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HYPERBOLIC GROUPS CONTAINING SUBGROUPS OF TYPE $\mathscr{F}_3$ NOT $\mathscr{F}_4$

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Abstract

We give examples of hyperbolic groups which contain subgroups that are of type $\mathscr{F}_3$ but not of type $\mathscr{F}_4$. These groups are obtained by Dehn filling starting from a non-uniform lattice in $\mathrm{PO}(8,1)$ which was previously studied by Italiano, Martelli and Migliorini.

1. Introduction

The purpose of this work is to prove the following theorem.

Theorem 1. *There exist infinitely many pairwise non-isomorphic hyperbolic groups G admitting a surjective homomorphism $\phi\colon G \to \mathbb{Z}$ whose kernel is of type $\mathscr{F}_3$ and has the property that $H_4(\ker(\phi), \mathbb{Z})$ is not finitely generated. In particular $\ker(\phi)$ is not of type $\mathscr{F}_4$.*

These hyperbolic groups will be obtained by *Dehn filling* starting from a certain non-uniform lattice in $\mathrm{PO}(8,1)$. Before describing our method, we provide some historical context for our result.

The study of finiteness properties of subgroups of hyperbolic groups has a long history. Rips gave the first example of a finitely generated subgroup of a hyperbolic group which is not finitely presented [**47**]. This was followed by many works on *coherence*, see e.g. [**9**, **28**, **31**, **32**, **33**] to name just a few of them. While these works lead to an ample supply of finitely generated subgroups of hyperbolic groups that are not themselves hyperbolic, finding finitely presented subgroups of hyperbolic groups that are not themselves hyperbolic turns out to be a much more subtle problem.

The first example of a subgroup of a hyperbolic group which is finitely presented but not of type $\mathscr{F}_3$, and thus not itself hyperbolic, was constructed by Brady [**10**] (see also [**11**] for a survey). Preceding Brady's work, Gromov [**24**] was the first to sketch the construction of a ramified covering of a 5-dimensional torus that seemed to produce a non-hyperbolic finitely presented subgroup of a hyperbolic group. However, it was later shown by Bestvina that Gromov's group cannot be hyper-

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bolic [**4**]. Brady's construction in [**10**] elegantly solves this problem, by taking a covering of a direct product of three graphs, ramified on a family of embedded graphs. More recently, other examples with the same property as Brady's group were constructed by Lodha [**38**] and by Kropholler [**37**], who generalises both Brady's and Lodha's constructions. Finally, Italiano, Martelli and Migliorini constructed in 2021 the very first examples of subgroups of hyperbolic groups which are not hyperbolic but have finite classifying spaces [**27**]. The existence of such subgroups was a well-known open problem before their work. To the best of our knowledge this provides a complete overview on known examples of non-hyperbolic finitely presented subgroups of hyperbolic groups. A variation of the construction in [**27**] was subsequently provided by [**22**].

In [**10**], Brady raised the following question.

Question 2. Do there exist hyperbolic groups containing subgroups of type $\mathscr{F}_{n-1}$ but not of type $\mathscr{F}_n$ for $n \geq 4$?

Theorem 1 gives a positive answer[1] to Brady's question for $n = 4$. We emphasize that parts of the methods we develop work in all dimensions and could thus also lead to examples of subgroups of hyperbolic groups of type $\mathscr{F}_{n-1}$ and not of type $\mathscr{F}_n$ for $n \geq 5$. This would require finding examples of non-uniform lattices in $\mathrm{PO}(2n, 1)$ endowed with homomorphisms onto $\mathbb{Z}$ whose kernels have strong enough finiteness properties.

Some evidence towards the existence of such lattices is provided by a recent theorem by Fisher [**20**], proving a conjecture of Kielak [**36**]. Kielak's conjecture was motivated by [**35**], and one of the consequences of its proof is the existence of subgroups of hyperbolic groups which are of type $\mathrm{FP}_{n-1}(\mathbb{Q})$ but not of type $\mathrm{FP}_n(\mathbb{Q})$ for each $n \geq 3$. We discuss this briefly in Section 3. Even though this does not seem to provide new examples of subgroups of hyperbolic groups of type $\mathscr{F}_{n-1}$ and not $\mathscr{F}_n$ for $n \geq 3$, it does suggest in conjunction with Theorem 1 that lattices in $\mathrm{PO}(2n, 1)$ are a good place to search for such groups.

We now describe the construction leading to the examples of Theorem 1. Let $\Gamma < \mathrm{PO}(n, 1)$ be a torsion-free nonuniform lattice where we assume that $n \geq 3$. Let $(H_i)_{i \in I}$ be the family of maximal parabolic subgroups of Γ. We assume that each H_i is purely unipotent, hence is isomorphic to $\mathbb{Z}^{n-1}$. We fix once and for all a collection $(B_i)_{i \in I}$ of open horoballs in the n-dimensional hyperbolic space $\mathbb{H}^n$, in such a way that:

1) B_i is invariant under H_i for each $i \in I$,
2) the B_i's are disjoint and the union $\cup_{i \in I} B_i$ is Γ-invariant,
3) the action of Γ on the closed set $\widetilde{C} = \mathbb{H}^n - \cup_{i \in I} B_i$ is cocompact.

[1] Note added in proof: a positive answer to Brady's question has now been given for all n, see C. Llosa Isenrich and P. Py, Subgroups of hyperbolic groups, finiteness properties and complex hyperbolic lattices, Invent. Math. 235, no. 1 (2024).

If Λ is a finite index subgroup of Γ we define $M_\Lambda := \widetilde{C}/\Lambda$. Hence

$$M_\Lambda \subset \mathbb{H}^n/\Lambda$$

is a compact submanifold whose boundary is a disjoint union of finitely many flat $(n-1)$-tori obtained by cutting the cusps of $\mathbb{H}^n/\Lambda$ along the projection of the horoballs B_i.

Let now $f\colon \Gamma \to \mathbb{Z}$ be a surjective homomorphism. Throughout the introduction, we make the following:

Standing assumption. The homomorphism f is nontrivial on each parabolic subgroup.

If $\Lambda < \Gamma$ is a finite index subgroup, we denote by $N_f(\Lambda)$ the normal subgroup of Λ generated by the union:

$$\bigcup_{i\in I} H_i \cap \Lambda \cap \ker(f)$$

and by $\overline{f}_\Lambda\colon \Lambda/N_f(\Lambda) \to \mathbb{Z}$ the homomorphism induced by f. The groups $H_i \cap \Lambda$ are isomorphic to $\mathbb{Z}^{n-1}$ and each intersection

$$H_i \cap \Lambda \cap \ker(f)$$

is isomorphic to $\mathbb{Z}^{n-2}$. Geometrically, $\Lambda/N_f(\Lambda)$ is the fundamental group of the space $M_{\Lambda,f}$ obtained from M_Λ by coning off in each boundary component a family of codimension 1 subtori parametrised by S^1. These subtori are determined by the condition that their fundamental group coincides with the kernel of the restriction of f to the fundamental group of the corresponding boundary component.

We will deduce from a result due to Fujiwara and Manning [**21**] about Dehn filling in cusped hyperbolic manifolds that the quotient $\Lambda/N_f(\Lambda)$ is hyperbolic if Λ is deep enough in Γ. We will describe this in more detail and give a precise definition of the expression "deep enough" in Section 2.1. The hyperbolicity of $\Lambda/N_f(\Lambda)$ can also be deduced from the more general theory of Dehn filling in relatively hyperbolic groups [**17**, **25**, **44**]. Our first result is the following:

Theorem 3. *Let $k \geq 1$. If the kernel of f is of type $\mathscr{F}_k$, then the kernel of the induced morphism $\overline{f}_\Lambda\colon \Lambda/N_f(\Lambda) \to \mathbb{Z}$ is also of type $\mathscr{F}_k$ for every deep enough finite index subgroup $\Lambda < \Gamma$.*

The main input to prove Theorem 3 will be Fujiwara and Manning's result [**21**, Theorem 2.7] stating that the coned-off space $M_{\Lambda,f}$ carries a locally CAT(-1) metric (for deep enough Λ), which implies that $M_{\Lambda,f}$ is aspherical. We spell out the particular case of Theorem 3 when $k = 2$:

Theorem 4. *If the kernel of f is finitely presented, then the kernel of the induced morphism $\overline{f}_\Lambda\colon \Lambda/N_f(\Lambda) \to \mathbb{Z}$ is also finitely presented for every deep enough finite index subgroup $\Lambda < \Gamma$.*

Theorem 3 leads to two natural questions. Can one find examples of pairs (Γ, f) to which one can apply this construction? Given such a pair, can one determine the exact finiteness properties of the kernel $\ker(\overline{f}_\Lambda)$ associated to a deep enough finite index subgroup $\Lambda < \Gamma$? We first give a partial answer to the second question when n is even.

Theorem 5. *Assume that $n = 2k$ and that the Betti numbers*

$$(b_i(\ker(f))_{1\leq i\leq k-1}$$

are finite. Then $b_k(\ker(f))$ is infinite. If, moreover, Λ is deep enough, then $b_k(\ker(\overline{f}_\Lambda))$ is also infinite and therefore $\ker(\overline{f}_\Lambda)$ is not of type $\mathscr{F}_k$.

The proof of Theorem 5 relies on classical arguments due to Milnor [**43**], combined with the fact that the Euler characteristic of finite volume (complete, oriented) even-dimensional hyperbolic manifolds is nonzero [**34**].

Let us also insist on the fact that the "deep enough" finite index subgroups appearing in Theorems 3, 4 and 5 can be chosen independently of the morphism f. One can actually pick once and for all a finite index subgroup $\Gamma_1 < \Gamma$ (depending only on the action of Γ on $\mathbb{H}^n$) with the property that these three theorems apply to any finite index subgroup $\Lambda < \Gamma_1$. This will be clear after the reader has gone through Section 2.1.

To deduce Theorem 1 from Theorems 3 and 5, we will use a certain hyperbolic manifold $M^8 = \mathbb{H}^8/\Gamma$ built from the dual of the 8-dimensional Euclidean Gosset polytope studied in [**19**], together with a continuous map $u\colon M^8 \to S^1$. The manifold M^8 and the corresponding map to the circle were constructed in [**26**]. There the authors prove that the kernel of the homomorphism

$$u_*\colon \Gamma \to \mathbb{Z}$$

induced by u is finitely presented. Here we go further and prove that $\ker(u_*)$ is actually of type $\mathscr{F}_3$ and not of type $\mathscr{F}_4$ (see Theorem 25). However, the restriction of the map u to some of the cusps is homotopic to a constant function. By considering a suitable rational perturbation of the cohomology class of u, we prove:

Theorem 6. *There exists a continuous map $v\colon M^8 \to S^1$ such that:*

1) *none of the restrictions of v to a cusp of M^8 is homotopic to a constant*
2) *the kernel of v_* is of type $\mathscr{F}_3$.*

For the part of our work concerning the specific properties of this 8-dimensional manifold we use a program written in Sage to perform some of the computations required in our argument.

Combining Theorems 3, 5 and 6, and applying them to $f = v_*$, we can prove the existence of one hyperbolic group satisfying the conclusion of Theorem 1: if $\Lambda < \Gamma$ is a deep enough finite index subgroup, the fundamental group $\Lambda/N_f(\Lambda)$ of the coned off space $M_{\Lambda,f}$ is hyperbolic and the kernel of the induced morphism

$$\overline{f}_\Lambda \colon \Lambda/N_f(\Lambda) \to \mathbb{Z}$$

has the properties stated in Theorem 1.

The only point that remains to be proved is then the fact that one can obtain from our methods infinitely many isomorphism classes of hyperbolic groups containing subgroups of type $\mathscr{F}_3$ and not $\mathscr{F}_4$. This will follow from a result due to Fujiwara and Manning [**21**, Theorem 2.13] by constructing a suitable infinite family of perturbations of u_* as in Theorem 6. This will require a few additional arguments and we will postpone this part of the proof to Section 5.

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2. Dehn fillings and finiteness properties

In this section we will prove Theorem 3. For this we first recall Fujiwara and Manning's results on Dehn fillings [**21**] of non-compact finite volume hyperbolic manifolds in Section 2.1. In Section 2.2 we then introduce different types of finiteness properties of groups and prove Theorem 3.

2.1. Dehn filling and asphericity. We keep the notations from the introduction: Γ is a torsion-free nonuniform lattice of isometries of $\mathbb{H}^n$, the family of its parabolic subgroups is denoted by $(H_i)_{i\in I}$ and we have fixed a family $(B_i)_{i\in I}$ of horoballs having the same properties as in the introduction. The complement of the union of the B_i's in $\mathbb{H}^n$ is denoted by $\widetilde{C}$ and we set $M_\Gamma = \widetilde{C}/\Gamma$. We pick a finite set

$$\{i_1, \ldots, i_r\} \subset I$$

with the property that each parabolic subgroup of Γ is conjugated to exactly one of the $(H_{i_s})_{1\leq s\leq r}$. The boundary of B_{i_s} projects onto a boundary torus T_s in M_Γ. The metric induced by the hyperbolic metric on T_s is flat. We now describe more precisely the coned-off space $M_{\Gamma,f}$ (see [**21**, §2.1] for a similar, more general description). Identify T_s with its flat metric with $\mathbb{R}^{n-1}/A_s$ for a lattice $A_s < \mathbb{R}^{n-1}$. Let $D_s < A_s$ be the intersection of the kernel of f with A_s. The span of D_s is a codimension 1 subspace $V \subset \mathbb{R}^{n-1}$ and the orthogonal projection $\mathbb{R}^n \to V^\perp$ induces a submersion $\pi_s \colon T_s \to S^1$ whose fibers are the translates of the subtorus V/D_s. The *partial cone* $C(T_s, V/D_s)$ is the space obtained from $T_s \times [0,1]$ by identifying $(p,1)$ to $(q,1)$ whenever $\pi_s(p) = \pi_s(q)$. The map $(p,t) \mapsto \pi_s(p)$ induces a map

$$C(T_s, V/D_s) \to S^1. \tag{1}$$

The space $M_{\Gamma,f}$ is obtained by gluing $C(T_s, V/D_s)$ to M_Γ, identifying $T_s \times \{0\}$ with T_s for each $s \in \{1, \ldots, r\}$. Clearly, one can perform this construction replacing Γ by any of its finite index subgroups.

We say that a torus equipped with a flat metric satisfies the 2π-condition if each of its closed geodesics has length greater than 2π. Similarly, we say that a totally geodesic subtorus of a flat torus satisfies the 2π-condition if each of the closed geodesics in the subtorus has length greater than 2π. Clearly if a torus satisfies the 2π-condition all of its totally geodesic subtori also satisfy it. The following theorem follows from Theorem 2.7 in [**21**].

Theorem 7. *Assume that for each $s \in \{1, \ldots, r\}$, V/D_s satisfies the 2π-condition. Then $M_{\Gamma,f}$ carries a locally* $\mathrm{CAT}(-1)$ *metric.*

A priori the 2π-condition for V/D_s need not be satisfied. However, there is always some finite index subgroup $\Lambda < \Gamma$ such that the boundary components of M_Λ satisfy the 2π-condition. We recall this classical argument. For $1 \leq s \leq r$, let $Z_s \subset H_{i_s}$ be the set of elements whose translation length for the action $H_{i_s} \curvearrowright \partial B_{i_s}$ is not greater than 2π. This is a finite set. Hence

$$Z\colon = Z_1 \cup \cdots \cup Z_r \subset \Gamma$$

is a finite set.

Definition 8. A finite index subgroup $\Lambda < \Gamma$ is *deep enough* if the intersection of Λ with the union $\cup_{\gamma\in\Gamma}\gamma Z\gamma^{-1}$ is trivial.

Deep enough finite index subgroups exist: indeed, since Γ is residually finite, there is a finite quotient $a\colon \Gamma \to F$ such that $a(z) \neq 1$ for all $z \in Z$. The kernel of such a morphism a is a deep enough finite index subgroup. A possibly nonnormal finite index subgroup of $\ker(a)$ also does the job. As a consequence of Theorem 7, applied to a finite index subgroup of Γ instead of Γ itself, we have:

Corollary 9. *If Λ is a deep enough finite index subgroup of Γ, then the space $M_{\Lambda,f}$ is aspherical and the group $\Lambda/N_f(\Lambda)$ is hyperbolic.*

Hence, if Λ is a deep enough finite index subgroup of Γ, each covering space $Y \to M_{\Lambda,f}$ is a classifying space for the corresponding subgroup of $\Lambda/N_f(\Lambda)$. This will allow us to study the finiteness properties of $\pi_1(Y)$ geometrically.

Remark 10. The asphericity of $M_{\Lambda,f}$ could also be established using the methods from [**16**]. That paper establishes the asphericity of certain spaces obtained by coning off totally geodesic submanifolds in closed negatively curved manifolds and the same techniques can also be applied in the context of Dehn fillings.

2.2. Finiteness properties. In this section, we recall the definitions of all the finiteness properties that we shall deal with and then prove Theorem 3. We start with the definition of property $\mathscr{F}_n$, introduced by Wall [**49**].

Definition 11. A group G is of type $\mathscr{F}_n$ if it admits a $K(G,1)$ which is a CW-complex with finite n-skeleton.

The above definition is a *geometric* finiteness property. We now define some *algebraic* finiteness properties, first introduced by Bieri [**6**].

Definition 12. Let R be a ring. A group G is of type $\mathrm{FP}_n(R)$ if R, considered as a trivial RG-module, admits a projective resolution

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to R \to 0$$

where P_i is a finitely generated RG-module for $i \leq n$.

In what follows, we will only consider the case when $R = \mathbb{Z}$ (in which case we abbreviate and write that G is of type FP_n instead of $\mathrm{FP}_n(\mathbb{Z})$) and the case $R = \mathbb{Q}$. If G is of type $\mathrm{FP}_n(R)$, one can actually construct a resolution

$$\cdots \to F_{n+1} \to F_n \to \cdots \to F_1 \to F_0 \to R \to 0$$

by *free* RG-modules F_i, where F_i is finitely generated for $i \leq n$ [**12**, Proposition VIII.4.3]. One has the implications

$$\mathscr{F}_n \Longrightarrow \mathrm{FP}_n \Longrightarrow \mathrm{FP}_n(\mathbb{Q}).$$

Furthermore, if a group G is of type $\mathrm{FP}_n(\mathbb{Q})$, its Betti numbers $b_i(G)$ are finite for $i \leq n$, where $b_k(G)$ is the k-th Betti number of G with coefficients in $\mathbb{Q}$. We also mention that all homology groups that we will consider will be taken with rational coefficients, if not specified otherwise.

We now return to the proof of Theorem 3. We assume that $\ker(f)$ is $\mathscr{F}_k$ and that we fixed a deep enough finite index subgroup $\Lambda < \Gamma$. In particular, the manifold M_Λ, together with the subtori of its boundary determined by $\ker(f)$, satisfies the 2π-condition. To simplify the presentation, we will use the following notations: $W := M_\Lambda$ is the hyperbolic manifold with boundary, $v\colon W \to S^1$ is a continuous map inducing the morphism $f\colon \Lambda \to \mathbb{Z}$ (obtained by restriction of $f\colon \Gamma \to \mathbb{Z}$), and W_f is the Dehn filling of W constructed in Section 2.1. The induced map and homomorphism are denoted by $\overline{v}\colon W_f \to S^1$ and $\overline{f}\colon \pi_1(W_f) \to \mathbb{Z}$. After passing to the finite covers of W and W_f corresponding to the preimage under f (or $\overline{f}$) of the finite index subgroup

$$\bigcap_{1 \leq s \leq r} f(A_s) \leq \mathbb{Z}$$

if necessary, we may moreover assume that $f|_{A_s}\colon A_s \to \pi_1(S^1) \cong \mathbb{Z}$ is surjective for $1 \leq s \leq r$. Since $\ker(f)$ and $\ker(\overline{f})$ remain the same under these finite covers of W and W_f, it suffices to consider this case. Finally we can and do assume that the map $\overline{v}$ coincides on $C(T_s, V/D_s)$ with the natural map $C(T_s, V/D_s) \to S^1$ induced by π_s, see (1). (This is possible after performing a suitable homotopy of the original map v.)

Let $\pi\colon W_{\mathbb{Z}} \to W$ and $\overline{\pi}\colon W_{f,\mathbb{Z}} \to W_f$ be the covering spaces induced by $\ker f$ and $\ker \overline{f}$. Our assumptions imply that for $1 \leq s \leq r$ the preimage $\overline{\pi}^{-1}(C(T_s, V/D_s))$ is connected and can canonically be identified with $C(V/D_s) \times \mathbb{R}$ via the $\mathbb{R}$-translates of the inclusion of the cone $C(V/D_s) \overset{\cong}{\to} v^{-1}(0) \cap C(T_s, V/D_s)$ of V/D_s in $W_{f,\mathbb{Z}}$. We obtain a commutative diagram

$$\begin{tikzcd}
W_{\mathbb{Z}} \arrow[r] \arrow[d, hook] & \mathbb{R} \arrow[r] \arrow[d, "\mathrm{id}_{\mathbb{R}}"] & S^1 \arrow[d, "\mathrm{id}_{S^1}"] \\
W_{f,\mathbb{Z}} \arrow[r] & \mathbb{R} \arrow[r] & S^1 \\
\bigcup_{1\leq s\leq r} C(V/D_s) \times \mathbb{R} \arrow[r] \arrow[u, hook] & \mathbb{R} \arrow[r] \arrow[u, "\mathrm{id}_{\mathbb{R}}"] & S^1, \arrow[u, "\mathrm{id}_{S^1}"]
\end{tikzcd}$$

with $W_{f,\mathbb{Z}} \backslash \left(\bigcup_{1\leq s\leq r} C(V/D_s) \times \mathbb{R} \right) = W_{\mathbb{Z}} \backslash \partial W_{\mathbb{Z}}$. It follows that there is a deformation retraction of the aspherical space $W_{f,\mathbb{Z}}$ onto the subspace

$$W_{\mathbb{Z}} \cup \left(\bigcup_{1\leq s\leq r} C(V/D_s) \times \{0\} \right).$$

In particular, up to homotopy equivalence, $W_{f,\mathbb{Z}}$ is an aspherical space that is obtained by attaching finitely many cells to the boundary of $W_{\mathbb{Z}}$. On the other hand, since the aspherical manifold $W_{\mathbb{Z}}$ is a $K(\ker(f), 1)$ for the group $\ker(f)$ of type $\mathscr{F}_k$, it is homotopy equivalent to an aspherical CW-complex with finitely many cells of dimension $\leq k$. We deduce that $W_{f,\mathbb{Z}}$ is homotopy equivalent to an aspherical CW-complex with finitely many cells of dimension $\leq k$. Thus, $\pi_1(W_{f,\mathbb{Z}}) = \ker(\overline{f})$ is of type $\mathscr{F}_k$. This completes the proof of Theorem 3.

We end this section by proving the following variation of Theorem 3, which we will require later.

Proposition 13. *Let $i \geq 0$. Let $\Lambda < \Gamma$ be a deep enough finite index subgroup, $f\colon \Lambda \to \mathbb{Z}$ and $\overline{f}_\Lambda\colon \Lambda/N_f(\Lambda) \to \mathbb{Z}$ be as in Section 2.1. Then $b_i(\ker(f))$ is finite if and only if $b_i(\ker(\overline{f}_\Lambda))$ is finite.*

Proof. We make the same assumptions and use the same notation as in the remainder of this section. From the decomposition

$$W_{f,\mathbb{Z}} = W_{\mathbb{Z}} \bigcup \left(\bigcup_{1\leq s\leq r} C(V/D_s) \times \mathbb{R} \right),$$

we obtain the Mayer–Vietoris sequence

$$\cdots \to H_i(\partial W_{\mathbb{Z}}) \to H_i(W_{\mathbb{Z}}) \oplus \bigoplus_{1\le s\le r} H_i(C(V/D_s)\times\mathbb{R}) \\ \to H_i(W_{f,\mathbb{Z}}) \to H_{i-1}(\partial W_{\mathbb{Z}}) \to \cdots .$$

We observe that $\partial W_{\mathbb{Z}}$ is homotopy equivalent to a disjoint union of r tori of dimension $(n-2)$ and that the spaces $C(V/D_s)\times\mathbb{R}$ are contractible for $1\le s\le r$. Thus, the exactness of the sequence implies that $b_i(W_{\mathbb{Z}})$ is finite if and only if $b_i(W_{f,\mathbb{Z}})$ is finite. q.e.d.

3. Middle-dimensional Betti number and finiteness properties

The aim of this section is to prove Theorem 5, showing in particular that when $n=2k$, the group $\ker(\overline{f}_\Lambda)$ is not of type $\mathscr{F}_k$ for deep enough Λ, assuming that $\ker(f)$ is of type $\mathscr{F}_{k-1}$. Before proving Theorem 5 in Section 3.2, we provide a brief discussion of the relevance of ℓ^2-Betti numbers for the study of finiteness properties in Section 3.1. We emphasize however that all the theorems stated in the introduction are proved without using ℓ^2-Betti numbers. This is mainly because we consider only groups obtained as kernels of homomorphism to $\mathbb{Z}$ rather than $\mathbb{Z}^k$ for some $k>1$, see Remark 16.

3.1. The use of ℓ^2-Betti numbers. Informally speaking, the ℓ^2-Betti numbers of a group G measure the dimension of the (reduced) G-invariant ℓ^2-homology of the universal covering space of a $K(G,1)$. When G has a $K(G,1)$ which is a finite complex, the relevant notion of dimension is the *von Neumann dimension*. In general, the definition of ℓ^2-Betti numbers is more involved and several approaches are available. See [**14**, **23**, **30**, **41**] for detailed (equivalent) definitions. We will only quickly recall the definition in the proof of Proposition 14 below. The i-th ℓ^2-Betti number of G is denoted by $b_i^{(2)}(G)$.

The *von Neumann algebra* $\mathcal{R}(G)$ of a discrete group G is the algebra of bounded operators on the Hilbert space

$$\ell^2(G)$$

which commute with the left-regular representation of G. We will use below that if M is an arbitrary $\mathcal{R}(G)$-module, there is a notion of dimension

$$\dim_{\mathcal{R}(G)} M \in [0,\infty]$$

which was introduced by Lück [**39**]; see also [**41**, Ch. 6]. When M is finitely generated and projective, one recovers the more classical notion of von Neumann dimension, whose definition can be found e.g. in [**41**, Ch. 1].

The next proposition is an elementary application of a theorem of Lück [**40**]. It already appears in the work of Fisher [**20**], in a more general context. We provide the proof for the reader's convenience and we shall comment further on Fisher's work below. In combination with Theorem 25 below, Proposition 14 allows us to determine the precise finiteness properties of the original infinite cyclic covering considered in [**26**].

Proposition 14. *Let $i \geq 1$ be an integer. Let Δ be a discrete group such that $b_i^{(2)}(\Delta) \neq 0$ and let $\phi\colon \Delta \to \mathbb{Z}^{k\geq 1}$ be a surjective homomorphism. Then the kernel of ϕ is not of type* $\mathrm{FP}_i(\mathbb{Q})$.

Proof. Since we are not assuming that the groups under consideration have finite classifying spaces, we must first recall the general definition of ℓ^2-Betti numbers.

Let G be a group and let X be a CW-complex which is a $K(G,1)$. Denote by $\widetilde{X}$ the universal cover of X. Let $C_*(\widetilde{X})$ be the cellular chain complex of $\widetilde{X}$ with $\mathbb{Z}$ coefficients. One now considers the tensor product

$$\mathcal{R}(G) \otimes_{\mathbb{Z}G} C_*(\widetilde{X}).$$

This gives a chain complex where the underlying modules, and the corresponding homology groups $H_m(\mathcal{R}(G) \otimes_{\mathbb{Z}G} C_*(\widetilde{X}))$ are $\mathcal{R}(G)$-modules. The m-th ℓ^2-Betti number of G is defined as the dimension

$$\dim_{\mathcal{R}(G)} H_m(\mathcal{R}(G) \otimes_{\mathbb{Z}G} C_*(\widetilde{X}))$$

of the homology group $H_m(\mathcal{R}(G) \otimes_{\mathbb{Z}G} C_*(\widetilde{X}))$ [**39**, §4].

Let us now consider the cellular chain complex $C_*(\widetilde{X},\mathbb{Q})$ of $\widetilde{X}$ with rational coefficients. One sees readily that the complexes

$$\mathcal{R}(G) \otimes_{\mathbb{Z}G} C_*(\widetilde{X})$$

and

$$\mathcal{R}(G) \otimes_{\mathbb{Q}G} C_*(\widetilde{X},\mathbb{Q})$$

are isomorphic as $\mathcal{R}(G)$-modules. Hence the m-th ℓ^2-Betti number of G coincides with

$$\dim_{\mathcal{R}(G)} H_m(\mathcal{R}(G) \otimes_{\mathbb{Q}G} C_*(\widetilde{X},\mathbb{Q})).$$

The complex $C_*(\widetilde{X},\mathbb{Q})$ with the natural augmentation map $C_0(\widetilde{X},\mathbb{Q}) \to \mathbb{Q}$ provides a natural free resolution of $\mathbb{Q}$ as a trivial $\mathbb{Q}G$-module. Hence if

$$\cdots \to F_1 \to F_0 \to \mathbb{Q} \to 0$$

is a free resolution of $\mathbb{Q}$ by $\mathbb{Q}G$-modules, there is a natural homotopy equivalence

$$C_*(\widetilde{X},\mathbb{Q}) \to F_*.$$

After applying $\mathcal{R}(G) \otimes_{\mathbb{Q}G} \cdot$, one sees that the group $H_m(\mathcal{R}(G) \otimes_{\mathbb{Q}G} C_*(\widetilde{X},\mathbb{Q}))$ is isomorphic to $H_m(\mathcal{R}(G) \otimes_{\mathbb{Q}G} F_*)$. If G is of type $\mathrm{FP}_i(\mathbb{Q})$, one can pick a resolution F_* where the F_m's are finitely generated and free for $m \leq i$. Hence for $m \leq i$, $\mathcal{R}(G) \otimes_{\mathbb{Q}G} F_m$ is isomorphic to $\mathcal{R}(G)^{k(m)}$ where $k(m)$ is the rank of F_m. Thus its dimension is equal to $k(m)$ [**39**, p. 140]. Since the dimension decreases by taking subquotients [**39**, Th. 06], this implies

$$\dim_{\mathcal{R}(G)} H_m(\mathcal{R}(G) \otimes_{\mathbb{Q}G} F_*) < \infty.$$

Hence the ℓ^2-Betti numbers of G are finite in degree $\leq i$.

The statement of the proposition now follows by combining this observation with [**40**, Th. 3.3 (4)]. Indeed, the latter says that if Δ is a discrete group and if $\psi\colon \Delta \to \mathbb{Z}^k$ is a surjective homomorphism, then the finiteness of all the numbers $(b_m^{(2)}(\ker(\psi)))_{0\leq m\leq i}$ implies the vanishing of all the numbers $(b_m^{(2)}(\Delta))_{0\leq m\leq i} = 0$. See also [**23**, Théorème 6.6] for a generalization of Lück's theorem which also applies here. q.e.d.

As a consequence we obtain:

Corollary 15. *Let* Γ *be a lattice (uniform or non-uniform) in the Lie group* $\mathrm{PO}(2n,1)$ *or* $\mathrm{PU}(n,1)$ *for* $n \geq 1$ *and let* $\phi\colon \Gamma \to \mathbb{Z}^{k\geq 1}$ *be a surjective homomorphism. Then* $\ker(\phi)$ *is not of type* $\mathrm{FP}_n(\mathbb{Q})$*.*

Note that the finiteness properties of kernels of homomorphisms from cocompact complex hyperbolic lattices to abelian groups were also studied in [**18**], in relation with the BNS invariant.

Proof of Corollary 15. By [**23**, Example 1.6 p. 109], $b_n^{(2)}(\Gamma) \neq 0$ for every lattice Γ in $\mathrm{PO}(2n,1)$ or $\mathrm{PU}(n,1)$ (see also [**29**, Theorem 3.3] or [**13**]). Thus, the assertion is an immediate consequence of Proposition 14. q.e.d.

Remark 16. When the image of ϕ is cyclic, Corollary 15 can be proved without using ℓ^2-Betti numbers. See Section 3.2.

In [**26**] Italiano, Martelli and Migliorini construct a $2n$-dimensional hyperbolic manifold M^{2n} and a homomorphism $f_n\colon \pi_1(M^{2n}) \to \mathbb{Z}$ induced by a circle-valued height function for $n \in \{2,3,4\}$ from duals of Euclidean Gosset polytopes. As a direct consequence of Corollary 15 we obtain:

Theorem 17. *The group* $\ker(f_n) \leqslant \pi_1(M^{2n})$ *is a subgroup of a relatively hyperbolic group that is not of type* $\mathrm{FP}_n(\mathbb{Q})$ *(therefore not* $\mathscr{F}_n$*).*

In Section 3.2 we will give a different argument showing that $\ker(v_*)$ is not $\mathscr{F}_4$ for the perturbed map $v\colon M^8 \to S^1$ from the introduction. More precisely, we will show that the 4th Betti number of $\ker(v_*)$ is infinite. This argument will make essential use of the fact that none of the restrictions of v to a cusp is null-homotopic (see the proof of Proposition 22). This different argument can thus not be applied to the homomorphism f_4 which is trivial on some of the parabolic subgroups of $\pi_1(M^8)$.

We now discuss the following very recent theorem of Fisher [20] that we alluded to in the introduction. To state it we use the notion of *Residually Finite Rationally Solvable (short: RFRS)* groups, introduced by Agol (see [1] for their definition).

Theorem 18 (Fisher)**.** *Let G be a virtually RFRS group of type* $\mathrm{FP}_n(\mathbb{Q})$. *Then there is a finite index subgroup $H < G$ admitting a homomorphism onto $\mathbb{Z}$ with kernel of type* $\mathrm{FP}_n(\mathbb{Q})$ *if and only if* $b_i^{(2)}(G) = 0$ *for* $i = 0, \dots, n$.

Remark 19. The proof of the "only if" part of Fisher's theorem above is given by Proposition 14. It does not use the RFRS hypothesis. However, Fisher's approach is more general and applies not only to "usual" ℓ^2-Betti numbers but also to a class of generalized Betti numbers with skew-field coefficients.

Consider now a torsion-free cocompact lattice $\Gamma < \mathrm{PO}(2k, 1)$ which is cubulable. We first recall why such groups exist: one can take Γ to be any torsion-free arithmetic lattice of the *simplest type*. This is a special family of arithmetic subgroups in $\mathrm{PO}(2k, 1)$ (which exists also in odd dimension). Their construction as well as explicit examples can be found in [42, §2] for instance. Thanks to the work of Bergeron, Haglund and Wise [3], arithmetic lattices of the simplest type are cubulable. Being hyperbolic and cubulable, such a lattice is virtually special due to Agol's Theorem [2, Theorem 1.1]. In particular Γ must be virtually RFRS. The ℓ^2-Betti numbers of any finite index subgroup Γ' of Γ vanish up to degree $k-1$ and $b_k^{(2)}(\Gamma') \neq 0$ [29, Theorem 3.3]. Applying Fisher's theorem we can thus find a finite index subgroup $\Gamma' < \Gamma$ and a surjective homomorphism $f \colon \Gamma' \to \mathbb{Z}$ such that $\ker(f)$ is of type $\mathrm{FP}_{k-1}(\mathbb{Q})$. The non-vanishing of $b_k^{(2)}(\Gamma')$ together with Proposition 14 implies that $\ker(f)$ is not of type $\mathrm{FP}_k(\mathbb{Q})$. This proves:

Proposition 20. *Let* $\Gamma < \mathrm{PO}(2k, 1)$ *be a torsion-free cocompact cubulable lattice. Then there exists a finite index subgroup* $\Gamma' < \Gamma$ *and a surjective homomorphism* $f \colon \Gamma' \to \mathbb{Z}$ *such that* $\ker(f)$ *is of type* $\mathrm{FP}_{k-1}(\mathbb{Q})$ *but not of type* $\mathrm{FP}_k(\mathbb{Q})$.

For every integer $k \geq 1$ this produces plenty of subgroups of hyperbolic groups which are $\mathrm{FP}_{k-1}(\mathbb{Q})$ but not $\mathrm{FP}_k(\mathbb{Q})$.

In the next section, we will prove Theorem 5 using arguments involving the homology with rational coefficients of the cyclic covering spaces associated to the characters under consideration. An alternative approach to prove Theorem 5 would be to compute the ℓ^2-Betti numbers of $\Lambda/N_f(\Lambda)$ and to appeal to Lück's theorem [40, Th. 3.3]. However, as far as we are aware, the ℓ^2-Betti numbers of the group $\Lambda/N_f(\Lambda)$ have not been computed. More generally, it would be interesting to study the behavior of ℓ^2-Betti numbers under Dehn fillings. We spell out one concrete question in this direction.

Question 21. Let $\Gamma < \mathrm{PO}(2k,1)$ be a torsion-free nonuniform lattice with purely unipotent parabolic subgroups. Let $(H_s)_{1\leq s\leq r}$ be pairwise nonconjugate maximal parabolic subgroups of Γ, representing all conjugacy classes of maximal parabolic subgroups. For every $s\in\{1,\dots,r\}$, pick a subgroup $A_s < H_s$.

Is it true that the k-th ℓ^2-Betti number of the quotient group

$$\Gamma/\langle\langle \bigcup_{1\leq i\leq N} A_i\rangle\rangle$$

of Γ by the normal closure of $\bigcup_{1\leq i\leq N} A_i$ is nonzero if the A_i's are deep enough in the H_i's?

This seems to be related to Lück's Approximation Conjecture [**41**, Ch. 13]. We would like to thank Jean Raimbault for pointing out this connection to us.

3.2. Middle-dimensional Betti number and infinite cyclic coverings. We resume with the notation of Section 2. In addition we now assume that $n=2k$ is even. By Corollary 15, $\ker(f)$ is not $\mathrm{FP}_k(\mathbb{Q})$. We will prove the following result, which does not require the use of ℓ^2-Betti numbers, but where instead we assume that $b_i(\ker(f))$ is finite for $i\leq k-1$. This is for instance the case when $\ker(f)$ is of type $\mathscr{F}_{k-1}$ or simply $\mathrm{FP}_{k-1}(\mathbb{Q})$. The proof will rely on our standing assumption that the restriction of f to each of the parabolic subgroups is non-trivial.

Proposition 22. *Assume that $b_i(\ker(f))$ is finite for $i\leq k-1$. Then the group $H_k(\ker(f))$ is infinite dimensional. In particular, the group $\ker(f)$ is not of type $\mathrm{FP}_k(\mathbb{Q})$ (hence not $\mathscr{F}_k$).*

This proposition implies the first statement in Theorem 5. Note that since the Euler characteristic of $\mathbb{H}^{2k}/\Gamma$ is nonzero [**34**], it follows from Milnor [**43**, Assertion 6] that $\mathbb{H}^{2k}/\ker(f)$ has *some* Betti number which is infinite. The proof of Proposition 22 consists in proving that the k-th Betti number of this space is infinite (and only this one).

We will require the following consequence of Poincaré–Lefschetz duality:

Lemma 23. *Let $(M,\partial M)$ be an n-dimensional compact oriented manifold with boundary and let $i\in\{1,\dots,n\}$. Then*

$$b_{n-i}(M)\leq b_{i-1}(\partial M)+b_i(M).$$

Proof. By Poincaré–Lefschetz duality and the Universal Coefficient Theorem there are isomorphisms

$$H_i(M)\cong H^{n-i}(M,\partial M)\cong H_{n-i}(M,\partial M)$$

for $0 \leq i \leq n$. From the long exact sequence

$$\cdots \to H_{n-i}(\partial M) \to H_{n-i}(M) \to H_{n-i}(M, \partial M) \to H_{n-i-1}(\partial M) \to \cdots$$

of the pair $(M, \partial M)$ in homology, we deduce that

$$b_{n-i}(M) \leq b_{n-i}(\partial M) + b_{n-i}(M, \partial M) = b_{i-1}(\partial M) + b_i(M),$$

where for the last equality we use Poincaré duality for the closed manifold ∂M. q.e.d.

Proof of Proposition 22. We write $M_{\mathbb{Z}} = \mathbb{H}^{2k}/\ker(f)$ and $M_\ell = \mathbb{H}^{2k}/N_\ell$, where N_ℓ is the kernel of the morphism

$$\Gamma \to \mathbb{Z}/\ell\mathbb{Z}$$

obtained by reducing f mod ℓ. Let $T\colon M_{\mathbb{Z}} \to M_{\mathbb{Z}}$ be a generator of the group of deck transformations of the cyclic covering space $M_{\mathbb{Z}} \to \mathbb{H}^{2k}/\Gamma$. Let $t\colon H_*(M_{\mathbb{Z}}) \to H_*(M_{\mathbb{Z}})$ be the transformation induced by T on homology. By applying Milnor's arguments [**43**, p. 118] to the cyclic covering $M_{\mathbb{Z}} \to M_\ell$ we obtain a long exact sequence:

$$\cdots \longrightarrow H_i(M_{\mathbb{Z}}) \xrightarrow{t^\ell - 1} H_i(M_{\mathbb{Z}}) \longrightarrow H_i(M_\ell) \longrightarrow H_{i-1}(M_{\mathbb{Z}}) \xrightarrow{t^\ell - 1} H_{i-1}(M_{\mathbb{Z}}) \longrightarrow \cdots$$

From it one deduces the following short exact sequence:

$$(2) \quad 0 \to H_i(M_{\mathbb{Z}})/(t^\ell - 1)H_i(M_{\mathbb{Z}}) \to H_i(M_\ell) \to \ker(t^\ell - 1\colon H_{i-1}(M_{\mathbb{Z}}) \to H_{i-1}(M_{\mathbb{Z}})) \to 0.$$

For $i \leq k-1$ the vector spaces appearing on the left and right in the above short exact sequence have finite dimension bounded by $b_i(M_{\mathbb{Z}})$ and $b_{i-1}(M_{\mathbb{Z}})$ respectively. We thus obtain

$$(3) \qquad b_i(M_\ell) \leq b_i(M_{\mathbb{Z}}) + b_{i-1}(M_{\mathbb{Z}})$$

for $i \leq k-1$ and all $\ell \geq 1$.

Since the restriction of f to each of the finitely many cusps of M is non-trivial, the number of boundary tori in M_ℓ is independent of ℓ. Therefore, for all i, the Betti numbers $b_{i-1}(\partial M_\ell)$ are also independent of ℓ. By Lemma 23 we have that $b_{2k-i}(M_\ell) \leq b_{i-1}(\partial M_\ell) + b_i(M_\ell)$ for $i \leq k-1$. Thus, we deduce from (3) that all Betti numbers $b_i(M_\ell)$, for $i \in \{0, \ldots, 2k\}$ with $i \neq k$, are uniformly bounded above (independently of ℓ).

Since the Euler characteristic of M_ℓ is equal to ℓ times that of M (which is nonzero [**34**]), we obtain that $b_k(M_\ell)$ grows roughly linearly with ℓ. In particular

$$b_k(M_\ell) \to \infty$$

as ℓ goes to ∞. Combining Equation (2), this time for $i = k$ with the fact that $b_{k-1}(M_{\mathbb{Z}})$ is finite, we deduce that the codimension of the image of the natural map

$$H_k(M_{\mathbb{Z}}) \to H_k(M_\ell)$$

is bounded above uniformly in ℓ. This implies that $H_k(M_{\mathbb{Z}})$ is infinite dimensional and completes the proof. q.e.d.

As a consequence we obtain the following corollary. It corresponds to the second assertion in Theorem 5, thus completing its proof.

Corollary 24. *We keep the asumptions from Proposition 22. Let* $\Lambda < \Gamma$ *be a deep enough finite index subgroup. Then the k-th Betti number of the group* $\ker(\overline{f}_\Lambda)$ *is infinite. In particular* $\ker(\overline{f}_\Lambda)$ *is not of type* $\mathrm{FP}_k(\mathbb{Q})$ *(hence not* $\mathscr{F}_k$*).*

Proof. This is an immediate consequence of Proposition 22 and Proposition 13. q.e.d.

4. The manifold M^8

In this section we describe the hyperbolic 8-manifold M^8 first studied in [**26**] and prove Theorem 6. We will use the letter M to denote a general hyperbolic manifold and will write M^8 when refering to the specific example built in [**26**].

4.1. Colourings. Let $P \subset \mathbb{X}^n$ be a finite-volume right-angled polytope, with $\mathbb{X}^n = \mathbb{R}^n$ or $\mathbb{H}^n$. A *facet* is a codimension-1 face of P. A *c-colouring* on P is the assignment of a colour $\lambda(F) \in \{1, \dots, c\}$ to each facet F of P, such that adjacent facets have distinct colours. We always suppose that each colour in the palette $\{1, \dots, c\}$ is assigned to at least one facet.

It is a standard fact, which goes back to [**48**], that a colouring produces a manifold M with the same geometry as $\mathbb{X}^n$, tessellated into 2^c copies of P. The construction goes as follows. The reflection group Γ_0 of P is generated by the reflections r_F along the facets F of P. A colouring defines a homomorphism $\Gamma_0 \to \mathbb{Z}_2^c$ which sends r_F to $e_{\lambda(F)}$, the generator of the $\lambda(F)$-th factor of $\mathbb{Z}_2^c$. The kernel Γ of this homomorphism is torsion-free, and hence defines a (flat or hyperbolic) manifold $M = \mathbb{X}^n/\Gamma$. We get an orbifold-covering $M \to P$ and M is tessellated into 2^c copies of P.

Here is a concrete description of the tessellation. For every $v \in \mathbb{Z}_2^c$ we pick a copy P_v of P. We glue these 2^c copies $\{P_v\}$ as follows: we identify every facet F of P_v with the same facet F (via the identity map) of the polytope P_{v+e_i} with $i = \lambda(F)$. The result is the tessellation of M.

If P is compact, then M also is. If $P \subset \mathbb{H}^n$ has some ideal vertices, the manifold M is finite-volume and cusped. Let v be an ideal vertex of P.

The link $\mathrm{lk}(v)$ of v in P is a right-angled Euclidean parallelepiped, which inherits a colouring (still denoted by λ) in the obvious way: every facet F of $\mathrm{lk}(v)$ is contained in a unique facet F' of P and we set $\lambda(F) = \lambda(F')$. The colouring λ on $\mathrm{lk}(v)$ produces an abstract flat compact $(n-1)$-manifold T which is topologically an $(n-1)$-torus [**26**, Proposition 7] tessellated into $2^{c'}$ copies of $\mathrm{lk}(v)$, where $c' \leq c$ is the number of distinct colours inherited by $\mathrm{lk}(v)$.

The preimage of $\mathrm{lk}(v)$ along the orbifold covering $M \to P$ consists of $2^{c-c'}$ copies of T. In particular all the cusps of M are toric. Some instructive examples are exposed in [**26**].

4.2. The manifold M^8. The 8-dimensional cusped hyperbolic manifold M^8 is constructed in [**26**] by taking the right-angled polytope $P = P^8 \subset \mathbb{H}^8$ dual to the Euclidean Gosset polytope 4_{21}, whose vertices are the 240 non-trivial elements of E_8 of smallest norm. The polytope P^8 has 240 facets, 2160 ideal vertices, and 17280 finite vertices. Its isometry group acts transitively on the facets.

Using octonions, the authors defined in [**26**] an extremely symmetric 15-colouring for P^8, where each colour is assigned to 16 distinct pairwise non-incident facets. It is also shown in [**26**, Proposition 10] that this colouring is minimal, that is, P^8 cannot be 14-coloured.

As shown in [**26**], there are two types of ideal vertices v with respect to this colouring: in total there are 1920 ideal vertices of the first type and 240 of the second type. The link $\mathrm{lk}(v)$ of a vertex v of the first type is a 14-coloured 7-cube, while the link of a vertex of the second type is a 7-coloured 7-cube. Each vertex v of the first type produces $2^{15-7} = 256$ cusps and each vertex of the second type produces $2^{15-14} = 2$ cusps. So the manifold M^8 has a total of $240 \cdot 256 + 1920 \cdot 2 = 65280$ cusps.

4.3. The combinatorial game. The combinatorial game of Jankiewicz, Norin and Wise [**28**] is then applied in [**26**] to construct a nice map $f\colon M^8 \to S^1$. We briefly explain the rules of the game, since we will need them below. We introduce here a slightly modified version that is more adapted to our purposes and has already appeared in [**27**].

We start with a right-angled polytope P, equipped with a colouring λ that produces a manifold M. A *state* for P is an assignment of a status I or O to each facet. The letters stand for In and Out. *A set of moves* is a partition of the colour palette $\{1, \ldots, c\}$. The individual sets of the partition are called *moves*.

The tessellation of M into 2^c copies of P is dual to a cubulation C, as explained in [**26**, Section 2.2]. The cubulation C has 2^c vertices dual to the copies of P in M, its edges are dual to their facets, and so on. If P has some ideal vertices, the cubulation is a compact object that embeds naturally in M as a spine: its complement $M \setminus C$ consists of the cusps. In particular M deformation retracts onto C. As explained in

[**27**, Section 1.6], the state and the set of moves determine an orientation of all the edges of the dual cubulation C. This can be done inductively as follows. The vertices of C are naturally identified with $\mathbb{Z}_2^c$. We start with the vertex $0 \in \mathbb{Z}_2^c$: the edges adjacent to 0 are dual to the facets of P, and an edge s connects 0 and e_i if the dual facet F is coloured by i. (There are multiple edges with the same endpoints if there are multiple facets having the same colour.) We orient the edge s towards 0 (inward) if F has status I, and towards e_i (outward) if it has status O. Now suppose that the edges adjacent to the vertex $v \in \mathbb{Z}_2^c$ are all oriented, and we must decide how to orient the edges adjacent to $v + e_i$ for some i. We note that the edges connecting v and $v + e_i$ are dual to the facets of P coloured with i.

The edges adjacent to $v + e_i$ are in natural 1-1 correspondence with those adjacent to v, since they are both in 1-1 correspondence with the facets of P. The rule is the following: when we pass from v to $v + e_i$, we invert the orientation (as seen from v and $v + e_i$ respectively) of all the edges whose dual facet in P has a colour that lies in the same move containing i. We can prove easily that these rules yield a well-defined orientation on all the edges of C.

We have oriented all the edges of the cubulation C, and we now examine the possible configurations at the squares. Every square of C is dual to a codimension-2 face of P that separates two facets F_1 and F_2 of P with distinct colours i_1 and i_2. There are three possibilities:

1) If i_1 and i_2 belong to two distinct moves, the edges are oriented as in Figure 1-(left); in this case we say that the pair F_1, F_2 is *very good*;
2) If i_1 and i_2 belong to the same move, and F_1 and F_2 have the same status, the edges are oriented as in Figure 1-(centre); we say that the pair F_1, F_2 is *good*;
3) if i_1 and i_2 belong to the same move and F_1 and F_2 have opposite statuses, the edges are oriented as in Figure 1-(right) and we say that the pair F_1, F_2 is *bad*.

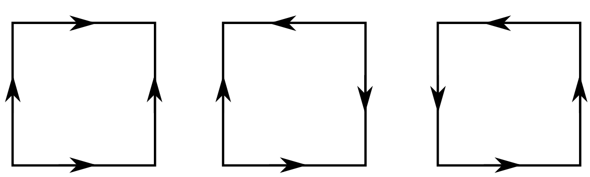

Figure 1. The possible configurations that may arise on a square of the cubulation C.

If every pair of adjacent facets in P is very good, we say that the orientation of the edges is *coherent*, and as explained in [**26**] we can define a *diagonal map* $f\colon C \to \mathbb{R}/\mathbb{Z} = S^1$ which is a Morse function in the sense of Bestvina and Brady [**5**]. We can combine this map with the deformation retraction of M onto C to get a map $f\colon M \to S^1$.

If every pair of adjacent facets in P is either good or very good, we do not get a diagonal map as above, but if we assign the same number 1 to each oriented edge we still get a 1-cocycle: indeed we see from Figure 1 that the contribution of the four edges sums to zero. This is a crucial point employed in [**27**].

If the set of moves is just the discrete partition

$$\{1\}, \dots, \{c\}$$

we deduce that distinct colours always belong to distinct moves, and hence all the squares are very good. This is the set of moves that was used implicitly in all the examples of [**26**]. We will need a different kind of set of moves at some point below.

4.4. The abelian cover $M^8_{\mathbb{Z}}$. Using octonions, the authors have defined in [**26**] a particularly symmetric state s for P^8. The state is *balanced*, in the sense that for each 16 facets sharing the same colour, half of them receive the status O and the other half receive the status I. They used as a set of moves the discrete partition, and hence they got a Morse function $f\colon C \to S^1$ inducing a map $f\colon M^8 \to S^1$.

The authors have then used a program written in Sage and available from [**50**] to analyse the ascending and descending links of all the vertices of the cubulation, see [**5**] for the terminology. The output of the program shows that all the ascending and descending links are simply connected, and hence by [**5**, Theorem 4.1] the subgroup $\ker(f)$ is finitely presented. Therefore the infinite cyclic manifold cover $M^8_{\mathbb{Z}} \to M^8$ determined by $\ker(f)$ is a complete (infinite volume, geometrically infinite) hyperbolic 8-manifold with finitely presented fundamental group $\ker(f)$. However $\ker(f)$ is not of type $\mathscr{F}_7$ because $b_7(M^8_{\mathbb{Z}}) = \infty$, see [**26**, Theorem 23].

In fact, more can be proved.

Theorem 25. *The group* $\ker(f)$ *is of type* $\mathscr{F}_3$ *and not of type* $\mathscr{F}_4$.

Proof. The same code from [**50**] shows that every ascending and descending link is also homologically 2-connected. From [**5**, Theorem 4.1, Lemma 3.7] we deduce that $\ker(f)$ is FP_3, which together with finite presentability implies $\mathscr{F}_3$ [**12**, Ch. VIII, Proof of Theorem 7.1]. On the other hand Theorem 17 shows that $\ker(f)$ is not of type $\mathscr{F}_4$. q.e.d.

4.5. Cusp homologies. We now turn back to the more general setting of a right-angled polytope $P \subset \mathbb{H}^n$ equipped with a colouring λ that produces a manifold M. As we said above, every ideal vertex v of P gives rise to some cusps in M that lie above v. That is, the preimage in M of a link $\mathrm{lk}(v)$ of an ideal vertex v in P consists of a union of disjoint flat $(n-1)$-tori. The orbifold covering $M \to P$ is regular, so the deck transformation group acts transitively on these tori. We pick one such cusp section $\iota\colon T \hookrightarrow M$ and we are interested in the induced map $\iota^*\colon H^1(M,\mathbb{R}) \to H^1(T,\mathbb{R})$ in cohomology.

Proposition 26. *Suppose that there are two opposite facets F_1, F_2 of the $(n-1)$-cube* $\mathrm{lk}(v)$ *such that one of the following holds:*

1) *The facets F_1 and F_2 have the same colour, or*
2) *The facets F_1 and F_2 have distinct colours i_1 and i_2, and they lie in distinct components of the subset $X \subset \partial P$ consisting of all the facets coloured with either i_1 or i_2.*

Then the map $\iota^*\colon H^1(M,\mathbb{R}) \to H^1(T,\mathbb{R})$ is non-trivial. If this holds for every pair of opposite facets of $\mathrm{lk}(v)$*, the map ι^* is surjective.*

Proof. Let $F_{1,i}, F_{2,i}$ denote the pairs of opposite facets in $\mathrm{lk}(v)$, for $i = 1,\dots,n-1$. Note that only opposite facets may share the same colour. Let $s_i \subset \mathrm{lk}(v)$ be the segment connecting the centers of $F_{1,i}$ and $F_{2,i}$. The preimage of $F_{1,i} \cup F_{2,i}$ in T consists of either two or four parallel connected components, depending on whether $F_{1,i}$ and $F_{2,i}$ share the same colour or not. A connected component of the preimage of s_i in T is a geodesic loop γ_i orthogonal to these, consisting of either two or four copies of s_i. See an example in Figure 2. We get $n-1$ geodesic loops $\gamma_1,\dots,\gamma_{n-1} \subset T$ that are pairwise orthogonal and generate $H_1(T,\mathbb{R})$. Let $\alpha_1,\dots,\alpha_{n-1} \in H^1(T,\mathbb{R})$ be the dual basis.

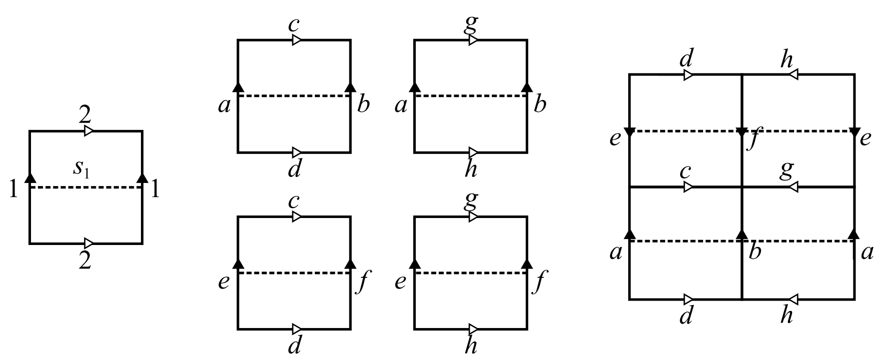

Figure 2. A square coloured with two colours 1 and 2 (left) produces a torus T tessellated into four squares (center and right). The preimage of the dashed segment s_1 consists of two parallel geodesic loops in T (right).

We now show that if the pair $F_{1,j}, F_{2,j}$ satisfies either (1) or (2) then α_j lies in the image of ι^*. This will conclude the proof.

In case (1), fix a state s for P such that $F_{1,j}$ and $F_{2,j}$ have opposite statuses, while all the other facets of $\mathrm{lk}(v)$ have status I. Use the discrete partition $\{1\},\dots,\{c\}$ as a set of moves. Every pair of facets in P is very good, so we get an orientation of the edges of the dual cubulation C which induces a 1-cocycle and therefore a class $\alpha \in H^1(M, \mathbb{R})$. The image $\iota^*(\alpha) \in H^1(T, \mathbb{R})$ sends γ_j to ± 2 and γ_i to zero for all $i \neq j$. This holds because each γ_i is isotopic to the cycle of two or four edges in the dual cubulation dual to the lifts of $F_{1,i}$ and $F_{2,i}$, and two consecutive edges are oriented in the same direction with respect to their common endpoint v (both inward or both outward with respect to v) if and only if the two statuses of $F_{1,i}$ and $F_{2,i}$ coincide. Therefore $\iota^*(\alpha) = \pm 2\alpha_j$ and we are done.

In case (2), fix a state s for P such that $F_{1,j}$ and $F_{2,j}$ have opposite statuses, and in each connected component of X all the facets have the same status (we can do this because $F_{1,j}$ and $F_{2,j}$ lie in distinct connected components). We also suppose as above that all the other facets of $\mathrm{lk}(v)$ have status I. We suppose that the colours of $F_{1,j}$ and $F_{2,j}$ are 1 and 2 for simplicity of notation and as a set of moves we pick $\{1, 2\}, \{3\}, \{4\}, \ldots, \{c\}$. By construction every pair of adjacent facets in P is either very good or good. Therefore the resulting orientation of the edges of the cubulation yields a cocycle α. As in (1) we conclude that $\iota^*(\alpha)$ sends γ_j to ± 4 and γ_i to 0 for all $i \neq j$. Therefore $\iota^*(\alpha) = \pm 4\alpha_j$. q.e.d.

We remark that it is also possible to prove the proposition using the work of Choi and Park on the cohomology of M, see [**15**]. We also note that there are some cases where ι^* is indeed trivial: pick a $2n$-gon $P \subset \mathbb{H}^2$ with $2n - 1$ right-angled real vertices and one ideal vertex, and colour P with 2 colours. None of the hypotheses of the proposition are fulfilled, and indeed, in this case, M is an orientable surface with a single cusp, and hence ι^* is trivial (here $T = S^1$).

We will need here the following consequence.

Proposition 27. *The cohomology group $H^1(M^8, \mathbb{R})$ has dimension 365. For every cusp section T the natural map $H^1(M^8, \mathbb{R}) \to H^1(T, \mathbb{R})$ is surjective.*

Proof. The number 365 was calculated in [**26**] using a formula of Choi–Park [**15**]. The 7-torus cusp section T projects to a 7-cube $\mathrm{lk}(v)$ of some ideal vertex v of P^8. There are two types of 7-cubes: for the 7-coloured ones, two opposite facets share the same colours and then Proposition 26 applies; for the 14-coloured ones, two opposite facets have distinct colours, but we have checked using Sage that the property stated in Proposition 26-(2) holds for all pairs of opposite facets. The code is available from [**50**]. q.e.d.

4.6. Constructing perturbations. We start this section with a short review of the theory of BNSR-invariants that will be useful to perturb certain characters while keeping control on the finiteness properties of their kernels. Let G be an arbitrary finitely generated group.

A *character* of G is an element of $H^1(G,\mathbb{R})$, i.e. a homomorphism $\chi\colon G \to \mathbb{R}$. We introduce the *character sphere* of equivalence classes of characters

$$S(G) = (H^1(G,\mathbb{R}) \setminus \{0\})/\sim,$$

where we call two characters χ_1 and χ_2 *equivalent* and write $\chi_1 \sim \chi_2$ if there is $\lambda \in \mathbb{R}_{>0}$ with $\lambda\cdot\chi_1 = \chi_2$. We denote by $[\chi]$ the equivalence class of a character χ in $S(G)$. We equip $S(G)$ with the topology induced by the choice of an auxiliary norm on the real vector space $H^1(G,\mathbb{R})$. Then $S(G)$ is a sphere of dimension $b_1(G)-1$. We call $[\chi]$ *rational* if it has a representative $\chi\colon G \to \mathbb{Q}$. Note that the rational equivalence classes of characters form a dense subset of $S(G)$.

In [**45**], Renz proves that the finiteness properties of the kernels of characters are determined by a descending sequence of subsets ${}^*\Sigma^i(G)$ of $S(G)$

$$S(G) = {}^*\Sigma^0(G) \supseteq {}^*\Sigma^1(G) \supseteq {}^*\Sigma^2(G) \supseteq \dots,$$

called the geometric (BNSR-)invariants of G. We refer to [**45**] for their precise definition.

More precisely Renz proves [**45**, Corollary AC] (see also [**46**, p.477, p.481], [**7**, Remark 6.5] and [**8**]):

Theorem 28. *Let G be a group of type $\mathscr{F}_k$ and let $S(G)$ be its character sphere. Then there is a descending chain of open subsets $\left\{{}^*\Sigma^i(G)\right\}_{0\le i\le k}$ of $S(G)$, such that the kernel of a rational character $\chi\colon G\to\mathbb{Q}$ is of type $\mathscr{F}_i$ for some $0\le i\le k$ if and only if $[\chi],[-\chi]\in {}^*\Sigma^i(G)$. In particular, for $0\le i\le k$, the condition that the kernel* $\ker(\chi)$ *is of type $\mathscr{F}_i$ is an open condition among rational equivalence classes of characters $[\chi]$, for the topology induced by $S(G)$.*

We now return to the setting of Section 4.5 and consider a hyperbolic manifold M with fundamental group Γ. The main result of this section is the following.

Proposition 29. *Let M and Γ be as above and let $\chi\colon\Gamma\to\mathbb{Z}$ be a character with kernel of type $\mathscr{F}_k$ for some $k\ge 0$. Let $T_1,\dots,T_r$ be cusp sections of the r cusps of M. Assume that for all $1\le s\le r$ the morphism $\iota_s^*\colon H^1(M,\mathbb{R})\to H^1(T_s,\mathbb{R})$ induced by the inclusion $\iota_s\colon T_s\hookrightarrow M$ is non-trivial.*

Then the set of all rational characters $U\subset H^1(M,\mathbb{Q})$ with the properties that

1) *$\forall\mu\in U$, the kernel ker(μ) is of type $\mathscr{F}_k$, and*
2) *$\forall\mu\in U$, the restriction $\mu|_{\pi_1(T_s)}$ to every cusp T_s, $1\le s\le r$, is non-trivial,*

is a non-empty open subset of $H^1(M,\mathbb{Q})\backslash\{0\}$ with respect to the subspace topology induced by $H^1(M,\mathbb{R})$. More precisely, U can be obtained by *intersecting* $H^1(M,\mathbb{Q}) \setminus \{0\}$ *with an infinite cone over a non-empty open subset of* $S(\Gamma)$.

Proof. First observe that the set

$$U_0 := H^1(M,\mathbb{Q}) \setminus \left(\bigcup_{1\leq s\leq r} (\iota_s^*)^{-1}(0) \right)$$

of all rational characters $\mu\colon \Gamma \to \mathbb{Q}$ which satisfy condition (2) is a dense open subset of $H^1(M,\mathbb{Q})$.

Since Γ is of type $\mathscr{F}_k$, the existence of a character χ with kernel of type $\mathscr{F}_k$ and Theorem 28 imply that the subset of $S(\Gamma)$ consisting of equivalence classes of characters with kernel of type $\mathscr{F}_k$ is non-empty and open. Combining this with the above description of U_0 yields the desired conclusion. q.e.d.

Proposition 29 allows us to perturb characters $\Gamma \to \mathbb{Z}$ so that they induce non-trivial homomorphisms on all parabolic subgroups of Γ. Thus, we can now complete the

Proof of Theorem 6. By Theorem 25 and Proposition 27 the manifold M^8 together with the character $u_*\colon \Gamma \to \mathbb{Z}$ satisfies the assumptions of Proposition 29 for $k = 3$. Thus there is a character $\chi\colon \Gamma \to \mathbb{Z}$ satisfying properties (1) and (2) of Proposition 29. Since every integral character on Γ is induced by a continuous map $v\colon M \to S^1$, this concludes the proof. q.e.d.

5. An infinite family of examples

In this section we will explain how to obtain an infinite family of pairwise non-isomorphic hyperbolic groups with subgroups of type $\mathscr{F}_3$ and not $\mathscr{F}_4$ from our construction.

Let $M = \mathbb{H}^n/\Gamma$ be a finite-volume hyperbolic n-manifold with r toric cusps, with sections $T_1, \ldots, T_r$.

Proposition 30. *Assume that for every cusp section* T_s *the natural map* $H^1(M,\mathbb{R}) \to H^1(T_s,\mathbb{R})$ *is surjective and that there is a character* $\chi\colon \Gamma \to \mathbb{Z}$ *with kernel of type* $\mathscr{F}_k$ *for some* $k \geq 0$. *Then there is a sequence of characters* $\mu_n\colon \Gamma \to \mathbb{Z}$, $n \in \mathbb{N}$, *with the following properties:*

1) *for every* $n \in \mathbb{N}$, *the kernel* $\ker(\mu_n)$ *is of type* $\mathscr{F}_k$,
2) *for every* $n \in \mathbb{N}$, *and every cusp section* T_s *of* M *the systole of the natural subtorus associated to the subgroup* $\ker(\mu_n) \cap \pi_1(T_s) < \pi_1(T_s)$ *equipped with the flat metric induced by* T_s *is* $\geq n$.

Proof. Let $V \subset H^1(M,\mathbb{Q})\setminus\{0\}$ be the open cone consisting of rational characters whose kernel is of type $\mathscr{F}_k$.

Note that for every $s \in \{1,\ldots,r\}$ and for every $n \in \mathbb{N}$ the set $Z_{n,s} \subset \pi_1(T_s)$ of non-trivial homotopy classes of loops in T_s that are homotopic to a loop of length $\leq n$ is finite. The surjectivity of $\iota_s^*\colon H^1(M,\mathbb{R}) \to H^1(T_s,\mathbb{R})$ thus allows us to choose characters $\mu_{n,s}\colon \Gamma \to \mathbb{Z}$ such that $Z_{n,s} \cap \ker(\mu_{n,s}) = \emptyset$ for $1 \leq s \leq r$. Using the openness of V and a simple induction on s we can choose arbitrarily small rational numbers $\lambda_{n,s} \in \mathbb{Q} \setminus \{0\}$ such that the character

$$\mu_n := \chi + \sum_{s=1}^{r} \lambda_{n,s}\mu_{n,s}\colon \Gamma \to \mathbb{Q}$$

is in V and satisfies $\ker(\mu_n) \cap Z_{n,s} = \emptyset$ for $1 \leq s \leq r$. Replacing μ_n by a representative of its equivalence class with values in $\mathbb{Z}$ completes the proof. q.e.d.

As a consequence we obtain:

Proposition 31. *With the same assumptions as in Proposition 30, assume now that* $\dim(M) = 2k$. *Then there is an infinite family of pairwise non-isomorphic hyperbolic groups, each of which has a subgroup which is of type* $\mathscr{F}_{k-1}$ *and not of type* $\mathscr{F}_k$.

Proof. Let $\mu_n\colon \Gamma \to \mathbb{Z}$ be a sequence of characters as in Proposition 30. By construction the intersection

$$\bigcap_{n\in\mathbb{N}} \ker\left(\mu_n \circ \iota_{s,*}\colon \pi_1(T_s) \to \mathbb{Z}\right) = \{1\} \tag{4}$$

is trivial for $1 \leq s \leq r$. As described in Section 2.1, we can use the methods of Fujiwara and Manning [**21**] to construct negatively curved Dehn fillings $\overline{M}_n$ for all characters μ_n with $n > 2\pi$ by coning off the family of translates over S^1 of the subtori of the T_s associated to the kernels $\ker(\mu_n)$. Note that for $n > 2\pi$ all fillings are 2π-fillings and therefore a passage to a finite index subgroup of Γ is not required before performing the Dehn filling in order to apply Theorem 7 and obtain that $\overline{M}_n$ is CAT(-1). All the spaces $\overline{M}_n$ are fillings of a *fixed* manifold, as required to apply the results from [**21**].

It then follows from (4) and [**21**, Proposition 2.12 and Theorem 2.13] that the family $\left\{\pi_1(\overline{M}_n) \mid n \in \mathbb{N}\right\}$ contains an infinite family of pairwise non-isomorphic hyperbolic groups. By combining Theorem 3 and Theorem 5, we obtain that the induced homomorphisms $\overline{\mu_n}\colon \pi_1(\overline{M}_n) \to \mathbb{Z}$ all have kernel of type $\mathscr{F}_{k-1}$ and not of type $\mathscr{F}_k$, thus completing the proof. q.e.d.

We can now conclude:

Proof of Theorem 1. By Theorem 25 and Proposition 27 the manifold M^8 satisfies the assumptions of Proposition 31 for $k = 4$. Theorem 1 is an immediate consequence. q.e.d.

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