\phi}$$

for $s \in H^0(kL)$.

Quantization from below.

→ for all k positive integers, ϕ_0 and ϕ_1 define (sup)norms $N_k(\phi_0)$, $N_k(\phi_1)$ on the space $H^0(kL)$ via

$$N_k(\phi)(s) = \sup_X |s|_{k\phi}$$

for $s \in H^0(kL)$.

→ those spaces of norms have distinguished geodesics for the $d_{1,k}$ distance (a distance on spaces of norms closely related to the one on psh metrics).

Quantization from below.

→ for all k positive integers, ϕ_0 and ϕ_1 define (sup)norms $N_k(\phi_0)$, $N_k(\phi_1)$ on the space $H^0(kL)$ via

$$N_k(\phi)(s) = \sup_X |s|_{k\phi}$$

for $s \in H^0(kL)$.

→ those spaces of norms have distinguished geodesics for the $d_{1,k}$ distance (a distance on spaces of norms closely related to the one on psh metrics). Let $t \mapsto \chi_{t,k}$ be the distinguished geodesic joining $N_k(\phi_0)$ and $N_k(\phi_1)$.

Quantization from below.

→ for all k positive integers, ϕ_0 and ϕ_1 define (sup)norms $N_k(\phi_0)$, $N_k(\phi_1)$ on the space $H^0(kL)$ via

$$N_k(\phi)(s) = \sup_X |s|_{k\phi}$$

for $s \in H^0(kL)$.

→ those spaces of norms have distinguished geodesics for the $d_{1,k}$ distance (a distance on spaces of norms closely related to the one on psh metrics). Let $t \mapsto \chi_{t,k}$ be the distinguished geodesic joining $N_k(\phi_0)$ and $N_k(\phi_1)$. Warning:

$$N_k(\phi_t) \neq \chi_{t,k}$$

for all $t \in (0,1)$ (but only "asymptotically").

Quantization from below.

$\rightarrow$ for all k positive integers, ϕ_0 and ϕ_1 define (sup)norms $N_k(\phi_0)$, $N_k(\phi_1)$ on the space $H^0(kL)$ via

$$N_k(\phi)(s) = \sup_X |s|_{k\phi}$$

for $s \in H^0(kL)$.

$\rightarrow$ those spaces of norms have distinguished geodesics for the $d_{1,k}$ distance (a distance on spaces of norms closely related to the one on psh metrics). Let $t \mapsto \chi_{t,k}$ be the distinguished geodesic joining $N_k(\phi_0)$ and $N_k(\phi_1)$. Warning:

$$N_k(\phi_t) \neq \chi_{t,k}$$

for all $t \in (0,1)$ (but only "asymptotically").

$\rightarrow$ building on previous work (R '20),

$$\begin{aligned} d_1(\phi_t, \phi_s) = \lim_k d_{1,k}(\chi_{t,k}, \chi_{s,k}) &= |t-s| \lim_k d_{1,k}(\chi_{0,k}, \chi_{1,k}) \\ &= |t-s| \cdot d_1(\phi_0, \phi_1). \end{aligned}$$

Thank you for your attention !