On Continuity of Quasimorphisms for Symplectic Maps

Michael Entov, Leonid Polterovich, Pierre Py, with an Appendix by Michael Khanevsky

Dedicated to Oleg Viro on the occasion of his 60th birthday

Abstract We discuss C^0-continuous homogeneous quasimorphisms on the identity component of the group of compactly supported symplectomorphisms of a symplectic manifold. Such quasimorphisms extend to the C^0-closure of this group inside the homeomorphism group. We show that for standard symplectic balls of any dimension, as well as for compact oriented surfaces other than the sphere, the space of such quasimorphisms is infinite-dimensional. In the case of surfaces, we give a user-friendly topological characterization of such quasimorphisms. We also present an application to Hofer's geometry on the group of Hamiltonian diffeomorphisms of the ball.

Keywords Symplectomorphism • Quasimorphism • Calabi homomorphism • Hofer metric

M. Entov (✉)
Department of Mathematics, Technion – Israel Institute of Technology, Israel
e-mail: entov@math.technion.ac.il

L. Polterovich
School of Mathematical Sciences, Tel Aviv University, Israel

Department of Mathematics, University of Chicago, USA
e-mail: polterov@post.tau.ac.il

P. Py
Department of Mathematics, University of Chicago, USA
e-mail: pierre.py@math.uchicago.edu

M. Khanevsky
School of Mathematical Sciences, Tel Aviv University, Israel
e-mail: khanev@post.tau.ac.il

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1 Introduction and Main Results

1.1 Quasimorphisms on Groups of Symplectic Maps

Let (Σ,ω) be a compact connected symplectic manifold (possibly with nonempty boundary $\partial\Sigma$). Denote by $\mathcal{D}(\Sigma,\omega)$ the identity component of the group of symplectic C^∞-diffeomorphisms of Σ whose supports lie in the interior of Σ. Write[1] $\mathcal{H}(\Sigma,\omega)$ for the C^0-closure of $\mathcal{D}(\Sigma,\omega)$ in the group of homeomorphisms of Σ supported in the interior of Σ. We always equip Σ with a distance d induced by a Riemannian metric on Σ, and view the C^0-topology on the group of homeomorphisms of Σ as the topology defined by the metric $dist(\phi,\psi) = \max_{x\in\Sigma} d(x,\psi^{-1}\phi(x))$.

The study of the algebraic structure of the groups $\mathcal{D}(\Sigma,\omega)$ was pioneered by Banyaga; see [2, 4]. For instance, when Σ is closed, he calculated the commutator subgroup of $\mathcal{D}(\Sigma,\omega)$ and showed that it is simple. However, the algebraic structure of the groups $\mathcal{H}(\Sigma,\omega)$ is much less understood. Even for the standard two-dimensional disk $\mathbb{D}^2$, it is still unknown whether $\mathcal{H}(\mathbb{D}^2)$ coincides with its commutator subgroup (see [10] for a comprehensive discussion). In the present paper, we focus on a particular algebraic feature of the groups $\mathcal{H}(\Sigma,\omega)$: homogeneous quasimorphisms.

Recall that a homogeneous quasimorphism on a group Γ is a map $\mu:\Gamma\to\mathbf{R}$ that satisfies the following two properties:

1. There exists a constant $C(\mu)\geq 0$ such that $|\mu(xy)-\mu(x)-\mu(y)|\leq C(\mu)$ for any x, y in Γ.
2. $\mu(x^n) = n\mu(x)$ for all $x\in\Gamma$ and $n\in\mathbf{Z}$.

Let us recall two well-known properties of homogeneous quasimorphisms that will be useful in the sequel: they are invariant under conjugation, and their restrictions to abelian subgroups are homomorphisms.

The space of all homogeneous quasimorphisms is an important algebraic invariant of the group. Quasimorphisms naturally appear in the theory of bounded cohomology and are crucial in the study of the commutator length [6]. We refer to [6, 14, 23] or [28] for a more detailed introduction to the theory of quasimorphisms.

Recently, several authors discovered that certain groups of diffeomorphisms preserving a volume or a symplectic form carry homogeneous quasimorphisms; see [5, 7, 17–19, 22, 41, 44, 45]. However, in many cases explicit constructions of nontrivial quasimorphisms on $\mathcal{D}(\Sigma,\omega)$ require a certain type of smoothness in an essential manner. Nevertheless, as we shall show below, some homogeneous quasimorphisms can be extended from $\mathcal{D}(\Sigma,\omega)$ to $\mathcal{H}(\Sigma,\omega)$.

Our first result deals with the case of the Euclidean unit ball $\mathbb{D}^{2n}$ in the standard symplectic linear space.

[1] We abbreviate $\mathcal{D}(\Sigma)$ and $\mathcal{H}(\Sigma)$ whenever the symplectic form ω is clear from the context.

Theorem 1. *The space of homogeneous quasimorphisms on $\mathcal{H}(\mathbb{D}^{2n})$ is infinite-dimensional.*

The proof is given in Sect. 2. Next, we focus on the case of a compact connected surface Σ equipped with an area form. Note that in this case, $\mathcal{H}(\Sigma)$ coincides with the identity component of the group of all area-preserving homeomorphisms supported in the interior of Σ; see [40] or [48].

Theorem 2. *Let Σ be a compact connected oriented surface other than the sphere $\mathbb{S}^2$, equipped with an area form. The space of homogeneous quasimorphisms on $\mathcal{H}(\Sigma)$ is infinite-dimensional.*

The proof is given in Sect. 4. This result is new, for instance, in the case of the 2-torus. The case of the sphere is still out of reach; see Sect. 5.2 for a discussion. Interestingly enough, for balls of any dimension and for the two-dimensional annulus, all our examples of homogeneous quasimorphisms on $\mathcal{H}$ are based on Floer theory. When Σ is of genus greater than one, the group $\mathcal{H}(\Sigma)$ carries plenty of homogeneous quasimorphisms, and the statement of Theorem 2 readily follows from the work of Gambaudo and Ghys [22].

As an immediate application, Theorems 1 and 2 yield that if Σ is a ball or a compact oriented surface other than the sphere, then the stable commutator length is unbounded on the commutator subgroup of $\mathcal{H}(\Sigma)$. This is a standard consequence of Bavard's theory [6].

1.2 Detecting Continuity

A key ingredient of our approach is the following proposition, due to Shtern [47]. It is a simple (nonlinear) analogue of the fact that linear forms on a topological vector space are continuous if and only if they are bounded in a neighborhood of the origin.

Proposition 1 ([47]). *Let Γ be a topological group and $\mu : \Gamma \to \mathbf{R}$ a homogeneous quasimorphism. Then μ is continuous if and only if it is bounded on a neighborhood of the identity.*

Proof. We only prove the "if" part. Assume that $|\mu|$ is bounded by $K > 0$ on an open neighborhood $\mathcal{U}$ of the identity. Let $g \in \Gamma$. For each $p \in \mathbf{N}$, define

$$\mathcal{V}_p(g) := \left\{h \in \Gamma \mid h^p \in g^p\mathcal{U}\right\}.$$

It is easy to see that $\mathcal{V}_p(g)$ is an open neighborhood of g. Pick any $h \in \mathcal{V}_p(g)$. Then $h^p = g^p f$ for some $f \in \mathcal{U}$. Therefore

$$|\mu(h^p) - \mu(g^p) - \mu(f)| \le C(\mu),$$

and hence

$$|\mu(h)-\mu(g)|\leq \frac{C(\mu)+K}{p},$$

which immediately yields the continuity of μ at g. □

Let us discuss in greater detail the extension problem for quasimorphisms. The next proposition shows that C^0-continuous homogeneous quasimorphisms on $\mathcal{D}(\Sigma)$ extend to $\mathcal{H}(\Sigma)$.

Proposition 2. *Let Λ be a topological group and let $\Gamma \subset \Lambda$ be a dense subgroup. Any continuous homogeneous quasimorphism on Γ extends to a continuous homogeneous quasimorphism on Λ.*

Proof. Since μ is continuous, it is bounded by a constant $C>0$ on an open neighborhood $\mathcal{U}$ of the identity in Γ. Since $\mathcal{U}$ is open in Γ, there exists $\mathcal{U}'$, open in Λ, such that $\mathcal{U}=\mathcal{U}'\cap\Gamma$. We fix an open neighborhood $\mathcal{O}$ of the identity in Λ such that $\mathcal{O}^2\subset\mathcal{U}'$ and $\mathcal{O}=\mathcal{O}^{-1}$. Given $g\in\Lambda$ and $p\in\mathbf{N}$, we define as before

$$\mathcal{V}_p(g):=\left\{h\in\Lambda \mid h^p\in g^p\mathcal{O}\right\}.$$

Pick a sequence $\{h_k\}$ in Γ such that each h_k lies in $\mathcal{V}_1(g)\cap\ldots\cap\mathcal{V}_k(g)$. For $k\geq p$, we can write $h_k^p=g^pg_{k,p}$ ($g_{k,p}\in\mathcal{O}$). If $k_1,k_2\geq p$, we can write

$$h_{k_1}^p=h_{k_2}^p g_{k_2,p}^{-1}g_{k_1,p},\quad g_{k_2,p}^{-1}g_{k_1,p}\in\mathcal{U}.$$

Hence, we have the inequality

$$\left|\mu(h_{k_1})-\mu(h_{k_2})\right|\leq\frac{C+C(\mu)}{p}\quad (k_1,k_2\geq p),$$

and $\{\mu(h_p)\}$ is a Cauchy sequence in $\mathbf{R}$. Denote its limit by $\mu'(g)$. One can check easily that the definition is correct and that for any sequence $g_i\in\Gamma$ converging to $g\in\Lambda$, one has $\mu(g_i)\to\mu'(g)$. This readily yields that the resulting function $\mu':\Lambda\to\mathbf{R}$ is a homogeneous quasimorphism extending μ. Its continuity follows from Proposition 1. □

In view of this proposition, all we need for the proof of Theorems 1 and 2 is to exhibit nontrivial homogeneous quasimorphisms on $\mathcal{D}(\Sigma)$ that are continuous in the C^0-topology. This leads us to the problem of continuity of homogeneous quasimorphisms, which is highlighted in the title of the present paper.

Remark 1. Note that all the concrete quasimorphisms that we know on groups of diffeomorphisms are continuous in the C^1-topology.

1.3 The Calabi Homomorphism and Continuity on Surfaces

It is a classical fact that the Calabi homomorphism is not continuous in the C^0-topology; see [21]. We will discuss the example of the unit ball in $\mathbf{R}^{2n}$ and then explain why the reason for the discontinuity of the Calabi homomorphism is, in a sense, universal.

First, let us recall the definition of the group of Hamiltonian diffeomorphisms of a symplectic manifold (Σ, ω). Given a smooth function $F : \Sigma \times S^1 \to \mathbf{R}$ supported in Interior$(\Sigma) \times S^1$, consider the time-dependent vector field sgradF_t given by $i_{\mathrm{sgrad}F_t}\omega = -\mathrm{d}F_t$, where $F_t(x)$ stands for $F(x,t)$. The flow f_t of this vector field is called the *Hamiltonian flow generated by the Hamiltonian function* F, and its time-one map f_1 is called the *Hamiltonian diffeomorphism generated by* F. Hamiltonian diffeomorphisms form a normal subgroup of $\mathcal{D}(\Sigma, \omega)$, denoted by $\mathrm{Ham}(\Sigma, \omega)$ or just by $\mathrm{Ham}(\Sigma)$. The quotient $\mathcal{D}(\Sigma)/\mathrm{Ham}(\Sigma)$ is isomorphic to a quotient of the group $H^1_{\mathrm{comp}}(\Sigma, \mathbf{R})$. In particular, $\mathcal{D}(\Sigma) = \mathrm{Ham}(\Sigma)$ for $\Sigma = \mathbb{D}^{2n}$ or for $\Sigma = \mathbb{S}^2$. We refer to [38] for the details.

Example 1. Let $\Sigma = \mathbb{D}^{2n}$ be the closed unit ball in $\mathbf{R}^{2n}$ equipped with the symplectic form $\omega = \mathrm{d}p \wedge \mathrm{d}q$. Take any diffeomorphism $f \in \mathrm{Ham}(\mathbb{D}^{2n})$ and choose a Hamiltonian F generating f. The value

$$\mathrm{Cal}(f) := \int_0^1 \int_{\mathbb{D}^{2n}} F(p,q,t)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}t$$

depends only on f and defines the *Calabi homomorphism* $\mathrm{Cal} : \mathcal{D}(\mathbb{D}^{2n}) \to \mathbf{R}$ [13].

Take a sequence of time-independent Hamiltonians F_i supported in balls of radii $\frac{1}{i}$ such that $\int_{\mathbb{D}^{2n}} F_i\,\mathrm{d}p\,\mathrm{d}q = 1$. The corresponding Hamiltonian diffeomorphisms f_i C^0-converge to the identity and satisfy $\mathrm{Cal}(f_i) = 1$. We conclude that the Calabi homomorphism is discontinuous in the C^0-topology.

In the remainder of this section, let us return to the case in which Σ is a compact connected surface equipped with an area form. Our next result shows, roughly speaking, that for a quasimorphism μ on $\mathrm{Ham}(\Sigma)$, its nonvanishing on a sequence of Hamiltonian diffeomorphisms f_i supported in a collection of shrinking balls is the only possible reason for discontinuity. The next remark is crucial for understanding this phenomenon. Observe that support$(f^N) \subset$ support(f) for any diffeomorphism f. Thus in the statement above, nonvanishing yields unboundedness: if $\mu(f_i) \neq 0$ for all i, then the sequence $\mu(f_i^{N_i}) = N_i \mu(f_i)$ is unbounded for an appropriate choice of N_i.

Theorem 3. *Let* $\mu : \mathrm{Ham}(\Sigma) \to \mathbf{R}$ *be a homogeneous quasimorphism. Then* μ *is continuous in the* C^0*-topology if and only if there exists* $a > 0$ *such that the following property holds: For any disk* $D \subset \Sigma$ *of area less than a, the restriction of* μ *to the group* $\mathrm{Ham}(D)$ *vanishes.*

Here, by a disk in Σ we mean the image of a smooth embedding $\mathbb{D}^2 \hookrightarrow \Sigma$. We view it as a surface with boundary equipped with the area form that is the restriction of the area form on Σ. The "only if" part of the theorem is elementary. It extends to certain four-dimensional symplectic manifolds (see Remark 2 below). The proof of the "if" part is more involved, and no extension to higher dimensions is available to us so far (see the discussion in Sect. 5.3 below).

Corollary 1. *Let $\mu : \mathcal{D}(\Sigma) \to \mathbf{R}$ be a homogeneous quasimorphism. Suppose that the following hold:*

(i) There exists $a > 0$ such that for any disk $D \subset \Sigma$ of area less than a, the restriction of μ to the group $\mathrm{Ham}(D)$ *vanishes.*
(ii) The restriction of μ to each one-parameter subgroup of $\mathcal{D}(\Sigma)$ is linear.

Then μ is continuous in the C^0-topology.

Note that assumption (ii) is indeed necessary, provided one believes in the axiom of choice. Indeed, assuming that Σ is not $\mathbb{D}^2$, $\mathbb{S}^2$, or $\mathbb{T}^2$, the quotient $\mathcal{D}(\Sigma)/\mathrm{Ham}(\Sigma)$ is isomorphic to the additive group of the vector space $V := H^1_{\mathrm{comp}}(\Sigma, \mathbf{R}) \neq \{0\}$. Define a quasimorphism $\mu : \mathcal{D}(\Sigma) \to \mathbf{R}$ as the composition of the projection $\mathcal{D}(\Sigma) \to V$ with a *discontinuous* homomorphism $V \to \mathbf{R}$. The homomorphism μ satisfies (i), since it vanishes on $\mathrm{Ham}(\Sigma)$, and it is obviously discontinuous.

The criteria of continuity stated in Theorem 3 and Corollary 1 are proved in Sect. 3. They will be used in Sect. 4 in order to verify C^0-continuity of a certain family of quasimorphisms on $\mathcal{D}(\mathbb{T}^2)$ introduced in [22] and explored in [46], which will enable us to complete the proof of Theorem 2.

1.4 An Application to Hofer's Geometry

Here we concentrate on the case of the unit ball $\mathbb{D}^{2n} \subset \mathbf{R}^{2n}$. For a diffeomorphism $f \in \mathrm{Ham}(\mathbb{D}^{2n})$, define its Hofer norm [26] as

$$\|f\|_H := \inf \int_0^1 \left(\max_{z \in \mathbb{D}^{2n}} F(z,t) - \min_{z \in \mathbb{D}^{2n}} F(z,t) \right) \mathrm{d}t,$$

where the infimum is taken over all Hamiltonian functions F generating f. Hofer's famous result states that $d_{\mathrm{H}}(f,g) := \|fg^{-1}\|_{\mathrm{H}}$ is a nondegenerate bi-invariant metric on $\mathrm{Ham}(\mathbb{D}^{2n})$. It is called *Hofer's metric* (see also [31, 42] for Hofer's metric on general symplectic manifolds). It turns out that the quasimorphisms that we construct in the proof of Theorem 1 are Lipschitz with respect to Hofer's metric. Hence, our proof of Theorem 1 yields the following result:

Proposition 3. *The space of homogeneous quasimorphisms on the group* $\mathrm{Ham}(\mathbb{D}^{2n})$ *that are both continuous for the C^0-topology and Lipschitz for Hofer's metric is infinite-dimensional.*

The relation between Hofer's metric and the C^0-metric on $\mathrm{Ham}(\Sigma)$ is subtle. First of all, the C^0-metric is never continuous with respect to Hofer's metric. Furthermore, arguing as in Example 1, one can show that Hofer's metric on $\mathrm{Ham}(\mathbb{D}^{2n})$ is not continuous in the C^0-topology. However, for $\mathbf{R}^{2n}$ equipped with the standard symplectic form $\mathrm{d}p \wedge \mathrm{d}q$ (informally speaking, this corresponds to the case of a ball of infinite radius), Hofer's metric is continuous for the C^0-Whitney topology [27].

An attempt to understand the relationship between Hofer's metric and the C^0-metric led Le Roux [34] to the following problem. Let $\mathscr{E}_C \subset \mathrm{Ham}(\mathbb{D}^{2n})$ be the complement of the closed ball (in Hofer's metric) of radius C centered at the identity:

$$\mathscr{E}_C := \left\{ f \in \mathrm{Ham}(\mathbb{D}^{2n}), d_H(f, \mathbb{1}) > C \right\}.$$

Le Roux asked the following: Is it true that $\mathscr{E}_C$ has nonempty interior in the C^0-topology for any $C > 0$?

The energy-capacity inequality [26] states that if $f \in \mathrm{Ham}(\mathbb{D}^{2n})$ displaces $\phi(\mathbb{D}^{2n}(r))$, where ϕ is any symplectic embedding of the Euclidean ball of radius r, then Hofer's norm of f is at least πr^2. (We say that f *displaces a set* U if $f(U) \cap \bar{U} = \emptyset$.) By Gromov's packing inequality [25], this could happen only when $r^2 \leq 1/2$. Since any Hamiltonian diffeomorphism that is C^0-close to f also displaces a slightly smaller ball $\phi(D^{2n}(r'))$ $(r' < r)$, we get that $\mathscr{E}_C$ indeed has nonempty interior in the C^0-sense for $C < \pi/2$. Using our quasimorphisms, we get an affirmative answer to Le Roux's question even for large values of C.

Corollary 2. *For any $C > 0$, the set $\mathscr{E}_C$ has nonempty interior in the C^0-topology.*

Proof. The statement follows simply from the existence of a nontrivial homogeneous quasimorphism $\mu : \mathrm{Ham}(\mathbb{D}^{2n}) \to \mathbf{R}$ that is both continuous in the C^0-topology and Lipschitz with respect to Hofer's metric. Indeed, choose a diffeomorphism f such that

$$\frac{|\mu(f)|}{\mathrm{Lip}(\mu)} \geq C + 1,$$

where $\mathrm{Lip}(\mu)$ is the Lipschitz constant of μ with respect to Hofer's metric. There is a neighborhood O of f in $\mathrm{Ham}(\mathbb{D}^{2n})$ in the C^0-topology on which $|\mu| > C \cdot \mathrm{Lip}(\mu)$. We get that $\|g\|_H > C$ for $g \in O$, and hence $O \subset \mathscr{E}_C$. This proves the corollary. $\square$

Note that Le Roux's question makes sense on any symplectic manifold. For certain closed symplectic manifolds with infinite fundamental group one can easily get a positive answer using the energy-capacity inequality in the universal cover (as in [32, 33]). However, for closed simply connected manifolds (and already for the case of the 2-sphere), the question is wide open.

2 Quasimorphisms for the Ball

In this section we prove Theorem 1. Denote by $\mathbb{D}^{2n}(r)$ the Euclidean ball $\{|p|^2+|q|^2 \le r^2\}$, so that $\mathbb{D}^{2n}=\mathbb{D}^{2n}(1)$. We say that a set U in a symplectic manifold (Σ,ω) is *displaceable* if there exists $\phi \in \mathrm{Ham}(\Sigma)$ that displaces it: $\phi(U)\cap\bar{U}=\emptyset$. A quasimorphism $\mu : \mathrm{Ham}(\Sigma) \to \mathbf{R}$ will be called *Calabi* if for any displaceable domain $U \subset M$ such that $\omega|_U$ is exact, one has $\mu|_{\mathrm{Ham}(U)} = \mathrm{Cal}|_{\mathrm{Ham}(U)}$.

We will use the following result, established in [18]: there exists $a>0$ such that the group $\mathrm{Ham}(\mathbb{D}^{2n}(1+a))$ admits an infinite-dimensional space of quasimorphisms that are Lipschitz in Hofer's metric, vanish on $\mathrm{Ham}(U)$ for every displaceable domain $U \subset \mathbb{D}^{2n}(1+a)$, and do not vanish on $\mathrm{Ham}(\mathbb{D}^{2n})$. These quasimorphisms are obtained by subtracting the appropriate multiple of the Calabi homomorphism from the Calabi quasimorphisms constructed in [9]. We claim that the restriction of each such quasimorphism, say η, to $\mathrm{Ham}(\mathbb{D}^{2n})$ is continuous in the C^0-topology. By Proposition 2, this yields the desired result. By Proposition 1, it suffices to show that for some $\epsilon>0$ the quasimorphism η is bounded on all $f \in \mathrm{Ham}(\mathbb{D}^{2n})$ such that

$$|f(x)-x| < \epsilon \;\; \forall x \in \mathbb{D}^{2n}. \tag{1}$$

For $c>0$ define the strip

$$\Pi(c) := \big\{(p,q)\in \mathbf{R}^{2n} \;:\; |q_n| < c\big\}.$$

Choose $\epsilon>0$ so small that $\Pi(2\epsilon)\cap\mathbb{D}^{2n}$ is displaceable in $\mathbb{D}^{2n}(1+a)$. Put $D_\pm := \mathbb{D}^{2n}\cap\{\pm q_n>0\}$. Observe that $D_\pm$ are displaceable in $\mathbb{D}^{2n}(1+a)$ by a Hamiltonian diffeomorphism that can be represented outside a neighborhood of the boundary as a small vertical shift along the q_n-axis (in the case of D_+, we take the shift that moves it up, and in the case of D_-, the shift that moves it down) composed with a 180° rotation in the (p_n,q_n)-plane. The desired boundedness result immediately follows from the following fragmentation-type lemma:

Lemma 1. *Assume that $f \in \mathrm{Ham}(\mathbb{D}^{2n})$ satisfies (1). Then f can be decomposed as $\theta\phi_+\phi_-$, where $\theta \in \mathrm{Ham}(\Pi(2\epsilon)\cap\mathbb{D}^{2n})$ and $\phi_\pm \in \mathrm{Ham}(D_\pm)$.*

Indeed, η vanishes on $\mathrm{Ham}(U)$ for every displaceable domain $U \subset \mathbb{D}^{2n}(1+a)$. Since $\Pi(2\epsilon)\cap\mathbb{D}^{2n}$ and $D_\pm$ are displaceable, $\eta(\theta)=\eta(\phi_\pm)=0$. Thus $|\eta(f)| \le 2C(\eta)$ for every $f \in \mathrm{Ham}(\mathbb{D}^{2n})$ lying in the ϵ-neighborhood of the identity with respect to the C^0-distance, and the theorem follows. It remains to prove the lemma.

Proof of Lemma 1: Denote by S the hyperplane $\{q_n=0\}$. For $c>0$ write R_c for the dilation $z \to cz$ of $\mathbf{R}^{2n}$. We assume that all compactly supported diffeomorphisms of $\mathbb{D}^{2n}$ are extended to the whole $\mathbf{R}^{2n}$ by the identity.

Take $f \in \mathrm{Ham}(\mathbb{D}^{2n})$ satisfying (1). Let $\{f_t\}_{0\le t\le 1}$ be a Hamiltonian isotopy supported in $\mathbb{D}^{2n}$ such that $f_t = \mathbb{1}$ for $t\in[0,\delta)$ and $f_t=f$ for $t \in (1-\delta,1]$ for some $\delta > 0$. Take a smooth function $c : [0,1] \to [1,+\infty)$ that equals 1 near 0 and 1 and satisfies $c(t) > (2\epsilon)^{-1}$ on $[\delta, 1-\delta]$. Consider the Hamiltonian isotopy $h_t = R_{1/c(t)} f_t R_{c(t)}$ of $\mathbf{R}^{2n}$. Note that $h_0 = \mathbb{1}$ and $h_1 = f$. Since $c(t) \geq 1$, we have $h_t z = z$ for $z \notin \mathbb{D}^{2n}$, and h_t is supported in $\mathbb{D}^{2n}$.

We claim that $h_t(S) \subset \Pi(2\epsilon)$. Observe that $R_{c(t)} S = S$. Take any $z \in S$. If $R_{c(t)} z \notin \mathbb{D}^{2n}$, we have that $h_t z = z$. Assume now that $R_{c(t)} z \in \mathbb{D}^{2n}$. Consider the following cases:

- If $t \in (1-\delta, 1]$, then $f_t R_{c(t)}(S) = f(S)$. Thus $f_t R_{c(t)} z \in f(S \cap \mathbb{D}^{2n}) \subset \Pi(2\epsilon)$, where the latter inclusion follows from (1). Therefore $h_t z \in \Pi(2\epsilon)$ since $c(t) \geq 1$.
- If $t \in [\delta, 1-\delta]$, then $h_t z \in \mathbb{D}^{2n}(2\epsilon) \subset \Pi(2\epsilon)$ by our choice of the function $c(t)$.
- If $t \in [0, \delta)$, then $h_t S = S \subset \Pi(2\epsilon)$.

This completes the proof of the claim.

By continuity of h_t, there exists $\kappa > 0$ such that $h_t(\Pi(\kappa)) \subset \Pi(2\epsilon)$ for all t. Cutting off the Hamiltonian of h_t near $h_t(\Pi(\kappa))$, we get a Hamiltonian flow θ_t supported in $\Pi(2\epsilon)$ that coincides with h_t on $\Pi(\kappa)$. Thus, $\theta_t^{-1} h_t$ is the identity on $\Pi(\kappa)$ for all t. It follows that $\theta_t^{-1} h_t$ decomposes into the product of two commuting Hamiltonian flows ϕ_-^t and ϕ_+^t supported in D_- and D_+ respectively. Therefore $f = \theta_1 \phi_-^1 \phi_+^1$ is the desired decomposition. □

3 Proof of the Criterion of Continuity on Surfaces

3.1 A C^0-Small Fragmentation Theorem on Surfaces

Before stating our next result, we recall the notion of *fragmentation* of a diffeomorphism. This is a classical technique in the study of groups of diffeomorphisms; see, e.g., [2, 4, 10]. Given a Hamiltonian diffeomorphism f of a connected symplectic manifold Σ and an open cover $\{U_\alpha\}$ of Σ, one can always write f as a product of Hamiltonian diffeomorphisms each of which is supported in one of the open sets U_α. It is known that the number of factors in such a decomposition is uniform in a C^1-neighborhood of the identity; see [2, 4, 10]. To prove our continuity theorem, we actually need to prove a similar result on surfaces in which one considers diffeomorphisms endowed with the C^0-topology. Such a result appears in [35] in the case when the surface is the unit disk. Observe also that the corresponding fragmentation result is known for volume-preserving homeomorphisms [20].

Theorem 4. *Let Σ be a compact connected surface (possibly with boundary), equipped with an area form. Then for every $a > 0$, there exist a neighborhood $\mathcal{U}$ of the identity in the group* $\mathrm{Ham}(\Sigma)$ *endowed with the C^0-topology and an integer $N > 0$ such that every diffeomorphism $g \in \mathcal{U}$ can be written as a product of at most N Hamiltonian diffeomorphisms supported in disks of area less than a.*

This result might be well known to experts and probably can be deduced from the corresponding result for homeomorphisms. However, since the proof is more difficult for homeomorphisms, and in order to keep this paper self-contained, we are going to give a direct proof of Theorem 4 in Sect. 6. Note that this last section is the most technical part of the text. Given the fragmentation result above, one obtains easily a proof of Theorem 3, as we will show now.

3.2 Proof of Theorem 3 and Corollary 1

1. We begin by proving that the condition appearing in the statement of the theorem is necessary for the quasimorphism μ to be continuous. Assume that μ is continuous for the C^0-topology. Then it is bounded on some C^0-neighborhood $\mathcal{U}$ of the identity in $\mathrm{Ham}(\Sigma)$. Choose now a disk D_0 in Σ. If D_0 has a sufficiently small diameter, then $\mathrm{Ham}(D_0) \subset \mathcal{U}$. But since $\mathrm{Ham}(D_0)$ is a subgroup and μ is homogeneous, μ must vanish on $\mathrm{Ham}(D_0)$.

   Now let $a = \mathrm{area}(D_0)$. If D is any disk of area less than a, the group $\mathrm{Ham}(D)$ is conjugate in $\mathrm{Ham}(\Sigma)$ to a subgroup of $\mathrm{Ham}(D_0)$, because for any two disks of the same area in Σ there exists a Hamiltonian diffeomorphism mapping one of the disks onto another; see, e.g., [1, Proposition A.1] for a proof (which, in fact, works for all Σ, though the claim there is stated only for closed surfaces). Hence, μ vanishes on $\mathrm{Ham}(D)$ as required.

Remark 2. This proof extends verbatim to higher-dimensional symplectic manifolds (Σ, ω) that admit a positive constant a_0 with the following property: for every $a < a_0$, all symplectically embedded balls of volume a in the interior of Σ are Hamiltonian isotopic. Here a symplectically embedded ball of volume a is the image of the standard Euclidean ball of volume a in $(\mathbf{R}^{2n}, \mathrm{d}p \wedge \mathrm{d}q)$ under a symplectic embedding. This property holds, for instance, for blowups of rational and ruled symplectic four-manifolds; see [8, 30, 36, 37].

2. We now prove the reverse implication. Assume that a homogeneous quasimorphism μ vanishes on all Hamiltonian diffeomorphisms supported in disks of area less than a. Take the C^0-neighborhood $\mathcal{U}$ of the identity and the integer N from Theorem 4. Then μ is bounded by $(N-1)C(\mu)$ on $\mathcal{U}$, and hence is continuous by Proposition 1. $\square$

   We now prove Corollary 1. Choose compactly supported symplectic vector fields $v_1, \ldots, v_k$ on Σ such that the cohomology classes of the 1-forms $i_{v_j}\omega$ generate $H^1_{\mathrm{comp}}(\Sigma, \mathbf{R})$. Denote by h_i^t the flow of v_i. Let $\mathcal{V}$ be the image of the following map:

$$(-\epsilon, \epsilon)^k \to \mathcal{D}$$

$$(t_1, \ldots, t_k) \mapsto \prod_{i=1}^{k} h_i^{t_i}.$$

Using assumption (i) and applying Theorem 3, we get that the quasimorphism μ is bounded on a C^0-neighborhood, say $\mathcal{U}$, of the identity in $\mathrm{Ham}(\Sigma)$. Thus by (ii) and the definition of a quasimorphism, μ is bounded on $\mathcal{U} \cdot \mathcal{V}$. But the latter set is a C^0-neighborhood of the identity in $\mathcal{D}$. Thus μ is continuous on $\mathcal{D}$ by Proposition 1. $\square$

4 Examples of Continuous Quasimorphisms

In this section we prove Theorem 2 case by case. The case of the disk has already been explained in Sect. 2. This construction generalizes verbatim to all closed surfaces of genus 0 with nonempty boundary, which proves Theorem 2 in this case.

When Σ is a closed surface of genus greater than one, Gambaudo and Ghys constructed in [22] an infinite-dimensional space of homogeneous quasimorphisms on the group $\mathcal{D}(\Sigma)$ satisfying the hypothesis of Theorem 3. These quasimorphisms are defined using 1-forms on the surface and can be thought of as some "quasifluxes." We refer to [22, Sect. 6.1] or to [23, Sect. 2.5] for a detailed description. The fact that these quasimorphisms extend continuously to the identity component of the group of area-preserving homeomorphisms of Σ can be checked easily without appealing to Theorem 3. This was already observed in [23].

In order to settle the case of surfaces of genus one, we shall apply the criterion given by Theorem 3. The quasimorphisms that we will use were constructed by Gambaudo and Ghys in [22]; see also [46]. We recall briefly this construction now.

The fundamental group $\pi_1(\mathbb{T}^2 \setminus \{0\})$ of the once-punctured torus is a free group on two generators, a and b, represented by a parallel and a meridian in $\mathbb{T}^2 \setminus \{0\}$. Let $\mu : \pi_1(\mathbb{T}^2 \setminus \{0\}) \to \mathbf{R}$ be a homogeneous quasimorphism. It is known that there are plenty of such quasimorphisms (see [11], for instance). We will associate to μ a homogeneous quasimorphism $\widetilde{\mu}$ on the group $\mathcal{D}(\mathbb{T}^2)$.

We fix a base point $x_* \in \mathbb{T}^2 \setminus \{0\}$. For all $v \in \mathbb{T}^2 \setminus \{0\}$ we choose a path $\alpha_v(t)$, $t \in [0,1]$, in $\mathbb{T}^2 \setminus \{0\}$ from x_* to v. We assume that the lengths of the paths α_v are uniformly bounded with respect to a Riemannian metric defined on the compact surface obtained by blowing up the origin on $\mathbb{T}^2$. Consider an element $f \in \mathcal{D}(\mathbb{T}^2)$ and fix an isotopy (f_t) from the identity to f. If x and y are distinct points in the torus, we can consider the curve

$$f_t(x) - f_t(y)$$

in $\mathbb{T}^2 \setminus \{0\}$. Its homotopy class depends only on f. We close it to form a loop:

$$\alpha(f,x,y) := \alpha_{x-y} * (f_t(x) - f_t(y)) * \overline{\alpha_{f(x)-f(y)}},$$

where $\overline{\alpha_{f(x)-f(y)}}(t) := \alpha_{f(x)-f(y)}(1-t)$. We have the cocycle relation

$$\alpha(fg,x,y) = \alpha(g,x,y) * \alpha(f,g(x),g(y)).$$

Define a function u_f on $\mathbb{T}^2 \times \mathbb{T}^2 \setminus \Delta$ (where Δ is the diagonal) by $u_f(x,y) = \mu(\alpha(f,x,y))$. From the previous relation and the fact that μ is a quasimorphism, we deduce the relation

$$\left|u_{fg}(x,y) - u_g(x,y) - u_f(g(x),g(y))\right| \leq C(\mu),\ \forall f,g \in \mathcal{D}(\mathbb{T}^2).$$

Moreover, it is not difficult to see that the function u_f is measurable and bounded on $\mathbb{T}^2 \times \mathbb{T}^2 \setminus \Delta$. Hence, the map

$$f \mapsto \int_{\mathbb{T}^2 \times \mathbb{T}^2} u_f(x,y)\mathrm{d}x\mathrm{d}y$$

is a quasimorphism. We denote by $\widetilde{\mu}$ the associated homogeneous quasimorphism

$$\widetilde{\mu}(f) = \lim_{p \to \infty} \frac{1}{p} \int_{\mathbb{T}^2 \times \mathbb{T}^2} u_{f^p}(x,y)\mathrm{d}x\mathrm{d}y.$$

One easily checks that $\widetilde{\mu}$ is linear on any 1-parameter subgroup. The following proposition was established in [46]:

Proposition 4. *Let $f \in \mathrm{Ham}(\mathbb{T}^2)$ be a diffeomorphism supported in a disk D. Then for any homogeneous quasimorphism $\mu : \pi_1(\mathbb{T}^2 \setminus \{0\}) \to \mathbf{R}$, one has*

$$\widetilde{\mu}(f) = 2\mu([a,b]) \cdot \mathrm{Cal}(f),$$

where $\mathrm{Cal} : \mathrm{Ham}(D) \to \mathbf{R}$ *is the Calabi homomorphism.*

By Corollary 1, we get that the quasimorphisms $\widetilde{\mu}$, where μ runs over the set of homogeneous quasimorphisms on $\pi_1(\mathbb{T}^2 \setminus \{0\})$ that take the value 0 on the element $[a,b]$, are all continuous in the C^0-topology. According to [22], this family spans an infinite-dimensional vector space. To complete the proof of Theorem 2 for surfaces of genus 1, we have only to check that the diffeomorphisms that were constructed in [22] in order to establish the existence of an arbitrary number of linearly independent quasimorphisms $\widetilde{\mu}$ can be chosen to be supported in any given subsurface of genus one. But this follows easily from the construction in [22, Sect. 6.2].

5 Discussion and Open Questions

5.1 Is $\mathcal{H}(\mathbb{D}^2)$ Simple? (Le Roux's Work)

Although the algebraic structure of groups of volume-preserving homeomorphisms in dimension greater than 2 is well understood [20], the case of area-preserving homeomorphisms of surfaces is still mysterious. In particular, it is unknown whether the group $\mathcal{H}(\mathbb{D}^2)$ is simple. Some normal subgroups of $\mathcal{H}(\mathbb{D}^2)$ were constructed by

Ghys, Oh, and more recently by Le Roux; see [10] for a survey. However, it is unknown whether any of these normal subgroups is a proper subgroup of $\mathcal{H}(\mathbb{D}^2)$. In [35], Le Roux established that the simplicity of the group $\mathcal{H}(\mathbb{D}^2)$ is equivalent to a certain fragmentation property. Namely, he established the following result (in the following, we assume that the total area of the disk is 1):

The group $\mathcal{H}(\mathbb{D}^2)$ is simple if and only if there exist numbers $\rho' < \rho$ in $(0,1]$ and an integer N such that any homeomorphism $g \in \mathcal{H}(\mathbb{D}^2)$ whose support is contained in a disk of area at most ρ can be written as a product of at most N homeomorphisms whose supports are contained in disks of area at most ρ'.

By a result of Fathi [20], see also [35], g can always be represented as such a product with some, a priori unknown, number of factors.

Remark 3. One can show that the property above depends only on ρ and not of the choice of ρ' smaller than ρ [35].

In the sequel we will denote by G_ε the set of homeomorphisms in $\mathcal{H}(\mathbb{D}^2)$ whose support is contained in an open disk of area at most ε. For an element g$\in \mathcal{H}(\mathbb{D}^2)$ we define (following [12, 35]) $|g|_\varepsilon$ as the minimal integer n such that g can be written as a product of n homeomorphisms of G_ε. Any homogeneous quasimorphism ϕ on $\mathcal{H}(\mathbb{D}^2)$ that vanishes on G_ε gives the following lower bound on $|\cdot|_\varepsilon$:

$$|g|_\varepsilon \geq \frac{|\phi(g)|}{C(\phi)} \quad (g \in \mathcal{H}(\mathbb{D}^2)).$$

In particular, if ϕ vanishes on G_ε but not on $G_{\varepsilon'}$ for some $\varepsilon' > \varepsilon$, then the norm $|\cdot|_\varepsilon$ is unbounded on $G_{\varepsilon'}$.

If $\phi : \mathcal{H}(\mathbb{D}^2) \to \mathbf{R}$ is a homogeneous quasimorphism that is continuous in the C^0-topology, we can define $a(\phi)$ to be the supremum of the positive numbers a satisfying the following property: ϕ vanishes on $\mathrm{Ham}(D)$ for any disk D of area less than or equal to a (for a homogeneous quasimorphism that is not continuous in the C^0-topology, one can define $a(\phi) = 0$). One can think of $a(\phi)$ as the *scale* at which one can detect the nontriviality of ϕ. According to the discussion above, the existence of a nontrivial quasimorphism with $a(\phi) > 0$ implies that the norm $|\cdot|_{a(\phi)}$ is unbounded on the set G_ρ (for any $\rho > a(\phi)$).

According to Le Roux's result, the existence of a sequence of continuous (for the C^0-topology) homogeneous quasimorphisms ϕ_n on $\mathcal{H}(\mathbb{D}^2)$ with $a(\phi_n) \to 0$ would imply that the group $\mathcal{H}(\mathbb{D}^2)$ is not simple. However, for all the examples of quasimorphisms on $\mathcal{H}(\mathbb{D}^2)$ that we know (coming from the continuous quasimorphisms described in Sect. 2), one has $a(\phi) \geq \frac{1}{2}$.

5.2 *Quasimorphisms on* $\mathbb{S}^2$

Consider the sphere $\mathbb{S}^2$ equipped with an area form of total area 1.

Question 1. (i) Does there exist a nonvanishing C^0-continuous homogeneous quasimorphism on $\mathrm{Ham}(\mathbb{S}^2)$?
(ii) If so, can it be made Lipschitz with respect to Hofer's metric?

If the answer to the first question is negative, this would imply that the Calabi quasimorphism constructed in [18] is unique. Indeed, the difference of two Calabi quasimorphisms is continuous in the C^0-topology according to Theorem 3. Note that for surfaces of positive genus, the examples of C^0-continuous quasimorphisms that we gave are related to the existence of many Calabi quasimorphisms [45, 46].

In turn, an affirmative answer to Question 1(ii) would yield the solution of the following problem posed by Misha Kapovich and the second author in 2006. It is known [43] that $\mathrm{Ham}(\mathbb{S}^2)$ carries a one-parameter subgroup, say $L := \{f_t\}_{t\in\mathbf{R}}$, that is a quasigeodesic in the following sense: $\|f_t\|_{\mathrm{H}} \geq c|t|$ for some $c > 0$ and all t. Given such a subgroup, put

$$A(L) := \sup_{\phi\in\mathrm{Ham}(\mathbb{S}^2)} d_{\mathrm{H}}(\phi, L).$$

Question 2. Is $A(L)$ finite or infinite?

The finiteness of $A(L)$ does not depend on the specific quasigeodesic one-parameter subgroup L. Intuitively, the finiteness of $A(L)$ would yield that the whole group $\mathrm{Ham}(\mathbb{S}^2)$ lies in a tube of finite radius around L.

We claim that if $\mathrm{Ham}(\mathbb{S}^2)$ admits a nonvanishing C^0-continuous homogeneous quasimorphism that is Lipschitz in Hofer's metric, then $A(L) = \infty$. Indeed, such a quasimorphism would be independent of the Calabi quasimorphism constructed in [18]. But the existence of two independent homogeneous quasimorphisms on $\mathrm{Ham}(\mathbb{S}^2)$ that are Lipschitz with respect to Hofer's metric implies that $A(L) = \infty$: otherwise, the finiteness of $A(L)$ would imply that Lipschitz homogeneous quasimorphisms are determined by their restriction to L.

5.3 Quasimorphisms in Higher Dimensions

Consider the following general question: given a homogeneous quasimorphism on $\mathrm{Ham}(\Sigma^{2n}, \omega)$, is it continuous in the C^0-topology?

The answer is positive, for instance, for quasimorphisms coming from the fundamental group $\pi_1(M)$ [22, 44]. It would be interesting to explore, for instance, the C^0-continuity of a quasimorphism μ given by the difference of a Calabi quasimorphism and the Calabi homomorphism [9, 18] (or more generally, by the difference of two distinct Calabi quasimorphisms). In order to prove the C^0-continuity of μ, one should establish a C^0-small fragmentation lemma with a controlled number of factors in the spirit of Lemma 1 for $\mathbb{D}^{2n}$ or Theorem 4 for surfaces. It is likely that the argument that we used for $\mathbb{D}^{2n}$ could go through without great complications for certain Liouville symplectic manifolds, that is, compact exact symplectic manifolds that admit a conformally symplectic vector field transversal to the boundary, such as the open unit cotangent bundle of the sphere.

Our result for $\mathbb{D}^{2n}$ should also allow the construction of continuous quasimorphisms for groups of Hamiltonian diffeomorphisms of certain symplectic manifolds symplectomorphic to "sufficiently large" open subsets of $\mathbb{D}^{2n}$ (for instance, the open unit cotangent bundle of a torus).

The C^0-small fragmentation problem on general higher-dimensional manifolds looks very difficult. Consider, for instance, the following toy case: find a fragmentation with a controlled number of factors for a C^0-small Hamiltonian diffeomorphism supported in a sufficiently small ball $D \subset \Sigma$. A crucial difference from the situation described in Sect. 2 is that we have no information about the Hamiltonian isotopy $\{f_t\}$ joining f with the identity: it can "travel" far away from D. In particular, when $\dim \Sigma \geq 6$, we do not know whether f lies in $\mathrm{Ham}(D)$. When $\dim \Sigma = 4$, the fact that $f \in \mathrm{Ham}(D)$ (and hence the fragmentation in our toy example) follows from a deep theorem by Gromov based on pseudoholomorphic curves techniques [25]. It would be interesting to apply powerful methods of four-dimensional symplectic topology to the C^0-small fragmentation problem.

6 Proof of the Fragmentation Theorem

In this section we prove Theorem 4. First, we need to recall a few classical results.

6.1 Preliminaries

In the course of the proof we will repeatedly use the following result:

Proposition 5. *Let Σ be a compact connected oriented surface, possibly with nonempty boundary $\partial\Sigma$, and let ω_1, ω_2 be two area forms on Σ. Assume that $\int_\Sigma \omega_1 = \int_\Sigma \omega_2$. If $\partial\Sigma \neq \emptyset$, we also assume that the forms ω_1 and ω_2 coincide on $\partial\Sigma$.*

Then there exists a diffeomorphism $f : \Sigma \to \Sigma$, isotopic to the identity, such that $f^*\omega_2 = \omega_1$. Moreover, f can be chosen to satisfy the following properties:

(i) If $\partial\Sigma \neq \emptyset$, then f is the identity on $\partial\Sigma$, and if ω_1 and ω_2 coincide near $\partial\Sigma$, then f is the identity near $\partial\Sigma$.
(ii) If Σ is partitioned into polygons (with piecewise smooth boundaries) such that $\omega_2 - \omega_1$ is zero on the 1-skeleton Γ of the partition and the integrals of ω_1 and ω_2 over each polygon are equal, then f can be chosen to be the identity on Γ.
(iii) The diffeomorphism f can be chosen arbitrarily C^0-close to $\mathbb{1}$, provided ω_1 and ω_2 are sufficiently C^0-close to each other (i.e., $\omega_2 = \chi\omega_1$ for a function χ sufficiently C^0-close to 1).

The existence of f in the case of a closed surface follows from a well-known theorem of Moser [39] (see also [24]). The method of the proof ("Moser's method") can be outlined as follows. Set $\omega_t := \omega_1 + t(\omega_2 - \omega_1)$ and note that the form $\omega_2 - \omega_1$ is exact. Choose a 1-form σ such that $d\sigma = \omega_2 - \omega_1$ and define f as the time-1 flow of the vector field ω_t-dual to σ. In order to prove (i) and (ii), one has to choose a primitive σ for $\omega_2 - \omega_1$ that vanishes near $\partial\Sigma$ or, respectively, on Γ; the construction of such a σ can be easily extracted from [3]. Property (iii) is essentially contained in [39]; it follows easily from the above construction of f, provided we can construct a C^0-small primitive σ for a C^0-small exact 2-form $\omega_2 - \omega_1$, but by [39, Lemma 1], it suffices to do so on a rectangle, and in this case σ can be constructed explicitly.

In fact, a stronger result than (iii) is true. It is known, see [40, 48], that f can be chosen C^0-close to the identity as soon as the two area forms (considered as measures) are close in the weak-$*$ topology. Note that if one of the two forms is the image of the other by a diffeomorphism C^0-close to the identity, the two forms are close in the weak-$*$ topology. However, to keep this text self-contained, we are not going to use this fact, but will prove again directly the particular cases we need.

We equip the surface Σ with a fixed Riemannian metric and denote by d the corresponding distance. For any map $f : X \to \Sigma$ (where X is a closed subset of Σ) we denote by $\|f\| := \max_x d(x, f(x))$ its C^0-norm. Accordingly, the C^0-norm of a smooth function u defined on a closed subset of Σ will be denoted by $\|u\|$.

The following lemmas are the main tools for the proof.

Lemma 2 (Area-preserving extension lemma for disks). *Let $D_1 \subset D_2 \subset D \subset \mathbf{R}^2$ be closed disks such that $D_1 \subset \text{Interior}\,(D_2) \subset D_2 \subset \text{Interior}\,(D)$. Let $\phi : D_2 \to D$ be a smooth area-preserving embedding (we assume that D is equipped with some area form). Then there exists $\psi \in \text{Ham}(D)$ such that*

$$\psi|_{D_1} = \phi \quad \text{and} \quad \|\psi\| \to 0 \text{ as } \|\phi\| \to 0.$$

Lemma 3 (Area-preserving extension lemma for rectangles). *Let $\Pi = [0,R] \times [-c,c]$ be a rectangle and let $\Pi_1 \subset \Pi_2 \subset \Pi$ be two smaller rectangles of the form $\Pi_i = [0,R] \times [-c_i, c_i]$ $(i = 1,2)$, $0 < c_1 < c_2 < c$. Let $\phi : \Pi_2 \to \Pi$ be an area-preserving embedding (we assume that Π is equipped with some area form) such that:*

- *ϕ is the identity near $0 \times [-c_2, c_2]$ and $R \times [-c_2, c_2]$.*
- *The area in Π bounded by the curve $[0,R] \times y$ and its image under ϕ is zero for some (and hence for all) $y \in [-c_2, c_2]$.*

Then there exists $\psi \in \text{Ham}(\Pi)$ such that

$$\psi|_{\Pi_1} = \phi \quad \text{and} \quad \|\psi\| \to 0 \text{ as } \|\phi\| \to 0.$$

The lemmas will be proved in Sect. 6.3. Let us mention that we implicitly assume in these lemmas that ϕ is close to the inclusion, i.e., that $\|\phi\|$ is small enough. Note that if one is interested only in the existence of ψ, without any control on its norm $\|\psi\|$, these results are standard.

6.2 Construction of the Fragmentation

We are now ready to prove the fragmentation theorem. In the case that Σ is the closed unit disk $\mathbb{D}^2$ in $\mathbf{R}^2$, the theorem has been proved by Le Roux [35, Proposition 4.2]. In general, our proof relies on the case of the disk.

For any $b > 0$ we fix a neighborhood $\mathscr{U}_0(b)$ of the identity in $\mathrm{Ham}(\mathbb{D}^2)$ and an integer $N_0(b)$ such that every element of $\mathscr{U}_0(b)$ is a product of at most $N_0(b)$ diffeomorphisms supported in disks of area at most b. We will prove the following assertion.

For any surface Σ there exist an integer N_1 and disks $(D_j)_{1\leq j\leq N_1}$ in Σ such that for any $\epsilon > 0$ there exists a neighborhood $\mathscr{V}(\epsilon)$ of the identity in $\mathrm{Ham}(\Sigma)$ *with the property that every diffeomorphism $f \in \mathscr{V}(\epsilon)$ can be written as a product $f = g_1 \cdots g_{N_1}$, where each g_i belongs to* $\mathrm{Ham}(D_j)$ *for one of the disks D_j and is ϵ-close to the identity. $(*)$*

Note that there is no restriction in $(*)$ on the areas of the disks D_j. Let us explain how to conclude the proof of Theorem 4 from this assertion. Fix $a > 0$. We can choose, for each i between 1 and N_1, a conformally symplectic diffeomorphism $\psi_i : \mathbb{D}^2 \to D_i$ such that the pullback of the area form on Σ by ψ_i equals the standard area form on the disk $\mathbb{D}^2$ times some constant $\lambda_i > 0$. Here we are using Proposition 5. If ϵ is sufficiently small, $\psi_i^{-1} g_i \psi_i$ is in $\mathscr{U}_0(\frac{a}{\lambda_i})$ for each i, and we can apply the result for the disk to it. This concludes the proof.

Remark 4. It is important that the disks D_i as well as the maps ψ_i are chosen in advance, since we need the neighborhoods $\psi_i \mathscr{U}_0(\frac{a}{\lambda_i})\psi_i^{-1}$ to be known in advance. They determine the neighborhood $\mathscr{V}(\epsilon)$.

We now prove $(*)$. The arguments we use are inspired by the work of Fathi [20]. Fix $\epsilon > 0$. We distinguish two cases: (1) Σ has a boundary, and (2) Σ is closed.

First case. Any compact connected surface with nonempty boundary can be obtained by gluing finitely many 1-handles to a disk. We prove the statement $(*)$ by induction on the number of 1-handles. We already know that $(*)$ is true for a disk (just take $N_1 = 1$ and let D_1 be the whole disk). Assume now that $(*)$ holds for any compact surface with boundary obtained by gluing l 1-handles to the disk. Let Σ be a compact surface obtained by gluing a 1-handle to a compact surface Σ_0, where Σ_0 is obtained from the disk by gluing l 1-handles.

Choose a diffeomorphism (singular at the corners) $\varphi : [-1,1]^2 \to \overline{\Sigma - \Sigma_0}$ sending $[-1,1] \times \{-1,1\}$ into the boundary of Σ_0. Let $\Pi_r = \varphi([-1,1] \times [-r,r])$. Let $\mathscr{V}_1(\epsilon)$

be the neighborhood of the identity in $\mathrm{Ham}(\Sigma_1)$ given by $(*)$ applied to the surface $\Sigma_1 := \Sigma_0 \cup \varphi([-1,1] \times \{s, |s| \geq \frac{1}{4}\})$, and let N_1 be the corresponding integer.

Let $f \in \mathrm{Ham}(\Sigma)$ close to the identity. We apply Lemma 3 to the chain of rectangles $\Pi_{\frac{1}{2}} \subset \Pi_{\frac{3}{4}} \subset \Pi_{\frac{7}{8}}$ and to the restriction of f to $\Pi_{\frac{3}{4}}$ (the hypothesis on the curve $[-1,1] \times \{y\}$ is met because f is Hamiltonian). Here again we are appealing to Proposition 5 to ensure that the pullback of the area form of Σ by φ can be identified with a fixed area form on $\Pi_{\frac{7}{8}}$. We obtain a diffeomorphism ψ supported in $\Pi_{\frac{7}{8}}$ and C^0-close to the identity that coincides with f on $\Pi_{\frac{1}{2}}$. Hence, we can write

$$f = \psi h,$$

where h is supported in Σ_1. Since $f \in \mathrm{Ham}(\Sigma)$ and $\psi \in \mathrm{Ham}(\Pi_{\frac{7}{8}})$, we get that h is Hamiltonian in Σ. Since $H^1_{\mathrm{comp}}(\Sigma_1, \mathbf{R})$ embeds in $H^1_{\mathrm{comp}}(\Sigma, \mathbf{R})$, it means that h actually belongs to $\mathrm{Ham}(\Sigma_1)$.

Define a neighborhood $\mathscr{V}(\epsilon)$ of the identity in $\mathrm{Ham}(\Sigma)$ by the following condition: $f \in \mathscr{V}(\epsilon)$ if first, $\|\psi\| < \epsilon$ (recall that when f converges to the identity, so does ψ) and second, $h \in \mathscr{V}_1(\epsilon)$. Hence, if $f \in \mathscr{V}(\epsilon)$, we can write it as a product of $N_1 + 1$ diffeomorphisms g_i, where each g_i is ϵ-close to the identity and belongs to $\mathrm{Ham}(D_j)$ for some disk $D_j \subset \Sigma$. This proves the claim $(*)$ for Σ in the first case.

Second case. The surface Σ is closed – we view it as a result of gluing a disk to a surface Σ_0 with one boundary component. Choose a diffeomorphism $\varphi : \mathbb{D}^2 \to \overline{\Sigma - \Sigma_0}$ sending the boundary of $\mathbb{D}^2$ into the boundary of Σ_0. Once again, by appealing to Proposition 5, we can assume that the pullback by φ of the area form of Σ is a given area form on $\mathbb{D}^2$. Denote by D_r the image by φ of the disk of radius $r \in [0,1]$ in $\mathbb{D}^2$. Let $\mathscr{V}_1(\epsilon)$ be the neighborhood of the identity given by $(*)$ applied to the surface $\Sigma_1 := \Sigma_0 \cup \varphi(\{z \in \mathbb{D}^2, |z| \geq \frac{1}{4}\})$ and let N_1 be the corresponding integer – recall that in the first case above, we have already proved $(*)$ for Σ_1, which is a surface with boundary.

Let $f \in \mathrm{Ham}(\Sigma)$ close to the identity. We apply Lemma 2 to the chain of disks $D_{\frac{1}{2}} \subset D_{\frac{3}{4}} \subset D_1$ and to the restriction of f to $D_{\frac{3}{4}}$. We obtain a diffeomorphism ψ supported in D_1 and close to the identity that coincides with f on $D_{\frac{1}{2}}$. Hence, we can write

$$f = \psi h,$$

where h is supported in Σ_1. Since $f \in \mathrm{Ham}(\Sigma)$ and $\psi \in \mathrm{Ham}(D_1)$, we get that h is Hamiltonian in Σ. Since Σ_1 has one boundary component, $H^1_{\mathrm{comp}}(\Sigma_1, \mathbf{R})$ embeds in $H^1_{\mathrm{comp}}(\Sigma, \mathbf{R})$, so h actually belongs to $\mathrm{Ham}(\Sigma_1)$. One concludes the proof as in the first case.

This finishes the proof of Theorem 4 (modulo the proofs of the extension lemmas).

6.3 Extension Lemmas

The area-preserving extension lemmas for disks and rectangles will be consequences of the following lemma.

Lemma 4 (Area-preserving extension lemma for annuli). *Let* $\mathbb{A} = S^1 \times [-3,3]$ *be a closed annulus and let* $\mathbb{A}_1 = S^1 \times [-1,1], \mathbb{A}_2 = S^1 \times [-2,2]$ *be smaller annuli inside* $\mathbb{A}$*. Let* ϕ *be an area-preserving embedding of a fixed open neighborhood of* $\mathbb{A}_1$ *into* $\mathbb{A}_2$ *(we assume that* $\mathbb{A}$ *is equipped with some area form* ω*) such that for some* $y \in [-1,1]$ *(and hence for all of them), the curves* $S^1 \times y$ *and* $\phi(S^1 \times y)$ *are homotopic in* $\mathbb{A}$ *and*

$$\text{the area in } \mathbb{A} \text{ bounded by } S^1 \times y \text{ and } \phi(S^1 \times y) \text{ is } 0. \tag{2}$$

Then there exists $\psi \in \mathrm{Ham}(\mathbb{A})$ *such that* $\psi|_{\mathbb{A}_1} = \phi$ *and* $\|\psi\| \to 0$ *as* $\|\phi\| \to 0$.

Moreover, if for some arc $I \subset S^1$ *we have that* $\phi = \mathbb{1}$ *outside a quadrilateral* $I \times [-1,1]$ *and* $\phi(I \times [-1,1]) \subset I \times [-2,2]$*, then* ψ *can be chosen to be the identity outside* $I \times [-3,3]$.

Once again, we assume in this lemma that $\|\phi\|$ is small enough. Let us show how this lemma implies the area-preserving extension lemmas for disks and rectangles.

Proof of Lemma 2. Up to replacing D_2 by a slightly smaller disk, we can assume that ϕ is defined in a neighborhood of D_2. Identify some small neighborhood of ∂D_2 with $\mathbb{A} = S^1 \times [-3,3]$ so that ∂D_2 is identified with $S^1 \times 0 \subset \mathbb{A}_1 \subset \mathbb{A}_2 \subset \mathbb{A}$ and $\phi(\mathbb{A}_1) \subset \text{Interior } (\mathbb{A}_2) \subset \mathbb{A} \subset \text{Interior } (D) \setminus \phi(D_1)$.

Apply Lemma 4 and find $h \in \mathrm{Ham}(\mathbb{A})$, $\|h\| \to 0$ as $\epsilon \to 0$, so that $h|_{\mathbb{A}_1} = \phi$. Set $\phi_1 := h^{-1} \circ \phi \in \mathrm{Ham}(D)$. Note that $\phi_1|_{D_1} = \phi$ and ϕ_1 is the identity on $\mathbb{A}_1$. Therefore we can extend $\phi_1|_{D_2 \cup \mathbb{A}_1}$ to D by the identity and obtain the required ψ. □

Proof of Lemma 3. Identify the rectangles $\Pi_1 \subset \Pi_2 \subset \Pi$, by a diffeomorphism, with quadrilaterals $I \times [-1,1] \subset I \times [-2,2] \subset I \times [-3,3]$ in the annulus $\mathbb{A} = S^1 \times [-3,3]$ for some suitable arc $I \subset S^1$ and apply Lemma 4. □

In order to prove Lemma 4, we first need to prove a version of the lemma concerning smooth (not necessarily area-preserving) embeddings.

Lemma 5 (Smooth extension lemma). *Let* $\mathbb{A}_1 \subset \mathbb{A}_2 \subset \mathbb{A}$ *be as in Lemma 4. Let* ϕ *be a smooth embedding of a fixed open neighborhood of* $\mathbb{A}_1$ *into* $\mathbb{A}_2$*, isotopic to the identity, such that* $\|\phi\| \leq \epsilon$ *for some* $\epsilon > 0$*. Then there exists* $\psi \in \mathrm{Diff}_{0,c}(\mathbb{A})$ *such that* ψ *is supported in* $\mathbb{A}_2$*,* $\psi|_{\mathbb{A}_1} = \phi$*, and* $\|\psi\| \leq C\epsilon$*, for some* $C > 0$*, independent of* ϕ.

Moreover, if $\phi = \mathbb{1}$ *outside a quadrilateral* $I \times [-1,1]$ *and* $\phi(I \times [-1,1]) \subset I \times [-2,2]$ *for some arc* $I \subset S^1$*, then* ψ *can be chosen to be the identity outside* $I \times [-3,3]$.

Lemma 5 will be proved in Sect. 6.4.

Proof of Lemma 4. As one can easily check using Proposition 5, we can assume without loss of generality that the area form on $\mathbb{A} = S^1 \times [-3,3]$ is $\omega = \mathrm{d}x \wedge \mathrm{d}y$, where x is the angular coordinate along S^1 and y is the coordinate along $[-3,3]$. All norms and distances are measured with the Euclidean metric on $\mathbb{A}$. Define $\mathbb{A}_+ := S^1 \times [1,2]$, $\mathbb{A}_- := S^1 \times [-2,-1]$.

Assume $\|\phi\| < \epsilon$. By Lemma 5, there exists $f \in \mathrm{Diff}_{0,c}(\mathbb{A}_2)$ such that $\|f\| \le C\epsilon$, and $f = \phi$ on a neighborhood of $\mathbb{A}_1$. Define $\Omega := f^*\omega$. By (2),

$$\int_{\mathbb{A}_+} \Omega = \int_{\mathbb{A}_+} \omega,\ \int_{\mathbb{A}_-} \Omega = \int_{\mathbb{A}_-} \omega. \tag{3}$$

Note that Ω coincides with ω on a neighborhood of $\partial\mathbb{A}_+$ and $\partial\mathbb{A}_-$. Let us find $h \in \mathrm{Diff}_{0,c}(\mathbb{A}_2)$ such that

- $h|_{\mathbb{A}_1} = \mathbb{1}$,
- $h^*\Omega = \omega$,
- $\|h\| \to 0$ as $\epsilon \to 0$.

Given such an h, we extend fh by the identity to the whole of $\mathbb{A}$. The resulting diffeomorphism of $\mathbb{A}$ is C^0-small (if ϵ is sufficiently small), preserves ω, and belongs to $\mathrm{Diff}_{0,c}(\mathbb{A})$, hence (see, e.g., [50]) also to $\mathcal{D}(\mathbb{A})$. It may not be Hamiltonian, but one can easily make it Hamiltonian by a C^0-small adjustment on $\mathbb{A} \setminus \mathbb{A}_2$. The resulting diffeomorphism $\psi \in \mathrm{Ham}(\mathbb{A})$ will have all the required properties.

Preparations for the construction of *h*. Since on $\mathbb{A}_1$ the map h is required to be the identity, we need to construct it on $\mathbb{A}_+$ and $\mathbb{A}_-$. We will construct $h_+ := h|_{\mathbb{A}_+}$, the case of $\mathbb{A}_-$ being similar. By a rectangle or a square in $\mathbb{A}$ we mean the product of a connected arc in S^1 and an interval in $[-3,3]$.

Let us divide $\mathbb{A}_+ = S^1 \times [1,2]$ into closed squares $K_1,\dots,K_N$, with a side of size $r = \epsilon^{1/4} > 3\epsilon$ (we assume that ϵ is sufficiently small). Denote by V the set of vertices that are not on the boundary and by E the set of edges that are not on the boundary. Finally, denote by Γ the 1-skeleton of the partition (i.e., the union of all the edges).

For each $v \in V$ denote by $B_v(\delta)$ the open ball in $\mathbb{A}_+$ of radius $\delta > 0$ with center at v. Fix a small positive $\delta_0 < r$ such that for $0 < \delta < \delta_0$, the balls $B_v(\delta)$, $v \in V$, are disjoint and each $B_v(\delta)$ intersects only the edges adjacent to v. Given such a δ, consider for each edge $e \in E$ a small open rectangle $U_e(\delta)$ covering $e \setminus \big(e \cap \cup_{v\in V} B_v(\delta)\big)$ such that

- $U_e(\delta) \cap B_v(\delta) \neq \emptyset$ if and only if v is adjacent to e.
- $U_e(\delta)$ does not intersect any other edge apart from e.
- All the rectangles $U_e(\delta)$, $e \in E$, are mutually disjoint.

Define a neighborhood $U(\delta)$ of Γ by

$$U(\delta) = (\cup_{v\in V} B_v(\delta)) \cup (\cup_{e\in E} U_e(\delta)).$$

For each $\varepsilon_1 > \varepsilon_2 > 0$ we pick a cut-off function $\chi_{\varepsilon_1,\varepsilon_2} : \mathbf{R} \to [0,1]$ that is equal to 1 on a neighborhood of $(-\varepsilon_2, \varepsilon_2)$ and vanishes outside $(-\varepsilon_1, \varepsilon_1)$. Finally, by $C_1, C_2, \ldots$ we will denote positive constants independent of ϵ. The construction of h_+ will proceed in several steps.

Adjusting Ω on Γ. We are going to adjust the form Ω by a diffeomorphism supported inside $U(\delta)$ to make it equal to ω on Γ. One can first construct $h_1 \in \mathrm{Diff}_{0,c}(\mathbb{A}_+)$ supported in $\cup_{v\in V} B_v(2\delta)$ such that $h_1^*\Omega = \omega$ on $\cup_{v\in V} B_v(\delta)$ for some $\delta < \delta_0$ (simply using Darboux charts for Ω and ω). Note that $\|h_1\| < 2\delta$. Write $\Omega' := h_1^*\Omega$. For each $e \in E$ we will construct a diffeomorphism h_e supported in $U_e(\delta)$ so that $h_e^*\Omega' = \omega$ on $l := U_e(\delta) \cap e$ (and thus on the whole e, since Ω' already equals ω on each $B_v(\delta)$).

Without loss of generality, let us assume that e does not lie on $\partial\mathbb{A}_+$ (since Ω' already coincides with ω there) and that $U_e(\delta)$ is of the form $(a,b) \times (-\delta, \delta)$. Write the restriction of Ω' on $l = (a,b) \times 0$ as $\beta(x)\mathrm{d}x \wedge \mathrm{d}y$, $\beta(x) > 0$.

Consider a cut-off function $\chi = \chi_{\delta,\delta/2} : \mathbf{R} \to [0,1]$ and define a vector field $\mathbf{w}(x,y)$ on $U_e(\delta)$ by

$$\mathbf{w}(x,y) = \chi(y)\log(\beta(x))y\frac{\partial}{\partial y}.$$

Note that $\mathbf{w}$ is zero on l and has compact support in $U_e(\delta)$ (the endpoints of l lie in the balls $B_v(\delta)$ on which $\Omega = \omega$ and thus $\beta = 1$ near these endpoints). Let φ_t be the flow of $\mathbf{w}$. A simple calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\omega = \varphi_t^* L_{\mathbf{w}}\omega = \log(\beta(x))\mathrm{e}^{t\log(\beta(x))}\mathrm{d}x \wedge \mathrm{d}y$$

at the point $\varphi_t((x,0)) = (x,0)$. Therefore $\varphi_1^*\omega = \Omega'$ on l. Thus setting $h_e := \varphi_1^{-1}$, we get that $h_e^*\Omega' = \omega$ on l and that $\|h_e\| \le 2\delta$, because h_e preserves the fibers $x \times (-\delta, \delta)$. Set

$$h_2 := \prod_{e\in E} h_e.$$

Since the rectangles $U_e(\delta)$ are disjoint, h_2 is supported in $U(\delta)$ and satisfies the conditions

- $h_2^*\Omega' = \omega$ on Γ.
- $\|h_2\| \le 2\delta$.

The diffeomorphism $h_3 := h_1 h_2 \in \mathrm{Diff}_{0,c}(\mathbb{A}_+)$ satisfies $\|h_3\| \le 4\delta$ and

$$h_3^*\Omega = h_2^*\Omega' = \omega \text{ on } \Gamma.$$

Consider the area form $\Omega'' := h_3^*\Omega$. It coincides with ω on the 1-skeleton Γ and near $\partial\mathbb{A}_+$. Moreover, $\int_{\mathbb{A}_+} \Omega'' = \int_{\mathbb{A}_+} \Omega'$, and hence by (3),

$$\int_{\mathbb{A}_+} \Omega'' = \int_{\mathbb{A}_+} \omega. \tag{4}$$

Adjusting the areas of the squares. In this paragraph we construct a C^0-perturbation $\rho\omega$ of ω that has the same integral as Ω'' on each square K_i.

Making δ sufficiently small, we can assume that $\|h_3\| < \epsilon$. Recall that $r = \epsilon^{1/4} > 3\epsilon$. Therefore the image of one of the squares K_i by h_3 contains a square of area $(r-\epsilon)^2$ and is contained in a square of area $(r+\epsilon)^2$. Hence,

$$\frac{(r-2\epsilon)^2}{r^2} \leq \frac{\int_{K_i}\Omega''}{\int_{K_i}\omega} \leq \frac{(r+2\epsilon)^2}{r^2}.$$

Since $\epsilon/r = \epsilon^{3/4} \to 0$ as $\epsilon \to 0$, we get that if ϵ is sufficiently small, there exists $C_1 > 0$ such that

$$1 - C_1\frac{\epsilon}{r} \leq \frac{\int_{K_i}\Omega''}{\int_{K_i}\omega} \leq 1 + C_1\frac{\epsilon}{r}. \tag{5}$$

Now set $s_i := \int_{K_i}\Omega''$ and $t_i = s_i/r^2 - 1$. By (5),

$$|t_i| \leq C_1\frac{\epsilon}{r} = C_1.\epsilon^{3/4}. \tag{6}$$

For each i we can choose a nonnegative function $\bar{\rho}_i$ supported in the interior of K_i such that $\int_{K_i}\bar{\rho}_i\omega = r^2$ and

$$\|\bar{\rho}_i\|_{C^0} \leq C_2\epsilon^{-1/2} \tag{7}$$

for some constant $C_2 > 0$ independent of i. Define a function ϱ on $\mathbb{A}$ by

$$\varrho := 1 + \sum_{i=1}^{N} t_i\bar{\rho}_i.$$

By (6) and (7), the function ϱ is positive, and the form $\varrho\omega$ converges to ω (in the C^0-sense) as ϵ goes to 0. Moreover, ϱ is equal to 1 on Γ, and the two area forms $\varrho\omega$ and Ω'' have the same integral on each K_i. By (4), one has

$$\int_{\mathbb{A}_+}\varrho\omega = \int_{\mathbb{A}_+}\Omega'' = \int_{\mathbb{A}_+}\omega. \tag{8}$$

Finishing the construction of h_+: Moser's argument. Let us apply Proposition 5, part (ii), to the forms Ω'' and $\varrho\omega$ on $\mathbb{A}_+$. These forms have the same integral over each K_i and coincide on Γ and near the boundary of $\mathbb{A}_+$; therefore, there exists a diffeomorphism $h_4 \in \mathrm{Diff}_{0,c}(\mathbb{A}_+)$ that is the identity on Γ and satisfies $h_4^*\Omega'' = \varrho\omega$. Since h_4 is the identity on Γ and maps each K_i into itself, its C^0-norm is bounded by the diameter of K_i, hence goes to 0 with ϵ.

Finally, apply Proposition 5 to the forms ω and $\varrho\omega$ on $\mathbb{A}_+$: By (8), their integrals over $\mathbb{A}_+$ are the same; they coincide on $\partial\mathbb{A}_+$ and are C^0-close. Therefore, there exists $h_5 \in \mathrm{Diff}_{0,c}(\mathbb{A}_+)$ such that $h_5^*(\varrho\omega) = \omega$ and

$$\|h_5\| \to 0 \text{ as } \epsilon \to 0. \tag{9}$$

Then $h_+ := h_3 h_4 h_5$ is the required diffeomorphism. This finishes the construction of h.

Final observation. Note that if $\phi = \mathbb{1}$ outside a quadrilateral $I \times [-1,1]$ for some arc $I \subset S^1$, then f can be chosen to have the same property. In such a case we need to construct $h_+ \in \mathrm{Diff}_{0,c}(\mathbb{A}_+)$ supported in $I \times [-3,3]$.

Let J be the complement of the interval I in the circle. The partition of $\mathbb{A}_+$ into squares can be chosen so that it extends a partition of $J \times [1,2] \subset \mathbb{A}_+$ into squares of the same size. Going over each step of the construction of h_+ above, we see that since $\Omega = \omega$ on $J \times [1,2]$, each of the maps h_1, h_2, h_3, h_4, h_5 can be chosen to be the identity on each of the squares in $J \times [1,2]$, hence on the whole $J \times [1,2]$. Therefore, h_+, hence h, hence $\psi = fh$, is the identity on $J \times [1,2]$. Moreover, ψ is automatically Hamiltonian in this case. $\square$

6.4 *Proof of the Smooth Extension Lemma*

As in the proof of Lemma 4, we assume that the Riemannian metric on $\mathbb{A} = S^1 \times [-3,3]$ used for the measurements is the Euclidean product metric. We can also assume that the neighborhood of $\mathbb{A}_1$ on which ϕ is defined is, in fact, an open neighborhood of $\mathbb{A}' := S^1 \times [-1.5, 1.5]$ and that $\epsilon \ll 0.5$.

Proof of Lemma 5. Applying Lemma 6 (see the appendix by M. Khanevsky below) to the two curves $S^1 \times \{\pm 1.5\}$ and their images under ϕ, we can find $\psi_1 \in \mathrm{Diff}_{0,c}(\mathbb{A})$, supported in $S^1 \times (-2,-1) \cup S^1 \times (1,2)$, such that ψ_1 coincides with ϕ^{-1} on the curves $\phi(S^1 \times \{\pm 1.5\})$. Moreover, it satisfies $\|\psi_1\| < C'\epsilon$. Define $\psi_2 := \psi_1\phi$. This map is defined on an open neighborhood of $\mathbb{A}' = S^1 \times [-1.5, 1.5]$ and has the following properties:

- The restriction of ψ_2 to $\mathbb{A}'$ is a diffeomorphism of $\mathbb{A}'$. It is the identity on $\partial\mathbb{A}'$ and coincides with ϕ on $\mathbb{A}_1 = S^1 \times [-1,1] \subset \mathbb{A}'$.
- $\|\psi_2\| < C''\epsilon$, where $C'' := C' + 1$.

We are going to modify ψ_2 (by a C^0-small perturbation) to make it the identity not only on $\partial\mathbb{A}'$ but on an open neighborhood of $\partial\mathbb{A}'$. Then we will extend it by the identity to a diffeomorphism of $\mathbb{A}$ with the required properties.

Since ψ_2 is the identity on $\partial\mathbb{A}'$, by perturbing it slightly near $\partial\mathbb{A}'$ (in the C^0-norm) we can assume that in addition to the properties listed above, near $\partial\mathbb{A}'$ the map ψ_2 preserves the foliation of $\mathbb{A}$ by the circles $S^1 \times y$. Let us explain briefly why (we describe how to perturb ψ_2 near the curve $y = 1.5$; the argument is the same near the other boundary component of $\mathbb{A}'$).

Fix $\alpha > 0$. Since $\psi_2(x, 1.5) = (x, 1.5)$, there exists $\delta > 0$ such that for $|y - 1.5| < \delta$, the curve $\psi_2(S^1 \times \{y\})$ is the graph of a function F_y (depending smoothly on y):

$$\psi_2(S^1 \times \{y\}) = \mathrm{graph}(F_y).$$

Note that $\frac{\partial F_y}{\partial y} > 0$. Choosing δ sufficiently small, we can assume that

$$\sup_{x \in S^1, |y-1.5| \leq \delta} |F_y(x) - 1.5| \leq \alpha \quad \text{and} \quad \delta < \alpha.$$

We can now extend the family of functions $(F_y)_{|y-1.5| \leq \delta}$ to a family of functions $(F_y)_{|y-1.5| \leq \alpha}$ such that $F_y(x) = y$ when $|y - 1.5|$ is close to α and such that we still have the conditions $\frac{\partial F_y}{\partial y} > 0$ and $|F_y(x) - 1.5| \leq \alpha$. By the implicit function theorem, we can now write

$$y = F_{c(x,y)}(x) \quad (x \in S^1, |y - 1.5| \leq \alpha),$$

with $\frac{\partial c}{\partial y} > 0$. Note that $c(x, y) = y$ when $|y - 1.5|$ is close to α. By composing ψ_2 with the C^0-small diffeomorphism h defined by $h(x, y) = (x, c(x, y))$, we obtain the desired perturbation.

The previous perturbation having been performed, we can now assume that for some sufficiently small $r > 0$ the restriction of ψ_2 to $S^1 \times [-1.5, -1.5 + r] \cup S^1 \times [1.5 - r, 1.5]$ has the form

$$\psi_2 : (x, y) \mapsto (x + u(x, y), y),$$

for some smooth function u such that $\|u\| < C''\epsilon$. Choose a cut-off function $\chi = \chi_{1.5, 1.5-r} : \mathbf{R} \to [0, 1]$ and define a map ψ_3 on $\mathbb{A}'$ as follows:

$$\psi_3 := \psi_2 \text{ on } S^1 \times [-1.5 + r, 1.5 - r],$$
$$\psi_3(x, y) := (x + \chi(y)u(x, y), y), \text{ when } |y| \geq 1.5 - r.$$

We now consider the diffeomorphism ψ that equals ψ_3 on $\mathbb{A}'$ and the identity outside $\mathbb{A}'$. It coincides with ϕ on $\mathbb{A}_1$ and satisfies $\|\psi\| < C''\epsilon$. Note that if ϵ is sufficiently small, ψ automatically belongs to the identity component $\mathrm{Diff}_{0,c}(\mathbb{A})$ (this can be easily deduced, for instance, from [15, 16] or [49]). This finishes the construction of ψ in the general case.

Let us now consider the case that $\phi = \mathbb{1}$ outside a quadrilateral $I \times [-1, 1]$ and $\phi(I \times [-1, 1]) \subset I \times [-2, 2]$ for some arc $I \subset S^1$. Then, by Lemma 6, we can assume that ψ_1 is supported in $I \times [-3, 3]$. Then ψ_2 is the identity outside $I \times [-1.5, 1.5]$. When we perturb ψ_2 near $\partial\mathbb{A}'$ to make it preserve the foliation by circles, we can choose the perturbation to be supported in $I \times [-1.5, 1.5]$. Thus $u(x,y)$ would be 0 outside $I \times [-1.5, 1.5]$. This yields that ψ_3, and consequently ψ, is the identity outside $I \times [-3, 3]$. □

7 Appendix by Michael Khanevsky: An Extension Lemma for Curves

For a diffeomorphism ϕ of a compact surface with a Riemannian distance d we write $\|\phi\| = \max d(x, \phi(x))$. The purpose of this appendix is to prove the following extension lemma, which was used in Sect. 6.4 above.

Lemma 6. *Let $A := S^1 \times [-1, 1]$ be an annulus equipped with the Euclidean product metric. Set $L = S^1 \times 0$. Assume that ϕ is a smooth embedding of an open neighborhood of L in A, so that L is homotopic to $\phi(L)$ and $\|\phi\| \leq \epsilon$ for some $\epsilon \ll 1$.*

Then there exists a diffeomorphism $\psi \in \mathrm{Diff}_{0,c}(A)$ such that $\psi = \phi$ on L and $\|\psi\| < C'\epsilon$ for some $C' > 0$ independent of ϕ.

Moreover, if $\phi = \mathbb{1}$ outside some arc $I \subset L$ and $\phi(I) \subset I \times [-1, 1]$, then ψ can be made the identity outside $I \times [-1, 1]$.

Proof. We view the coordinate x on A along S^1 as a horizontal one, and the coordinate y along $[-1, 1]$ as a vertical one. If $a, b \in L$ are not antipodal, we denote by $[a, b]$ the shortest closed arc in L between a and b. The proof consists of a few steps. By $C_1, C_2, \ldots$ we will denote some universal positive constants.

Step 1. Shift the curve $\phi(L)$ by 3ϵ upward by a diffeomorphism $\psi_1 \in \mathrm{Diff}_{0,c}(A)$ with $\|\psi_1\| \leq C_1\epsilon$, so that $K := \psi_1(\phi(L))$ lies strictly above L (see Fig. 1).

Step 2. Let $x_1, \ldots, x_N$ be points on L chosen in a cyclic order so that the distance between any two consecutive points x_i and x_{i+1} is at most ϵ (here and below, $i+1$ is taken to be 1, if $i = N$).

For each $i = 1, \ldots, N$, consider a vertical ray originating at x_i and assume, without loss of generality, that it is transversal to K and that K is parallel to L near its intersection points with the ray. Among the intersection points of the ray with K choose the closest one to L and denote it by y_i. Denote by r_i the closed vertical interval between x_i and y_i. Choose small disjoint open rectangles U_i, of width at most $\epsilon/3$ and of height at most 4ϵ around each of the intervals r_i.

For each $i = 1, \ldots, N$, it is easy to construct a diffeomorphism $\psi_{2,i}$ supported in U_i that moves a connected arc of $K \cap U_i$ containing y_i by a parallel shift downward

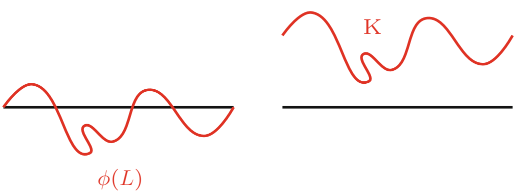


Fig. 1 Shifting L

Fig. 2 $\tilde{K}$ coincides with L near x_i

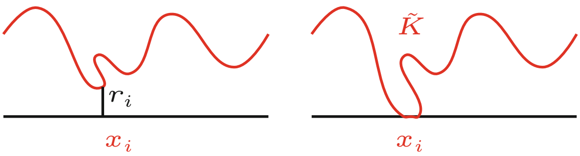
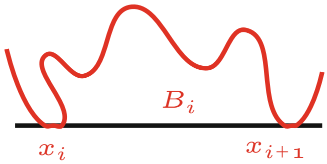


Fig. 3 The open set B_i



into an arc of L containing x_i so that $\psi_{2,i}(K)$ lies completely in $\{y \geq 0\}$. Set $\psi_2 := \prod_{i=1}^{N} \psi_{2,i}$. Clearly, $\|\psi_{2,i}\| \leq C_2\epsilon$ for each i, and therefore, since the supports of all the diffeomorphisms $\psi_{2,i}$ are disjoint, $\|\psi_2\| \leq C_2\epsilon$ as well. Set (see Fig. 2)

$$\tilde{\psi} := \psi_2\psi_1 \in \mathrm{Diff}_{0,c}(A), \quad \tilde{K} := \tilde{\psi}(\phi(L)).$$

Note that $\|\tilde{\psi}\| \leq C_3\epsilon$.

Step 3. Note that the points x_i, $i = 1, \ldots, N$, lie on $\tilde{K}$ and that

$$\tilde{K} \subset \{y \geq 0\}.$$

An easy topological argument shows that in such a case, since the points x_i lie on L in cyclic order, they also lie in the same cyclic order on $\tilde{K}$.

For each i there are two arcs in $\tilde{K}$ connecting x_i and x_{i+1}. Denote by K_i the one homotopic with fixed endpoints to the arc $[x_i, x_{i+1}] \subset L$. Since the points x_i lie on $\tilde{K}$ in the same cyclic order as on L, we see that $K_1, \ldots, K_N$ are precisely the closures of the N open arcs in $\tilde{K}$ obtained by removing the points $x_1, \ldots, x_N$ from $\tilde{K}$.

Let B_i be the open set bounded by K_i and $[x_i, x_{i+1}]$ (see Fig. 3). The B_i are disjoint and have diameter at most $C_4\epsilon$. Let B_i' be disjoint open neighborhoods of the B_i of diameter at most $C_5\epsilon$. Now for each i, the two arcs K_i and $[x_i, x_{i+1}]$ are homotopic in B_i', hence isotopic. Thus, one can find a diffeomorphism $\psi_{3,i} \in \mathrm{Diff}_{0,c}(B_i')$ such that $\psi_{3,i}(K_i) = [x_i, x_{i+1}]$. Since $\psi_{3,i}$ is supported in B_i', we have $\|\psi_{3,i}\| \leq C_5\epsilon$. Set $\psi_3 := \prod_{i=1}^{N} \psi_{3,i}$. Since the supports of all $\psi_{3,i}$ are disjoint, we get $\|\psi_3\| \leq C_5\epsilon$.

Step 4. Define $\psi_4 := \psi_3\tilde{\psi} = \psi_3\psi_2\psi_1$. Clearly, $\psi_4 \in \mathrm{Diff}_{0,c}(A)$ and $\|\psi_4\| \leq C_6\epsilon$. Recall that for each i we have $\psi_3(K_i) = [x_i, x_{i+1}]$ and that each K_i is the shortest arc between x_i and x_{i+1} in $\tilde{K} = \psi_2\psi_1(L)$. Thus ψ_4 maps K into L. The diffeomorphism ψ_4^{-1} satisfies $\psi_4^{-1}(L) = \phi(L)$. We now obtain easily the required ψ by a C^0-small perturbation of ψ_4^{-1}. $\square$

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