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ON THE REPRESENTATION THEORY OF THE SYMMETRIC GROUPS

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We present here a new approach to the description of finite-dimensional complex irreducible representations of the symmetric groups due to A. Okounkov and A. Vershik. It gives an alternative construction to the combinatorial one, which uses tabloids, polytabloids, and Specht modules. Its aim is to show how the combinatorial objects of the theory (Young diagrams and tableaux) arise from the internal structure of the symmetric group. Bibliography: 9 *titles.*

1. Introduction

The purpose of this note is to present a new approach, due to Andrei Okounkov and Anatoly Vershik [2], to the description of finite-dimensional complex irreducible representations of the symmetric groups. Its aim is to give an alternative construction to the combinatorial one, which uses tabloids, polytabloids, and Specht modules, and to show how the combinatorial objects we introduce (Young diagrams and tableaux) arise from the internal structure of the symmetric groups. For this reason, this method is more abstract than the classical one. We use a few results about representations of finite groups in general, as they are set out in [9]. The main result we obtain is the branching rule for the representations of S_n. (Which irreducible representations of S_{n-1} are contained in a given irreducible representation of S_n?) Moreover, using the results presented, it is easy to show that the representations of the symmetric groups are defined over $\mathbb{Q}$ and to describe the characters of S_n.

2. Multiplicities of representations of the symmetric groups

In this paper, we denote by S_n the group of permutations of the set $\{1, \cdots, n\}$. For $i \leq n$, unless otherwise stated, we consider S_i as the subgroup of S_n acting on the set $\{1, \cdots, i\}$. From this we deduce an injection of the group algebra $\mathbb{C}[S_i]$ into $\mathbb{C}[S_n]$. We denote by $S_n^\wedge$ the set of finite-dimensional irreducible complex representations of S_n, and, by definition, $S_1^\wedge = \{\lambda_1\}$. If λ is in $S_n^\wedge$, we denote by V^λ the space of the representation λ. The main object we are going to deal with is an infinite graph, called the *Bratteli diagram*, whose vertices are

$$\coprod_{n\geq 1} S_n^\wedge.$$

We put k directed edges between a representation $\lambda \in S_n^\wedge$ and a representation $\mu \in S_{n-1}^\wedge$ if μ is contained k times in λ, regarding λ as a representation of S_{n-1} (k equals the dimension of the space $\mathrm{Hom}_{S_{n-1}}(V^\mu, V^\lambda)$). We write

$$\mu \nearrow \lambda$$

in this case. We will see that the symmetric groups are multiplicity-free, that is, if $\mu \in S_{n-1}^\wedge$ and $\lambda \in S_n^\wedge$, then

$$\dim \mathrm{Hom}_{S_{n-1}}(V^\mu, V^\lambda) \in \{0, 1\}.$$

Our aim is to describe this graph, that is, to answer the following question:

Given an irreducible representation of S_n, which irreducible representations of S_{n-1} does it contain?

Answering this question leads to combinatorial objects such as Young diagrams and tableaux. In the classical approach of the representation theory of the symmetric groups, we construct some S_n-modules using these combinatorial objects and observe *a posteriori* that they contain all the irreducible S_n-modules. In the present approach, we start from the branching of the irreducible representations and try to understand it.

We denote by Z_n the center of the algebra $\mathbb{C}[S_n]$, and by X_n the element

$$(1, n) + \cdots + (n-1, n)$$

(with $X_1 = 0$). We then define A_n as the algebra generated by $Z_1, \cdots, Z_n$ in $\mathbb{C}[S_n]$. This algebra is commutative and contains the elements X_i, because

$$X_i = \sum_{S_i} \text{transpositions} - \sum_{S_{i-1}} \text{transpositions}.$$

We are going to prove the following theorem.

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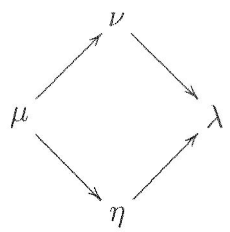
Theorem 2.1. *Let V^λ be an irreducible representation of S_{l+k}, and let V^μ be an irreducible representation of S_l. Then the multiplicity of V^μ in V^λ is at most $k!$.*

In particular, any irreducible representation of S_{n-1} is contained at most once in an irreducible representation of S_n. If $k = 2$, there are three possible situations:

- The representation μ is not contained in λ.
- The multiplicity of μ in λ is equal to 1. Then the part of the Bratteli diagram between λ and μ is of the form (with $\nu \in S_{l+1}$)
$$\mu \to \nu \to \lambda.$$
- The multiplicity of μ in λ is equal to 2. Since μ is contained only once in each irreducible representation of S_{l+1}, we have (with $\eta, \nu \in S_{l+1}$):



Assume that G is a finite group, H is a subgroup of G, and $\rho_1 : G \to GL(V)$ and $\rho_2 : H \to GL(U)$ are some irreducible representations. Then the centralizer

$$\mathbb{C}[G]^H = \{x \in \mathbb{C}[S_n],\, hxh^{-1} = x,\, \forall h \in H\}$$

acts on $\mathrm{Hom}_H(U, V)$ by $f^g = \rho_1(g) \circ f$ ($f \in \mathrm{Hom}_H(U, V)$, $g \in \mathbb{C}[G]^H$), with ρ_1 extended linearly to $\mathbb{C}[G]$. We can easily check that $\mathrm{Hom}_H(U, V)$ is an irreducible $\mathbb{C}[G]^H$-module. We can apply this observation with $G = S_{l+k}$ and $H = S_l$ and write $Z_{k,l} = \mathbb{C}[S_{l+k}]^{S_l}$. We use the following theorem proved in [6] and [2].

Theorem 2.2. *The algebra $Z_{k,l}$ is generated by*

(1) *the elements $X_{l+1}, \cdots, X_{l+k}$;*
(2) *the group S_k regarded as the group of permutations of the set $\{l+1, \cdots, l+k\}$;*
(3) *the algebra Z_l.*

Let us now prove Theorem 2.1. Since the X_i's commute, they have a common eigenvector v in $\mathrm{Hom}_{S_l}(V^\mu, V^\lambda)$. We consider the space V spanned by the vectors

$$s(v) \quad \text{with} \quad s \in S_k.$$

Since Z_l is in the center of $Z_{k,l}$, it acts linearly on the irreducible $Z_{k,l}$-module. Then Z_l stabilizes V. Moreover, the elements of S_k stabilize V, and the relations

$$s_i X_j = X_j s_i, \quad j \notin \{i, i+1\},$$
$$X_{i+1} = s_i X_i s_i + s_i$$

combined with the choice of v show that the X_i's also stabilize V. Hence we have

$$V = \mathrm{Hom}_{S_l}(V^\mu, V^\lambda)$$

and the inequality on the dimension stated in Theorem 2.1.

3. Canonical basis of V^λ

Here we will see how to construct a particular basis of the spaces V^λ. Since the irreducible representations of S_{n-1} are multiplicity-free in V^λ, the decomposition

$$V^\lambda = \bigoplus_{\mu \in S^\wedge_{n-1}, \mu \nearrow \lambda} V^\mu$$

is canonical. Decomposing each space V^μ into irreducible representations of S_{n-2} and iterating this process, we obtain a decomposition of V^λ into lines

$$V^\lambda = \bigoplus_T V_T,$$

where $T = \lambda_1 \nearrow \cdots \nearrow \lambda_n = \lambda$ runs over the set of paths from λ_1 to λ in the Bratteli diagram. A *Young basis* of V^λ is any basis corresponding to this decomposition into one-dimensional subspaces. Applying this procedure to each irreducible representation of S_n, we obtain a particular basis (defined up to scalars) of the space

$$\bigoplus_{\lambda \in S_n^\wedge} V^\lambda.$$

We still call it the *Young basis* and denote it by $\{v_T\}_T$. Here T runs over the set of all paths of length n starting from λ_1 in the Bratteli diagram. Using the isomorphism

$$\mathbb{C}[S_n] \simeq \bigoplus_{\lambda \in S(n)^\wedge} \mathrm{End}\,(V^\lambda),$$

we can identify $\mathbb{C}[S_n]$ with a subalgebra of the algebra of endomorphisms of the space $\bigoplus_{\lambda \in S(n)^\wedge} V^\lambda$. Then we have the following proposition.

Proposition 3.1. *The algebra A_n is the algebra of diagonal operators in the Young basis.*

Proof. Let $\mathcal{A}$ be the algebra of diagonal operators in the Young basis. Clearly, it is a maximal commutative subalgebra of $\mathrm{End}\,(\oplus_\lambda V^\lambda)$. Since A_n is commutative, it suffices to prove that $\mathcal{A} \subset A_n$. We choose a path $T = \lambda_1 \nearrow \cdots \nearrow \lambda_n$ of length n in the Bratteli diagram. Let

$$p_{\lambda_i} = \frac{\dim(V^{\lambda_i})}{i!} \sum_{g \in S(i)} \chi_{\lambda_i}(g) g$$

(χ_i being the character of the representation λ_i). The endomorphism p_{λ_n} is the projection from $\bigoplus_{\lambda \in S(n)^\wedge} V^\lambda$ onto V^{λ_n}, and for each i, the restriction of p_{λ_i} to $V^{\lambda_{i+1}}$ is the projection onto V^{λ_i}. Hence the product

$$p = p_{\lambda_1} \cdots p_{\lambda_n}$$

is the projection onto V_T. Moreover, $p_{\lambda_i} \in Z_i$, and thus p is in A_n. Since the projections onto the lines V_T span $\mathcal{A}$, this finishes the proof. $\square$

Another important consequence of Theorem 2.2 is the following result.

Proposition 3.2. *The elements $X_1, \cdots, X_n$ generate A_n.*

Proof. We prove this result by induction. It is obvious if $n = 2$. Then $X_1, \cdots, X_{n-1}$ generate A_{n-1}, and in particular Z_{n-1}. By Theorem 2.2, X_n and Z_{n-1} generate $Z_{1,n}$, which contains Z_n. $\square$

Now we can consider the following map on the set of all paths in the Bratteli diagram leading from λ_1 to an element of $S_n^\wedge$ (equivalently, on the one-dimensional subspaces corresponding to the Young basis of $\bigoplus_{\lambda \in S(n)^\wedge} V^\lambda$):

$$T \mapsto \phi(T) = (a_1, \cdots, a_n) \in \mathbb{C}^n,$$

where $X_i(v_T) = a_i v_T$. This map is injective. Indeed, if two paths T and T' have the same image under ϕ, then for every f in $\mathcal{A}$ we have

$$f(v_T) = c v_T,$$
$$f(v_{T'}) = c v_{T'}.$$

Taking f equal to the projection onto V_T, we see that the lines V_T and $V_{T'}$ coincide, and thus the paths T and T' are the same. We denote by $\mathrm{Spec}(n)$ the image of ϕ. If $\alpha \in \mathrm{Spec}(n)$, we denote by v_α a vector (which is defined only up to scalar factor) whose image under ϕ is α. Finally, we define an equivalence relation on the set $\mathrm{Spec}(n)$ as follows: $\alpha \sim \beta$ if and only if the vectors v_α and v_β are in the same irreducible representation of S_n, or, in other words, if the corresponding paths in the Bratteli diagram have the same end. In what follows, we describe the form of the set $\mathrm{Spec}(n)$.

We now denote by $s_i = (i, i+1)$ the Coxeter generators of the symmetric group S_n. Let v_T be a vector of the Young basis, and let s_i be one of these generators. If $i \le k-1$, then s_i is in S_k, and the vectors v_T and $s_i(v_T)$ are in the same isotypical component for S_k. If $i \ge k+1$, then s_i commutes with the elements of S_k, so $s_i(v_T)$ and v_T are still in the same isotypical component. Since a vector $v_{T'}$ ($T' = \lambda'_1 \nearrow \cdots \nearrow \lambda'_n$) lies in the isotypical component λ'_i of S_i, this establishes the following result.

Proposition 3.3. *The vector $s_i(v_T)$ is a linear combination of vectors $v_{T'}$ such that*

$$\lambda_j' = \lambda_j \quad \text{if} \quad j \neq k.$$

By Theorem 2.1, there are only two possibilities: there are at most two paths that satisfy the condition of the previous proposition, because the multiplicity of λ_{k-1} in λ_{k+1} is at most two. Hence if v_T is not an eigenvector for the action of s_i, then it is a linear combination of two vectors of the Young basis.

In $\mathbb{C}[S_n]$, the elements s_i, X_i, X_{i+1} generate an algebra denoted by A. The representation of A in a subspace of V^λ is always totally reducible. Indeed, in $\mathbb{C}[S_n]$, the transpositions act by left translations as symmetric operators (for the inner product $\langle g_1, g_2\rangle = \delta_{g_1 g_2}$; g_1, $g_2 \in S_n$). The generators of A are sums of transpositions; hence they are symmetric with respect to this inner product, which implies that the representation of A in $\mathbb{C}[S_n]$ is totally reducible. Since $\mathbb{C}[S_n]$ contains each V^λ, its representations in each V^λ are totally reducible.

Proposition 3.4. *Let $\alpha = (a_1, \cdots, a_n)$ be an element of* Spec(n). *It satisfies the following conditions:*

(1) $a_i \neq a_{i+1}$;

(2) $a_{i+1} = a_i \pm 1 \Leftrightarrow s_i(v_\alpha) = \pm v_\alpha$;

(3) *If $a_{i+1} \neq a_i \pm 1$, then $\beta = s_i(\alpha) = (a_1, \cdots, a_{i+1}, a_i, \cdots, a_n)$ is still an element of* Spec(n) *and $\beta \sim \alpha$. Moreover,*

$$v_\beta = \left(s_i - \frac{1}{a_{i+1} - a_i}\right) v_\alpha.$$

In the basis (v_α, v_β), the elements s_i, X_i, X_{i+1} act as follows:

$$\begin{pmatrix} \frac{1}{a_{i+1}-a_i} & 1 - \frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i - a_{i+1}} \end{pmatrix}, \quad \begin{pmatrix} a_i & 0 \\ 0 & a_{i+1} \end{pmatrix}, \quad \begin{pmatrix} a_{i+1} & 0 \\ 0 & a_i \end{pmatrix}.$$

Proof.

- First assume that v_α and $s_i(v_\alpha)$ are linearly independent. The relations that define the algebra A show that the plane they span is stable under the actions of s_i, X_i, and X_{i+1}. In this basis, these elements act as follows:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} a_i & -1 \\ 0 & a_{i+1} \end{pmatrix}, \quad \begin{pmatrix} a_{i+1} & 1 \\ 0 & a_i \end{pmatrix}.$$

  If $a_i = a_{i+1}$, then X_i is not diagonalizable in the subspace under consideration, which is impossible (this element lies in V^λ and hence in each stable subspace). Thus $a_i \neq a_{i+1}$.
- If v_α and $s_i(v_\alpha)$ are not linearly independent, then $s_i(v_\alpha) = \pm v_\alpha$, and the relation $X_{i+1} = s_i X_i s_i + s_i$ shows that $a_{i+1} = a_i \pm 1$.

This proves the first claim.

Assume that $s_i(v_\alpha)$ and v_α are linearly independent and $a_{i+1} = a_i + \epsilon$ with $\epsilon = \pm 1$. In this case, a simple computation shows that there is only one line stable under the action of A in the plane spanned by these two vectors (it is generated by $s_i(v_\alpha) - \epsilon v_\alpha$). This is impossible because of the total reducibility of the representation of A. Hence if $a_{i+1} = a_i + \epsilon$, then $s_i(v_\alpha)$ and v_α are proportional, and we can check that $s_i(v_\alpha) = \epsilon v_\alpha$. This proves the second claim.

Conversely, if $a_{i+1} - a_i \neq \pm 1$, then the vectors $s_i(v_\alpha)$ and v_α are linearly independent in view of the above argument. Then $v_\beta = \left(s_i - \frac{1}{a_{i+1}-a_i}\right) v_\alpha$ satisfies

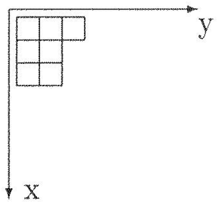
$$X_k(v_\beta) = b_k v_\beta$$

with $b_i = a_{i+1}$, $b_{i+1} = a_i$, and $b_k = a_k$ otherwise. This implies that v_β is a vector of the Young basis lying in the same irreducible representation as v_α (since $i \leq n-1$). Hence $\beta = (a_1, \cdots, a_{i+1}, a_i, \cdots, a_n)$ is an element of Spec(n) and $\alpha \sim \beta$.

In the latter case, the transposition s_i is called *admissible* for α.

4. Young diagrams

A partition of n is a decreasing sequence of integers whose sum equals n. We can represent it by a Young diagram. For instance, the following diagram stands for the partition $(l^1, l^2, l^3) = (3, 2, 2)$ of 7:



A *box* in a Young diagram is described by its coordinates (x, y); by definition, the *content* of a box (x, y) is the integer $c((x, y)) = y - x$. We can now introduce a second graph, called the *Young graph.* Its vertices are all the Young diagrams (of any size). A diagram α with $n-1$ boxes is connected to a diagram β with n boxes if $\alpha \subset \beta$. We denote by λ_1 the unique diagram with one box. We will see that the Bratteli diagram and the Young graph are isomorphic.

If λ is a Young diagram with n boxes, a *Young tableau* (associated to λ) is a path from λ_1 to λ in the Young graph. Equivalently, to obtain a tableau, we have to enumerate the boxes of λ in such a way that for every k, the k first boxes form a Young diagram associated to some partition of k. Here we refer to the terminology of [2], different from the classical one used in other books on the symmetric groups. For instance, in [7], a tableau is just an enumeration of the boxes of a diagram, whereas a tableau with the previous property is called a standard tableau.

Thus a Young tableau of length n can be denoted by $(\nu_1, \cdots, \nu_n)$, ν_i being a Young diagram with i boxes, $\nu_{i-1} \subset \nu_i$. In this case $\nu_i \setminus \nu_{i-1}$ is just one box. Thus we can consider the element

$$(c(\nu_1), c(\nu_2 \setminus \nu_1), \cdots, c(\nu_n \setminus \nu_{n-1}))$$

of $\mathbb{Z}^n$. The first box of any diagram always has coordinates $(1, 1)$, so we always have $c(\nu_1) = 0$. Moreover, when we place the ith box, the space above and to the left of this box is already occupied by the previous boxes. In particular, there is a box immediately above or immediately to the left of the ith box. This implies $\{c(\nu_i/\nu_{i-1}) - 1, c(\nu_i/\nu_{i-1}) + 1\} \cap \{c(\nu_1), \cdots, c(\nu_{i-1}/\nu_{i-2})\} \neq \emptyset$. Finally, if a box has coordinates $(b, b+a)$ and thus lies on the line $y - x = a$, and if we want to put a second box on the same line, we must first fill one of the following two boxes: $(b+1, b+a)$ or $(b, b+a+1)$. This leads us to the following definition.

Definition 4.1. *We define* $\mathrm{Cont}(n)$ *as the set of elements* $\alpha = (a_1, \cdots, a_n) \in \mathbb{C}^n$ *such that the following conditions hold:*

(1) $a_1 = 0$;

(2) $\{a_i + 1, a_i - 1\} \cap \{a_1, \cdots, a_{i-1}\} \neq \emptyset$ *if* $i \geq 2$;

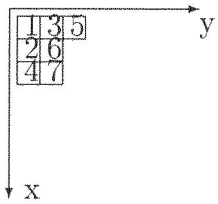
(3) *If* $a_p = a_q = a$ *with* $p < q$, *then*

$$\{a+1, a-1\} \subset \{a_{p+1}, \cdots, a_{q-1}\}.$$

We denote by $\mathrm{Tab}(n)$ the set of all Young tableaux of length n, that is, the set of all paths of length n (from λ_1) in the Young graph. With the previous observations, we have a map

$$\mathrm{Tab}(n) \to \mathrm{Cont}(n) \subset \mathbb{Z}^n,$$
$$(\nu_1, \cdots, \nu_n) \mapsto (c(\nu_1), c(\nu_2/\nu_1), \cdots, c(\nu_n/\nu_{n-1})).$$

We can easily check that this map is a bijection. For instance, to the tableau



we associate the element $(0, -1, 1, -2, 2, 0, -1)$ of $\mathbb{Z}^7$.

Now we are able to prove the following theorem.

Theorem 4.2. $\operatorname{Spec}(n) \subset \operatorname{Cont}(n)$.

Lemma 4.3. *Let $\alpha = (a_1, \cdots, a_n) \in \mathbb{C}^n$ be such that for some i,*

$$a_i = a_{i+1} + 1 = a_{i+2} \quad \text{or} \quad a_i = a_{i+1} - 1 = a_{i+2}.$$

Then α is not in the set $\operatorname{Spec}(n)$.

Proof of the lemma. Assume that α is in $\operatorname{Spec}(n)$. We can use Proposition 3.4 and the Coxeter relation

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

If $a_i = a_{i+1} + 1 = a_{i+2}$, then the left-hand side of the equation above multiplies v_α by -1, whereas the right-hand side multiplies it by 1. The argument is the same if $a_i = a_{i+1} - 1 = a_{i+2}$. $\square$

Proof of the theorem. Let us consider an element $\alpha = (a_1, \cdots, a_n)$ in $\operatorname{Spec}(n)$. It is obvious that $a_1 = 0$, since X_1 is zero. We check the two other properties by induction on n.

• The value a_2 is in $\{\pm 1\}$, because it is an eigenvalue of an element of order two. Assume now that the property is true for $i \leq n-1$, and assume that $\{a_n + 1, a_n - 1\} \cap \{a_1, \cdots, a_{n-1}\} = \emptyset$. Then $(n-1, n)$ is admissible, so that $(a_1, \cdots, a_{n-2}, a_n)$ is in $\operatorname{Spec}(n-1)$. But the hypothesis we have made implies, in particular, $\{a_n+1, a_n-1\} \cap \{a_1, \cdots, a_{n-2}\} = \emptyset$, which contradicts the induction hypothesis on $\operatorname{Spec}(n-1)$. This establishes the second claim.

• First we observe that a_3 cannot be zero, since otherwise α contains the sequence $(0, \pm 1, 0)$. Assume now that $a_p = a_n = a$ with $p < n$. We may choose p to be the maximum integer such that $a \notin \{a_{p+1}, \cdots, a_{n-1}\}$; this does not change the result we have to prove. First assume that $a + 1 \notin \{a_{p+1}, \cdots, a_{n-1}\}$. The integer $a - 1$ is contained at most once in $\{a_{p+1}, \cdots, a_{n-1}\}$, since otherwise, by the induction hypothesis, a would appear between two occurrences of $a - 1$, which contradicts our choice of p.

Now, applying admissible transpositions, we obtain an element of $\operatorname{Spec}(n)$ that contains one of the following sequences:

$$(\cdots a, a \cdots), \quad (\cdots, a, a-1, a, \cdots).$$

This is impossible by Proposition 3.4 and Lemma 4.3. In the same way we prove that $a - 1 \in \{a_{p+1}, \cdots, a_{n-1}\}$. $\square$

Remark. We know by the previous theorem that $\operatorname{Spec}(n) \subset \mathbb{Z}^n$. Earlier we have proved that we always have $s_i(v_\alpha) = \pm v_\alpha$ or $s_i(v_\alpha) = v_\beta + \frac{1}{a_{i+1} - a_i} v_\alpha$. The coefficient 1 in front of v_β is in fact arbitrary, because vectors of the Young basis are defined only up to scalar. But these formulas allow us to think that we can define the representation over the field of rational numbers, provided that the generator of each line V_T of the Young basis is chosen appropriately. It is shown in [2] how we can arbitrarily choose a vector v_T for a certain path T and then deduce a particular choice for all the other vectors so that the representation will be well-defined over $\mathbb{Q}$.

The notion of admissible transposition for an element α of $\operatorname{Spec}(n)$ can be extended to an element α of $\operatorname{Cont}(n)$: a transposition s_i is admissible for α if it exchanges two consecutive coordinates whose difference is not ± 1. In terms of paths in the Young graph, an admissible transposition exchanges two consecutive boxes in a path that are neither in the same column nor in the same row.

We now reach our first aim and prove that the Young graph and the Bratteli diagram are isomorphic. To do this, let us introduce an equivalence relation on $\operatorname{Cont}(n)$ as follows:

$$(a_1, \cdots, a_n) \approx (b_1, \cdots, b_n) \Leftrightarrow \exists\, \sigma \in S_n \text{ such that } a_i = b_{\sigma(i)}.$$

We can interpret this relation in terms of tableaux: two tableaux are equivalent if and only if they lie on the same diagram. In terms of paths in the Young graph, we have exactly the same interpretation as for the Bratteli diagram and the relation $\sim$ (two paths of length n are equivalent if they have the same end).

Lemma 4.4. *Let T_1 and T_2 be two paths from λ_1 to λ in the Young graph. Then we can obtain T_2 from T_1 by admissible transpositions.*

Proof. It suffices to prove that any path from λ_1 to $\lambda = (l^1, l^2, \cdots)$ can be transformed by admissible transpositions into the path associated to the following tableau:

$$\begin{array}{ccccc} 1 & 2 & \cdots & \cdots & l^1 \\ l^1+1 & \cdots & l^1+l^2 & & \\ \cdots & & & & \end{array}$$

It is sufficient to prove that we can put the number n in the last box of the last row of the diagram (and then use induction). We denote by i the number that is initially in the last box of the last row of the tableau under consideration. The numbers $\geq i+1$ cannot be in the same column as the number i: it is at the end of the column, and the other boxes of the column contain numbers $\leq i-1$. For the same reason, the numbers $i+1, \cdots, n$ cannot be in the same row as i. Hence the transpositions $(i, i+1), \cdots, (n-1, n)$ are admissible. $\square$

Assume now that $\alpha \in \mathrm{Spec}(n)$ and $\beta \in \mathrm{Cont}(n)$ with $\alpha \approx \beta$. To say that $\alpha \approx \beta$ means that the tableaux associated to α and β are on the same diagram. By the previous lemma, we can pass from one tableau to the other one by admissible transpositions. Since these transpositions preserve the set $\mathrm{Spec}(n)$ (by Proposition 3.4) and also preserve the equivalence classes for $\sim$, we have $\beta \in \mathrm{Spec}(n)$ and $\alpha \sim \beta$.

Now we can observe that $\mathrm{Spec}(n)/\sim$ and $\mathrm{Cont}(n)/\approx$ have the same cardinality, which is the number of partitions of n. The above argument shows that an equivalence class for $\approx$ either contains no element of $\mathrm{Spec}(n)$, or is contained in an equivalence class for $\sim$. This implies that the classes for $\approx$ and $\sim$ coincide and, moreover, that

$$\mathrm{Spec}(n) = \mathrm{Cont}(n) \text{ and the relations } \sim \text{ and } \approx \text{ coincide.}$$

Put differently, this means that for each integer n, the set of paths from the unique vertex of the first level to the vertices of level n in the Young graph and in the Bratteli diagram are in a bijection. Hence we have proved the following theorem.

Theorem 4.5. *The sets* $\mathrm{Spec}(n)$ *and* $\mathrm{Cont}(n)$ *coincide. The Young graph and the Bratteli diagram are isomorphic, and the relations* $\sim$ *and* $\approx$ *are the same.*

Remark. This theorem contains the branching rule for the restriction of representations:

An irreducible representation of S_n *contains an irreducible representation of* S_{n-1} *if and only if the corresponding diagrams are contained one in the other.*

Combining this with the Frobenius reciprocity formula, we can obtain the same result for the induction of representations.

Let V^μ (respectively, V^λ) be an irreducible representation of S_{n-1} (respectively, S_n). Then V^λ is contained in $\mathrm{Ind}_{S_{n-1}}^{S_n} V^\mu$ if and only if the diagram associated to μ is contained in the diagram associated to λ.

The approach to the representation theory of S_n developed in [2] presents many advantages. First, the results we have presented here allow us to establish the branching rule for representations simultaneously with the description of representations themselves. Another point is that the description of representations is recursive: to describe representations of S_n, we use the descriptions of representations of $S_{n-1}, S_{n-2} \ldots$.

5. An application

As an application, we can now see how this method allow us to describe the spectrum of the element $X_n = (1, n) + \cdots + (n-1, n)$, which acts on $\mathbb{C}[S_n]$ by left translations.

If λ is a partition of n, we now describe the spectrum of X_n in the isotypical component $(\dim V^\lambda)V^\lambda$ of $\mathbb{C}[S_n]$. We denote by λ^- any partition of $n-1$ obtained by deleting a box of λ (the box λ/λ^- is called an "inner corner" in [7]). Then λ/λ^- is the last box of $\dim V^{\lambda^-}$ paths from λ_1 to λ in the Bratteli diagram. Hence the integer $c(\lambda/\lambda^-)$ is an eigenvalue for X_n with multiplicity $\dim V^{\lambda^-}$ in V^λ. All the eigenvalues of X_n are integers, and since they have the form

$$y - x, \quad x, y \in \{1, \cdots, n\},$$

they are integers from $\{1-n, \cdots, n-1\}$. Using the partitions

$$(n-k, \underbrace{1, \cdots, 1}_{k\,\text{times}}),$$

it is easy to see that all the integers from $\{1-n, \cdots, n-1\}$ are eigenvalues of X_n.

But computing the multiplicity in $\mathbb{C}[S_n]$ of a given integer is rather difficult: first, because we have to determine all the partitions λ of n for which the integer occurs in V^λ, and, second, because explicitly computing the multiplicity inside V^λ requires knowing the dimensions of irreducible representations of S_n (for which there is no simple formula). For instance, we can compute the multiplicities of the two highest eigenvalues: the only representation in which the value $n-1$ can occur is the one-dimensional representation associated to the partition

(n) (the corresponding representation is the trivial one: $S_n \to \{1\}$); hence $n-1$ has multiplicity one in $\mathbb{C}[S_n]$. To compute the multiplicity of $n-2$, we have to see for which integers from $\{1, \cdots, n\}$ we can have

$$y - x = n - 2.$$

The case $(x, y) = (2, n)$ is impossible (the corresponding Young diagram would have more than $2n$ boxes); the case $(x, y) = (1, n-1)$ can occur only in the representation associated to the partition $(n-1, 1)$, which is of dimension $n-1$. The representation of S_{n-1} that we obtain by deleting the box with coordinates $(n-1, 1)$ is $(n-2)$-dimensional. Finally, the multiplicity of $n-2$ in $\mathbb{C}[S_n]$ is $(n-1)(n-2)$.

Remark. If we consider the Laplacian $\Delta = n - 1 - X_n$, which acts on $\mathbb{C}[S_n]$, then the above argument shows that its first positive eigenvalue is one with multiplicity $(n-1)(n-2)$. This result was already proved in [3] and [4] without using representation theory. Moreover, these computations were already carried out with the help of classical methods of representation theory in [1, 8, 5].

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