Some remarks on area-preserving actions of lattices

Pierre Py

April 2008

Abstract

In the spirit of Zimmer's study of large groups actions on closed manifolds, we discuss the existence of area-preserving actions of higher rank lattices on surfaces. We explain why certain vanishing results in bounded cohomology, due to Burger and Monod, combined with some constructions by Gambaudo and Ghys, give some constraints on the measure theoretical properties of such actions.

1 Introduction

The aim of this note is to discuss the existence of area-preserving actions of higher rank lattices on surfaces. All the diffeomorphisms and vector fields we will consider in the text will be of class C^∞. Although much of our discussion would be valid with a lower regularity, we will not be concerned with this issue here.

We consider a lattice Γ in a connected simple Lie group G, with finite center and of real rank greater than 1. See [5] for some classical constructions of lattices. Let Σ be a closed oriented surface endowed with an area form ω. According to a well-known conjecture of Zimmer [48], any homomorphism $\rho : \Gamma \to \mathrm{Diff}(\Sigma, \omega)$ from Γ to the group of area-preserving diffeomorphisms of Σ should have finite image. Note that this conjecture can be seen as part of a more general program proposed by Zimmer [48] for the study of (not necessarily volume preserving) actions of higher rank lattices on closed manifolds. For recent advances on this program, the reader might consult [7, 10, 11, 16, 22, 23, 26, 27, 37, 38, 46, 47, 49].

From now on, we will assume that a homomorphism $\rho : \Gamma \to \mathrm{Diff}(\Sigma, \omega)$ is given. We will say that the associated action of Γ on Σ is trivial if $\rho(\Gamma)$ is a finite group. We will also denote by μ the measure associated to the 2-form ω on Σ, and will suppose that $\int_\Sigma d\mu = 1$.

Although the conjecture is still open, a certain number of results already show that such an action of Γ have poor dynamical properties. For instance, Zimmer [47, 49] showed that in this situation Γ must preserve a measurable Riemannian metric on Σ. As a consequence, all the elements in the group $\rho(\Gamma)$ have zero metric entropy. Another consequence is that the action has discrete spectrum [49]: the space $\mathrm{L}^2(\Sigma, \mu)$ breaks down as a direct sum of finite-dimensional subrepresentations for the natural Γ-action. One approach to the conjecture (explained in [18] for instance) would be to study the regularity of the invariant Riemannian metric provided by Zimmer's theorem. Here we will follow a different route and give a few more evidences that such an action of Γ, if not trivial, has simple dynamical properties (at least from the measure theoretical point of view).

As we have just said, a consequence of Zimmer's results is that the group $\rho(\Gamma)$ does not contain any area-preserving diffeomorphism with positive metric entropy for the measure μ. On the other side of the spectrum of possible dynamical behaviours are area-preserving diffeomorphisms which are contained in a Hamiltonian flow. A consequence of our discussion will be that this kind of diffeomorphisms cannot appear in the group $\rho(\Gamma)$ (see section 3).

Observe first that we can compose ρ with the projection π from $\mathrm{Diff}(\Sigma, \omega)$ to the mapping class group of Σ, denoted by $\Lambda(\Sigma)$, to get a homomorphism $\pi \circ \rho : \Gamma \to \Lambda(\Sigma)$. According to a result of Farb and Masur [15], such a homomorphism has finite image. Hence, up to replacing Γ by a subgroup of finite index, we can assume that the image of ρ lies in the group $\mathrm{Diff}_0(\Sigma, \omega)$ of area-preserving diffeomorphisms of Σ which are isotopic to the identity. In fact, we can do one more reduction and assume that the image of ρ lies in the group of *Hamiltonian diffeomorphisms* of Σ. We recall its definition now.

Rotation vectors and Hamiltonian diffeomorphisms. There is a canonical homomorphism a from the group $\mathrm{Diff}_0(\Sigma, \omega)$ to an abelian group A (which depends on Σ). Its kernel is the group $\mathrm{Ham}(\Sigma, \omega)$ of Hamiltonian diffeomorphisms of Σ. When Σ is the 2-sphere, the group A is trivial and the groups $\mathrm{Diff}_0(\mathbf{S}^2, \omega)$ and $\mathrm{Ham}(\mathbf{S}^2, \omega)$ coincide (in that case, these two groups also coincide with the whole group of area-preserving diffeomorphisms of the sphere, which is connected [43]). When $\Sigma = \mathbf{T}^2$ is the torus, the group A equals $\mathbf{R}^2/\mathbf{Z}^2$ and the homomorphism a is defined as follows. For each diffeomorphism $f \in \mathrm{Diff}_0(\mathbf{T}^2, \omega)$, choose a lift $F : \mathbf{R}^2 \to \mathbf{R}^2$ of f. Since f is isotopic to the identity, the map F commutes with integral translations. Hence, the map $x \mapsto F(x) - x \in \mathbf{R}^2$ is invariant under integral translations and defines a map from $\mathbf{T}^2$ to $\mathbf{R}^2$. One defines:

$$a(f) = \int_{\mathbf{T}^2} (F(x) - x) d\mu(x) \bmod \mathbf{Z}^2.$$

As suggested by the notation, $a(f) \in \mathbf{T}^2$ only depends on f: any lift of f differs from F by a translation by an element of $\mathbf{Z}^2$, hence the integral above is well defined modulo $\mathbf{Z}^2$. It is often called the *rotation vector* of the diffeomorphism f. The map

$$a : \mathrm{Diff}_0(\mathbf{T}^2, \omega) \to \mathbf{R}^2/\mathbf{Z}^2$$

is a homomorphism. When Σ has genus greater than 1, the group A equals the first homology group $H_1(\Sigma, \mathbf{R})$ of Σ and one can construct the homomorphism a in the same spirit as above, see [20, 33]. Note that, as opposed to the case of the torus, we do not need to divide by the group $H_1(\Sigma, \mathbf{Z})$ since the group $\mathrm{Diff}_0(\Sigma, \omega)$ is simply connected [12]. The homomorphism $a : \mathrm{Diff}_0(\Sigma, \omega) \to H_1(\Sigma, \mathbf{R})$ is dual to the classical *flux homomorphism* from $\mathrm{Diff}_0(\Sigma, \omega)$ to $H^1(\Sigma, \mathbf{R})$, see for instance [35] and section 2 of [21].

It can be shown that the group $\mathrm{Ham}(\Sigma, \omega)$ is exactly the group of diffeomorphisms which are time 1 maps of a *Hamiltonian isotopy* $(f_t)_{t \in [0,1]}$. Recall that this means that the isotopy $(f_t)_{t \in [0,1]}$ satisfies the following differential equation

$$\begin{cases} f_0(x) = x \\ \frac{d}{dt}(f_t(x)) = X_t(f_t(x)), \end{cases}$$

for all $x \in \Sigma$, for a time-dependent vector field X_t which is the *symplectic gradient* of a smooth time-dependent function H_t on Σ. This means by definition that X_t satisfies the relation:

$$dH_t(u) = \omega(X_t, u)$$

for any tangent vector $u \in \mathrm{T}\Sigma$.

Coming back to our lattice, we see that the homomorphism $a \circ \rho$ has finite image, Γ having Kazhdan's property (T) (see [31]). Hence, up to considering once again a subgroup of finite index, one can assume that the homomorphism ρ has its image contained in the group of Hamiltonian diffeomorphisms of Σ. *The study of area-preserving actions of higher rank lattices on closed surfaces is thus reduced to the study of actions by Hamiltonian diffeomorphisms.*

Let us mention that when Γ is a *non-uniform* lattice, and Σ has positive genus, Zimmer's conjecture has been proved by Polterovich [38], using methods from symplectic topology. His results have been reproved by Franks and Handel [22, 23] (and extended to the case where $\Sigma = \mathbf{S}^2$ for a particular class of non-uniform lattices) using methods from 2-dimensional dynamics. Note that one of the central theorems established by Polterovich in [38] to study actions of lattices is the following (in the statement below, we assume that Σ has genus at least 2).

If Λ is a finitely generated subgroup of $\mathrm{Ham}(\Sigma, \omega)$ endowed with any word metric $|\cdot|$, and if $\gamma \in \Lambda$ is different from the identity, there exists $\varepsilon > 0$ such that $|\gamma^n| \geq \varepsilon n$ ($n \in \mathbf{N}$).

This allows one to exclude the existence of non-trivial actions of non-uniform lattices thanks to the presence of unipotent elements, see [34]. But it also has different applications: for instance, any finitely generated nilpotent subgroup of $\mathrm{Ham}(\Sigma, \omega)$ is in fact Abelian.

Quasi-morphisms and bounded cohomology. Recall that a *quasi-morphism* on a group Λ is a map $\phi : \Lambda \to \mathbf{R}$ for which there exists a constant $C > 0$ such that:

$$|\phi(xy) - \phi(x) - \phi(y)| \leq C$$

for any $x, y \in \Lambda$. The quasi-morphism ϕ is called *homogeneous* if moreover it satisfies $\phi(x^n) = n\phi(x)$ ($n \in \mathbf{Z}$, $x \in \Lambda$) i.e. ϕ is a true homomorphism when restricted to cyclic subgroups of Λ. For any quasi-morphism ϕ, the limit

$$\phi_h(x) = \lim_{n\to\infty} \frac{\phi(x^n)}{n}$$

exists. The map ϕ_h is the unique homogeneous quasi-morphism such that $\phi - \phi_h$ is bounded. We will denote by $\mathrm{QM}(\Lambda, \mathbf{R})$ the vector space of all quasi-morphisms on Λ, and by $\mathrm{QM}_h(\Lambda, \mathbf{R})$ the subspace of homogeneous quasi-morphisms. For more details on quasi-morphisms and bounded cohomology see [4, 6] as well as Gromov's seminal paper [30]. We will return to the link between quasi-morphisms and bounded cohomology in the next section.

According to a result by Burger and Monod (see [7] or [8]), any homogeneous quasi-morphism on a higher rank lattice Γ as above is a homomorphism, and hence is identically zero since Γ is a Kazhdan group. On the other hand, the group of Hamiltonian diffeomorphisms of a closed surface (which is simple and hence admits no non-trivial homomorphism to $\mathbf{R}$ [3]) admits many quasi-morphisms: Gambaudo and Ghys [25] proved that the vector space $\mathrm{QM}_h(\mathrm{Ham}(\Sigma, \omega), \mathbf{R})$ is infinite-dimensional. Their proof consists in constructing explicitly a family of linearly independent homogeneous quasi-morphisms. More constructions of quasi-morphisms on groups of Hamiltonian diffeomorphisms can be found in the work of Entov and Polterovich [13] and of the author [40]. It is now well-known (see [29]) that one could try to use this contrast between the lattice Γ and groups of Hamiltonian diffeomorphisms of surfaces to attack the problem of the existence of non-trivial homomorphisms $\rho : \Gamma \to \mathrm{Ham}(\Sigma, \omega)$. According to the discussion above, for any homogeneous quasi-morphism $\phi : \mathrm{Ham}(\Sigma, \omega) \to \mathbf{R}$ the invariant $\phi \circ \rho : \Gamma \to \mathbf{R}$ vanishes identically. This should give some constraints on the homomorphism ρ. This circle of ideas is nicely discussed in Ghys' survey [29].

Here, we will explain how one can use more subtly the vanishing results of Burger and Monod in bounded cohomology to obtain much more precise constraints on the homomorphism ρ. Namely, we will use the vanishing of certain bounded cohomology groups with unitary coefficients.

Note that the use of bounded cohomology for the study of group actions is not new. Bounded cohomology already appears in the study of groups acting on the circle, see [7, 28].

2 Vanishing results and consequences

We begin with a brief reminder on the *second bounded cohomology group* of a discrete group Λ, with unitary coefficients (see for instance [36] for more details).

We consider a unitary representation π of Λ, i.e. a homomorphism from Λ to the group of unitary operators of a Hilbert space $\mathscr{H}$. We will write $||v||$ for the norm of a vector $v \in \mathscr{H}$. All the unitary representations we will consider are obtained in the following way: the group Λ is acting by measure preserving transformations on a probability space (X, μ) and we consider the representation π of Λ on the Hilbert space $\mathscr{H} = \mathrm{L}^2(X, \mu)$ of square integrable functions on X, defined by $\pi(\gamma)(f) = f \circ \gamma^{-1}$ ($f \in \mathrm{L}^2(X, \mu)$). We will say that a map $c : \Lambda^j \to \mathscr{H}$ is bounded if the quantity

$$|c|_{\infty, \Lambda^j} := \sup_{(\gamma_1, \ldots, \gamma_j) \in \Lambda^j} ||c(\gamma_1, \ldots, \gamma_j)||$$

is finite. The space $Z^2(\Lambda, \pi)$ of 2-cocycles on Λ with values in $\mathscr{H}$ is the space of maps $c : \Lambda^2 \to \mathscr{H}$ such that:

$$\pi(\gamma_1)(c(\gamma_2, \gamma_3)) - c(\gamma_1\gamma_2, \gamma_3) + c(\gamma_1, \gamma_2\gamma_3) - c(\gamma_1, \gamma_2) = 0.$$

We will denote by $Z_b^2(\Lambda, \pi) \subset Z^2(\Lambda, \pi)$ the subspace of bounded 2-cocycles. In the same way, the space $B^1(\Lambda, \pi) \subset Z^2(\Lambda, \pi)$ of coboundaries consists of the maps $c : \Lambda^2 \to \mathscr{H}$ which satisfy the equation $c(\gamma_1, \gamma_2) = \pi(\gamma_1)(v(\gamma_2)) + v(\gamma_1) - v(\gamma_1\gamma_2)$ for some map $v : \Lambda \to \mathscr{H}$. The subspace of coboundaries c for which the map v in the equation above can be chosen bounded will be denoted by $B_b^1(\Lambda, \pi)$. The second bounded cohomology group of Λ with coefficients in $\mathscr{H}$ is the quotient

$$H_b^2(\Lambda, \pi) = Z_b^2(\Lambda, \pi)/B_b^1(\Lambda, \pi),$$

and the second usual cohomology group of Λ with coefficients in $\mathscr{H}$ is the quotient $H^2(\Lambda, \pi) = Z^2(\Lambda, \pi)/B^1(\Lambda, \pi)$. Of course, there is a natural map from $H_b^2(\Lambda, \pi)$ to

$H^2(\Lambda, \pi)$ which sends the class of a bounded cocycle to its usual cohomology class. We will denote by $EH_b^2(\Lambda, \pi)$ the kernel of this map.

In the following proposition $\mathscr{C}(\Lambda, \mathscr{H})$ is the space of all maps from Λ to $\mathscr{H}$, $\mathscr{C}_b(\Lambda, \mathscr{H})$ the subspace of those maps which are bounded and $\mathfrak{d}^1 u$ is the coboundary of a map $u \in \mathscr{C}(\Lambda, \mathscr{H})$:

$$\mathfrak{d}^1 u(\gamma_1, \gamma_2) = \pi(\gamma_1)(u(\gamma_2)) + u(\gamma_1) - u(\gamma_1\gamma_2).$$

Proposition 2.1 *The space $EH_b^2(\Lambda, \pi)$ is isomorphic to :*

$$\{u \in \mathscr{C}(\Lambda, \mathscr{H}), |\mathfrak{d}^1 u|_{\infty,\Lambda^2} < \infty\} / \left(Z^1(\Lambda, \pi) + \mathscr{C}_b(\Lambda, \mathscr{H})\right).$$

Proof. If $u \in \mathscr{C}(\Lambda, \mathscr{H})$ is such that $|\mathfrak{d}^1 u|_{\infty,\Lambda^2} < \infty$, the coboundary $\mathfrak{d}^1 u$ of u is a bounded 2-cocycle, hence defines a class $[\mathfrak{d}^1 u] \in H_b^2(\Lambda, \pi)$ which is obviously trivial in usual cohomology. We have thus constructed a map $u \mapsto [\mathfrak{d}^1 u]$ from $\{u \in \mathscr{C}(\Lambda, \mathscr{H}), |\mathfrak{d}^1 u|_{\infty,\Lambda^2} < \infty\}$ to $EH_b^2(\Lambda, \pi)$, which is (by definition) surjective. Let us examine its kernel. The class $[\mathfrak{d}^1 u]$ vanishes in bounded cohomology precisely if there exists a bounded map $v : \Lambda \to \mathscr{H}$ such that $\mathfrak{d}^1 u = \mathfrak{d}^1 v$. In that case $w = u - v$ is a 1-cocycle and we have $u = w + v \in Z^1(\Lambda, \pi) + \mathscr{C}_b(\Lambda, \mathscr{H})$. □

When $\mathscr{H} = \mathbf{R}$, with the trivial representation π_0 of Λ, the previous proposition simply proves the following classical fact [4, 6]: the group $EH_b^2(\Lambda, \pi_0)$ is isomorphic to the quotient

$$\mathrm{QM}(\Lambda, \mathbf{R}) / \left(\mathrm{Hom}(\Lambda, \mathbf{R}) \oplus \mathscr{C}_b(\Lambda, \mathbf{R})\right).$$

Note that this last quotient is also isomorphic to the space $\mathrm{QM}_h(\Lambda, \mathbf{R})/\mathrm{Hom}(\Lambda, \mathbf{R})$. Let us come back to the case of a higher rank lattice Γ, as in the introduction. In that case, we have already said that the group $EH_b^2(\Gamma, \pi_0)$ is trivial, according to a result of Burger and Monod [7, 8]. In fact, their result is much more general: they prove that the group $EH_b^2(\Gamma, \pi)$ vanishes, for any unitary representation π of Γ on a Hilbert space $\mathscr{H}$ (the result even holds for more general Banach spaces, but we will not really discuss this here). This can be seen as a strengthening of property (T): the group Γ has the property that for any unitary representation π, the group $H^1(\Gamma, \pi)$ as well as the group $EH_b^2(\Gamma, \pi)$ vanishes. Following Monod [36], we will say that a discrete group with this property has property (TT). We now discuss the dynamical consequences of property (TT).

Suppose that $\Lambda \curvearrowright (X, \mu)$ is a measure preserving action of Λ on a probability space. We will say that a map $u : \Lambda \to \mathrm{L}^2(X, \mu)$ is a *quasi-cocycle* if there exists a constant $C > 0$ such that

$$|u(\gamma_1\gamma_2) - \pi(\gamma_1)(u(\gamma_2)) - u(\gamma_1)| \leq C$$

almost everywhere, for all $\gamma_1, \gamma_2 \in \Lambda$. We will give some examples of quasi-cocycles defined on groups of Hamiltonian diffeomorphisms of surfaces in the next section. According to the subadditive ergodic theorem (see [32] for instance), if u is a quasi-cocycle and $\gamma \in \Lambda$, the sequence of functions

$$\frac{u(\gamma^n)}{n}$$

converges almost everywhere to a function $\widehat{u}(\gamma)$. The reader should observe that since the action of Λ on X is measure-preserving, the map

$$\gamma \mapsto \phi(\gamma) = \int_X u(\gamma) d\mu$$

is a quasi-morphism on Λ. It is not difficult to check that the map $\gamma \mapsto \phi_u(\gamma) = \int_X \widehat{u}(\gamma) d\mu$ is the unique homogeneous quasi-morphism at a bounded distance from ϕ. Note that all the quasi-morphisms defined by Gambaudo and Ghys [25] and by the author [40] are obtained in this way: by integration of a quasi-cocycle.

Remark 1 The quasi-morphisms constructed by Entov and Polterovich [13] are defined in a completely different way, see [39] for a survey. It is not à priori clear how to relate them (in a non-trivial way) to some bounded cohomology class with non-trivial coefficients.

Remark 2 A weaker definition of quasi-cocycle already appears in the litterature, see [44]. There, a map $u : \Lambda \to \mathrm{L}^2(X, \mu)$ is called a quasi-cocycle if

$$|\mathfrak{d}^1 u|_{\infty,\Lambda^2} = \sup_{(\gamma_1,\gamma_2)\in\Lambda^2} |\mathfrak{d}^1 u(\gamma_1, \gamma_2)|_{\mathrm{L}^2}$$

is finite. In our definition, we require the stronger condition that

$$\sup_{(\gamma_1,\gamma_2)\in\Lambda^2} |\mathfrak{d}^1 u(\gamma_1, \gamma_2)|_{\mathrm{L}^\infty}$$

is finite, to ensure that $(\frac{u(\gamma^n)}{n})_{n\geq 0}$ converges almost everywhere for all γ. The notion of quasi-cocycle is also related to the notion of *rough action* (see [36], chapter V).

We now see the advantage of using the full result of Burger and Monod: if Λ is a group with property (TT) acting on X and u is a quasi-cocycle, the vanishing result with trivial coefficients tells us that for any $\gamma \in \Lambda$ the integral

$$\phi_u(\gamma) = \int_X \widehat{u}(\gamma) d\mu$$

is zero, while the full vanishing result tells us that the function $\widehat{u}(\gamma)$ is zero almost everywhere, as the following proposition shows.

Proposition 2.2 *Suppose Λ has property (TT). Then, for any measure preserving action $\Lambda \curvearrowright (X, \mu)$ on a probability space and for any quasi-cocycle $u : \Lambda \to \mathrm{L}^2(X, \mu)$ we have :*

$$\widehat{u}(\gamma) = 0,$$

almost everywhere (for all $\gamma \in \Lambda$).

Proof. If u is a quasi-cocycle, the map $\mathfrak{d}^1 u$ is a bounded 2-cocycle. Since the group $EH_b^2(\Lambda, \pi)$ is trivial, there exists, according to the previous proposition, a bounded map $v : \Lambda \to \mathrm{L}^2(X, \mu)$ and a 1-cocycle $w : \Lambda \to \mathrm{L}^2(X, \mu)$ such that $u = v + w$. We write $D := \sup_{\gamma\in\Lambda} ||v(\gamma)||$. Since Λ has property (T), there exists a map $\varphi \in \mathrm{L}^2(X, \mu)$ such that $w(\gamma) = \varphi \circ \gamma - \varphi$. We obtain:

$$\frac{u(\gamma^n)}{n} = \frac{\varphi \circ \gamma^n - \varphi}{n} + \frac{v(\gamma^n)}{n}.$$

We already know that the sequence $\frac{u(\gamma^n)}{n}$ converges almost everywhere to $\widehat{u}(\gamma)$. The function

$$\frac{\varphi \circ \gamma^n - \varphi}{n} = \frac{1}{n} \sum_{j=0}^{n-1} (\varphi \circ \gamma - \varphi) \circ \gamma^j$$

converges almost everywhere by Birkhoff's theorem; it is a classical lemma in ergodic theory that its limit is 0 almost everywhere. Finally, $\frac{v(\gamma^n)}{n}$ converges almost everywhere to $\widehat{u}(\gamma)$. Since the norm of $\frac{v(\gamma^n)}{n}$ in $\mathrm{L}^2(X,\mu)$ is less than $\frac{D}{n}$, there exists a subsequence $\frac{v(\gamma^{n_k})}{n_k}$ which converges almost everywhere to 0. Hence $\widehat{u}(\gamma) = 0$ almost everywhere. □

3 Some examples

We now describe some examples of concrete quasi-cocycles on groups of Hamiltonian diffeomorphisms of surfaces, taken from [17, 20, 24, 25, 33, 40]. These functions are constructed in the spirit of Schwartzman's classical asymptotic cycle [42] or of Arnold's asymptotic Hopf invariant, see [1]. The interested reader will find much more examples in the work of Gambaudo and Ghys [25].

Since we are dealing with diffeomorphisms which are isotopic to the identity, we will always make use of isotopies $(f_t)_{t\in[0,1]}$ from the identity $\mathbb{1} = f_0$ to a given diffeomorphism $f = f_1$ and will consider various (kind of) rotation numbers associated to the orbit $(f_t(x))$ of a point x or to a pair of orbits $(f_t(x))$, $(f_t(y))$ $(x \neq y)$. To ensure that the number we define does not depend on the choice of the isotopy $(f_t)_{t\in[0,1]}$ but only on f we will often appeal to some results on the topology of the groups $\mathrm{Ham}(\Sigma,\omega)$ or $\mathrm{Diff}_0(\Sigma,\omega)$, see [12].

Example 1 [17, 24] We first give an example of a 1-cocycle defined on the group $\mathrm{Diff}_c(\mathbf{D}^2,\omega)$ of area-preserving diffeomorphisms of the disc $\mathbf{D}^2 = \{(x,y) \in \mathbf{R}^2,\ |x|^2 + |y|^2 \leq 1\}$, which coincide with the identity near the boundary. This group is torsion free (see [39]). Although there is no non-trivial homomorphism from a Kazhdan group to $\mathrm{Diff}_c(\mathbf{D}^2,\omega)$ (thanks to Thurston's stability theorem), it is worth studying this first example. Consider a diffeomorphism $f \in \mathrm{Diff}_c(\mathbf{D}^2,\omega)$ and choose an isotopy (f_t) from $f_0 = \mathbb{1}$ to $f_1 = f$. For any two distinct points x, y in the disc, consider the non-zero vector $f_t(x) - f_t(y) \in \mathbf{R}^2$. When t goes from 0 to 1 the argument $e^{2i\pi u(t)}$ of this vector varies of a quantity $\mathrm{angle}_f(x,y) := u(1) - u(0)$. This defines a continuous function

$$\mathrm{angle}_f : \mathbf{D}^2 \times \mathbf{D}^2 - \Delta \to \mathbf{R},$$

where $\Delta = \{(x,x), x \in \mathbf{D}^2\}$ is the diagonal. One easily establishes that this function does not depend on the choice of the isotopy $(f_t)_{t\in[0,1]}$ but only on f, and that it is bounded, see [24]. If $f, g \in \mathrm{Diff}_c(\mathbf{D}^2,\omega)$ and (f_t) and (g_t) are two isotopies from the identity to f and g respectively, we can consider the isotopy $(h_t) = (g_t) * (f_t g)$ from the identity to fg. It allows us to establish the relation:

$$\mathrm{angle}_{fg}(x,y) = \mathrm{angle}_g(x,y) + \mathrm{angle}_f(g(x), g(y)).$$

Hence the map $f \mapsto \mathrm{angle}_{f^{-1}}$ defines a 1-cocycle on the group $\mathrm{Diff}_c(\mathbf{D}^2,\omega)$ with values in the space $\mathrm{L}^2(\mathbf{D}^2 \times \mathbf{D}^2, \mathbf{R})$.

Problem. Suppose $\Lambda \subset \mathrm{Diff}_c(\mathbf{D}^2,\omega)$ is a finitely generated group. Can we give a condition on Λ which ensures that the class of the cocycle angle is non-zero in $H^1(\Lambda, \mathrm{L}^2(\mathbf{D}^2 \times \mathbf{D}^2, \mu^2))$?

The following proposition shows that the class of the cocycle $f \mapsto \mathrm{angle}_{f^{-1}}$ is always non-zero in the space $H^1(\Lambda, \mathscr{C}^0(\mathbf{D}^2 \times \mathbf{D}^2 - \Delta))$, unless Λ is trivial (here $\mathscr{C}^0(\mathbf{D}^2 \times \mathbf{D}^2 - \Delta)$ is the space of continuous functions on $\mathbf{D}^2 \times \mathbf{D}^2 - \Delta$).

Proposition 3.1 *Fix a diffeomorphism $f \in \mathrm{Diff}_c(\mathbf{D}^2, \omega)$. Assume that there exists a continuous function $\varphi : \mathbf{D}^2 \times \mathbf{D}^2 - \Delta \to \mathbf{R}$ such that* $\mathrm{angle}_f = \varphi \circ f - \varphi$*. Then f is the identity.*

Proof. We assume that f is not the identity. Then, it is enough to find an integer n and two fixed points x, y of f^n such that $\mathrm{angle}_{f^n}(x, y) \neq 0$. Indeed the relation $\mathrm{angle}_f = \varphi \circ f - \varphi$ implies $\mathrm{angle}_{f^n} = \varphi \circ f^n - \varphi$. If φ is continuous this forces the equality $\mathrm{angle}_{f^n}(x, y) = 0$, for any pair (x, y) of distinct fixed points of f^n.

We will see at the end of this section (see Lemma 3.2 and the paragraph following it) that if f is distinct from the identity there exists a fixed point x_0 and a point $y_0 \neq x_0$ such that the number

$$\widetilde{\mathrm{angle}}_f(x_0, y_0) := \lim_{n \to \infty} \frac{1}{n} \mathrm{angle}_{f^n}(x_0, y_0)$$

exists and is non-zero. Consider now the compact annulus $\mathbf{A}$ obtained from the disc by blowing up the fixed point x_0 thanks to the action of the differential df_{x_0}. The diffeomorphism f naturally extends to a homeomorphism of $\mathbf{A}$, still denoted f. Let F be the lift of f to the universal cover $\widetilde{\mathbf{A}}$ of the annulus which pointwise fixes the component of the boundary of $\widetilde{\mathbf{A}}$ corresponding to the boundary of the disc. If z is in the interior of $\mathbf{A}$ (identified with the interior of the disc, minus the point x_0), the limit

$$\lim_{n \to \infty} \frac{1}{n} \mathrm{angle}_{f^n}(x_0, z),$$

if it exists, is the rotation number $\rho_F(z)$ (determined by the lift F) of the point z in the annulus. We know that:

- the point y_0 has a non-zero rotation number,
- points close to the boundary have zero rotation number.

Now, let us recall a result of Franks (this is Corollary 2.4 of [19]). Let $g : \mathbf{A} \to \mathbf{A}$ be a homeomorphism isotopic to the identity and G be a lift of g to $\widetilde{\mathbf{A}}$. If x is a point of $\mathbf{A}$ we will denote by $\rho_G(x)$ the rotation number of x determined by G, if it exists. Let $K \subset \mathbf{A}$ be a chain transitive compact invariant set for g. See [19] for the notion of chain transitivity; this is a very weak form of recurrence. Assume that there exist two points x_1 and x_2 in K whose rotation numbers $\rho_G(x_1)$ and $\rho_G(x_2)$ exist and satisfy $\rho_G(x_1) < \rho_G(x_2)$. Then, for any rational number $\frac{p}{q} \in]\rho_G(x_1), \rho_G(x_2)[$, g has a periodic point with rotation number $\frac{p}{q}$.

Let us apply the above result to f. Since f is area-preserving, the chain transitivity hypothesis will be easily satisfied. Almost every point of the interior of $\mathbf{A}$ is recurrent, hence every point of $\mathbf{A}$ is non-wandering. In particular every point of $\mathbf{A}$ is chain recurrent. The compact set $K = \mathbf{A}$ is connected, f-invariant, and all its points are chain recurrent, hence it is chain transitive (this is Proposition 1.2 in [19]). Franks' result applies with $K = \mathbf{A}$. The existence of the point y_0 and of points with zero rotation number implies that there exists a periodic point y_1 with non-zero rotation number. If the period of y_1 is n, this exactly means that the quantity $\mathrm{angle}_{f^n}(x_0, y_1)$ is non-zero. $\square$

Example 2 [20, 33] If f is a Hamiltonian diffeomorphism of the torus, we have seen in the introduction that for any lift $F : \mathbf{R}^2 \to \mathbf{R}^2$ of f, one has $\int_{\mathbf{T}^2} (F(x) - x) d\mu(x) \in \mathbf{Z}^2$. We can therefore choose a particular lift f_* for f: the unique one for which $\int_{\mathbf{T}^2}(f_*(x)-x)d\mu(x) = 0$. Since the map

$$\begin{array}{ccc} \mathbf{R}^2 & \to & \mathbf{R}^2 \\ x & \mapsto & f_*(x) - x \end{array}$$

is invariant under integral translations, there exists a map $v_f : \mathbf{T}^2 \to \mathbf{R}^2$ such that $v_f(p(x)) = f_*(x) - x$ (where $p : \mathbf{R}^2 \to \mathbf{T}^2$ is the natural projection). From the relation $(f \circ g)_* = f_* \circ g_*$, one deduce the cocycle relation $v_{f\circ g} = v_g + v_f \circ g$. The map $f \mapsto v_{f^{-1}} \in \mathrm{L}^2(\mathbf{T}^2, \mathbf{R}^2)$ is therefore a 1-cocycle.

Supppose now that $\Lambda \subset \mathrm{Ham}(\mathbf{T}^2, \omega)$ is a finitely generated subgroup on which the previous cocycle is cohomologically trivial: there exists a measurable map $\varphi : \mathbf{T}^2 \to \mathbf{R}^2$ such that $v_f = \varphi \circ f - \varphi$ ($f \in \Lambda$). Recall first that for almost every point x in the torus, we have:

$$\liminf_n |\varphi(f^n(x)) - \varphi(x)| = 0.$$

This is an easy consequence of Poincaré's recurrence theorem and Lusin's theorem. But we also have the following relation: $f_*^n(x) - x = \varphi(f^n(p(x))) - \varphi(p(x))$ ($x \in \mathbf{R}^2$). Hence, we get that almost every point of the plane is recurrent for the dynamics of the diffeomorphism f_* (for all $f \in \Lambda$). In fact the function

$$\begin{array}{rcl} \psi : \mathbf{R}^2 & \to & \mathbf{R}^2 \\ x & \mapsto & x - \varphi(p(x)) \end{array}$$

is invariant under the lifted action of Λ on the plane (through the diffeomorphisms f_*). The domain $\psi^{-1}([0,1[^2) \subset \mathbf{R}^2$, as well as its translates by integral vectors, has finite measure and is invariant under Λ. Therefore the vanishing of the cohomology class of the cocycle $f \mapsto v_{f^{-1}}$ give some constraints on the (measurable) dynamics of Λ.

Example 3 [25] We assume here that the genus of Σ is greater than 1. Endow Σ with a hyperbolic metric and identify its universal cover with the Poincaré disc Δ. For each 1-form η on Σ, we will define a quasi-cocycle u_η with values in the space of continuous functions on Σ. This quasi-cocycle is a true cocycle if η is closed, and a coboundary if η is exact.

If $f : \Sigma \to \Sigma$ is a Hamiltonian diffeomorphism, we will name $f_* : \Delta \to \Delta$ the unique lift of f which commutes with the action of the fundamental group of Σ on Δ. If $\widetilde{x} \in \Delta$, one can consider the unique geodesic arc $\alpha(\widetilde{x})$ between $\widetilde{x}$ and $f_*(\widetilde{x})$. Let $\widetilde{\eta}$ denote the lift of the 1-form η to Δ. The map

$$\begin{array}{ccc} \Delta & \to & \mathbf{R} \\ \widetilde{x} & \mapsto & \int_{\alpha(\widetilde{x})} \widetilde{\eta} \end{array}$$

descends to a function $u_\eta(f, \cdot)$ on Σ. If $x \in \Sigma$ and $\widetilde{x} \in \Delta$ is any lift of x, the quantity

$$u_\eta(fg, x) - u_\eta(g, x) - u_\eta(f, g(x))$$

equals the integral of the 2-form $d\widetilde{\eta}$ on the geodesic triangle of Δ with vertices $\widetilde{x}$, $g_*(\widetilde{x})$, $f_* \circ g_*(\widetilde{x})$. Since the 2-form $d\widetilde{\eta}$ is bounded (it is invariant by the action of the fundamental group of Σ) this quantity is bounded by the norm of $d\widetilde{\eta}$ times the area of a hyperbolic triangle. We obtain:

$$|u_\eta(fg, x) - u_\eta(g, x) - u_\eta(f, g(x))| \leq \pi \cdot |d\eta|_\infty.$$

Hence the map $f \mapsto u_\eta(f^{-1}, \cdot)$ is a quasi-cocycle. The reader will find an alternative description of this quasi-cocycle in [39].

Example 4 [25] We now give an example of a quasi-cocycle defined on the group $\mathrm{Diff}_0(\mathbf{T}^2, \omega)$. We will make use of the group structure of the torus.

We fix a point x_* in $\mathbf{T}^2 - \{0\}$ and choose a homogeneous quasi-morphism

$$\phi : \pi_1(\mathbf{T}^2 - \{0\}, x_*) \to \mathbf{R}$$

(there are many since $\pi_1(\mathbf{T}^2 - \{0\}, x_*)$ is a free group, see [6] as well as [14] for generalizations). For each point $v \in \mathbf{T}^2 - \{0\}$ we fix a path $(\alpha_v(t))_{t\in[0,1]}$ from x_* to v in $\mathbf{T}^2 - \{0\}$ whose length is bounded for a Riemannian metric defined on the compact surface with boundary $\overline{\mathbf{T}^2 - \{0\}}$ obtained by blowing-up the origin on the torus. If $(f_t)_{t\in[0,1]}$ is an area-preserving isotopy on $\mathbf{T}^2$ and x and y two distinct points on $\mathbf{T}^2$, we can consider the curve

$$f_t(x) - f_t(y) \in \mathbf{T}^2 - \{0\}.$$

We close it to form a loop $\alpha(f, x, y) = \alpha_{x-y} * (f_t(x) - f_t(y)) * \overline{\alpha_{f_1(x)-f_1(y)}}$. Then, we define a function $u(\phi, f, \cdot, \cdot)$ on $\mathbf{T}^2 \times \mathbf{T}^2 - \Delta$ by: $u(\phi, f, x, y) = \phi(\alpha(f, x, y))$. One can easily check that this function does not depend on the choice of the isotopy $(f_t)_{t\in[0,1]}$ but only on f, is measurable, and bounded, see [25]. From the relation

$$\alpha(fg, x, y) = \alpha(g, x, y) * \alpha(f, g(x), g(y))$$

and the fact that ϕ is a quasi-morphism, we deduce:

$$|u(\phi, fg, x, y) - u(\phi, g, x, y) - u(\phi, f, g(x), g(y))| \leq C.$$

This means precisely that the map $f \mapsto u(\phi, f^{-1}, \cdot, \cdot) \in \mathrm{L}^2(\mathbf{T}^2 \times \mathbf{T}^2, \mu \otimes \mu)$ is a quasi-cocycle.

Example 5 [40] Consider the 2-sphere, with an area form ω of total area 2. Let $\Lambda(\mathbf{S}^2)$ be the space of oriented tangent lines to $\mathbf{S}^2$ and $p : \Lambda(\mathbf{S}^2) \to \mathbf{S}^2$ the canonical projection. There is a natural action of the circle $\mathbf{R}/\mathbf{Z}$ on $\Lambda(\mathbf{S}^2)$: if $\ell \in \Lambda(\mathbf{S}^2)$ is a tangent line at $x \in \mathbf{S}^2$, $e^{2i\pi s} \cdot \ell$ is the tangent line at x whose angle with ℓ is $2\pi s$. Let X be the (periodic) vector field which generates this action of $\mathbf{R}/\mathbf{Z}$ on $\Lambda(\mathbf{S}^2)$. We can find a 1-form α on $\Lambda(\mathbf{S}^2)$, invariant by rotations, such that

$$\alpha(X) = 1 \quad \text{and} \quad d\alpha = p^*\omega.$$

This implies that α is a contact form whose Reeb vector field is X. Let f be an area-preserving diffeomorphism of $\mathbf{S}^2$ and (f_t) a Hamiltonian isotopy from the identity to f, generated by the Hamiltonian (H_t). Let X_t be the symplectic gradient of H_t. The isotopy (f_t) induces an isotopy on $\Lambda(\mathbf{S}^2)$ through the action of the differential $df_t : \Lambda(\mathbf{S}^2) \to \Lambda(\mathbf{S}^2)$ of f_t. The key point of our last example is that there exists a second isotopy on $\Lambda(\mathbf{S}^2)$ which lifts (f_t). It is constructed as follows. Let $\widehat{X}_t$ be the horizontal lift of X_t to $\Lambda(\mathbf{S}^2)$, i.e. the vector field on $\Lambda(\mathbf{S}^2)$ defined by the equations:

$$\alpha(\widehat{X}_t) = 0 \ \text{ and } \ p_*(\widehat{X}_t) = X_t.$$

The time-dependent vector field $\widehat{X}_t - (H_t \circ p)X$ generates an isotopy $\theta(f_t) : \Lambda(\mathbf{S}^2) \to \Lambda(\mathbf{S}^2)$ which preserves α and whose isotopy class only depends on the isotopy class of (f_t), see [2].

We can now measure a kind of rotation number associated to any tangent line $\ell \in \Lambda(\mathbf{S}^2)$. Indeed for all t, $\theta(f_t)(\ell)$ and $df_t(\ell)$ are tangent lines at the point $f_t(p(\ell)) \in \mathbf{S}^2$. Hence, we can write $df_t(\ell) = e^{2i\pi u(t)} \cdot \theta(f_t)(\ell)$ where $e^{2i\pi u(t)}$ is the angle between $\theta(f_t)(\ell)$ and $df_t(\ell)$. The quantity $w(f,\ell) = u(1) - u(0)$ only depends on f and ℓ. Then $w(f,\cdot) : \Lambda(\mathbf{S}^2) \to \mathbf{R}$ is a continuous map which satisfies:

$$w(fg,\ell) = w(g,\ell) + w(f, dg(\ell)).$$

This is not yet exactly what we need: since the action $(f,\ell) \mapsto df(\ell)$ of the group $\mathrm{Ham}(\mathbf{S}^2,\omega)$ on $\Lambda(\mathbf{S}^2)$ does not have an invariant measure we can not consider the function $w(f,\cdot)$ as a vector in some unitary representation of $\mathrm{Ham}(\Sigma,\omega)$. Nevertheless, we can prove the following inequality, see [40]: if ℓ_1 and ℓ_2 are two tangent lines at the same point $x = p(\ell_1) = p(\ell_2) \in \mathbf{S}^2$ then:

$$|w(f,\ell_1) - w(f,\ell_2)| \leq 10.$$

Thus, if we define $\underline{w}(f,x) = \max_{p(\ell)=x} w(f,\ell)$, $\underline{w}(f,\cdot)$ is a bounded measurable function on $\mathbf{S}^2$ and we have:

$$|\underline{w}(fg,x) - \underline{w}(g,x) - \underline{w}(f,g(x))| \leq 30$$

The map $f \mapsto \underline{w}(f^{-1},\cdot)$ is therefore a quasi-cocycle.

$\star\star\star$

Motivated by these examples and by the vanishing results of the previous section, we can raise the following question.

Question 1 *If $f : \Sigma \to \Sigma$ is a Hamiltonian diffeomorphism of a closed surface, distinct from the identity, can we find a quasi-cocycle u, defined on the group* $\mathrm{Ham}(\Sigma,\omega)$*, with values in the space* $\mathrm{L}^2(\Sigma,\mu)$*, or* $\mathrm{L}^2(\Sigma\times\Sigma,\mu\otimes\mu)$*, or even* $\mathrm{L}^2(\Sigma^n,\mu^n)$ *for some integer n, such that the function $\widehat{u}(f)$ is not zero almost everywhere? In other words, can we find a set of positive measure (in Σ or $\Sigma\times\Sigma$...) of points for which some kind of rotation number or asymptotic linking number is non-zero?*

A positive answer to this question would yield a positive answer to Zimmer's conjecture. Note however that we should certainly add some hypothesis on f to get a positive answer. For instance, if f is a rotation of the sphere, $\widehat{u}(f)$ vanishes for all known quasi-cocycles. It is probably possible to construct more subtle counter-examples. Assume for instance that we can construct a Hamiltonian diffeomorphism f with the following two properties: f is weakly mixing, and, for some algebraic reason, $\phi(f) = 0$ for any homogeneous quasi-morphism defined on the group of Hamiltonian diffeomorphisms. In that case, for any quasi-cocycle u with values in $\mathrm{L}^2(\Sigma^n,\mu^n)$ one has $\widehat{u}(f) = 0$ almost everywhere. Indeed, since the diagonal action of f on Σ^n is ergodic for any n, one has $\widehat{u}(f) = \phi_u(f) = 0$ almost everywhere on Σ^n. Note however that this kind of counter-example cannot appear inside the image $\rho(\Gamma)$ of a higher rank lattice, since all the diffeomorphisms in the group $\rho(\Gamma)$ have discrete spectrum.

We now give two small hints that tend to show that a non-trivial Hamiltonian diffeomorphism of a compact surface contains somewhere a set of positive measure of points which "rotate around each other".

Consider first a diffeomorphism $f \in \mathrm{Diff}_c(\mathbf{D}^2, \omega)$, distinct from the identity. We will write

$$\widetilde{\mathrm{angle}}_f(x,y) = \lim_{n\to\infty} \frac{1}{n} \mathrm{angle}_{f^n}(x,y)$$

when this limit exists. Note that, according to Birkhoff's theorem, $\widetilde{\mathrm{angle}}_f(x,y)$ exists for $\mu \otimes \mu$-almost every point $(x,y) \in \mathbf{D}^2 \times \mathbf{D}^2$, and if x_0 is a fixed point of f, $\widetilde{\mathrm{angle}}_f(y,x_0)$ exists for μ-almost every point $y \in \mathbf{D}^2$. A result of Viterbo [45] implies that there exists a fixed point $x_0 \in \mathbf{D}^2$ for f and a Borel set $B \subset \mathbf{D}^2$ of positive measure such that:

$$\widetilde{\mathrm{angle}}_f(y,x_0) \neq 0,$$

for all $y \in B$. We now explain this fact. Recall first that one can define the *symplectic action* $\mathcal{A}(x_0)$ of a fixed point x_0 of f in the following way. If $(f_t)_{t\in[0,1]}$ is a Hamiltonian isotopy from the identity to f, generated by a Hamiltonian (H_t) one defines:

$$\mathcal{A}(x_0) = \int_\gamma \lambda + \int_0^1 H_t(f_t(x))dt,$$

where γ is the loop $(f_t(x_0))_{t\in[0,1]}$ and λ is a primitive of the area form ω on $\mathbf{D}^2$. This number does not depend on the choice of the isotopy (f_t) from the identity to f. The next lemma, which I learnt from Patrice Le Calvez, relates the symplectic action of a fixed point to the cocycle angle_f.

Lemma 3.2 *For any fixed point x_0 of f one has:*

$$\mathcal{A}(x_0) = \int_{\mathbf{D}^2} \mathrm{angle}_f(y,x_0)d\mu(y) = \int_{\mathbf{D}^2} \widetilde{\mathrm{angle}}_f(y,x_0)d\mu(y).$$

Proof. To compute the action $\mathcal{A}(x_0)$, we can choose an isotopy $(f_t)_{t\in[0,1]}$ such that $f_t(x_0) = x_0$ for all t. We denote by X_t the vector field which generates the isotopy and H_t the corresponding compactly supported Hamiltonian. Let $d\theta$ be a closed 1-form on the annulus $\mathbf{D}^2 - \{x_0\}$ whose integral over a generator of the first homology of the annulus equals 1. The integral of the function $\mathrm{angle}_f(\cdot, x_0)$ over $\mathbf{D}^2$ equals $\int_{\mathbf{D}^2}\int_0^1 d\theta(X_t)dt\,\omega$. Using the fact that the 3-form $d\theta \wedge \omega$ is 0 we get:

$$\begin{aligned} d\theta(X_t)\,\omega &= d\theta \wedge \omega(X_t, \cdot) \\ &= -d(H_t d\theta). \end{aligned}$$

Choose now a small disc D_ε around x_0. We have:

$$\begin{aligned} \int_{\mathbf{D}^2} \mathrm{angle}_f(y,x_0)d\mu(y) &= \lim_{\varepsilon\to 0} \int_{\mathbf{D}^2 - D_\varepsilon} \int_0^1 d\theta(X_t)dt\,\omega \\ &= \lim_{\varepsilon\to 0} \int_{\partial D_\varepsilon} \int_0^1 H_t dt\, d\theta. \end{aligned}$$

Here we have used Stokes' theorem and the fact that H_t vanishes near the boundary of $\mathbf{D}^2$. The last limit in the equality above coincides with the action $\mathcal{A}(x_0) = \int_0^1 H_t(x_0)dt$. We have therefore proved the first equality of the lemma. The second equality follows from Birkhoff's theorem. □

Now, using symplectic methods, Viterbo shows that if f is distinct from the identity, there exists a fixed point x_0 with non-zero action. Thanks to the interpretation of the

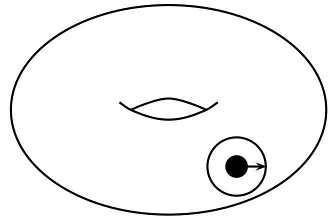

Figure 1: Linking of points in an inessential annulus on the torus

action of x_0 as the average linking number around x_0, we deduce the existence of the set B. We get that the map $\widetilde{\text{angle}}_f$ is non-zero on the set $B \times \{x_0\} \subset \mathbf{D}^2 \times \mathbf{D}^2$. However, from our point of view, it would be more natural to obtain a set of positive measure in $\mathbf{D}^2 \times \mathbf{D}^2$ for the *product measure*.

Second, we will consider the simplest examples of Hamiltonian diffeomorphisms: integrable ones. For instance, let us take a smooth function $H : \mathbf{T}^2 \to \mathbf{R}$ and consider the associated Hamiltonian flow $\varphi_H^t : \mathbf{T}^2 \to \mathbf{T}^2$. Assume that H is non-constant. Then, we will show that there is a quasi-cocycle u on $\mathrm{Ham}(\mathbf{T}^2, \omega)$ with values in $\mathrm{L}^2(\mathbf{T}^2, \mathbf{R}^2)$ or in $\mathrm{L}^2(\mathbf{T}^2 \times \mathbf{T}^2, \mathbf{R})$ such that $\widehat{u}(\varphi_H^1)$ is not zero almost everywhere (for the measure μ on $\mathbf{T}^2$ in the first case, for the measure $\mu \otimes \mu$ in the second one). As H is non-constant there is an embedding of the annulus

$$i : A = \mathbf{S}^1 \times [0,1] \to \mathbf{T}^2$$

on which the flow φ_H^t reads: $\varphi_H^t(i(\theta, s)) = i(\theta + t\vartheta(s), s)$ for a non-vanishing function $\vartheta : [0,1] \to \mathbf{R}^*$. To prove this fact, one just consider a connected component $\mathcal{C}$ of a regular level of H. In a neighborhood of $\mathcal{C}$ all orbits of φ_H^t are periodic hence $\mathcal{C}$ is contained in an annulus which is invariant under the flow φ_H^t and in which each orbit is periodic. From this one easily deduces the existence of the embedding i.

We will write $\hat{i} : \mathbf{R} \times [0,1] \to \mathbf{R}^2$ for a lift of the map i. Assume first that the annulus A is essential in $\mathbf{T}^2$. In that case there exists a non zero vector $w \in \mathbf{Z}^2$ such that $\hat{i}(\theta + k, s) = \hat{i}(\theta, s) + kw$. Consequently, if $v_{\varphi_H^1}$ is the cocycle from Example 2 we have:

$$\frac{v_{\varphi_H^n}(i(\theta, s))}{n} \underset{n\to\infty}{\to} \vartheta(s) w \neq 0.$$

Assume now that the annulus A is inessential. We will make use of the quasi-cocycle from Example 4. We can assume that the curve $i(\mathbf{S}^1 \times \{0\})$ bounds an embedded disc D which is disjoint from $i(\mathbf{S}^1 \times]0,1])$ (up to changing the parameter s by $1-s$). Assume that $s_2 > s_1$ and consider the trajectories of the two points $i(\theta_1, s_1)$ and $i(\theta_2, s_2)$. The family of curves $(\varphi_H^t(i(\theta_2, s_2)) - \varphi_H^t(i(\theta_1, s_1)))_{t\in[0,n]}$ winds around 0 in the disc $D \cup i(A)$ (see the figure). The loops $\alpha(\varphi_H^n, i(\theta_2, s_2), i(\theta_1, s_1))$ are therefore almost equal to a power of the commutator $[a, b]$, where a and b are the standard generators of the group $\pi_1(\mathbf{T}^2 - \{0\}, x_*)$. More precisely, we can write:

$$\alpha(\varphi_H^n, i(\theta_2, s_2), i(\theta_1, s_1)) = \alpha_n * [a, b]^{[n\vartheta(s_2)]} * \beta_n$$

where α_n and β_n stay in a fixed finite subset of $\pi_1(\mathbf{T}^2 - \{0\}, x_*)$ and $[n\vartheta(s_2)]$ is the integer part of $n\vartheta(s_2)$. From this we deduce that $u(\phi, \varphi_H^n, i(\theta_2, s_2), i(\theta_1, s_1)) - \phi([a, b]^{[n\vartheta(s_2)]})$

is bounded independently of n. Therefore, for any homogeneous quasi-morphism ϕ on $\pi_1(\mathbf{T}^2 - \{0\}, x_*)$ we have:

$$\frac{u(\phi, \varphi_H^n, i(\theta_2, s_2), i(\theta_1, s_1))}{n} \to \vartheta(s_2)\phi([a,b]).$$

To conclude we now choose a quasi-morphism ϕ such that $\phi([a,b])$ is non-zero and observe that the set $\{(i(\theta_1, s_1), i(\theta_2, s_2)), s_2 > s_1\} \subset \mathbf{T}^2 \times \mathbf{T}^2$ has positive $\mu \otimes \mu$-measure. Hence we have proved:

Theorem *Let $\rho : \Gamma \to \mathrm{Ham}(\mathbf{T}^2, \omega)$ be a homomorphism, where Γ is a higher rank lattice, as above. If a diffeomorphism $f \in \rho(\Gamma)$ is contained in a Hamiltonian flow, then f is the identity.*

In fact, this result could have been deduced directly from the existence of Zimmer's invariant Riemannian metric. Indeed, in the situation above, one can always find an embedding i such that the function ϑ is not constant. This implies that for an open set of points x the sequence of differentials $(d\varphi_H^n(x))_{n\geq 0}$ tends to infinity. This contradicts the existence of Zimmer's metric. Yet, the above proof was presented in order to illustrate the possible use of the quasi-cocycles that we described.

Also, it is not difficult to establish a similar result on a surface of genus greater than 1, or on the sphere if we assume that the flow is not conjugated to a 1-parameter group of rotations, see [41].

4 Final remarks

We have already mentioned that, according to a result of Zimmer, the group $\rho(\Gamma)$ preserves (almost everywhere) a measurable Riemannian metric. In the case where the surface is the torus $\mathbf{T}^2$, we can slightly improve this fact: the group $\rho(\Gamma)$ preserves a pair (X^1, X^2) of linearly independent measurable vector fields. If g is a Hamiltonian diffeomorphism of $\mathbf{T}^2$ let us consider a Hamiltonian isotopy (g_t) from the identity to g. We identify the differential $dg_t(x)$ of the diffeomorphism g_t at a point x with a 2×2 matrix. Hence, the path of matrices $dg_t(x)$ defines an element $\widetilde{dg}(x) \in \widetilde{\mathrm{SL}}_2(\mathbf{R})$ in the universal cover of $\mathrm{SL}_2(\mathbf{R})$, which depends only on g and x. The map

$$\begin{array}{rcl} \mathrm{Ham}(\mathbf{T}^2) \times \mathbf{T}^2 & \to & \widetilde{\mathrm{SL}}_2(\mathbf{R}) \\ (g, x) & \mapsto & \widetilde{dg}(x) \end{array}$$

is a cocycle. If p denotes the projection from $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ to $\mathrm{SL}_2(\mathbf{R})$, the map $(g,x) \mapsto dg(x) = p(\widetilde{dg}(x))$ is the usual derivative cocycle. Suppose $\Gamma \hookrightarrow \mathrm{Ham}(\mathbf{T}^2, \omega)$ is a Kazhdan group and consider the restriction of the previous cocycle to Γ. According to Zimmer [47, 49], there exists a measurable map $\varphi : \mathbf{T}^2 \to \mathrm{SL}_2(\mathbf{R})$ such that

$$\varphi(\gamma \cdot x)^{-1} \circ d\gamma(x) \circ \varphi(x) \in \mathrm{SO}(2),$$

almost everywhere (for $\gamma \in \Gamma$). We denote by $\tau : \widetilde{\mathrm{SL}}_2(\mathbf{R}) \to \mathbf{R}$ the translation number associated to the action of $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ on the line (see [28]) and by T the generator of the center of $\widetilde{\mathrm{SL}}_2(\mathbf{R})$. Let $\widetilde{\varphi} : \mathbf{T}^2 \to \widetilde{\mathrm{SL}}_2(\mathbf{R})$ be a measurable lift of φ chosen in such a way that the map $x \mapsto \tau(\widetilde{\varphi}(x))$ is bounded. This is always possible since one has $\tau(a \cdot T^k) = \tau(a) + k$ ($a \in \widetilde{\mathrm{SL}}_2(\mathbf{R})$, $k \in \mathbf{Z}$). The map $(\gamma, x) \mapsto \widetilde{\varphi}(\gamma \cdot x)^{-1} \circ \widetilde{d\gamma}(x) \circ \widetilde{\varphi}(x)$ is a cocycle whose image is contained in the inverse image of SO(2) in $\widetilde{\mathrm{SL}}_2(\mathbf{R})$, which is canonically isomorphic to $\mathbf{R}$.

Lemma 4.1 *For each $\gamma \in \Gamma$ the map $x \mapsto s_\gamma(x) := \widetilde{\varphi}(\gamma \cdot x)^{-1} \circ \widetilde{d\gamma}(x) \circ \widetilde{\varphi}(x) \in \mathbf{R}$ is bounded. Hence the cocycle $\gamma \mapsto s_\gamma$ determines a class in $H^1(\Gamma, \mathrm{L}^2(\mathbf{T}^2, \mathbf{R}))$.*

Proof. Since the restriction of τ to the inverse image of SO(2) in $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ is the identity, it is enough to show that the map $x \mapsto \tau(s_\gamma(x))$ is bounded. Since the two maps $x \mapsto \tau(\widetilde{\varphi}(x))$ and $x \mapsto \tau(\widetilde{\varphi}(\gamma \cdot x)^{-1})$ are bounded thanks to the choice of the lift $\widetilde{\varphi}$ and since τ is a quasi-morphism, it is enough to check that the map $x \mapsto \tau(\widetilde{d\gamma}(x))$ is bounded. But this is obvious since this is a continuous map from $\mathbf{T}^2$ to $\mathbf{R}$. $\square$

From the lemma, we deduce that there exists a measurable (in fact L^2) map $\psi : \mathbf{T}^2 \to \mathbf{R}$ such that: $s_\gamma(x) = \psi(\gamma \cdot x) - \psi(x)$. Coming back to $\mathrm{SL}_2(\mathbf{R})$, we obtain:

$$\varphi(\gamma \cdot x)^{-1} \circ d\gamma(x) \circ \varphi(x) = e^{2i\pi(\psi(\gamma(x)) - \psi(x))},$$

almost everywhere, where $e^{2i\pi u}$ stands for the rotation of angle $2\pi u$ in $\mathrm{SL}_2(\mathbf{R})$. This tells us that the derivative cocycle is cohomologous to the constant cocycle (equal to the identity) which is equivalent to the existence of the vector fields X^1 and X^2.

Although the results we have discussed here give more evidence toward Zimmer's conjecture, it seems difficult to exploit them. In particular, the quest for a set of positive measure of points that "rotates" under the action of a diffeomorphism seems delicate. Note also that we have simply used global cohomological properties of the lattice Γ to give some constraints on the dynamics of each single diffeomorphism in the group $\rho(\Gamma)$ but we were not able to investigate the dynamics of the whole group Γ. Rather than studying measure theoretical properties, it might be useful to study the topological dynamics of the action. In this direction, the work of Calegari [9] contains many promising ideas.

Acknowledgements. This work is part of my thesis realised at École Normale Supérieure de Lyon. I would like to thank Étienne Ghys for all the discussions we had around the problems discussed in this text and for his warm encouragements. I would also like to thank François Béguin, Patrice Le Calvez, Frédéric Le Roux and Leonid Polterovich for the numerous conversations we had around Zimmer's conjecture. This text was completed while I was enjoying the hospitality of Tel-Aviv University. This work was also partly supported by the French Agence Nationale de la Recherche.

Finally, I would like to thank the referee for his/her constructive comments on this text.

References

[1] V.I. Arnold and B.A. Khesin, *Topological methods in hydrodynamics*, Applied Mathematical Sciences **125**, Springer-Verlag, New York, (1998).

[2] A. Banyaga, *The group of diffeomorphisms preserving a regular contact form*, Topology and algebra (Proc. Colloq., Eidgenöss. Tech. Hochsch., Zurich, 1977), 47–53, Monograph. Enseign. Math. **26**, Univ. Genève (1978).

[3] A. Banyaga, *The structure of classical diffeomorphism groups*, Mathematics and its applications **400**, Kluwer Academic Publishers Group, Dordrecht, 1997.

[4] C. Bavard, *Longueur stable des commutateurs*, Enseign. Math. (2) **37**, No. 1-2 (1991), 109–150.

[5] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963), 111–122.

[6] R. Brooks, *Some remarks on bounded cohomology*, Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), 53–63, Ann. of Math. Stud. **97**, Princeton University Press, New Jersey, (1981).

[7] M. Burger and N. Monod, *Bounded cohomology of lattices in higher rank Lie groups*, J. Eur. Math. Soc. (JEMS) **1**, No. 2 (1999), 199–235.

[8] M. Burger and N. Monod, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. **12**, No. 2 (2002), 219–280.

[9] D. Calegari, *Circular groups, planar groups, and the Euler class*, Proceedings of the Casson Fest, Geom. Topol. Monogr. **7**, Geom. Topol. Publ., Coventry (2004), 431–491.

[10] S. Cantat, *Version kählérienne d'une conjecture de Robert J. Zimmer*, Ann. Sci. École Norm. Sup. (4) **37**, No. 5 (2004), 759–768.

[11] J. Déserti, *Groupe de Cremona et dynamique complexe : une approche de la conjecture de Zimmer*, Int. Math. Res. Not. (2006).

[12] C.J. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*, J. Diff. Geometry **3**, 19–43, (1969).

[13] M. Entov and L. Polterovich, *Calabi quasimorphism and quantum homology*, Int. Math. Res. Not., No. 30 (2003), 1635–1676.

[14] D. Epstein and K. Fujiwara, *The second bounded cohomology of word-hyperbolic groups*, Topology **36**, No. 6 (1997), 1275–1289.

[15] B. Farb and H. Masur, *Superrigidity and mapping class groups*, Topology **37**, No. 6 (1998), 1169-1176.

[16] B. Farb and P. Shalen, *Real-analytic actions of lattices*, Invent. Math. **135**, No. 2 (1999), 273–296.

[17] A. Fathi, *Transformations et homéomorphismes préservant la mesure. Systèmes dynamiques minimaux*, Thèse, Orsay, (1980).

[18] D. Fisher, *Talk at the conference "Geometry, Rigidity, and Group Actions", in honor of R. J. Zimmer's 60th birthday*, University of Chicago (september 2007).

[19] J. Franks, *Recurrence and fixed points of surface homeomorphisms*, Ergodic Theory Dynam. Systems **8***, Charles Conley Memorial Issue (1988), 99–107.

[20] J. Franks, *Rotation vectors for surface diffeomorphisms*, Proceedings of the International Congress of Mathematicians, Vol.1, 2, (Zürich, 1994), Birkhäuser, Basel (1995), 1179-1186.

[21] J. Franks and M. Handel, *Periodic points of Hamiltonian surface diffeomorphisms*, Geom. Topol. **7**, (2003), 713–756.

[22] J. Franks and M. Handel, *Area preserving group actions on surfaces*, Geom. Topol. **7**, (2003), 757–771.

[23] J. Franks and M. Handel, *Distortion elements in group actions on surfaces*, Duke Math. J. **131**, No. 3 (2006), 441-468.

[24] J.-M. Gambaudo and É. Ghys, *Enlacements asymptotiques*, Topology **36**, No. 6 (1997), 1355–1379.

[25] J.-M. Gambaudo and É. Ghys, *Commutators and diffeomorphisms of surfaces*, Ergodic Theory Dynam. Systems **24**, No. 5 (2004), 1591-1617.

[26] É. Ghys, *Sur les groupes engendrés par des difféomorphismes proches de l'identité*, Bol. Soc. Brasil. Mat. (N.S.) **24**, No. 2 (1993), 137–178.

[27] É. Ghys, *Actions de réseaux sur le cercle*, Invent. Math. **137**, No. 1 (1999), 199–231.

[28] É. Ghys, *Groups acting on the circle*, Enseign. Math. (2) **47**, No. 3-4 (2001), 329–407.

[29] É. Ghys, *Knots and Dynamics*, International Congress of Mathematicians, Vol. I, Eur. Math. Soc., Zürich (2007), 247–277.

[30] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **56**, (1983), 5–99.

[31] P. de la Harpe and A. Valette, *La propriété* (T) *de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger)*, Astérisque No. 175 (1989).

[32] J.F.C. Kingman, *Subadditive processes*, École d'Été de Probabilités de Saint-Flour, V 1975, Lecture Notes in Math. **539**, (1976), Springer, Berlin, 167–223.

[33] P. Le Calvez, *Identity isotopies on surfaces*, Dynamique des difféomorphismes conservatifs des surfaces : un point de vue topologique, Panor. Synthèse **21**, Soc. Math. France, Paris (2006), 105–143.

[34] A. Lubotzky, S. Mozes and M. S. Raghunathan, *The word and Riemannian metrics on lattices of semisimple groups*, Inst. Hautes Études Sci. Publ. Math., No. 91 (2000), 5–53.

[35] D. McDuff and D. Salamon, *Introduction to symplectic topology*, Second edition, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, (1998).

[36] N. Monod, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Mathematics **1758**, Springer-Verlag, Berlin, (2001).

[37] A. Navas, *Actions de groupes de Kazhdan sur le cercle*, Ann. Sci. École Norm. Sup. (4) **35**, No. 5 (2002), 749–758.

[38] L. Polterovich, *Growth of maps, distortion in groups and symplectic geometry*, Invent. Math. **150**, No. 3 (2002), 655–686.

[39] L. Polterovich, *Floer homology, dynamics and groups*, Morse theoretic methods in nonlinear analysis and in symplectic topology, NATO Sci. Ser. II Math. Phys. Chem. **217**, Springer, Dordrecht (2006), 417–438.

[40] P. Py, *Quasi-morphismes et invariant de Calabi*, Ann. Sci. École Norm. Sup. (4) **39**, No. 1 (2006), 177–195.

[41] P. Py, *Quasi-morphismes et difféomorphismes hamiltoniens*, Ph.D thesis, École Normale Supérieure de Lyon (2008), available at `http://tel.archives-ouvertes.fr/tel-00263607/fr/`.

[42] S. Schwartzman, *Asymptotic cycles*, Ann. of Maths (2) **66**, (1957), 270–284.

[43] S. Smale, *Diffeomorphisms of the* 2*-sphere*, Proc. Amer. Math. Soc. **10** (1959), 621–626.

[44] A. Thom, *Low degree bounded cohomology and* L^2*-invariants for negatively curved groups*, to appear in Groups, Geometry and Dynamics, arXiv 0710.4207.

[45] C. Viterbo, *Symplectic topology as the geometry of generating functions*, Math. Ann. **292**, No. 4 (1992), 685-710.

[46] D. Witte, *Arithmetic groups of higher* $\mathbf{Q}$*-rank cannot act on* 1*-manifolds*, Proc. Amer. Math. Soc. **122**, No. 2 (1994), 333-340.

[47] R. J. Zimmer, *Kazhdan groups acting on compact manifolds*, Invent. Math. **75**, No. 3 (1984), 425–436.

[48] R. J. Zimmer, *Actions of semisimple groups and discrete subgroups*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 1247–1258, Amer. Math. Soc., Providence, RI (1987).

[49] R. J. Zimmer, *Spectrum, entropy, and geometric structures for smooth actions of Kazhdan groups*, Israel J. Math. **75**, No. 1 (1991), 65–80.

Pierre Py
Université de Lyon, UMPA, UMR 5669 CNRS, École Normale Supérieure de Lyon
46, Allée d'Italie, 69364 Lyon, Cedex 07 FRANCE
Pierre.Py@umpa.ens-lyon.fr