p-Adic Confluence of q-Difference Equations

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Abstract

We develop the theory of p-adic confluence of q-difference equations. The main result is the fact that, in the p-adic framework, a function is (Taylor) solution of a differential equation if and only if it is solution of a q-difference equation. This fact implies an equivalence, called *Confluence*, between the category of differential equations and those of q-difference equations. We develop this theory by introducing a category of "*sheaves*" on the disk $\mathrm{D}^-(1,1)$, for which the stalk at 1 is a differential equation, the stalk at q is a q-difference equation if q is not a root of unity, and the stalk at a root of unity ξ is a mixed object, formed by a differential equation and an action of σ_ξ.

Contents

Introduction 2

Index of categories 8

1 Notation 8

1.1 Rings of functions . . . 9

1.2 Affinoids . . . 9

1.3 Norms . . . 9

1.4 Generic points . . . 10

2 Discrete or analytic $\sigma-$modules and $(\sigma,\delta)-$modules 11

2.1 Discrete $\sigma-$modules . . . 11

2.2 Discrete $(\sigma,\delta)-$modules . . . 12

2.3 Analytic $\sigma-$modules . . . 13

2.4 Analytic $(\sigma,\delta)-$modules . . . 14

3 Solutions (formal definition) 15

3.1 Discrete $\sigma-$algebras and $(\sigma,\delta)-$algebras . . . 15

3.2 Constant Solutions . . . 16

4 C-Constant Confluence 17

4.1 C-Constant modules . . . 17

4.2 C-Constant deformation and C-constant confluence . . . 18

5 Taylor solutions 20

5.1 The $q-$algebras $\Omega\{T-c\}_{q,R}$ and $\Omega[\![T-c]\!]_q$ . . . 20

5.2 q-invariant Affinoid . . . 23

5.3 The generic Taylor solution . . . 23

5.4 Transfer Principle . . . 24

5.5 Properties of the generic Taylor solution . . . 25

5.6 Taylor solutions of $(\sigma_q,\delta_q)-$modules . . . 27

5.7 Twisted Taylor formula for $(\sigma,\delta)-$modules, and rough estimate of radius . . . 27

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6 Generic radius of convergence and solvability **28**
6.1 Solvability over an annulus and over the Robba ring 29
6.2 Generic radius for discrete and analytic objects over $\mathcal{R}_K$ and $\mathcal{H}_K^\dagger$ 31
7 The Propagation Theorem **32**
7.1 Taylor admissible modules 32
7.2 Propagation Theorem 33
7.3 Extending the Confluence Functor to the case $|q-1| = |q| = 1$ 35
7.4 Propagation Theorem over $\mathcal{H}_K^\dagger$ and $\mathcal{R}_K$ 35
7.5 Classification of solvable rank one $q-$difference equations over $\mathcal{R}_{K_\infty}$ 36
8 Quasi unipotence and $p-$adic local monodromy theorem **37**
8.1 Frobenius Functor and Frobenius Structure 38
8.2 Special coverings of $\mathcal{H}_K^\dagger$ 40
8.3 Quasi unipotence of differential equations and canonical extension 41
8.4 Quasi unipotence of $\sigma-$modules and $(\sigma,\delta)-$modules with Frobenius structure 43
8.5 The confluence of André-Di Vizio 45
8.6 The theory of slopes 48

Introduction

The main aim of this paper is to provide a *theory of confluence* for q-difference equations in the p-adic framework.

A motivation : the rough idea of the confluence

Heuristically we say that a family of $q-$difference equations $\{\ \sigma_q(Y_q) = A(q,T)\cdot Y_q\ \}_{q\in D^-(1,\epsilon)-\{1\}}$ (where σ_q is the automorphism $f(T)\mapsto f(qT)$), is confluent to the differential equation $\delta_1(Y_q) = G(1,T)\cdot Y_q$, with $\delta_1 := T\frac{d}{dT}$, if one has $\lim_{q\to 1}\frac{A(q,T)-1}{q-1} = G(1,T)$ and, in some suitable meaning

$$\lim_{q\to 1} Y_q = Y_1\ . \tag{0.0.1}$$

Roughly speaking, in this paper we show that in the p-adic framework, if a differential equation is given, then, for ϵ sufficiently small, one may choose the family $\{G(q,T)\}_q$ in order to have $Y_q = Y_1$, for all $q\in \mathrm{D}^-(1,\epsilon)$. Conversely if q_0 is not a root of unity, and if a single equation $\sigma_{q_0}(Y_{q_0}) = A(q_0,T)\cdot Y_{q_0}$ is given, then, under some assumptions on the radius of convergence of its *generic* Taylor solution Y_{q_0}, one can find a differential equation, and family as above with the property that $Y_q = Y_{q_0} = Y_1$, for all $q\in \mathrm{D}^+(1,|q_0-1|)$. In this sense, in the p-adic context, the solutions of q-difference equations are not simply a "*discretization*" of the solutions of differential equations, but they are actually equal. We want now to state these facts more precisely.

The work of Y.André and L.Di Vizio

In [ADV04] the authors initiated the study of the phenomena of confluence in a $p-$adic setting. For K a complete discrete valuation field of mixed characteristic, they found an equivalence between the category of $q-$difference equations with Frobenius structure over the Robba ring $\mathcal{R}_{K^{\mathrm{alg}}}$ (here called $\sigma_q-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$), and the category of differential equations with Frobenius structure over the Robba ring $\mathcal{R}_{K^{\mathrm{alg}}}$ (here called $\delta_1-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$).

One of the restrictions of [ADV04] is that the number q is required to satisfy $|q-1|<|p|^{\frac{1}{p-1}}$. Indeed, in the annulus $|q-1| = |p|^{\frac{1}{p-1}}$ one encounters the $p-$th root of unity and, if $\xi^p = 1$, then the category $\sigma_\xi-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is different in nature from the category of differential equation, since it is not $K^{\mathrm{alg}}-$linear.

The equivalence of [ADV04] is obtained as follows. In [And02] one proves that the Tannakian group of $\delta_1 - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is $\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a$, where k is the perfect residue field of K, and $\mathcal{I}_{k^{\mathrm{alg}}((t))}$ is the absolute Galois group of $k^{\mathrm{alg}}((t))$. On the other hand in [ADV04] one shows that $\sigma_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ has the same Tannakian group $\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a$. By composition with the respective Tannakian equivalences (T_q and T_1 below), one obtains then the so called the *confluence functor* " Conf_q " (in the notations of [ADV04] one has $T_1 = \mathrm{V}_d^{(\phi)}$ and $T_q = \mathrm{V}_{\sigma_q}^{(\phi)}$):

$$\begin{array}{ccc} \sigma_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} & \overset{\mathrm{Conf}_q}{\underset{\cong}{\dashrightarrow}} & \delta_1 - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} \\ & {}_{T_q}\searrow^{\cong} \quad {}^{\cong}\swarrow_{T_1} & \\ & \underline{\mathrm{Rep}}_{K^{\mathrm{alg}}}(\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a) & \end{array} \tag{0.0.2}$$

The strategy of [ADV04] consists in showing that, as in the case of differential equations (cf. [And02]), every object M in $\sigma_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is quasi-unipotent, i.e. becomes unipotent after a special extension of $\mathcal{R}_K$ (cf. section 8.3). Once a basis of M is fixed, this means that M admits a complete basis of solutions $\widetilde{Y} \in \mathrm{GL}_n(\widetilde{\mathcal{R}_K}[\log(T)])$, where $\widetilde{\mathcal{R}_K}$ is the union of all special extensions of $\mathcal{R}_K$ (it is a sort of lifting of $k((t))^{\mathrm{alg}}$). We will call "*étale*" solutions the solutions of M in $\widetilde{\mathcal{R}_K}[\log(T)]$. The proof of this relevant result needs a substantial effort, and is actually not less complicated than the classical $p-$adic local monodromy theorem for differential equations itself (i.e. the fact that T_1 is an equivalence). Thanks to the fact that this important, but also very peculiar, class of q-difference and differential equations are trivialized by $\widetilde{\mathcal{R}_K}[\log(T)]$, one can define the functor T_1 (resp. T_q) associating to a differential (resp. q-difference) equation $(\mathrm{M}, \delta_1^{\mathrm{M}})$ (resp. $(\mathrm{M}, \sigma_q^{\mathrm{M}})$) the K^{alg}-vector space $T_1(\mathrm{M}, \delta_1^{\mathrm{M}})$ (resp. $T_q(\mathrm{M}, \sigma_q^{\mathrm{M}})$) of its "étale" solutions in $\widetilde{\mathcal{R}_K}[\log(T)]$.[1] The action of $\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a$ on the space of the "étale" solutions arises from its action on $\widetilde{\mathcal{R}_K}[\log(T)]$ by $\mathcal{R}_K$-linear automorphisms commuting with δ_1 and σ_q on $\widetilde{\mathcal{R}_K}[\log(T)]$.

Hence one sees for the first time in [ADV04] the fact that the "étale" solutions of a q-difference equation with Frobenius structure, are also the "étale" solutions of a differential equation. Moreover the functor Conf_q is nothing but the functor sending a q-difference equation (with (strong) Frobenius structure) into the differential equation having the same solutions.

In the present paper we prove that this "permanence" of the solutions holds also for *Taylor solutions* (see below). We develop then a p-adic theory of Confluence using, as a unique tool, this fact, here called *propagation principle*. We prove indeed that this principle is sufficient to define the Confluence and Deformation equivalences, over almost all p-adic ring of functions, with very basic assumptions on the equations. This theory requires only the definition and the formal properties of the *generic Taylor solution* $Y(x,y)$. For this reason it is not a consequence of the heretofore developed theory. Conversely we deduce, as a special case, the confluence of [ADV04] by comparing Taylor solutions and "étale" solutions (cf. the end of the introduction).

The generic q-Taylor solution

Let now K be an arbitrary ultrametric complete valued field of mixed characteristic $(0,p)$. Let $X = \mathrm{D}^+(c_0, R_0) - \cup_{i=1,\ldots,n} \mathrm{D}^-(c_i, R_i)$ be an affinoid, where $\mathrm{D}^-(c,R)$ denotes the open disc centered at c of radius R. Let $\mathcal{H}_K(X)$ be the ring of analytic elements on X. Consider a q-difference equation

$$\sigma_q(Y) = A(q,T) \cdot Y \ , \quad A(q,T) \in GL_n(\mathcal{H}_K(X)) \tag{0.0.3}$$

[1] Following the definition section 3.2, $\mathrm{V}_d^{(\phi)}(\mathrm{M}) := (\mathrm{M} \otimes_{\mathcal{R}_K} \widetilde{\mathcal{R}_K}[\log(T)])^{\delta_1 = 0}$ is actually the dual of the space of solutions $\mathrm{Hom}_{\mathcal{R}_K}^{\delta_1}(\mathrm{M}, \widetilde{\mathcal{R}_K}[\log(T)])$ (resp. same remark for $\mathrm{V}_{\sigma_q}^{(\phi)}(\mathrm{M}) := (\mathrm{M} \otimes_{\mathcal{R}_K} \widetilde{\mathcal{R}_K}[\log(T)])^{\sigma_q = \mathrm{Id}}$ and $\mathrm{Hom}_{\mathcal{R}_K}^{\sigma_q}(\mathrm{M}, \widetilde{\mathcal{R}_K}[\log(T)])$).

on X. Denote by $(\mathrm{M}, \sigma_q^{\mathrm{M}})$ the q-difference module over X defined by this equation.

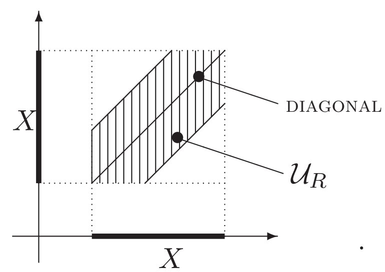
A major difference between the complex and the p-adic settings is that in the latter there are disks (not centered at 0) which are q-invariant. A disk $\mathrm{D}^-(c,R) \subset X(K)$ is q-invariant (i.e. the map $x \mapsto qx$ is a bijection of $\mathrm{D}^-(c,R)$) if and only if $|q-1||c| < R$, and $|q| = 1$ (cf. Lemma 5.1). Starting from this consideration, in [DV04] the author define, for q-difference equations, the q-analogue of the generic Taylor solution of a differential equation (cf. Def. 5.11):

$$Y(x,y) := \sum_{n \geqslant 0} H_n(q,T) \frac{(x-y)_{q,n}}{[n]_q^!} \ , \tag{0.0.4}$$

where $H_n(q,T)$ is obtained by iterating the equation (0.0.3): $d_q^n(Y) = H_n(q,T) \cdot Y$, where $d_q := \frac{\sigma_q - 1}{(q-1)T}$. For a large class of equations it happens that, for all $c \in X(K)$, the series $Y(x,c)$ represents a function which converges on a disk $\mathrm{D}^-(c,R)$, with $|q-1||c| < R$. More precisely $Y(x,y)$ converges in a neighborhood of the diagonal of the type $\mathcal{U}_R := \{(x,y) \in X \times X \mid |x-y| < R\}$, with

$$|q-1| \cdot \mathfrak{s}_X < R \ , \tag{0.0.5}$$

where $\mathfrak{s}_X := \sup_{c \in X} |c|$ as shown in the following picture (one easily sees that $\mathfrak{s}_X = \max(|c_0|, R_0)$):



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We call such equations *Taylor admissible*. The matrix function $Y(x,y) : \mathcal{U}_R \to GL_n(K)$ is invertible and satisfies the cocycle conditions: $Y(x,y) \cdot Y(y,z) = Y(x,z)$ and $Y(x,y)^{-1} = Y(y,x)$, for all $(x,y),(y,z),(x,z) \in \mathcal{U}_R$. Moreover $Y(qx,y) = A(q,x)Y(x,y)$ and, for all $c \in X(K)$, the matrix $Y(x,c) \in GL_n(\mathcal{A}_K(c,R))$ is a fundamental basis of solutions of the equation (0.0.3). In particular the q-difference algebra $\mathcal{A}_K(c,R)$ of analytic functions over the disk $\mathrm{D}^-(c,R)$, trivializes $(\mathrm{M}, \sigma_q^{\mathrm{M}})$.

The following fact is the main point of this paper (cf. Theorem 7.7). If now $q' \neq q$ belongs to the disk $\mathrm{D}^-(q, R/\mathfrak{s}_X) = \mathrm{D}^-(1, R/\mathfrak{s}_X)$, then the matrix

$$A(q',x) := Y(q'x,y) \cdot Y(x,y)^{-1} = Y(q'x,y) \cdot Y(y,x) = Y(q'x,x) \tag{0.0.6}$$

is an analytic function of x on all of X. Indeed $(q'x,x) \in \mathcal{U}_R$, for all $x \in X$, and hence the matrix $A(q',x)$ maps $x \mapsto (q'x,x) \mapsto Y(q'x,x) = A(q',x)$. One shows easily that $A(q',x) \in GL_n(\mathcal{H}_K(X))$, for all $q' \in \mathrm{D}^-(1, R/\mathfrak{s}_X)$, since $Y(x,y)$ is invertible. This fact implies that $Y(x,y)$ *is simultaneously the Taylor solution of every equation of the family $\{\sigma_{q'}(Y) = A(q',T)Y\}_{q'}$, for all $q' \in \mathrm{D}^-(1, R/\mathfrak{s}_X)$.* Equivalently, this means that the q-difference module $(\mathrm{M}, \sigma_q^{\mathrm{M}})$ is canonically endowed with an action of $\sigma_{q'}$, for all $q' \in \mathrm{D}^-(1, R/\mathfrak{s}_X)$. This remarkable fact will be called *Propagation Principle*. As one can see, this happens actually under the following weak assumptions on $(\mathrm{M}, \sigma_q^{\mathrm{M}})$:

i) q is not a root of unity; (0.0.7)

ii) $Y(x,y)$ converges on some $\mathcal{U}_R$ with $|q-1| \cdot \mathfrak{s}_X < R \leqslant r_X$; (0.0.8)

where $r_X = \min(R_0, R_1, \ldots, R_n)$ is a number depending on the geometry of X. The category of q-difference modules $(\mathrm{M}, \sigma_q^{\mathrm{M}})$ satisfying these two properties for a suitable unspecified R satisfying $|q-1|\mathfrak{s}_X < r \leqslant R \leqslant r_X$ will be denoted by $\sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]}$.

The assumption $|q-1|\mathfrak{s}_X < R$ assures that the image of the map $x \mapsto (qx, x) : X \mapsto X \times X$ is contained in $\mathcal{U}_R$. The bound $R \leqslant r_X$ assures that the function $Y(x,y)$ does not converge outside X. Indeed the properties of $Y(x,y)$ outside X are not invariant under $\mathcal{H}_K(X)$-base changes in M. Finally condition ii) also assures that the map $x \mapsto qx$ is a bijection of X globally fixing each individual hole of X (cf. section 5.2). Since $r_X \leqslant \mathfrak{s}_X$, we are assuming implicitly that $|q-1| < 1$. But no restrictive assumptions on X or on K are made.

Obviously this process works just as well if the initial function $Y(x,y)$ is the generic Taylor solution of a differential equation. The category of *differential* equations whose Taylor solution converges on $\mathcal{U}_R$, for an unspecified R satisfying $r \leqslant R \leqslant r_X$, will be denoted by $\delta_1-\mathrm{Mod}(\mathcal{H}_K(X))^{[r]}$.

Discrete and analytic σ-modules

Let $\mathcal{Q}(X)$ be the set of $q \in K$ for which $x \mapsto qx$ is a bijection of X. Then $\mathcal{Q}(X)$ is a topological subgroup of $K^\times$, and the disk $\mathrm{D}^-(1, R/\mathfrak{s}_X)$, with $R \leqslant r_X$, is an open subgroup of $\mathcal{Q}(X)$. The group $\mathcal{Q}(X)$ acts continuously on $\mathcal{H}_K(X)$ via $q \mapsto \sigma_q$. The data of M, together with the simultaneous σ_q-semi-linear action of σ_q^{M}, for all $q \in \mathrm{D}^-(1, R/\mathfrak{s}_X)$, is then a *semi-linear representation of the sub-group* $\mathrm{D}^-(1, R/\mathfrak{s}_X) \subseteq \mathcal{Q}(X)$. This representation has three remarkable properties:

(a) The map $(q', x) \mapsto A(q', x)$ is analytic in (q', x). In particular the representation is continuous;

(b) The group $\mathrm{D}^-(1, R/\mathfrak{s}_X)$ depends on R, and hence on M;

(c) The matrix $Y(x,y)$ is simultaneously the generic Taylor solution of the q-difference module $(\mathrm{M}, \sigma_q^{\mathrm{M}})$, for all $q \in \mathrm{D}^-(1, R/\mathfrak{s}_X)$.

Inspired by the first two properties we define a new class of objects called *discrete or analytic σ-modules* as follows. Consider a subset $S \subset \mathcal{Q}(X)$. A *discrete σ-module* on S is nothing but a $\mathcal{H}_K(X)$ semi-linear representation of the group $\langle S \rangle$ generated by S. If $S = U$ is an open subset of $\mathcal{Q}(X)$, we define *analytic σ-modules on U* to be a discrete σ-modules over U together with a certain condition of analyticity of σ_q^{M} with respect to q. These categories are denoted by $\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_S^{\mathrm{disc}}$ and $\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{an}}$ respectively. In this paper the words "*discrete*" or "*analytic*" will be referred to the discreteness or analyticity of σ_q^{M} with respect to q. We heuristically imagine the analytic σ-modules as *semi-linear representations of the (co-variant) sheaf of groups $U \mapsto \langle U \rangle$*.

Remark 0.1. It is important to notice that morphisms between analytic σ-modules over U are morphisms of representations. More precisely once a basis of M (resp. N) is fixed, we have a family of operators $\{\sigma_q(Y) = A(q,T)Y\}_{q \in \langle U \rangle}$ (resp. $\{\sigma_q(Y) = \widetilde{A}(q,T)Y\}_{q \in \langle U \rangle}$) such that $A(q,T)$ (resp. $\widetilde{A}(q,T)$) depends analytically on (q,T).[2] A morphism $\alpha : \mathrm{M} \to \mathrm{N}$ then must simultaneously commute with σ_q^{M} and σ_q^{N}, for all $q \in \langle U \rangle$. In other words the matrix B of α must simultaneously verify $A(q,T)B = \sigma_q(B)\widetilde{A}(q,T)$, for all $q \in \langle U \rangle$. Actually there are *non isomorphic* analytic σ-modules over U defining isomorphic q-difference equations at every $q \in \langle U \rangle$ (see example 2.6). This is analogous to have *non isomorphic* sheaves having isomorphic stalks at every point.

Taylor admissible σ-modules

We now want to analyse property (c): the constancy of the solutions. If $S \not\subseteq \boldsymbol{\mu}_{p^\infty}$, we call *Taylor admissible σ-modules over S* those σ-modules for which the q-Taylor solution $Y(x,y)$ is the same for all $q \in \langle S \rangle$, and satisfy the condition ii), for all $q \in S$ (cf. (0.0.8)). If $S = U$ is open, by the Propagation Principle, Taylor admissible σ-modules are *automatically* analytic on U (cf. Remark 7.8). This category is denoted by $\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}} \subseteq \sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{an}}$. We heuristically

[2] The data of an analytic σ-module is actually nothing but "*a family of q-difference equations depending analytically on q*".

imagine Taylor admissible σ-modules as *semi-linear representations of the (co-variant) sheaf of groups $U \mapsto \langle U\rangle$*, which are *locally constant*.

Taylor admissibility is a particular case of a more classical notion. If $\mathrm{C}/\mathcal{H}_K(X)$ is an algebra admitting an action of the group $\langle S\rangle$ extending that on $\mathcal{H}_K(X)$, then a semi-linear representation of $\langle S\rangle$ over $\mathcal{H}_K(X)$ is called C-*admissible* if it is trivialized by C. For a discrete σ-module M over S to be trivialized by C means exactly that there exists $Y \in GL_n(\mathrm{C})$ which is a simultaneous solution of all operators defined by M. If M is trivialized by C we will say that M is C-*constant*. We observe that if $S = q^{\mathbb{Z}}$, then C is nothing but a q-difference algebra over $\mathcal{H}_K(X)$. So the constancy of the solutions does not depend on the analyticity of M, rather it is a *discrete* fact.

In section 3 we define *discrete σ-algebras*, and we develop a basic differential/difference Galois theory for discrete σ-algebras. The analogue of the Picard-Vessiot theorem providing the existence of a discrete σ-algebra trivializing a given discrete σ-module *is missing*. We are thus obliged to work with the category of discrete σ-modules trivialized by a fixed discrete σ-algebra C. In section 4 we develop formally the theory of C-Confluence and C-Deformation, which will also depend on the chosen discrete σ-algebra C.

Remark 0.2. Notice that solutions will be defined formally as morphisms $\mathrm{M} \to \mathrm{C}$ commuting simultaneously with the actions of σ_q for all $q \in S$ (cf. Section 3.2). This fact, together with Remark 0.1, explains why the notion of C-*constant* σ-module implies the constancy of the solutions (with respect to q).

The Confluence functor

Let $(\mathrm{M}, \sigma^{\mathrm{M}})$ be an analytic σ-module over U. By analyticity we also have an action of the *Lie algebra* of $\langle U\rangle$ (here systematically identified with $K \cdot \delta_1$). In other words the following limit converges to a connection $\delta_1^{\mathrm{M}} : \mathrm{M} \to \mathrm{M}$ (cf. section 2.4):

$$\delta_1^{\mathrm{M}} \quad := \quad \lim_{q \in \langle U\rangle, q \to 1} \frac{\sigma_q^{\mathrm{M}} - 1}{q - 1} \quad \in \quad \mathrm{End}_K^{\mathrm{cont}}(\mathrm{M}) \;, \tag{0.0.9}$$

where q runs over the (open) group $\langle U\rangle$ generated by U. In terms of matrices, the matrix $G(1,T)$ of δ_1^{M} is $G(1,T) = q\frac{\partial}{\partial q}\Big(A(q,T)\Big)_{|q=1}$ (cf. equation (2.4.5)). By continuity, morphisms of analytic σ-modules also commute with the connection (cf. remark 2.4.1). Hence we obtain a functor called $\mathrm{Conf}_U : \sigma - \mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{an}} \longrightarrow \delta_1 - \mathrm{Mod}(\mathcal{H}_K(X))$, sending $(\mathrm{M}, \sigma^{\mathrm{M}})$ into $(\mathrm{M}, \delta_1^{\mathrm{M}})$ (cf. Remark 2.13). This functor is not an equivalence, but it does induce an equivalence:

$$\mathrm{Conf}_U^{\mathrm{Tay}} \;:\; \sigma - \mathrm{Mod}(\mathcal{H}_K(X))_U^{[r]} \;\xrightarrow{\;\sim\;}\; \delta_1 - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \;, \tag{0.0.10}$$

where $\mathrm{Conf}_U^{\mathrm{Tay}}$ simply denotes the restriction of Conf_U to the category $\sigma - \mathrm{Mod}(\mathcal{H}_K(X))_U^{[r]} \subseteq \sigma - \mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}}$ of Taylor admissible σ-modules verifying condition ii) with $r \leqslant R \leqslant r_X$ (cf. (0.0.8)), where $r > 0$ is large enough to have $U \subset \mathrm{D}^-(1, r/\mathfrak{s}_X)$ (cf. Corollary 7.9). The Propagation Principle gives a quasi inverse functor (cf. Remark 2.13 for a formal presentation).

On the other hand let $q \in U - \boldsymbol{\mu}_{p^\infty}$. An analytic σ-module over U defines a q-difference module by forgetting the action of $\sigma_{q'}^{\mathrm{M}}$, for all $q' \neq q$. Again the Propagation Principle provides an equivalence

$$\mathrm{Res}_q^U \;:\; \sigma - \mathrm{Mod}(\mathcal{H}_K(X))_U^{[r]} \;\xrightarrow{\;\sim\;}\; \sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \;, \tag{0.0.11}$$

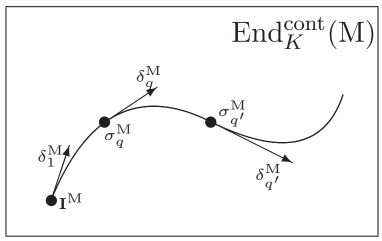
where $r \leqslant r_X$ is sufficiently large to have $U \subseteq \mathrm{D}^-(1, r/\mathfrak{s}_X)$ (cf. Cor. 7.9). We call the composite equivalence $\mathrm{Conf}_q^{\mathrm{Tay}}$. Thus we have

$$\mathrm{Conf}_q^{\mathrm{Tay}} := \mathrm{Conf}_U^{\mathrm{Tay}} \circ (\mathrm{Res}_q^U)^{-1} \;:\; \sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\;\sim\;} \delta_1 - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \;. \tag{0.0.12}$$

The equivalence $\mathrm{Conf}_q^{\mathrm{Tay}}$ sends a q-difference equation satisfying conditions i) and ii) (cf. (0.0.7),(0.0.8)), into the differential equation having the same generic Taylor solution.

Roots of unity and q-tangent operators

In this last equivalence the number q must not belong to $\boldsymbol{\mu}_{p^\infty}$. If $q' = \xi$, with $\xi^{p^n} = 1$, the category of σ_ξ-difference equations is not K-linear and cannot be equivalent to the category of differential equations. Nevertheless, if, for $q \notin \boldsymbol{\mu}_{p^\infty}$, the radius R of the q-Taylor solution is large, the Propagation Principle gives an operator $\sigma_\xi^{\mathrm{M}} : \mathrm{M} \to \mathrm{M}$ acting on M. The idea is to replace the category $\sigma_\xi - \mathrm{Mod}(\mathcal{H}_K(X))$ with another category. The expected object "at ξ" should also be endowed with an action of the Lie algebra, *as we have just done in the case* $\xi = 1$. For all $q \in \langle U \rangle$ the action of the Lie algebra of $\langle U \rangle$ is given by the limit $\delta_q^{\mathrm{M}} := \lim_{q' \to q} \frac{\sigma_{q'}^{\mathrm{M}} - \sigma_q^{\mathrm{M}}}{q'-q} \in \mathrm{End}_K^{\mathrm{cont}}(\mathrm{M})$, for $q, q' \in \langle U \rangle$, as shown in the picture:



Clearly $\delta_q^{\mathrm{M}} = \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}}$, so to give δ_q^{M} is equivalent to give δ_1^{M}. In a root of unity the "limit object" is a mixed data $(\mathrm{M}, \sigma_\xi^{\mathrm{M}}, \delta_1^{\mathrm{M}})$, i.e. a connection δ_1^{M} on M together with an action of σ_ξ^{M} on M. We call these new objects (σ_ξ, δ_ξ)*-modules*. In the sequel every terminology is given simultaneously for σ-modules and (σ, δ)-modules. The additional data of δ_ξ^{M} makes the category of (σ_ξ, δ_ξ)-modules K-linear. Moreover δ_ξ^{M} preserve the "*information*" in a neighborhood of ξ, indeed we find equivalences

$$\mathrm{Conf}_\xi^{\mathrm{Tay}} := \mathrm{Conf}_U^{\mathrm{Tay}} \circ (\mathrm{Res}_\xi^U)^{-1} \ : \ (\sigma_\xi, \delta_\xi) - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\ \sim\ } \delta_1 - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \ , \tag{0.0.13}$$

$$\mathrm{Def}_{\xi,q}^{\mathrm{Tay}} := \mathrm{Res}_q^U \circ (\mathrm{Res}_\xi^U)^{-1} \ : \ (\sigma_\xi, \delta_\xi) - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\ \sim\ } (\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \tag{0.0.14}$$

If q is not a root of unity, then the data of δ_1^{M} is superfluous, indeed if the module is Taylor admissible the Propagation Principle allows one to re-construct δ_1^{M} from σ_q^{M}.

In the classical setting over the complex numbers $\mathbb{C}$, understanding of the case $q = \xi \in \boldsymbol{\mu}_{p^\infty}$ remains an open problem.

Quasi unipotence and comparison with André-Di Vizio's Confluence

Up to a correct definition for the notion of Taylor admissibility, the previous theory can be generalized to more general rings of functions. From section 7.4 on we obtain the theory over $\mathcal{R}_K$. We prove that every q-difference equations with Frobenius Structure over $\mathcal{R}_K$, is quasi unipotent (i.e. is trivialized by $\widetilde{\mathcal{R}_K}[\log(T)]$), for all $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$, generalizing the main result of [ADV04]. We actually prove this theorem in the more general context of σ-modules, and (σ, δ)-modules. We deduce it by the quasi unipotence of p-adic *differential* equations with Frobenius Structure over $\mathcal{R}_K$, and by deformation. The idea is the following. As already mentioned, we are obliged to work with σ-modules trivialized by a fixed discrete σ-algebra C, and the C-Confluence and C-Deformations functors depend on C. In the "quasi unipotent" context this algebra is $\mathrm{C} := \widetilde{\mathcal{R}_K}[\log(T)]$, while in the context of the propagation theorem $\mathrm{C} := \mathcal{A}_K(c, R)$, for an arbitrary point $c \in X$, and suitable $R > 0$. To compare Taylor solutions to the "étale solutions" in $GL_n(\widetilde{\mathcal{R}_K}[\log(T)])$, the idea is to find a discrete σ-algebra of *functions over a disk* containing $\widetilde{\mathcal{R}_K}[\log(T)]$. Actually such an algebra does not exist. Thus we use a theorem of S.Matsuda (cf. Th. 8.13) providing an equivalence between $\delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$ with the sub-category of $\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$ formed by Special objects. Special objects are trivialized by a special extension of $\mathcal{H}_K^\dagger$ (cf. Section 8.3). The ring $\mathcal{A}_K(1,1)$ is a discrete σ-algebra over $\mathcal{H}_K^\dagger$. We prove then that the algebra $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]$ generated over $\mathcal{H}_K^\dagger$ by all the "étale solutions" of Special objects admits an embedding $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)] \subset \mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ commuting with δ_1, with the Frobenius, and with σ_q^{M}, for all $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$ (cf. Lemma 8.24). This will prove that the C-Confluence and the C-Deformation functors defined by using $\mathrm{C} = \mathcal{A}_K(1,1)$, or $\mathrm{C} = \widetilde{\mathcal{R}_K}[\log(T)]$ are actually the same (cf. Cor. 8.26). Moreover it proves also that the confluence of André-Di Vizio coincides with our $\mathrm{Conf}_q^{\mathrm{Tay}}$ (cf. Section 8.5), thus it is independent on the Frobenius.

Structure of the paper

Section 1 is devoted to notation. In section 2, we give definitions and basic facts on *discrete/analytic $\sigma-$modules, and $(\sigma,\delta)-$modules*. In section 3 we define *discrete σ-algebras and (σ,δ)-algebras*, and we give the abstract definition of *solutions*. In section 4 we give the formal notion of *confluence*. In section 5 we introduce *generic Taylor solutions* and *generic radius of convergence*. In section 7 we define *Taylor admissible objects* and obtain the main *Propagation Theorem* 7.7. In the last section 8 we apply the previous theory to the Robba ring, and to the *$p-$adic local monodromy theorem*.

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Index of categories

$\sigma - \mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_S$	11
$\sigma_q - \mathrm{Mod}(\mathrm{B})$	11
$(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_S$	12
$(\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B})$	13
$\delta_1 - \mathrm{Mod}(\mathrm{B})$	13
$\sigma - \mathrm{Mod}(\mathrm{B})^{\mathrm{an}}_U$	14
$\sigma - \mathrm{Mod}(\mathcal{R}_K)^{\mathrm{an}}_U$	14
$\sigma - \mathrm{Mod}(\mathcal{R}_K)^{\mathrm{disc}}_S$	14
$\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{an}}_U$	14
$\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{disc}}_S$	14
$(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})^{\mathrm{an}}_U$	14
$\sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S$	17
$(\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S$	17
$\sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_U$	17
$(\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_U$	17
$\sigma_q - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S$	19
$\sigma_q - \mathrm{Mod}(B,\mathrm{C})^{\mathrm{an}}_U$	19
$\sigma - \mathrm{Mod}(\mathcal{R}_K)^{[r]}_S$	31
$\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{[r]}_S$	31
$(\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_K)^{[r]}_S$	31
$(\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{[r]}_S$	31
$\sigma - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]}_S$	32
$\sigma - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_S$	32
$(\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]}_S$	32
$(\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_S$	32
$\sigma - \mathrm{Mod}(\mathcal{R}_K)^{\mathrm{adm}}_S$	32
$\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{adm}}_S$	32
$\sigma - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}_S$	39
$\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}_S$	39
$\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$	41
$\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}_S$	42
$(\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}_S$	42
$\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{\mathrm{conf}(\phi)}$	46
$\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{\mathrm{conf}}$	48

1. Notation

We refer to [DM] for the definitions concerning Tannakian categories. In the sequel when we say that a given category $\mathcal{C}$ is (or is not) K-linear, we mean that the ring of endomorphisms of the unit object is (or is not) exactly *equal* to K. We set $\mathbb{R}_\geqslant := \{r \in \mathbb{R} \mid r \geqslant 0\}$, and $\delta_1 := T\frac{d}{dT}$.

1.1 Rings of functions

Let $R > 0$ and $c \in K$. The ring of analytic functions on the disk $\mathrm{D}^-(c, R)$ is

$$\mathcal{A}_K(c,R) := \{\sum_{n\geqslant 0} a_n(T-c)^n \mid a_n \in K, \liminf_n |a_n|^{-1/n} \geqslant R\} . \tag{1.1.1}$$

Its topology is given by the family of norms $|\sum a_i(T-c)^i|_{(c,\rho)} := \sup |a_i|\rho^i$, for all $\rho < R$. Let $\emptyset \neq I \subseteq \mathbb{R}_{\geqslant 0}$ be some interval. We denote the annulus relative to I by $\mathcal{C}_K(I) := \{x \in K \mid |x| \in I\}$. By $\mathcal{C}(I)$, without the index K, we mean the annulus itself and not its $K-$valued points. The ring of analytic functions on $\mathcal{C}(I)$ is

$$\mathcal{A}_K(I) := \{\sum_{i\in\mathbb{Z}} a_iT^i \mid a_i \in K, \lim_{i\to\pm\infty} |a_i|\rho^i = 0, \text{ for all } \rho \in I\} . \tag{1.1.2}$$

We set $|\sum_i a_iT^i|_\rho := \sup_i |a_i|\rho^i < +\infty$, for all $\rho \in I$. The ring $\mathcal{A}_K(I)$ is complete for the topology given by the family of norms $\{|.|_\rho\}_{\rho\in I}$. Set $I_\varepsilon :=]1-\varepsilon, 1[$, $0 < \varepsilon < 1$. The Robba ring is defined as

$$\mathcal{R}_K := \bigcup_{\varepsilon>0} \mathcal{A}_K(I_\varepsilon) , \tag{1.1.3}$$

and is complete with respect to the limit Frechet topology.

1.2 Affinoids

Definition 1.1. A $K-$*affinoid* is an analytic subset of $\mathbb{P}^1$ defined by

$$X := \mathrm{D}^+(c_0, R_0) - \bigcup_{i=1}^{n} \mathrm{D}^-(c_i, R_i) , \tag{1.2.1}$$

for some $R_0, \dots, R_n > 0$, $c_0, \dots, c_n \in K$, $c_1, \dots, c_n \in \mathrm{D}^+_K(c_0, R)$. We denote by X the $K-$affinoid itself, and for all ultrametric valued $K-$algebras $(L, |.|)$, we denote by $X(L)$ its $L-$rational points.

Let $H^{\mathrm{rat}}_K(X)$ be the ring of rational fractions $f(T)$ in $K(T)$, without poles in $X(K^{\mathrm{alg}})$, and let $\|.\|_X$ be the norm on $H^{\mathrm{rat}}_K(X)$ given by $\|f(T)\|_X := \sup_{x\in X(K^{\mathrm{alg}})} |f(x)|$. We denote by

$$\mathcal{H}_K(X) \tag{1.2.2}$$

the completion of $(H^{\mathrm{rat}}_K(X), \|\cdot\|_X)$. It is known that if $\rho_1, \rho_2 \in |K^{\mathrm{alg}}|$, and if $X = \mathrm{D}^+(0, \rho_2) - \mathrm{D}^-(0, \rho_1)$, then $\mathcal{H}_K(X) = \mathcal{A}_K([\rho_1, \rho_2])$. Let now $\varepsilon > 0$. If $X = \mathrm{D}^+(c_0, R_0) - \bigcup_{i=1}^n \mathrm{D}^-(c_i, R_i)$, we set $X_\varepsilon := \mathrm{D}^+(c_0, R_0 + \varepsilon) - \bigcup_{i=1}^n \mathrm{D}^-(c_i, R_i - \varepsilon)$. We then set

$$\mathcal{H}^\dagger_K(X) := \bigcup_{\varepsilon>0} \mathcal{H}_K(X_\varepsilon) . \tag{1.2.3}$$

The ring $\mathcal{H}^\dagger_K(X)$ is complete with respect to the limit topology. Let $X_1 := \{x \mid |x| = 1\}$, we set

$$\mathcal{H}_K := \mathcal{H}_K(X_1) , \quad \mathcal{H}^\dagger_K := \mathcal{H}^\dagger_K(X_1) . \tag{1.2.4}$$

1.3 Norms

Every semi-norm $|.|_{\mathrm{B}}$ on a ring B will be extended to a semi-norm on $M_{n\times n}(\mathrm{B}) = M_n(\mathrm{B})$, by setting $|(b_{i,j})_{i,j}|_{\mathrm{B}} := \max_{i,j} |b_{i,j}|_{\mathrm{B}}$.

Definition 1.2. Let X be an affinoid. A *bounded multiplicative semi-norm* on $\mathcal{H}_K(X)$ is a function $|.|_* : \mathcal{H}_K(X) \to \mathbb{R}_{\geqslant 0}$, such that $|0|_* = 0$, $|1|_* = 1$, $|f - g|_* \leqslant \max(|f|_*, |g|_*)$, $|fg|_* = |f|_*|g|_*$, and $|.|_* \leqslant C\|.\|_X$, for some constant $C > 0$.

1.3.1 Let $(L,|.|)/(K,|.|)$ be an extension of valued fields. Let $c \in X(L)$, then $|.|_c : f \mapsto |f(c)|_L$ is a bounded multiplicative semi-norm on $\mathcal{H}_K(X)$. If $\mathrm{D}^+(c,R) \subseteq X$, then $|f|_{(c,R)} := \sup_{x\in \mathrm{D}^+_{L^{\mathrm{alg}}}(c,R)}|f(x)|$ is a bounded multiplicative semi-norm on $\mathcal{H}_K(X)$. Moreover if $f = \sum_{i\geqslant 0} a_i(T-c)^i$, $a_i \in L$ is the Taylor expansion of f at $c \in X(L)$, then $|f|_{(c,R)} = \sup_i |a_i|R^i$.

Definition 1.3. Let $f(T) = \sum_{i\in\mathbb{Z}} a_i(T-c)^i$, $a_i \in K$, be a formal power series. We set $|f|_{(c,\rho)} := \sup_i |a_i|\rho^i$, this number can be equal to $+\infty$.

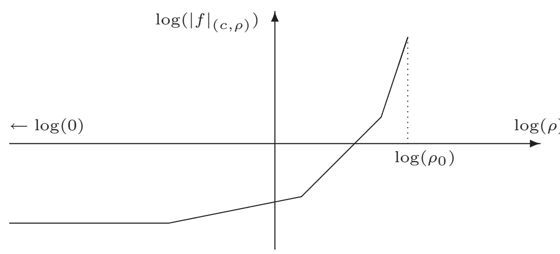
Definition 1.4. Let $r \mapsto N(r) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a function. The log-function attached to N is defined by $\widetilde{N}(t) := \log(N(\exp(t)))$ (i.e. $\widetilde{N} : \mathbb{R}\cup\{-\infty\} \xrightarrow[\sim]{\exp} \mathbb{R}_{\geqslant 0} \xrightarrow{N} \mathbb{R}_{\geqslant 0} \xrightarrow[\sim]{\log} \mathbb{R}\cup\{-\infty\}$). We will say that N has a given property logarithmically if $\widetilde{N}$ has that property.

Definition 1.5. Let $f(T) = \sum_{i\geqslant 0} a_i(T-c)^i$, $a_i \in K$ be a formal power series. The radius of convergence of $f(T)$ at c is $Ray(f(T),c) := \liminf_{i\geqslant 0}|a_i|^{-1/i}$. If $F(T) = (f_{h,k}(T))_{h,k}$, is a matrix, then we set $Ray(F(T),c) := \min_{h,k}\ Ray(f_{h,k}(T),c)$.

Lemma 1.6 ([CR94, ch.II]). *Let $f(T) \in K[[T-c]]$. Suppose that $|f|_{(c,\rho_0)} < \infty$, for some $\rho_0 > 0$. Then:*

i) *For all $\rho < \rho_0$ one has $Ray(f(T),c) \geqslant \rho$, and $|f|_{(c,\rho)} < \infty$;*

ii) *the function $\rho \mapsto |f|_{(c,\rho)} : [0,\rho_0] \longrightarrow \mathbb{R}_{\geqslant 0}$ is log-convex, piecewise log-affine, and log-increasing:*



iii) *One has $|f(T)|_{(c,\rho)} = \sup_{|x-c|\leqslant\rho, x\in K^{\mathrm{alg}}}|f(x)|_{K^{\mathrm{alg}}} = \lim_{r\to\rho^-}\sup_{|x-c|=r, x\in K^{\mathrm{alg}}}|f(x)|_{K^{\mathrm{alg}}}$;*

iv) *All zeros of $f(T)$ are algebraic. Moreover $f(T)$ has a zero $\zeta \in K^{\mathrm{alg}}$, with $|\zeta - c| = \rho < \rho_0$, if and only if the previous graph has a break at $\log(\rho)$.* □

1.4 Generic points

Let $(\Omega,|.|)/(K,|.|)$ be a complete field such that $|\Omega| = \mathbb{R}_{\geqslant 0}$, and that k_Ω/k is not algebraic.

Proposition 1.7 ([CR94, 9.1.2]). *For every disk $\mathrm{D}^+(c,\rho)$, $c \in K$, there exists a point $t_{c,\rho} \in \Omega$, called* generic point of $\mathrm{D}^+(c,\rho)$ *such that $|\,t_{c,\rho} - c\,|_\Omega = \rho$, and that $\mathrm{D}^-_\Omega(\,t_{c,\rho}\,,\ \rho\,) \cap K^{\mathrm{alg}} = \emptyset$.* □

1.4.1 A generic point defines a bounded multiplicative semi-norm on $\mathcal{H}_K(X)$, and hence defines a Berkovich point (cf. [Ber90]). The reader knowing the language of Berkovich will not find difficulties in translating the contents of this paper into the language of Berkovich.

For all $f(T) \in \mathcal{H}_K(\mathrm{D}^+(c,\rho))$, one has

$$|f(t_{c,\rho})|_\Omega \;=\; |f(T)|_{(c,\rho)} \;=\; \sup_{\substack{|x-c|\leqslant\rho\\ x\in K^{\mathrm{alg}}}}|f(x)| \;=\; \lim_{r\to\rho^-}\sup_{\substack{|x-c|=r\\ x\in K^{\mathrm{alg}}}}|f(x)|\,. \tag{1.4.1}$$

Hence, although the point $t_{c,\rho}$ is not uniquely determined by the fact that $\mathrm{D}^-_\Omega(t_{c,\rho},\rho)\cap K^{\mathrm{alg}} = \emptyset$, the norm $|\cdot|_{(c,\rho)}$ (i.e. the Berkovich point $|\cdot|_{(c,\rho)}$) does not depend on the choice of $t_{c,\rho}$.

By point iii) of Lemma 1.6, if $\rho \in |K|$ (resp. $\rho \in |K^{\mathrm{alg}}|$; $\rho \notin |K^{\mathrm{alg}}|$), then one also has $|f(t_{c,\rho})| = \max_{\substack{|x|=\rho\\ x\in K}}|f(x)|$ (resp. $|f(t_{c,\rho})| = \max_{\substack{|x|=\rho\\ x\in K^{\mathrm{alg}}}}|f(x)|$; $|f(t_{c,\rho})| = \lim_{r\to\rho^-}\max_{\substack{|x-c|=r\in|K^{\mathrm{alg}}|\\ x\in K^{\mathrm{alg}}}}|f(x)|$).

Proposition 1.8 ([Ber90]). *Let $X = \mathrm{D}^+(c_0,R_0) - \cup_{i=1,\dots,n}\mathrm{D}^-(c_i,R_i)$ be an affinoid. Let $t_{c_i,R_i} \in X(\Omega)$ be the generic point of $\mathrm{D}^+(c_i,R_i)$. Then, for all $f \in \mathcal{H}_K(X)$, one has*

$$\|f(T)\|_X = \max(|f(t_{c_0,R_0})|_\Omega, \dots, |f(t_{c_n,R_n})|_\Omega)\ . \tag{1.4.2}$$

Lemma 1.9. *Let $X = \mathrm{D}^+(c_0,R_0) - \cup_{i=1,\dots,n}\mathrm{D}^-(c_i,R_i)$ be an affinoid. Let $r_X := \min(R_0,\dots,R_n)$. Then $\|\frac{d}{dT}f(T)\|_X \leqslant r_X^{-1}\|f(T)\|_X$.*

Proof. This follows easily from the Mittag-Leffler decomposition of $f(T)$ together with the observations that $\|f(T)\|_X = \max_{i=0,\dots,n}(|f(t_{c_i,R_i})|)$ (cf. (1.4.2)), and $|f'(t_{c_i,R_i})| \leqslant R_i^{-1}|f(t_{c_i,R_i})|$, $\forall\ i$. □

2. Discrete or analytic $\sigma-$modules and $(\sigma,\delta)-$modules

Definition 2.1. Let B be one of the rings of section 1.1. We denote by

$$\mathcal{Q}(\mathrm{B}) = \{q \in K \mid \sigma_q : f(T) \mapsto f(qT) \text{ is an automorphism of B}\} \tag{2.0.3}$$

$$\mathcal{Q}_1(\mathrm{B}) = \mathcal{Q}(\mathrm{B}) \cap \mathrm{D}^-(1,1)\ . \tag{2.0.4}$$

We will write $\mathcal{Q}$ and $\mathcal{Q}_1$ when no confusion is possible.

Notice that $\mathcal{Q}(\mathrm{B}) \subset (K^\times, |.|)$ is a topological group and always contains a disk $\mathrm{D}^-(1,\tau_0)$, for some $\tau_0 > 0$. One has $\mathcal{Q}(\mathcal{A}_K(I)) = \mathcal{Q}(\mathcal{R}_K) = \mathcal{Q}(\mathcal{H}_K^\dagger) = \{q \in K \mid |q| = 1\}$. One sees easily that $\mathcal{Q}(\mathcal{H}_K(X)) \subset \{q \in K \mid |q| = 1\}$ (cf. section 5.2, and Lemma 5.1).

Definition 2.2. Let $S \subseteq \mathcal{Q}$ be a subset. We denote by $\langle S\rangle$ the subgroup of $\mathcal{Q}$ generated by S. Let $\boldsymbol{\mu}(\mathcal{Q})$ be the set of all roots of unity belonging to $\mathcal{Q}$. Then we set

$$S^\circ := S - \boldsymbol{\mu}(Q)\ . \tag{2.0.5}$$

2.1 Discrete $\sigma-$modules

By assumption, every finite dimensional free B$-$module M has the product topology.

Definition 2.3 (discrete $\sigma-$modules). Let $S \subset \mathcal{Q}$ be an arbitrary subset. An object of

$$\sigma\text{–}\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}} \tag{2.1.1}$$

is a finite dimensional free B$-$module M, together with a group morphism

$$\sigma^{\mathrm{M}} : \langle S\rangle \to \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M})\ , \tag{2.1.2}$$

sending $q \mapsto \sigma_q^{\mathrm{M}}$, such that, for all $q \in S$, the operator σ_q^{M} is σ_q-semi-linear, that is

$$\sigma_q^{\mathrm{M}}(fm) = \sigma_q(f)\cdot\sigma_q^{\mathrm{M}}(m)\ , \tag{2.1.3}$$

for all $f \in \mathrm{B}$, and all $m \in \mathrm{M}$. Objects (M,σ^{M}) in $\sigma-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ will be called *discrete $\sigma-$modules over S*. A morphism between $(\mathrm{M},\sigma^{\mathrm{M}})$ and $(\mathrm{N},\sigma^{\mathrm{N}})$ is a B$-$linear map $\alpha : \mathrm{M} \to \mathrm{N}$ such that

$$\alpha \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{N}} \circ \alpha\ , \tag{2.1.4}$$

for all $q \in S$. We will denote the $K-$vector space of morphisms by $\mathrm{Hom}_S^\sigma(\mathrm{M},\mathrm{N})$.

Notation 2.4. If $S = \{q\}$ is reduced to a point, then the category of discrete $\sigma-$modules over $\{q\}$ is the usual category of $q-$difference modules. We will therefore use a simplified notation:

$$\sigma_q - \mathrm{Mod}(\mathrm{B}) \ := \ \sigma - \mathrm{Mod}(\mathrm{B})_{\{q\}}^{\mathrm{disc}}\ . \tag{2.1.5}$$

Remark 2.5. 1.— Conditions (2.1.3) and (2.1.4) for $q \in S$ imply the same conditions for every $q \in \langle S\rangle$.

2.— If $\mathrm{M} \neq 0$, the map $\sigma^{\mathrm{M}} : \langle S \rangle \to \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M})$ is injective. Indeed, since B is a domain and M is free, the equality $\sigma_q^{\mathrm{M}}(fm) = \sigma_{q'}^{\mathrm{M}}(fm)$, $\forall\, f \in \mathrm{B}$, $\forall\, m \in \mathrm{M}$, implies $\sigma_q(f)\sigma_q^{\mathrm{M}}(m) = \sigma_{q'}(f)\sigma_{q'}^{\mathrm{M}}(m)$, and hence the contradiction: $\sigma_q(f) = \sigma_{q'}(f), \forall f \in \mathrm{B}$.

3.— The morphism σ^{M} on $\langle S \rangle$ is determined by its restriction to the set S. Conversely, if a map $S \to \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M})$ is given, then this map extends to a group morphism $\langle S \rangle \to \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M})$ if and only if the following conditions are verified:

i. $\sigma_q^{\mathrm{M}} \circ \sigma_{q'}^{\mathrm{M}} = \sigma_{q'}^{\mathrm{M}} \circ \sigma_q^{\mathrm{M}}$, for all $q, q' \in S$;
ii. If $\exists\, n, m \in \mathbb{Z}$, and $q_1, q_2 \in S$, such that $q_1^n = q_2^m$, then $(\sigma_{q_1}^{\mathrm{M}})^n = (\sigma_{q_2}^{\mathrm{M}})^m$;
iii. If $1 \in S$, then $\sigma_1^{\mathrm{M}} = \mathrm{Id}$.

2.1.1 *Matrices of* σ^{M}. Let $\mathbf{e} = \{\mathrm{e}_1, \ldots, \mathrm{e}_n\} \subset \mathrm{M}$ be a basis over B. If $\sigma_q^{\mathrm{M}}(\mathrm{e}_i) = \sum_j a_{i,j}(q,T) \cdot \mathrm{e}_j$, then in this basis σ_q^{M} acts as

$$\sigma_q^{\mathrm{M}}(f_1, \ldots, f_n) = (\sigma_q(f_1), \ldots, \sigma_q(f_n)) \cdot A(q,T) \;, \tag{2.1.6}$$

where $A(q,T) := (a_{i,j}(q,T))_{i,j}$. By definition $A(1,T) = \mathrm{Id}$, and one has

$$A(qq', T) = A(q', qT) \cdot A(q,T) \;. \tag{2.1.7}$$

In particular $A(q^n, T) = A(q, q^{n-1}T) \cdot A(q, q^{n-2}T) \cdots A(q,T)$.

2.1.2 *Internal* Hom *and* $\otimes$. Let $(\mathrm{M}, \sigma^{\mathrm{M}})$, $(\mathrm{N}, \sigma^{\mathrm{N}})$ be two discrete $\sigma-$modules over S. We define a structure of discrete $\sigma-$module on $\mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$ by setting $\sigma_q^{\mathrm{Hom}(\mathrm{M},\mathrm{N})}(\alpha) := \sigma_q^{\mathrm{N}} \circ \alpha \circ (\sigma_q^{\mathrm{M}})^{-1}$, for all $q \in S$, and all $\alpha \in \mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$. We define on $\mathrm{M} \otimes_{\mathrm{B}} \mathrm{N}$ a structure of discrete $\sigma-$module over S by setting $\sigma_q^{\mathrm{M}\otimes\mathrm{N}}(m \otimes n) := \sigma_q^{\mathrm{M}}(m) \otimes \sigma_q^{\mathrm{N}}(n)$, for all $q \in S$, and all $m \in \mathrm{M}$, $n \in \mathrm{N}$.

2.1.3 If $S^{\circ} \neq \emptyset$ (cf. (2.0.5)), then the category $\sigma - \mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ is $K-$linear. If B is a Bezout ring (i.e. every finitely generated ideal of B is principal), then $\sigma - \mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ is Tannakian (cf. [ADV04, 12.3]). The ring $\mathcal{H}_K(X)$ is always principal. If K is spherically closed, then $\mathcal{A}_K(I)$, $\mathcal{R}_K$, $\mathcal{H}_K^{\dagger}$ are Bezout rings.

2.1.4 As already mentioned in the introduction, the following is an example of two *non isomorphic* analytic σ-modules over X, having isomorphic "stalks" at every $q \in U \subset \mathcal{Q}(X)$. This is analogous to have *non isomorphic* sheaves having isomorphic stalks at every point.

Example 2.6. Let $X = \{|x| = 1\}$, then $\mathcal{Q}(X) = \{x \in K \mid |x| = 1\}$. Let $U := \mathrm{D}^-(1,1)$, and let $\pi \in K$ satisfy $|\pi| = |p|^{\frac{1}{p-1}}$. Put then $A(q,x) := \exp(\pi(q-1)x)$, and $\widetilde{A}(q,x) := \exp(\pi q(q-1)x)$. Let M (resp. N) be the discrete σ-module over U defined by the family $\{\sigma_q(Y) = A(q,x) \cdot Y\}_{q \in U}$ (resp. $\{\sigma_q(Y) = \widetilde{A}(q,x) \cdot Y\}_{q \in U}$). In this fixed basis of M and N, the matrices of every isomorphism between $(\mathrm{M}, \sigma_q^{\mathrm{M}})$ and $(\mathrm{N}, \sigma_q^{\mathrm{N}})$ are of the form $B(q,x) = \lambda \cdot \exp(\pi(1-q)x) \in \mathcal{H}_K(X)^{\times}$, with $\lambda \in K^{\times}$. Hence for all $q \in U$ the equation $\sigma_q(Y) = A(q,x)Y$ is isomorphic to $\sigma_q(Y) = \widetilde{A}(q,x)Y$. But since $B(q,x)$ depends on q, M *and* N *are not isomorphic as analytic σ-modules over U.*

2.2 Discrete $(\sigma, \delta)-$modules

Let $S \subset \mathcal{Q}(\mathrm{B})$ be an arbitrary subset.

Definition 2.7 (discrete $(\sigma, \delta)-$modules). An object of

$$(\sigma, \delta)\text{–}\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}} \tag{2.2.1}$$

is a discrete $\sigma-$module over S, together with a connection[3] $\delta_1^{\mathrm{M}} : \mathrm{M} \to \mathrm{M}$. Objects $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta_1^{\mathrm{M}})$ of $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ will be called *discrete* $(\sigma,\delta)-$*modules over* S. A morphism between $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta_1^{\mathrm{M}})$ and $(\mathrm{N}, \sigma^{\mathrm{N}}, \delta_1^{\mathrm{N}})$ is a morphism $\alpha : (\mathrm{M}, \sigma^{\mathrm{M}}) \to (\mathrm{N}, \sigma^{\mathrm{N}})$ of discrete $\sigma-$modules satisfying

$$\alpha \circ \delta_1^{\mathrm{M}} = \delta_1^{\mathrm{N}} \circ \alpha \ . \tag{2.2.2}$$

We will denote the $K-$vector space of morphisms by $\mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{N})$.

Notation 2.8. By analogy with (2.1.5), if $S = \{q\}$, then we set:

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B}) \ := \ (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_{\{q\}} \ . \tag{2.2.3}$$

If $q = 1$ we denote it by $\delta_1 - \mathrm{Mod}(\mathrm{B})$.

As already mentioned in the introduction, we introduce the operator

$$\delta_q^{\mathrm{M}} := \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}} \ . \tag{2.2.4}$$

For all $f \in \mathrm{B}$, all $m \in \mathrm{M}$, and all $q \in \langle S \rangle$, one has

$$\delta_q^{\mathrm{M}}(f \cdot m) = \sigma_q(f) \cdot \delta_q^{\mathrm{M}}(m) + \delta_q(f) \cdot \sigma_q^{\mathrm{M}}(m) \ . \tag{2.2.5}$$

Moreover, for all $\alpha \in \mathrm{Hom}^{(\sigma,\delta)}(\mathrm{M},\mathrm{N})$, and all $q \in \langle S \rangle$, one has $\alpha \circ \delta_q^{\mathrm{M}} = \delta_q^{\mathrm{N}} \circ \alpha$. Heuristically we imagine M as endowed with the map $q \mapsto \delta_q^{\mathrm{M}} : \langle S \rangle \to \mathrm{End}_K^{\mathrm{cont}}(\mathrm{M})$. This justifies notations (2.2.1) and (2.2.3).

2.2.1 *Matrices of* δ_q^{M}. Let $\mathbf{e} = \{\mathrm{e}_1, \ldots, \mathrm{e}_n\} \subset \mathrm{M}$ be a basis over B. Let $A(q,T) \in GL_n(\mathrm{B})$ be the matrix of σ_q^{M} in the basis $\mathbf{e}$ (cf. (2.1.6)). If $\delta_q^{\mathrm{M}}(\mathrm{e}_i) = \sum_j g_{i,j}(q,T) \cdot \mathrm{e}_j$, and if $G(q,T) = (g_{i,j}(q,T))_{i,j}$, then δ_q^{M} acts in the basis $\mathbf{e}$ as:

$$\delta_q^{\mathrm{M}}(f_1, \ldots, f_n) = (\delta_q(f_1), \ldots, \delta_q(f_n)) \cdot A(q,T) + (\sigma_q(f_1), \ldots, \sigma_q(f_n)) \cdot G(q,T) \ . \tag{2.2.6}$$

One has moreover

$$G(q' \cdot q, T) = G(q', qT) \cdot A(q,T) \ . \tag{2.2.7}$$

2.2.2 *Internal* Hom *and* $\otimes$. Let $(\mathrm{M}, \sigma^{\mathrm{M}}, \delta^{\mathrm{M}})$, $(\mathrm{N}, \sigma^{\mathrm{N}}, \delta^{\mathrm{N}})$ be two discrete $(\sigma,\delta)-$modules over S. We define a structure of discrete $(\sigma,\delta)-$module on $\mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$ by setting

$$\delta_q^{\mathrm{Hom(M,N)}}(\alpha) := \Big(\delta_q^{\mathrm{N}} \circ \alpha - \sigma_q^{\mathrm{Hom(M,N)}}(\alpha) \circ \delta_q^{\mathrm{M}}\Big) \circ (\sigma_q^{\mathrm{M}})^{-1} \ . \tag{2.2.8}$$

This definition gives the relation $\delta_q^{\mathrm{N}}(\alpha \circ m) = \sigma_q^{\mathrm{H}}(\alpha) \circ \delta_q^{\mathrm{M}}(m) + \delta_q^{\mathrm{H}}(\alpha) \circ \sigma_q^{\mathrm{M}}(m)$, for all $\alpha \in \mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$, and all $m \in \mathrm{M}$, where $\mathrm{H} := \mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{N})$. We define on $\mathrm{M} \otimes_{\mathrm{B}} \mathrm{N}$ a structure of discrete $(\sigma,\delta)-$module over S by setting

$$\delta_q^{\mathrm{M}\otimes\mathrm{N}}(m \otimes n) := \delta_q^{\mathrm{M}}(m) \otimes \sigma_q^{\mathrm{N}}(n) + \sigma_q^{\mathrm{M}}(m) \otimes \delta_q^{\mathrm{N}}(n) \ , \tag{2.2.9}$$

for all $q \in S$, and all $m \in \mathrm{M}$, $n \in \mathrm{N}$.

2.2.3 If B is Bezout, then $(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ is $K-$linear and Tannakian.

2.3 Analytic $\sigma-$modules

Analytic $\sigma-$modules are defined only if the ring B is equal to one of the following rings: $\mathcal{A}_K(I)$, $\mathcal{H}_K(A)$, $\mathcal{H}_K^{\dagger}(X)$, $\mathcal{H}_K$, $\mathcal{H}_K^{\dagger}$, $\mathcal{R}_K$. Notice that if $U \subset \mathcal{Q}(\mathrm{B})$ is an open subset, then the subgroup $\langle U \rangle \subseteq \mathcal{Q}(\mathrm{B})$ generated by U is open, i.e. $\langle U \rangle$ contains a disk $\mathrm{D}_K^-(1,\tau)$, for some $\tau > 0$.

---

[3] i.e. δ_1^{M} verifies $\delta_1^{\mathrm{M}}(fm) = \delta_1(f) \cdot m + f \cdot \delta_1^{\mathrm{M}}(m)$, $\forall\, f \in \mathrm{B}$, $\forall\, m \in \mathrm{M}$. Recall that $\delta_1 := T\frac{d}{dT}$.

Definition 2.9. Let $\mathrm{B} := \mathcal{H}_K(X)$. Let $(\mathrm{M}, \sigma^{\mathrm{M}})$ be a discrete $\sigma-$module over U. Let $A(q,T) \in GL_n(\mathrm{B})$ be the matrix of σ_q^{M} in a fixed basis. We will say that $(\mathrm{M}, \sigma^{\mathrm{M}})$ is an *analytic* $\sigma-$*module* if, for all $q \in U$, there exist a disk $\mathrm{D}^-(q,\tau_q) = \{q' \mid |q'-q| < \tau_q\}$, with $\tau_q > 0$, and a matrix $A_q(Q,T)$ such that:

i) $A_q(Q,T)$ is an analytic element on the domain $(Q,T) \in \mathrm{D}^-(q,\tau_q) \times X$;

ii) For all $q' \in \mathrm{D}_K^-(q,\tau_q)$, one has $A_q(Q,T)_{|Q=q'} = A(q',T)$.

This definition does not depend on the choice of basis $\mathbf{e}$. We define

$$\sigma - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \tag{2.3.1}$$

as the full sub-category of $\sigma - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{disc}}$, whose objects are analytic $\sigma-$modules. Let $I \subset \mathbb{R}_{\geqslant 0}$ be an interval. We give the same definition over the ring $\mathrm{B} := \mathcal{A}_K(I)$, namely, if $\mathcal{C}(I) := \{|T| \in I\}$, the point i) is replaced by

i′) $A_q(Q,T)$ is an analytic function on the domain $(Q,T) \in \mathrm{D}^-(q,\tau_q) \times \mathcal{C}(I)$.

Example 2.10. The discrete σ-modules appearing in the Example 2.6, are actually analytic.

2.3.1 *Analyticity of* $\mathrm{Hom}(\mathrm{M},\mathrm{N})$ *and* $\mathrm{M}\otimes\mathrm{N}$. If $(\mathrm{M}, \sigma^{\mathrm{M}})$ and $(\mathrm{N}, \sigma^{\mathrm{N}})$ are two analytic $\sigma-$modules over U, then $(\mathrm{Hom}(\mathrm{M},\mathrm{N}), \sigma^{\mathrm{Hom}(\mathrm{M},\mathrm{N})})$ and $(\mathrm{M}\otimes\mathrm{N}, \sigma^{\mathrm{M}\otimes\mathrm{N}})$ are analytic. This follows from the explicit dependence of the matrices of $\sigma^{\mathrm{Hom}(\mathrm{M},\mathrm{N})}$ and $\sigma^{\mathrm{M}\otimes\mathrm{N}}$ on the matrices of σ^{M} and σ^{N}.

2.3.2 *Discrete and analytic $\sigma-$modules over $\mathcal{A}_K(I)$, $\mathcal{R}_K$ and $\mathcal{H}_K^\dagger(X)$.* If $I_1 \subset I_2$, then the restriction functor $\sigma - \mathrm{Mod}(\mathcal{A}_K(I_2))_U^{\mathrm{an}} \to \sigma - \mathrm{Mod}(\mathcal{A}_K(I_1))_U^{\mathrm{an}}$ is fully faithful. Indeed the equality $f_{|I_1} = g_{|I_1}$ implies $f = g$, for all $f,g \in \mathcal{A}_K(I_2)$ (analytic continuation [CR94, 5.5.8]).

Definition 2.11. Let $S \subseteq \mathcal{Q}$ be a subset, and let $U \subseteq \mathcal{Q}$ be an open subset. We set

$$\sigma - \mathrm{Mod}(\mathcal{R}_K)_U^{\mathrm{an}} := \bigcup_{\varepsilon>0} \sigma - \mathrm{Mod}(\mathcal{A}_K(]1-\varepsilon,1[))_U^{\mathrm{an}} \ ; \tag{2.3.2}$$

$$\sigma - \mathrm{Mod}(\mathcal{R}_K)_S^{\mathrm{disc}} := \bigcup_{\varepsilon>0} \sigma - \mathrm{Mod}(\mathcal{A}_K(]1-\varepsilon,1[))_S^{\mathrm{disc}} \ . \tag{2.3.3}$$

Similarly, one can define $\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger(X))_U^{\mathrm{an}}$ and $\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger(X))_S^{\mathrm{disc}}$.

Remark 2.12. Since U is open, one has $U^\circ \neq \emptyset$ (cf. (2.0.5)). By Section 2.1.3, if B is one of the previous rings (and if it is a Bezout ring), then $\sigma - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}}$ is $K-$linear and Tannakian.

2.4 Analytic $(\sigma,\delta)-$modules

We maintain the previous notations. In section 2.4.1 below we define a *fully faithful* functor

$$(\text{Forget } \delta)^{-1} : \ \ \sigma - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \longrightarrow (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{disc}} \ , \tag{2.4.1}$$

which is a "local" section of the functor Forget $\delta : (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{disc}} \to \sigma - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{disc}}$. The essential image of the functor $(\text{Forget } \delta)^{-1}$ will be denoted by

$$(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \ . \tag{2.4.2}$$

By definition, the functor which "forgets" the action of δ is therefore an equivalence

$$(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \xrightarrow[\sim]{\text{Forget } \delta} \sigma - \mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}} \ . \tag{2.4.3}$$

Notice that a morphism between analytic $(\sigma,\delta)-$modules is, by definition, a morphism of *discrete* $(\sigma,\delta)-$modules.

2.4.1 *Construction of* δ. Let $(\mathrm{M},\sigma^{\mathrm{M}})$ be an analytic $\sigma-$module. We shall define a (σ,δ)-module structure on M. It follows from definitions 2.9 and 2.11 that the map $q\mapsto\sigma_q^{\mathrm{M}}:\langle U\rangle\to\mathrm{Aut}_K(\mathrm{M})$ is *derivable*, in the sense that, for all $q\in\langle U\rangle$, the limit

$$\delta_q^{\mathrm{M}}:=q\cdot\lim_{q'\to q}\frac{\sigma_{q'}^{\mathrm{M}}-\sigma_q^{\mathrm{M}}}{q'-q}=\text{“ }(q\frac{d}{dq}\sigma^M)(q)\text{ ”}\tag{2.4.4}$$

exists in $\mathrm{End}_K^{\mathrm{cont}}(\mathrm{M})$, with respect to the simple convergence topology (cf. (2.4.5)). Moreover, for all $q\in\langle U\rangle$, the rule (2.2.5) holds, and $\delta_q^{\mathrm{M}}=\sigma_q^{\mathrm{M}}\circ\delta_1^{\mathrm{M}}$.

Let $\alpha:(\mathrm{M},\sigma^{\mathrm{M}})\to(\mathrm{N},\sigma^{\mathrm{N}})$ be a morphism of analytic $\sigma-$modules, that is $\alpha\circ\sigma_q^{\mathrm{M}}=\sigma_q^{\mathrm{N}}\circ\alpha$, for all $q\in U$. Passing to the limit in the definition (2.4.4), one shows that α commutes with δ_q^{M}, for all $q\in U$. Hence the inclusion $\mathrm{Hom}_U^{(\sigma,\delta)}(\mathrm{M},\mathrm{N})\subseteq\mathrm{Hom}_U^{\sigma}(\mathrm{M},\mathrm{N})$ is an equality. If $\mathbf{e}=\{\mathrm{e}_1,\ldots,\mathrm{e}_n\}\subset\mathrm{M}$ is a basis in which the matrix of σ_q^{M} is $A(q,T)$, then the matrix of δ_q^{M} is (cf. (2.2.6), Def. 2.9 and 2.11)

$$G(q,T)\;:=\;q\cdot\lim_{q'\to q}\frac{A(q',T)-A(q,T)}{q'-q}\;=\;\Big(\partial_Q\big(A_q(Q,T)\big)\Big)_{|Q=q}\;,\tag{2.4.5}$$

where ∂_Q is the derivation $Q\frac{d}{dQ}$, and $A_q(Q,T)$ is the matrix of Definition 2.9.

Remark 2.13. By the above definitions, there is an obvious functor

$$\mathrm{Conf}_U:\sigma-\mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}}\longrightarrow\delta_1-\mathrm{Mod}(\mathrm{B})\;,\tag{2.4.6}$$

obtained by composing $(\mathrm{Forget}\ \delta)^{-1}$ (cf. (2.4.3)) with $\mathrm{Forget}\ \sigma:(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_U^{\mathrm{an}}\longrightarrow\delta_1-\mathrm{Mod}(\mathrm{B})$.

3. Solutions (formal definition)

3.1 Discrete $\sigma-$algebras and $(\sigma,\delta)-$algebras

Let $S\subseteq\mathcal{Q}(\mathrm{B})$ be a subset.

Definition 3.1 (Discrete $\sigma-$algebra over S). A B-*discrete* $\sigma-$*algebra over* S, or simply a *discrete* $\sigma-$*algebra over* S is a B$-$algebra C such that:

i) C is an *integral domain*,

ii) there exists a group morphism $\sigma^{\mathrm{C}}:\langle S\rangle\to\mathrm{Aut}_K(\mathrm{C})$ such that σ_q^{C} is a ring automorphism extending σ_q^{B}, for all $q\in\langle S\rangle$;

iii) one has $\mathrm{C}_S^{\sigma}=K$, where $\mathrm{C}_S^{\sigma}:=\{c\in\mathrm{C}\mid\sigma_q(c)=c\ ,\ \text{for all }q\in S\}$.

We will call C_S^{σ} *the sub-ring of* $\sigma-$*constants of* C. We will write σ_q instead of σ_q^{C}, when no confusion is possible.

Observe that no topology is required on C. The word *discrete* is employed, here and later on, to emphasize that we do not ask “continuity” with respect to q. Notice also that if a discrete $\sigma-$algebra C is free and of finite rank as B$-$module, then it is a discrete $\sigma-$module.

3.1.1 If $S^{\circ}\neq\emptyset$ (cf. (2.0.5)), then $\mathrm{B}_S^{\sigma}=K$, and B itself is a discrete $\sigma-$algebra over S. On the other hand, If $S=\{\xi\}$ is reduced to a root of unity $\xi\in\boldsymbol{\mu}(\mathcal{Q})$, since $\mathrm{B}_S^{\sigma}=\mathrm{B}^{\sigma_\xi}\neq K$, it follows that B itself is not a discrete $\sigma-$algebra over S. Hence there is no discrete $\sigma-$algebra over $S=\{\xi\}$. To deal with this problem we introduce the following

Definition 3.2 (Discrete $(\sigma,\delta)-$algebra over S). A *discrete* $(\sigma,\delta)-$*algebra* C *over* S is a B$-$algebra such that:

i) C satisfies properties i) and ii) of Definition 3.1,

ii) there exists a derivation δ_1^{C}, extending $\delta_1 = T\frac{d}{dT}$ on B, and commuting with σ_q^{C}, for all $q \in \langle S\rangle$,

iii) one has $\mathrm{C}_S^{(\sigma,\delta)} = K$, where $\mathrm{C}_S^{(\sigma,\delta)} := \{f \in \mathrm{C} \mid f \in \mathrm{C}_S^{\sigma}\ ,\ \text{and}\ \ \delta_1(f) = 0\}$.

We will call $\mathrm{C}_S^{(\sigma,\delta)}$ the *sub-ring of* $(\sigma,\delta)-$*constants of* C. We will write δ_1 instead of δ_1^{C}, if no confusion is possible.

The operator $\delta_q^{\mathrm{C}} := \sigma_q^{\mathrm{C}} \circ \delta_1^{\mathrm{C}}$ satisfies property (2.2.5). Since $\mathrm{B}_S^{(\sigma,\delta)} = K$, it follows that B is always a $(\sigma,\delta)-$algebra over S, for an arbitrary sub-set $S \subseteq \mathcal{Q}(\mathrm{B})$, even for $S = \{\xi\}$, with $\xi \in \boldsymbol{\mu}(\mathcal{Q}(\mathrm{B}))$.

3.2 Constant Solutions

Definition 3.3 (Constant solutions on S). Let $(\mathrm{M},\sigma^{\mathrm{M}})$ (resp.$(\mathrm{M},\sigma^{\mathrm{M}},\delta^{\mathrm{M}})$) be a *discrete* $\sigma-$module (resp. $(\sigma,\delta)-$module) over S, and let C be a discrete $\sigma-$algebra (resp. $(\sigma,\delta)-$algebra) over S. A *constant solution* of M, with values in C, is a $B-$linear morphism

$$\alpha : \mathrm{M} \longrightarrow \mathrm{C}$$

such that $\alpha \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{C}} \circ \alpha$, for all $q \in S$ (resp. α simultaneously satisfies $\alpha \circ \delta_1^{\mathrm{M}} = \delta_1^{\mathrm{C}} \circ \alpha$, and $\alpha \circ \sigma_q^{\mathrm{M}} = \sigma_q^{\mathrm{C}} \circ \alpha$, for all $q \in S$). We denote by $\mathrm{Hom}_S^{\sigma}(\mathrm{M},\mathrm{C})$ (resp. $\mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{C})$) the $K-$vector space of the solutions of M in C.

3.2.1 *Matrices of solutions.* Let M be a discrete $\sigma-$module (resp. $(\sigma,\delta)-$module). Let C be a discrete $\sigma-$algebra (resp. $(\sigma,\delta)-$algebra) over S. Recall that, if $S = \{\xi\}$, with $\xi^n = 1$, then there is no discrete $\sigma-$algebra, over S (cf. Section 3.1.1).

Let $\mathbf{e} = \{\mathrm{e}_1,\ldots,\mathrm{e}_n\}$ be a basis of M, and let $A(q,T)$ (resp. $G(q,T)$) be the matrix of σ_q^{M} (resp. δ_q^{M}) in this basis (cf. (2.2.6)). We identify a morphism $\alpha : \mathrm{M} \to \mathrm{C}$ with the vector $(y_i)_i \in \mathrm{C}^n$, given by $y_i := \alpha(\mathrm{e}_i)$. In this way constant solutions become solutions in the usual *vector form*. Indeed

$$\begin{pmatrix} \sigma_q(y_1) \\ \vdots \\ \sigma_q(y_n) \end{pmatrix} = A(q,T) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \ , \quad \text{for all } q \in S\ , \tag{3.2.1}$$

$$\Bigg(\text{resp. } \begin{pmatrix} \delta_q(y_1) \\ \vdots \\ \delta_q(y_n) \end{pmatrix} = G(q,T) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \ , \quad \text{for all } q \in S \ \Bigg)\ . \tag{3.2.2}$$

Definition 3.4. By a *fundamental matrix of solutions* of M (in the basis $\mathbf{e}$) we mean a matrix $Y \in GL_n(\mathrm{C})$ satisfying *simultaneously*

$$\sigma_q(Y) = A(q,T) \cdot Y\ , \qquad \text{for all } q \in S\ , \tag{3.2.3}$$

(resp. satisfying *simultaneously*

$$\left\{ \begin{array}{rcll} \sigma_q(Y) & = & A(q,T) \cdot Y\ , & \text{for all } q \in S\ , \\ \delta_1(Y) & = & G(1,T) \cdot Y\ . & \end{array} \right. \quad)\ . \tag{3.2.4}$$

3.2.2 *Unit object and σ-constants.* Let $\mathbb{I} = \mathrm{B}$ be the unit object. By the description given above, every solution $\alpha \in \mathrm{Hom}_S^{\sigma}(\mathbb{I},\mathrm{C})$ (resp. $\alpha \in \mathrm{Hom}_S^{(\sigma,\delta)}(\mathbb{I},\mathrm{C})$) can be identified with $y := \alpha(1) \in \mathrm{C}_S^{\sigma}$ (resp. $y := \alpha(1) \in \mathrm{C}_S^{(\sigma,\delta)}$). We obtain $\mathrm{C}_S^{\sigma} \cong \mathrm{Hom}_S^{\sigma}(\mathbb{I},\mathrm{C})$ (resp. $\mathrm{C}_S^{(\sigma,\delta)} \cong \mathrm{Hom}_S^{(\sigma,\delta)}(\mathbb{I},\mathrm{C})$). In particular B_S^{σ} (resp. $\mathrm{B}_S^{(\sigma,\delta)}$) is identified with $\mathrm{End}_S^{\sigma}(\mathbb{I})$ (resp. $\mathrm{End}_S^{(\sigma,\delta)}(\mathbb{I})$), and the category is K-linear if and only if $\mathrm{B}_S^{\sigma} = K$ (resp. $\mathrm{B}_S^{(\sigma,\delta)} = K$).

3.2.3 *Dimension of the space of solutions.* Let $F := \mathrm{Frac}(\mathrm{C})$ be the fraction field of C, then both σ_q and δ_1 extend to F (cf. [vdPS03, Ex.1.5]).

Lemma 3.5 (Wronskian Lemma). *Let* M *be a* $(\sigma,\delta)-$*module (resp.* $\sigma-$*module) over* S*, and let* C *be a discrete* $(\sigma,\delta)-$*algebra (resp.* $\sigma-$*algebra) over* S*. One has*

$$\dim_K \mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{C}) \leqslant \mathrm{rk}_{\mathrm{B}}(\mathrm{M}) \; . \tag{3.2.5}$$

(resp. if $S^\circ \neq \emptyset$ *(cf. (2.0.5)), then* $\dim_K \mathrm{Hom}_S^\sigma(\mathrm{M},\mathrm{C}) \leqslant \mathrm{rk}_{\mathrm{B}}(\mathrm{M})$*.)*

Proof. One has $\dim_K \mathrm{Hom}_S^{(\sigma,\delta)}(\mathrm{M},\mathrm{C}) \leqslant \dim_K \mathrm{Hom}^{\delta_1}(\mathrm{M},\mathrm{C}) \leqslant \mathrm{rk}_{\mathrm{B}}(\mathrm{M})$. On the other hand, if $q \in S^\circ$, then $\mathrm{Hom}^{\sigma_q}(\mathrm{M},\mathrm{C}) \leqslant \mathrm{rk}_{\mathrm{B}}(\mathrm{M})$ (cf. [DV02, Lemma 1.1.11]). Hence $\dim_K \mathrm{Hom}_S^\sigma(\mathrm{M},\mathrm{C}) \leqslant \dim_K \mathrm{Hom}^{\sigma_q}(\mathrm{M},\mathrm{C}) \leqslant \mathrm{rk}_{\mathrm{B}}(\mathrm{M})$. □

4. C-Constant Confluence

In this section we state the formal results regarding confluence. We introduce the notion of C-constant modules. As explained in the introduction, this notion is an adaptation of the notion of C-admissibility in the sense of representation theory. On the other hand it can be interpreted as a generalization of the Galois theory for differential and q-difference equations. According to this point of view, in our context we have the problem that the analogue of the Picard-Vessiot algebra trivializing a given object M does not exist for arbitrary objects M. Also the uniqueness of the Picard-Vessiot algebra remains an open problem. We avoid these problems by working with the category of modules trivialized by a given algebra C which is fixed once and for all. We hope that this problem will be overcome in the future.

4.1 C-Constant modules

Let B be one of the rings of Sections 1.1 and 1.2, let $S \subset \mathcal{Q}(\mathrm{B})$ be a subset, and let $U \subset \mathcal{Q}(\mathrm{B})$ be an open subset.

Definition 4.1 (C-Constant modules). Let M be a discrete $\sigma-$module over S. We will say that M is C-*constant* on S, or equivalently that M *is trivialized by* C, if there exists a discrete $\sigma-$algebra C over S such that

$$\dim_K \mathrm{Hom}_S^\sigma \;\; (\mathrm{M},\mathrm{C}) = \mathrm{rk}_{\mathrm{B}}\mathrm{M} \; . \tag{4.1.1}$$

We give the analogous definition for (σ,δ)-modules. The full sub-category of $\sigma - \mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$ (resp. $(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{\mathrm{disc}}$), whose objects are trivialized by C, will be denoted by

$$\sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}} \qquad (\text{resp. } \; (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}} \;\;) \; . \tag{4.1.2}$$

The full subcategory of $\sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{const}}$ (resp. $(\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{const}}$) whose objects are analytic will be denoted by

$$\sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}} \qquad (\text{resp. } \; (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}} \;\;) \; . \tag{4.1.3}$$

Notice that M is trivialized by C if there exists $Y \in GL_n(\mathrm{C})$, $n := \mathrm{rk}_{\mathrm{B}}\mathrm{M}$, such that Y is *simultaneously* a solution, for all $q \in S$, of the family of equations (3.2.3) (resp. both the conditions of (3.2.4)). Roughly speaking, M is C-constant on S if it admits a basis of $q-$solutions in $GL_n(\mathrm{C})$ which "does not depend on $q \in S$".

Lemma 4.2. *Let* M, N *be two discrete* $\sigma-$*modules (resp.* $(\sigma,\delta)-$*modules). If* M, N *are both trivialized by* C*, then* $\mathrm{M}\otimes\mathrm{N}$, $\mathrm{Hom}(\mathrm{M},\mathrm{N})$, $M^\vee$, $N^\vee$ *are trivialized by* C.

Proof. The fundamental matrix solution of $\mathrm{M}\otimes\mathrm{N}$ (resp. $\mathrm{Hom}(\mathrm{M},\mathrm{N})$) is obtained by taking products of entries of the two matrices of solutions of M and N respectively. Hence "it does not depend on $q \in S$". The assertion on $\mathrm{M}^\vee$, $\mathrm{N}^\vee$ is a particular case of the previous one. □

Lemma 4.3. *Let $S' \subseteq S$ be a non empty subset. Let* C *be a discrete* $(\sigma,\delta)-$*algebra over* S. *Then the restriction functor* $\mathrm{Res}^S_{S'}$, *sending* $(\mathrm{M},\sigma^{\mathrm{M}},\delta_1^{\mathrm{M}})$ *into* $(\mathrm{M},\sigma^{\mathrm{M}}_{|_{\langle S'\rangle}},\delta_1^{\mathrm{M}})$*:*

$$\mathrm{Res}^S_{S'} : (\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \longrightarrow (\sigma,\delta)-\mathrm{Mod}(\mathrm{B})^{\mathrm{disc}}_{S'} \tag{4.1.4}$$

is fully faithful and its image is contained in the category $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_{S'}$. *The same fact is true for discrete* $\sigma-$*modules under the assumption:* $(S')^\circ \neq \emptyset$.

Proof. The proof is the same in both cases: here we give the proof in the case of $(\sigma,\delta)-$modules. We must show that the inclusion $\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M},\mathrm{N}) \to \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M},\mathrm{N})$ is an isomorphism, for all M, N in $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S$. In other words, we have to show that if $\alpha : \mathrm{M}\to\mathrm{N}$ commutes with $\sigma_{q'}$, for all $q'\in S'$, then it commutes also with σ_q, for all $q\in S$. One has

$$\begin{aligned}\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M},\mathrm{N}) &= \mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{B})\ ; \\ \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M},\mathrm{N}) &= \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{B})\ .\end{aligned} \tag{4.1.5}$$

Observe that $\mathrm{M}\otimes\mathrm{N}^\vee$ is the dual of the "internal hom" $\mathrm{Hom}(\mathrm{M},\mathrm{N})$. By Lemma 4.2, $\mathrm{M}\otimes\mathrm{N}^\vee$ is trivialized by C. The restriction of $\mathrm{M}\otimes\mathrm{N}^\vee$ to S' is obviously C-constant on S', since it is trivialized by C. This implies that

$$\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{C}) = \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{C})\ . \tag{4.1.6}$$

This shows that a morphism with values in $\mathrm{B}\subseteq\mathrm{C}$ commutes with all σ_q and δ_q, for all $q\in S$, if and only if it commutes with all σ_q and δ_q, for all $q\in S'$. Hence

$$\mathrm{Hom}^{(\sigma,\delta)}_S(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{B}) = \mathrm{Hom}^{(\sigma,\delta)}_{S'}(\mathrm{M}\otimes\mathrm{N}^\vee,\mathrm{B})\ . \qquad \square \tag{4.1.7}$$

4.1.1 *Restriction to a roots of unity.* By the previous lemma, if $\xi\in S\cap\boldsymbol{\mu}(\mathcal{Q})$, then

$$\mathrm{Res}^S_{\{\xi\}} : (\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \longrightarrow (\sigma_\xi,\delta_\xi)-\mathrm{Mod}(\mathrm{B}) \tag{4.1.8}$$

is again fully faithful. On the other hand, if $S^\circ\neq\emptyset$, then the restriction

$$\mathrm{Res}^S_{\{\xi\}} : \sigma-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \longrightarrow \sigma_\xi-\mathrm{Mod}(\mathrm{B}) \tag{4.1.9}$$

is *not* fully faithful, since $\sigma-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S$ is $K-$linear, while $\sigma_\xi-\mathrm{Mod}(\mathrm{B})$ is not $K-$linear (i.e. $K\subset\mathrm{End}(\mathbb{I})$, but $K\neq\mathrm{End}(\mathbb{I})$, cf. Section 1).

4.1.2 *The case of an open subset.* We observe that if U is open, then the condition $U^\circ\neq\emptyset$ is automatically verified. Hence, by Lemma 4.3, if $S\subset U$ is a (non empty) subset, the restriction

$$\mathrm{Res}^U_S : (\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_U \longrightarrow (\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \tag{4.1.10}$$

is fully faithful. The same is true for $\sigma-$modules, under the assumption $S^\circ\neq\emptyset$. In particular, if $U'\subset U$ is an open subset, then the restriction functor is fully faithful:

$$\mathrm{Res}^U_{U'} : (\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_U \longrightarrow (\sigma,\delta)-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{an,const}}_{U'}\ . \tag{4.1.11}$$

4.2 C-Constant deformation and C-constant confluence

In this section we give the formal definition of the confluence and deformation functors. As usual $S\subseteq\mathcal{Q}(\mathrm{B})$ is an arbitrary subset, and $U\subseteq\mathcal{Q}(\mathrm{B})$ is an open subset.

Definition 4.4 (Extensible objects). Let $q\in S$. Let C be a discrete $\sigma-$algebra over S. A $q-$difference module M is said to be C-*extensible to* S if it belongs to the essential image of the restriction functor

$$\mathrm{Res}^S_{\{q\}} : \sigma-\mathrm{Mod}(\mathrm{B},\mathrm{C})^{\mathrm{const}}_S \longrightarrow \sigma_q-\mathrm{Mod}(\mathrm{B})\ .$$

The *full sub-category* of $\sigma_q - \mathrm{Mod}(\mathrm{B})$ whose objects are C-extensible to S, will be denoted by $\sigma_q - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S$. If U is open, and if $q \in U$, we will denote by

$$\sigma_q - \mathrm{Mod}(B,\mathrm{C})_U^{\mathrm{an}} \tag{4.2.1}$$

the full sub-category of $\sigma_q - \mathrm{Mod}(\mathrm{B})_U$ whose objects belong to the essential image of $\sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}}$. We give analogous definitions for (σ,δ)-modules.

Lemma 4.3 and Definition 4.4 easily give the following formal statement:

Corollary 4.5. *With the notation of Lemma 4.3, one has an equivalence*

$$\mathrm{Res}^S_{\{q\}} \;:\; (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S^{\mathrm{const}} \xrightarrow{\sim} (\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S\;. \tag{4.2.2}$$

The same fact is true for $\sigma-$modules, under the additional hypothesis that $q \in S^\circ$. □

Definition 4.6. 1.– Let $S \subseteq \mathcal{Q}(\mathrm{B})$ be a subset and let $q, q' \in \langle S \rangle$. We will call the C-*constant deformation functor*, denoted by

$$\mathrm{Def}^{\mathrm{C}}_{q,q'} : (\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S \xrightarrow{\;\sim\;} (\sigma_{q'},\delta_{q'}) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S\;, \tag{4.2.3}$$

the equivalence obtained by composition of the restriction functor (4.2.2):

$$\mathrm{Def}^{\mathrm{C}}_{q,q'} := \mathrm{Res}^S_{\{q'\}} \circ (\mathrm{Res}^S_{\{q\}})^{-1}\;. \tag{4.2.4}$$

2.– We will call the C-*constant confluence functor*, the equivalence

$$\mathrm{Conf}^{\mathrm{C}}_q := \mathrm{Def}^{\mathrm{C}}_{q,1} : (\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S \xrightarrow{\sim} (\sigma_1,\delta_1) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S\;. \tag{4.2.5}$$

3.– Suppose that $q \in S^\circ$ and $q' \in S$, then we will call again the C-*constant deformation functor*, denoted again by

$$\mathrm{Def}^{\mathrm{C}}_{q,q'} : \sigma_q - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S \longrightarrow \sigma_{q'} - \mathrm{Mod}(\mathrm{B},\mathrm{C})_S\;, \tag{4.2.6}$$

the functor obtained by composition of the restriction functor (4.2.2): $\mathrm{Def}^{\mathrm{C}}_{q,q'} := \mathrm{Res}^S_{\{q'\}} \circ (\mathrm{Res}^S_{\{q\}})^{-1}$. If $q' \in S^\circ$, then $\mathrm{Def}^{\mathrm{C}}_{q,q'}$ is an equivalence.

It follows from Corollary 4.5, that if $q, q' \in U$, one has an *equivalence*, again called $\mathrm{Def}^{\mathrm{C}}_{q,q'}$

$$\mathrm{Def}^{\mathrm{C}}_{q,q'} \;:\; (\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an}} \underset{\sim}{\longrightarrow} (\sigma_{q'},\delta_{q'}) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an}}\;. \tag{4.2.7}$$

The same fact is true for analytic $\sigma-$modules under the condition $q, q' \notin \boldsymbol{\mu}(\mathcal{Q})$.

4.2.1 Notice that the functor $\mathrm{Res}^S_{\{q\}}$ does not depend on C, but $(\mathrm{Res}^S_{\{q\}})^{-1}$ is a particular section of $\mathrm{Res}^S_{\{q\}}$ with values in the category of objects trivialized by C (cf. Corollary (4.5)). Hence $(\mathrm{Res}^S_{\{q\}})^{-1}$, $\mathrm{Conf}^{\mathrm{C}}_q$ and $\mathrm{Def}^{\mathrm{C}}_{q,q'}$ actually depend on C.

4.2.2 According to Definition 4.4 (cf. also 2.1.5 and 2.2.3), if $q \in U \subset U'$, then, by Lemma 4.3 (cf. Section 4.1.2), the following restriction functors are fully faithful immersions:

$$\begin{array}{lllll} & \mathrm{Res}^{U'}_U : \sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_{U'} & \longrightarrow & \sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U & \\ & \mathrm{Res}^{U'}_U : \sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_{U'}^{\mathrm{an,const}} & \longrightarrow & \sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}} & \\ \Big(\text{resp.} & \mathrm{Res}^{U'}_U : (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_{U'} & \longrightarrow & (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U & \\ & \mathrm{Res}^{U'}_U : (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_{U'}^{\mathrm{an,const}} & \longrightarrow & (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}} & \Big)\;. \end{array} \tag{4.2.8}$$

We can then consider the following diagram in which we heuristically imagine categories appearing in the first two lines as the stalks at q of suitable corresponding stacks over $\mathcal{Q}(X)$:

$$\begin{array}{ccc}
\bigcup_U \ \sigma - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}} & \overset{\mathrm{Def.}\ 2.4.3}{=\!=\!=} & \bigcup_U \ (\sigma,\delta) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U^{\mathrm{an,const}} \\
\Big\downarrow {\scriptstyle \bigcup_U \ \mathrm{Res}^U_{\{q\}}} & \odot & \wr \Big\downarrow {\scriptstyle \bigcup_U \ \mathrm{Res}^U_{\{q\}}} \\
\bigcup_U \ \sigma_q - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U & \xleftarrow[\text{Forget } \delta_q]{} & \bigcup_U \ (\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B},\mathrm{C})_U \\
\Big\downarrow {\scriptstyle i_\sigma} & \odot & \Big\downarrow {\scriptstyle i_{(\sigma,\delta)}} \\
\sigma_q - \mathrm{Mod}(\mathrm{B}) & \xleftarrow[\text{Forget } \delta_q]{} & (\sigma_q,\delta_q) - \mathrm{Mod}(\mathrm{B})
\end{array} \tag{4.2.9}$$

where U runs over the set of open neighborhoods of q, and where i_σ and $i_{(\sigma,\delta)}$ are the trivial inclusions of full sub-categories. In the sequel we will study the full subcategory of $\sigma_q - \mathrm{Mod}(\mathrm{B})$ (resp. $(\sigma_q, \delta_q) - \mathrm{Mod}(\mathrm{B})$) formed by *Taylor admissible objects*, this category is contained in the essential image of i_σ (resp. $i_{(\sigma,\delta)}$) (see Th. 7.6). In this case we will obtain an analogous diagram (see Cor. 7.9)) in which $i_{(\sigma,\delta)}$ is an equivalence (for all $q \in U$), and i_σ is an equivalence only if q is not a root of unity.

If q is not a root of unity, then all the arrows of this diagram will be equivalences, hence giving δ_q is superfluous. If q is a root of unity, then the right hand side vertical arrows will be equivalences, while the arrow on the left hand side will not. In this last case the $q-$tangent operator is necessary to "*preserve the information in the neighborhood of* q". In this case the good notion of "*stalk at* q" of an analytic σ-module is the notion of (σ_q, δ_q)-module and not simply that of σ_q-module.

One may have the feeling that the functor "Forget δ_q" contains "information" if q is a root of unity, but we will see (Prop. 8.6) that, if $\mathrm{B} = \mathcal{R}_K$ or if $\mathrm{B} = \mathcal{H}_K^\dagger$, then this functor sends every $(\sigma, \delta)-$module with Frobenius structure into a direct sum of copies of the unit object.

4.2.3 *Dependence on* C. Let $\mathrm{C}_1 \subseteq \mathrm{C}_2$ be two algebras as above. Then clearly $\mathrm{Def}^{\mathrm{C}_2}_{q,q'}$ extends $\mathrm{Def}^{\mathrm{C}_1}_{q,q'}$ to the larger category of modules trivialized by C_2. One of the main problems of the theory is that, if there are no inclusions between C_1 and C_2, then it is not clear whether there exists a discrete σ-algebra (resp. (σ, δ)-algebra) C_3 containing both C_1 and C_2. For this reason, if the same object is trivialized by C_1, and also by C_2, it is not clear whether its deformations with respect to C_1 and C_2 are equal. We will encounter this problem in section 8.4.

5. Taylor solutions

In this section $\mathrm{B} = \mathcal{H}_K(X)$, for some affinoid $X = \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i)$, and $S = \{q\} \in \mathcal{Q}(\mathcal{H}_K(X)) \subseteq \{q \in K \mid |q| = 1\}$ is reduced to a point. Let $(\Omega, |.|)/(K, |.|)$ be an arbitrary extension of complete valued fields. Let $c \in X(\Omega)$ and let $\rho_{c,X} > 0$ be the largest real number such that $\mathrm{D}^-_{\Omega'}(c, \rho_{c,X}) \subseteq X(\Omega')$, for all complete valued field extensions $(\Omega', |.|)/(\Omega, |.|)$. One has

$$\rho_{c,X} = \min(R_0, |c - c_1|, |c - c_2|, \cdots, |c - c_n|). \tag{5.0.10}$$

Notice that c can be equal to a generic point (cf. Definition 1.7). We want to find solutions of $q-$difference equations converging in a disc centered at c, i.e. matrix solutions in the form (3.2.3), with values in the σ_q-algebra $\mathrm{C} := \mathcal{A}_K(c, R)$, for some $0 < R \leqslant \rho_{c,X}$.

5.1 The $q-$algebras $\Omega\{T - c\}_{q,R}$ and $\Omega[\![T - c]\!]_q$

Unless we explicitly state the contrary, we will not assume that $q \notin \boldsymbol{\mu}(\mathcal{Q})$. The following results generalize the analogous constructions of [DV04] to the case of a root of unity.

Lemma 5.1. *Let $0 < R \leqslant \rho_{c,X}$. The algebra $\mathcal{A}_\Omega(c,R)$ is an $\mathcal{H}_\Omega(X)$-discrete $\sigma-$algebra over $S = \{q\}$, if and only if both of the following conditions hold:*

$$|q-1||c| < R\ , \qquad \text{and} \qquad |q| = 1\ . \tag{5.1.1}$$

Definition 5.2. Let $q \in K^\times$ be an arbitrary number. Following [DV04] and [ADV04] we set

$$(T-c)_{q,n} := (T-c)(T-qc)(T-q^2c)\cdots(T-q^{n-1}c)\ , \tag{5.1.2}$$

$$[n]_q := 1+q+q^2+\cdots+q^{n-1}\ , \tag{5.1.3}$$

$$[n]_q^! := \frac{(q-1)(q^2-1)(q^3-1)\cdots(q^n-1)}{(q-1)^n}\ . \tag{5.1.4}$$

5.1.1 *q-binomial.* For all $q \in K^\times$, we define the q-binomial $\binom{n}{i}_q$ by the relation

$$(1-T)(1-qT)\cdots(1-q^{n-1}T) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}_q q^{\frac{i(i-1)}{2}}T^i\ , \tag{5.1.5}$$

where, if $i=0$, the symbol $q^{\frac{i(i-1)}{2}}$ is by definition equal to 1. This extends the definition given in [DV04] (cf. eq. (5.1.7) below) to the case of root of unity. If $1 \leqslant i \leqslant n-1$, by induction one has

$$\binom{n}{i}_q = \binom{n-1}{i-1}_q + q^i\binom{n-i}{i}_q = q^{n-i}\binom{n-1}{i-1}_q + \binom{n-i}{i}_q\ . \tag{5.1.6}$$

If q is not a root of unity, then one can write

$$\binom{n}{i}_q = \frac{[n]_q\cdot[n-1]_q\cdots[n-i+1]_q}{[i]_q^!}\ . \tag{5.1.7}$$

If q is an $m-$th root of 1, then $[n]_q^! = 0$, for all $n \geqslant m$. The family $\{(T-c)_{q,n}\}_{n\geqslant 0}$ is adapted to the $q-$derivation

$$d_q := \frac{\sigma_q - 1}{(q-1)T} = \frac{\Delta_q}{T} \tag{5.1.8}$$

in the sense that for all $n \geqslant 1$ one has $d_q((T-c)_{q,n}) = [n]_q\cdot(T-c)_{q,n-1}$. One has always the relation $d_q(fg) = \sigma_q(f)d_q(g) + d_q(f)g$. More generally our definition of q-binomials allow us to generalize the proof of [DV04, Lemma 1.2, (1.2.2)] to the case of a root of unity. We obtain the formula

$$d_q^n(fg)(T) = \sum_{i=0}^{n}\binom{n}{i}_q d_q^{n-i}(f)(q^iT)d_q^i(g)(T)\ . \tag{5.1.9}$$

5.1.2 The following Lemma extends [DV04, Section 1.3] to the case of a root of unity.

Lemma 5.3. *Let $(\Omega,|.|)/(K,|.|)$ be a complete extension of valued fields. Let $|q-1||c| < R$, $|q| = 1$, and let $f(T) = \sum_{n\geqslant 0} a_n(T-c)^n \in \mathcal{A}_\Omega(c,R)$. Then:*

i) *$f(T)$ can be written uniquely as the following series of functions:*

$$f(T) = \sum_{n\geqslant 0}\widetilde{a}_n(T-c)_{q,n} \in \mathcal{A}_\Omega(c,R)\ , \tag{5.1.10}$$

with $\widetilde{a}_n \in \Omega$ satisfying $\sup_n|\widetilde{a}_n|\rho^n < \infty$, for all $\rho < R$;

ii) *for all $|q-1||c| < \rho < R$ one has $|f(T)|_{(c,\rho)} = \sup_{n\geqslant 0}|a_n|\rho^n = \sup_{n\geqslant 0}|\widetilde{a}_n|\rho^n$;*

iii) *one has $Ray(f(T),c) = \liminf_n|a_n|^{-1/n} = \liminf_n|\widetilde{a}_n|^{-1/n}$;*

iv) *if moreover $q \notin \boldsymbol{\mu}(\mathcal{Q})$, then one has the so called q-Taylor expansion (cf. [DV04]):*

$$f(T) = \sum_{n\geqslant 0} d_q^n(f)(c)\frac{(T-c)_{q,n}}{[n]_q^!} \ . \tag{5.1.11}$$

Proof. Since $\mathcal{A}_\Omega(c,R) = \varprojlim_{r\to R^-} \mathcal{H}_\Omega(\mathrm{D}^+(c,r))$, we need only prove the proposition for $\mathcal{H}_\Omega(\mathrm{D}^+(c,r))$, with $|q-1||c| < r < R$. We recall that a series of functions $\sum_{n\geqslant 0} f_n$, $f_n \in \mathcal{H}_K(\mathrm{D}^+(c,r))$ converges to a function $f \in \mathcal{H}_K(\mathrm{D}^+(c,r))$ if and only if $\lim_n |f_n|_{(c,r)} = 0$. Writing $(T-q^ic) = (1-q^i)c+(T-c)$, one sees easily that $(T-c)_{q,n} = \sum_{i=0}^n \widetilde{b}_{n,i}(T-c)^i$, with $\widetilde{b}_{i,j}$ satisfying i) $\widetilde{b}_{0,0} = 1$, ii) $\widetilde{b}_{0,i} = 0 \ \forall\, i \geqslant 1$, iii) $\widetilde{b}_{n,n} = 1 \ \forall\, n \geqslant 0$, iv) $\widetilde{b}_{n,i} = 0 \ \forall\, i > n$, and v) for all $0 \leqslant i < n$:

$$\widetilde{b}_{n,i} = c^{n-i} \cdot \sum_{0\leqslant k_1<\cdots<k_{n-i}\leqslant n-1} (1-q^{k_1})(1-q^{k_2})\cdots(1-q^{k_{n-i}}) \ . \tag{5.1.12}$$

In other words $[1,(T-c)_{q,1},(T-c)_{q,2},\ldots,(T-c)_{q,n}]^t = \widetilde{B}\cdot[1,(T-c),(T-c)^2,\ldots,(T-c)^n]^t$ where $\widetilde{B} = (\widetilde{b}_{n,i})_{n,i=0,\ldots,n}$ is an $(n+1)\times(n+1)$ lower triangular matrix satisfying i),ii),iii),iv),v). Since $|q^i-1| \leqslant |q-1|$, one has also the property vi) $|\widetilde{b}_{n,i}| \leqslant (|q-1||c|)^{n-i} < r^{n-i}$, for all $0 \leqslant i < n$. Hence for all $n \geqslant 0$, one has $(T-c)_{q,n} = (T-c)^n + g_n(T)$, with $|g_n(T)|_{(c,r)} < r^n$, so $|(T-c)_{q,n}|_{(c,r)} = |(T-c)^n|_{(c,r)} = r^n$. It is easy to prove that also the matrix $B := \widetilde{B}^{-1} = (b_{n,i})_{n,i=0,\ldots,n}$ satisfies the properties i),ii),iii),iv),v),vi). Consider now $f(T) = \sum_{n\geqslant 0} a_n(T-c)^n$. Writing $f_m(T) := \sum_{n=0}^m a_n(T-c)^n = \sum_{n=0}^m a_n \sum_{i=0}^n b_{n,i}(T-c)_{q,i}$ and rearranging terms one finds $f_m(T) = \sum_{n=0}^m \widetilde{a}_{n,m}(T-c)_{q,n}$, with $\widetilde{a}_{n,m} = \sum_{k=0}^{m-n} a_{n+k}b_{n+k,n}$. By property vi) and by the assumption that $\lim_n |a_n|r^n = 0$ the sum $\widetilde{a}_n := \sum_{k\geqslant 0} a_{n+k}b_{n+k,n}$ converges in Ω. Moreover

$$|\widetilde{a}_n|r^n \ \leqslant \ \max_{k\geqslant 0} |a_{n+k}||b_{n+k,n}|\cdot r^n \ \leqslant \ \max_{k\geqslant 0} |a_{n+k}|r^{n+k} \ . \tag{5.1.13}$$

This proves that $\lim_n |\widetilde{a}_n|r^n = 0$, and hence that the series of functions $\sum_{n\geqslant 0}\widetilde{a}_n(T-c)_{q,n}$ is convergent in $\mathcal{H}_\Omega(\mathrm{D}^+(c,r))$. If $f_m^0(T) := \sum_{n=0}^m \widetilde{a}_n(T-c)_{q,n}$, one sees that $|f_m^0 - f_m|_{(c,r)} \leqslant \sup_{k\geqslant 0}|a_{m+k}|r^{m+k}$ which tends to 0, so $\lim_m f_m^0(T) = \lim_m f_m(T) = f(T)$ in $\mathcal{H}_\Omega(\mathrm{D}^+(c,r))$. Now the inequality (5.1.13) shows that $\max_{n\geqslant 0}|\widetilde{a}_n|r^n \leqslant \max_{n\geqslant 0}|a_n|r^n$, and a symmetric argument using the matrix $\widetilde{B}$ instead of B proves the opposite inequality so $\max_{n\geqslant 0}|\widetilde{a}_n|r^n = \max_{n\geqslant 0}|a_n|r^n = |f(T)|_{(c,r)}$. This last equality shows the uniqueness of the coefficients $\{\widetilde{a}_n\}_n$ since if $\sum_{n\geqslant 0}\widetilde{a}_n(T-c)_{q,n} = \sum_{n\geqslant 0}\widetilde{a}'_n(T-c)_{q,n}$, then $\sum_{n\geqslant 0}(\widetilde{a}_n - \widetilde{a}'_n)(T-c)_{q,n} = 0$, and hence $\sup_n(|\widetilde{a}_n - \widetilde{a}'_n|r^n) = 0$, so that $\widetilde{a}_n = \widetilde{a}'_n$, for all $n \geqslant 0$. Clearly the radius of convergence of $f(T)$ is equal to both $\sup_{n\geqslant 0}\{r \geqslant 0 \mid |a_n|r^n \text{ is bounded}\}$ and $\sup_{n\geqslant 0}\{r \geqslant 0 \mid |\widetilde{a}_n|r^n \text{ is bounded}\}$. Hence, by classical arguments on the radius of convergence, one has $Ray(f(T),c) = \liminf_n |a_n|^{-1/n} = \liminf_n |\widetilde{a}_n|^{-1/n}$. The assertion iv) is proved in [DV04]. □

REMARK 5.4. If $f(T) = \sum_{n\geqslant 0} f_n(T-c)_{q,n}$, and if $g(T) = \sum_{n\geqslant 0} g_n(T-c)_{q,n}$, then $f(T)g(T) = \sum_{n\geqslant 0} h_n(T-c)_{q,n}$, where $h_n = h_n(q;c;f_0,\ldots,f_n;g_0,\ldots,g_n)$ is a polynomial in $\{q,c,f_0,\ldots,f_n,g_0,\ldots,g_n\}$. Indeed one has $(T-c)_{q,n}\cdot(T-c)_{q,m} = \sum_{k=\max(n,m)}^{n+m}\alpha_k^{(n,m)}(T-c)_{q,k}$, with $\alpha_k^{(n,m)} = \alpha_k^{(n,m)}(q,c) \in \Omega$. This also shows that if $v_{q,c}(f) := \min\{n \mid f_n \neq 0\}$, then one has

$$v_{q,c}(fg) \geqslant \max(v_{q,c}(f), v_{q,c}(g)) \ . \tag{5.1.14}$$

If moreover $q \notin \boldsymbol{\mu}(\mathcal{Q})$, then, by using equation (5.1.9) and (5.1.11), one has

$$h_n = \sum_{j=0}^n \sum_{s=0}^j \frac{[n]_q^![j]_q^![s+n-j]_q^!}{([s]_q^!)^2[n-j]_q^!}\cdot q^{\frac{s(s-1)}{2}}(q-1)^s c^s f_{s+n-j}g_j \ . \tag{5.1.15}$$

5.1.3 *The algebras $\Omega[\![T-c]\!]_q$, and $\Omega\{T-c\}_{q,R}$.*

Definition 5.5. For all $q \in \mathcal{Q}(X)$ we set

$$\Omega[\![T-c]\!]_q := \{\sum_{n\geqslant 0} f_n(T-c)_{q,n} \mid f_n \in \Omega\} \;. \tag{5.1.16}$$

$$\Omega\{T-c\}_{q,R} := \{ \sum_{n\geqslant 0} f_n(T-c)_{q,n} \mid f_n \in \Omega,\; \liminf_n |f_n|^{-1/n} \geqslant R \} \;. \tag{5.1.17}$$

We define a multiplication on $\Omega[\![T-c]\!]_q$ and $\Omega\{T-c\}_{q,R}$ by the rule given in Remark 5.4.

Lemma 5.6. *$\Omega[\![T-c]\!]_q$ and $\Omega\{T-c\}_{q,R}$ are commutative Ω-algebras, $\forall\, q \in \mathcal{Q}$.*

Proof. We prove only the associativity, the others verifications are similar. We have to prove that $(fg)h = f(gh)$. By Lemma 5.3 the assertion is proved if $f, g, h \in \Omega\{T-c\}_{q,R}$, with $|q-1||c| < R$, since in this case $\Omega\{T-c\}_{q,R} \cong \mathcal{A}_\Omega(c,R)$. On the other hand one can assume that f, g, h are polynomials since, by Remark 5.4, the n-th coefficient of $(fg)h$ and of $f(gh)$ is a polynomial in q and in the first n coefficients of f, g, h. □

Remark 5.7. If there exists a (smallest) integer k_0 such that $|\overline{q}^{k_0}-1||c| < R$, then one shows that $\Omega\{T-c\}_{q,R} = \prod_{i=0}^{k_0-1} \mathcal{A}_\Omega(q^i c, \widetilde{R})$, where $\widetilde{R}$ depends explicitly on R, c, and q (cf. [DV04, 15.3]). In this case $\Omega\{T-c\}_{q,R}$ is not a domain and hence is not a $\mathcal{H}_\Omega(X)-$discrete $\sigma-$algebra over $S = \{q\}$.

Remark 5.8. If x, y are variables, then $\Omega[\![x-y]\!]_q$ is not an algebra, but merely a vector space. Indeed the multiplication law involves y in the coefficients "h_n" of Remark 5.4. This minor mistake occurs occasionally in [DV04], but it is an irrelevant inaccuracy and does not jeopardize any proposition of [DV04]. The matrix $Y(x,y)$ always seems to be used there under the assumption (5.5.6).

5.2 q-invariant Affinoid

Let $|q| = 1$, $q \in K$. Let $X := \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i)$, $c_1, \ldots, c_n \in \mathrm{D}^+_K(c_0, R_0)$, $c_0 \in K$, be a K-affinoid. Then X is q-invariant if and only if $|q-1||c_0| < R_0$, and the map $x \mapsto qx$ permutes the family of disks $\{\ \mathrm{D}^-(c_i, R_i)\ \}_{i=1,\ldots,n}$. This happens if and only if for all $i = 1, \ldots, n$ there exists (a smallest) $k_i \geqslant 1$, such that $|q^{k_i}-1||c_i| < R_i$, and moreover the family of disks $\{\ \mathrm{D}^-(q^k c_i, R_i)\ \}_{k=1,\ldots,k_i}$ is finite and contained in $\{\ \mathrm{D}^-(c_i, R_i)\ \}_{i=1,\ldots,n}$. If k_0 is the minimum common multiple of the k_i's, then $x \mapsto q^{k_0}x$ leaves every disk globally fixed and, by Lemmas 5.1 and 5.3, one has

$$\|d_{q^{k_0}}(f)\|_X \;\leqslant\; r_X^{-1}\|f\|_X \;, \tag{5.2.1}$$

for all $f \in \mathcal{H}(X)$ (cf. Lemma 1.9). Indeed the by Mittag-Lefler decomposition [CR94], we reduce to showing that every series $f = \sum_{j\leqslant -1} a_j(T-c_i)^j$, such that $|a_j|R_i^j$ tends to zero, satisfies $|d_{q^{k_0}}(f)|_{(c_i,R_i)} \leqslant R_i^{-1} \cdot |f|_{(c_i,R_i)}$, and this is true by Lemma 5.3.

Such a bound does not exist for d_q itself. One can easily construct counterexamples via the Mittag-Leffler decomposition.

5.3 The generic Taylor solution

We recall the definition of the classical Taylor solution of a differential equation

Definition 5.9. Let $\delta_1 - G(1,T)$, be a differential equation. Let $G_{[n]}(T)$ be the matrix of $(d/dx)^n$. We set

$$Y_{G(1,T)}(x,y) := \sum_{n\geqslant 0} G_{[n]}(y)\frac{(x-y)^n}{n!} \;. \tag{5.3.1}$$

By induction on the rule $G_{[n+1]} = G'_{[n]} + G_{[n]}G_{[1]}$, one finds $\|G_{[n]}\|_X \leqslant \max(\|G_{[1]}\|_X, r_X^{-1})^n$, hence

$$Ray(Y_G(T,c),c) = \liminf_n \Big(\frac{|G_{[n]}(c)|_\Omega}{|n!|}\Big)^{-1/n} \geqslant \frac{|p|^{\frac{1}{p-1}}}{\max(r_X^{-1}, \|G_{[1]}\|_X)} \;. \tag{5.3.2}$$

In other words $Y_G(x,y)$ is an analytic element over a neighborhood $\mathcal{U}_R$ of the diagonal of the type

$$\mathcal{U}_R := \{(x,y) \in X \times X \mid |x-y| < R\} \;, \tag{5.3.3}$$

for some $R > 0$.

Lemma 5.10. *One has $Y_G(x,x) = \mathrm{Id}$, and for all $(x,y) \in \mathcal{U}_R$:*

$$d/dy\ (Y_G(x,y)) = -Y_G(x,y)\cdot G_{[1]}(y) \;, \tag{5.3.4}$$

$$Y_G(x,y)^{-1} = Y_G(y,x) \;, \tag{5.3.5}$$

$$Y_G(x,y)\cdot Y_G(y,z) = Y_G(x,z) \;, \tag{5.3.6}$$

$$d/dx\ (Y_G(x,y)) = G_{[1]}(x)\cdot Y_G(x,y) \;. \tag{5.3.7}$$

Proof. See [CM02a, p.137] (cf. Lemma 5.16). The proof is analogous to that of Lemma 5.16. □

Definition 5.11. Let $q \in \mathcal{Q} - \boldsymbol{\mu}(\mathcal{Q})$. Consider the $q-$difference equation

$$\sigma_q(Y) = A(q,T)\cdot Y \;, \qquad A(q,T) \in GL_n(\mathcal{H}_K(X)) \;. \tag{5.3.8}$$

Let H_n be defined by $d_q^n(Y) = H_n \cdot Y$. We formally set

$$Y_{A(q,T)}(x,y) \;=\; \sum_{n\geqslant 0} H_n(y)\frac{(x-y)_{q,n}}{[n]_q^!} \;. \tag{5.3.9}$$

We will omit the index $A(q,T)$ if no confusion is possible. Observe that $Y_{A(q,T)}(x,y)$ is a symbol and does not necessarily define a convergent function.

Example 5.12. With the notations of example 2.6, the generic Taylor solution of the equations $\sigma_q(Y) = A(q,x)Y$, $\sigma_q(Y) = \widetilde{A}(q,x)Y$, are $Y_{A(q,x)}(x,y) = \exp(\pi(x-y))$ and $Y_{\widetilde{A}(q,x)}(x,y) = \exp(\pi q(x-y))$ respectively. Notice that $Y_{A(q,x)}(x,y)$ is *constant* with q.

Definition 5.13. For all (not necessarily bounded nor multiplicative) semi-norms $|.|_*$ on $\mathcal{H}_K(X)$ extending the absolute value of K we set

$$Ray(Y_{A(q,T)}(x,y),|.|_*) \;:=\; \liminf_n (|H_n(y)|_*/|[n]_q^!|)^{-1/n} \;. \tag{5.3.10}$$

If $Y_{A(q,T)}(x,y)$ is a convergent function on some neighborhood of the diagonal of $X\times X$, then, for $|f(T)|_* := |f(c)|_\Omega$, $c \in X(\Omega)$, one finds Definition 1.5, namely $Ray(Y(x,y),|.|_c) = Ray(Y(x,c),c)$. In this case we will write $Ray(Y(x,y),c) := Ray(Y(x,y),|.|_c)$ (cf. Section 1.3.1). If $X' \subseteq X$ is a sub-affinoid we simply write $Ray(Y(x,y),X') := Ray(Y(x,y),\|.\|_{X'})$.

5.4 Transfer Principle

As in the differential setting, if $X' := \mathrm{D}^+(c'_0,R'_0) - \cup_{i=1}^n \mathrm{D}^-(c'_i,R'_i) \subseteq X$ is a $q-$invariant sub-affinoid, such that every disk $\mathrm{D}^-(c'_i,R'_i)$ is also $q-$invariant, then the estimate (5.2.1) holds (cf. Remark 7.12). Then, by induction on the rule $H_{n+1} = d_q(H_n) + \sigma_q(H_n)H_1$, one shows that $\|H_n\|_{X'} \leqslant \max(\|H_1\|_{X'}, r_{X'}^{-1})^n$, hence

$$\begin{aligned} Ray(Y(x,y),X') &:= \liminf_n (\|H_n\|_{X'}/[n]_q^!)^{-1/n} = \min_{c\in X'(\Omega)} Ray(Y(x,y),c) \\ &\geqslant \frac{\liminf_n([n]_q^!)^{1/n}}{\max(r_{X'}^{-1}, \|H_1\|_{X'})} \;, \end{aligned} \tag{5.4.1}$$

where $(\Omega, |.|)/(K, |.|)$ is an algebraically closed extension of complete valued field such that $|\Omega| = \mathbb{R}_{\geqslant 0}$. Observe that the second equality follows by the fact that $\|.\|_X = \sup_{c\in X(\Omega)} |.|_c$. In particular if $X' = \mathrm{D}^+(c,\rho) \subseteq X$, with $|q-1||c| < \rho \leqslant \rho_{c,X}$, is a $q-$invariant disk, then $Ray(Y(x,y),c)$ is greater than or equal to

$$Ray(Y(x,y), \mathrm{D}^+(c,\rho)) = \min_{c'\in\, \mathrm{D}^+_{\Omega}(c,\rho)} Ray(Y(x,y),c') \ \geqslant\ \frac{\liminf_n([n]^!_q)^{1/n}}{\max(\rho^{-1}, |H_1|_{(c,\rho)})}\ . \tag{5.4.2}$$

Notice that if $|q-1||c| < R_c := Ray(Y(x,y),c)$, then $Y(x,c) \in M_n(\mathcal{A}_\Omega(c,R_c))$, but $Y(x,c)$ is invertible only in $GL_n(\mathcal{A}_\Omega(c,\widetilde{R}))$, with $\widetilde{R} := \min(\rho_{c,X}, Ray(Y(x,y),c))$ (cf. Lemmas 5.15 and 5.16).

5.5 Properties of the generic Taylor solution

The formal matrix solution $Y_A(x,y)$ is not always a function in a neighborhood of type $\mathcal{U}_R$ of the diagonal of $X\times X$. But if for all $c\in X(K^{\mathrm{alg}})$ one has $|q-1||c| < R \leqslant \min(\rho_{c,X}, Ray(Y(x,y),c))$, then, by Lemma 5.3, and by the Transfer Principle (cf. Equation (5.4.2)), $Y_A(x,y)$ actually defines an *invertible* function on $\mathcal{U}_R$ (cf. Lemmas 5.15 and 5.16). If $X = \mathrm{D}^+(c_0,R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i,R_i)$, the condition $|q-1||c| < R \leqslant \min(\rho_{c,X}, Ray(Y(x,y),c))$, for all $c\in X(K^{\mathrm{alg}})$, implies

$$|q-1|\sup(R_0,|c_0|) \;=\; |q-1|\max_{c\in X}|c| \;<\; R \;\leqslant\; \min_{c\in X}\rho_{c,X} \;=\; \min(R_0,\dots,R_n) \;=\; r_X\ . \tag{5.5.1}$$

In particular, since $r_X = \min(R_0,\dots,R_n) \leqslant \sup(|c_0|,R_0)$, this is possible only if

$$|q-1| < 1\ ,\quad \text{i.e. if}\ \ q\in\mathcal{Q}_1(X)\ . \tag{5.5.2}$$

Hypothesis 5.14. From now on, without explicit mention to the contrary, we will assume that

$$q\in\mathcal{Q}_1(X)\ . \tag{5.5.3}$$

Lemma 5.15. *Let $q\in\mathcal{Q}_1(X) - \boldsymbol{\mu}(\mathcal{Q}_1(X))$. Let $f(x,y)$ be an analytic function in a neighborhood of type $\mathcal{U}_R\subset X\times X$ of the diagonal of $X\times X$. Assume that*[4]

$$|q-1|\max(|c_0|,R_0) < R \leqslant r_X\ . \tag{5.5.4}$$

If moreover $f(x,y)$ satisfies $f(x,qy) = a(y)\cdot f(x,y)$, with $a(y)\in\mathcal{H}_K(X)^\times$, then $f(x,y)$ is invertible.

Proof. Since f is an analytic function, it is sufficient to prove that f has no zeros in $\mathcal{U}_R$. We need only show that for all $c\in X(\Omega)$, the function $g_c(y) := f(c,y)$ has no zeros in $\mathrm{D}^-(c,R)$. One has $d_q(g_c(y)) = h(y)\cdot g_c(y)$, with $h(y) = \frac{a(y)-1}{(q-1)y}$. Assume that $g_c(\widetilde{c}) = 0$, for some $\widetilde{c}\in\mathrm{D}^-(c,R) = \mathrm{D}^-(\widetilde{c},R)$, then, by Lemma 5.3, $g_c(y) = \sum_{n\geqslant 0} a_k(y-\widetilde{c})_{q,n}$, with $a_0 = 0$. Since $q\notin\boldsymbol{\mu}(\mathcal{Q})$, we have $[n]_q a_n = 0$ if and only if $a_n = 0$. Hence, by Remark 5.4 one has $v_{q,\widetilde{c}}(d_q(g_c)) = v_{q,\widetilde{c}}(g_c) - 1$. On the other hand, $v_{q,\widetilde{c}}(hg_c) \geqslant v_{q,\widetilde{c}}(g_c)$, which contradicts $d_q(g_c) = hg_c$. □

Lemma 5.16. *Let $q\in\mathcal{Q}_1(X) - \boldsymbol{\mu}(Q_1(X))$, and let*

$$\begin{array}{llll} \sigma_q^x : & f(x,y)\mapsto f(qx,y)\ , & \sigma_q^y : & f(x,y)\mapsto f(x,qy)\ , \\ d_q^x := & \frac{\sigma_q^x - 1}{(q-1)x}\ , & d_q^y := & \frac{\sigma_q^y-1}{(q-1)y}\ . \end{array} \tag{5.5.5}$$

Suppose that $Y_A(x,y)$ converges on $\mathcal{U}_R$, with (cf. Section 5.5)

$$|q-1|\max(|c_0|,R_0) \;<\; R \;\leqslant\; r_X\ . \tag{5.5.6}$$

[4] i.e. assume that $|q-1|\max(|c_0|,R_0) < R \leqslant \rho_{c,X}$ for all $c\in X(K^{\mathrm{alg}})$.

Then $Y_A(x,y)$ is invertible on $\mathcal{U}_R$ and satisfies $Y_A(x,x) = \mathrm{Id}$ and:

$$d_q^y\, Y_A(x,y) = -\sigma_q^y(Y_A(x,y)) \cdot H_1(y) \;, \tag{5.5.7}$$
$$\sigma_q^y\, Y_A(x,y) = Y_A(x,y) \cdot A(q,y)^{-1} \;, \tag{5.5.8}$$
$$Y_A(x,y) \cdot Y_A(y,z) = Y_A(x,z) \;, \tag{5.5.9}$$
$$Y_A(x,y)^{-1} = Y_A(y,x) \;, \tag{5.5.10}$$
$$d_q^x\, Y_A(x,y) = H_1(x) \cdot Y_A(x,y) \;. \tag{5.5.11}$$
$$\sigma_q^x\, Y_A(x,y) = A(q,x) \cdot Y_A(x,y) \;. \tag{5.5.12}$$

Proof. The relation $Y(x,x) = \mathrm{Id}$ is evident, while (5.5.7) is easy to compute explicitly, and is equivalent to (5.5.8). Since $Y(x,y)$ converges on $\mathcal{U}_R$, equation (5.5.8) implies that the determinant $d(x,y)$ of $Y(x,y)$ satisfies $d(x,qy) = a(y)d(x,y)$, with $a(y) = \det(A(q,y)^{-1}) \in \mathcal{H}_K(X)^\times$. By Lemma 5.15, $d(x,y)$ is then invertible on $\mathcal{U}_R$, and hence also $Y(x,y)$ is invertible. By equation (5.1.9), and since $q \notin \boldsymbol{\mu}(\mathcal{Q})$, the relation $d_q^y(Y(x,y)Y(x,y)^{-1}) = 0$ gives

$$d_q^y(Y(x,y)^{-1}) = -\sigma_q^y(Y(x,y)^{-1}) \cdot d_q^y(Y(x,y)) \cdot Y(x,y)^{-1} \;. \tag{5.5.13}$$

Hence, for all x,y,z such that $|x-y|, |z-y| < R$, the relation (5.5.13), together with relation (5.5.7), give $d_q^y(Y(x,y) \cdot Y(z,y)^{-1}) = 0$. Since $q \notin \boldsymbol{\mu}(\mathcal{Q})$, this implies, by Lemma 5.3, that the function $Y(x,y)Y(z,y)^{-1}$ does not depend on y. Specializing for $y = x$, and $y = z$, one finds $Y(x,z) = Y(z,x)^{-1}$, and $Y(x,y) \cdot Y(y,z) = Y(x,z)$. Then, by the above expression for $d_q^x(Y(y,x)^{-1}) = d_q^x(Y(x,y))$, the relations (5.5.11) and (5.5.12) follow from (5.5.10) and (5.5.7). □

5.5.1 *The case $|q-1| = 1$, $|q| = 1$.* If for a $c \in X$ one has $|q-1||c| \geqslant Ray(Y_{A(q,T)}(x,y), c)$, then Lemma 5.16 does not apply (cf. [DV04, Section 15]). It may happen (cf. Remark 7.12) that there exists a (smallest) $k_0 \geqslant 0$ such that condition (5.5.6) holds for q^{k_0} instead of q, and for $Y_{A(q^{k_0},T)}(x,y)$ instead of $Y_{A(q,T)}(x,y)$. There then exists a Taylor solution $Y_c \in M_n(\mathcal{A}_\Omega(c,R))$ of the iterated system $\sigma_{q^{k_0}}(Y_c) = A(q^{k_0},T)Y_c$. In this case, for all $c \in X(\Omega)$, we can recover a solution Y^{big} of the system $\sigma_q(Y^{\mathrm{big}}) = A(q,T)Y^{\mathrm{big}}$ itself in the algebra of analytic functions over the disjoint union of disks $\bigcup_{i=0}^{k_0-1} \mathrm{D}^-(q^i c, R)$. Indeed σ_q acts on the algebra $\prod_{i\in\mathbb{Z}/k_0\mathbb{Z}} M_n(\mathcal{A}_K(q^i c,R))$ by $\sigma_q((M_{q^i c}(T))_{i\in\mathbb{Z}/k_0\mathbb{Z}}) = (M_{q^{i+1}c}(qT))_{i\in\mathbb{Z}/k_0\mathbb{Z}}$, and so one has

$$Y^{\mathrm{big}}(T) = (\ Y^{\mathrm{big}}_{q^i c}(T)\ )_i := (\ A(q^i, q^{-i}T) \cdot Y_c(q^{-i}T)\ )_{i\in\mathbb{Z}/k_0\mathbb{Z}} \;. \tag{5.5.14}$$

In fact $A(q^{i+1}, q^{-i}T) = A(q,T)A(q^i,q^{-i}T)$. This and related matters are very well explained in [DV04].

5.5.2 Notice that the relations of Lemma 5.16 hold for $Y_A(x,y)$ as a function on $\mathcal{U}_R$, and not for $Y^{\mathrm{big}}(T)$ (cf. (5.5.14)). In other words the expression $Y_A^{\mathrm{big}}(x,y)$ has no meaning if $|x-y| \geqslant R$. In particular the expression (5.5.9), which is the main tool of the Propagation Theorem 7.7, holds only if $|x-y|, |z-y| < R$.

5.5.3 *The case of a root of unity.* If $q \in \boldsymbol{\mu}(\mathcal{Q})$ is a root of unity, then even when a solution $Y \in GL_n(\mathcal{A}_\Omega(c,R))$ exists, *the radius is not defined* since we may have another solution with different radius (cf. Example 5.17 below). For this reason, the radius of convergence of the system (5.3.8) will be not defined if $q \in \boldsymbol{\mu}(\mathcal{Q})$.

Example 5.17. Let $q = \xi$ be a $p-$th root of unity, with $\xi \neq 1$. The solutions of the unit object at $t^p \in \Omega$ are the functions $y \in \mathcal{A}_\Omega(t^p, R)$ such that $y(\xi T) = y(T)$. Every function in T^p has this property. For example the family of functions $\{\ y_\alpha := \exp(\alpha(T^p - t^p))\ \}_{\alpha\in\Omega}$, is such that for different values of α one has different radii.

5.6 Taylor solutions of $(\sigma_q, \delta_q)-$modules

In this subsection q may be a root of unity. We preserve the previous notations. We consider now a system (the notion of solution of such a system have been defined in section 3.2):

$$\begin{cases} \sigma_q(Y) = A(q,T)\cdot Y \ , & A(q,T)\in GL_n(\mathcal{H}_K(X))\ , \\ \delta_q(Y) = G(q,T)\cdot Y \ , & G(q,T)\in M_n(\mathcal{H}_K(X))\ . \end{cases} \tag{5.6.1}$$

It can happen that a solution of σ_q^{M} is not a solution of δ_q^{M} as shown by the following example:

Example 5.18. Suppose that $q\in \mathrm{D}^-(1,1)$ is not a root of unity. Let $X := \mathrm{D}^+(0,|p|^{\frac{1}{p-1}})$, $A(q,T) := \exp((q-1)T)\in\mathcal{H}_K(X)^\times$, $G(q,T) := 0$. Let $c = 0$, and $R < |p|^{\frac{1}{p-1}}$. Then every solution $y(T)\in\mathcal{A}_K(0,R)$ of the operator $\sigma_q - A(q,T)$ is of the form $y(T) = \lambda\cdot\exp(T)$, with $\lambda\in K$. If $\delta_q(y) = 0$, then $y = 0$. Hence, the $(\sigma_q,\delta_q)-$module defined by $A(q,T)$ and $G(q,T)$ has no (non trivial) solutions in $\mathcal{A}_K(0,R)$.

To guarantee the existence of solutions we need a *compatibility condition* between σ_q and δ_q, which should be written explicitly in terms of matrices of σ_q^n and δ_1^n. This obstruction will not appear in the sequel of the paper since this condition is automatically satisfied by analytic σ-modules (cf. Lemma 5.19). This fact will follows from that a solution $\alpha : \mathrm{M}\to\mathcal{A}_\Omega(c,R)$ is continuous (see the proof of Lemma 5.19). Observe that Lemma 5.19 below is not a formal consequence of the previous theory. Indeed, by Definition 3.2, the general $(\sigma,\delta)-$algebra C used in Definition 3.2 has the discrete topology, hence the morphism $\alpha : \mathrm{M}\to \mathrm{C}$ defining the solution is not continuous in general.

Lemma 5.19. *Let $U\subseteq\mathcal{Q}(\mathcal{H}_K(X))$ be an open subset, and let* M *be an analytic $(\sigma,\delta)-$module on U, representing the family of equations $\{\sigma_q(Y) = A(q,T)\cdot Y\}_{q\in U}$, with $A(q,T)\in GL_n(\mathcal{H}_K(X))$, for all $q\in U$. Let $Y_c(T)\in GL_n(\mathcal{A}_\Omega(c,R))$, $|q-1||c| < R\leqslant\rho_{c,X}$, be a simultaneous solution of every equation of this family. Then $Y_c(T)$ is also solution of the equation*

$$\delta_q(Y) = G(q,T)\cdot Y\ , \tag{5.6.2}$$

where $G(q,T) := q\frac{d}{dq}(A(q,T))$ (cf. (2.4.5)). Hence $Y_c(T)$ is solution of the differential equation defined in section 2.4.1:

$$\delta_1(Y_c(T)) = G(1,T)\cdot Y_c(T)\ , \tag{5.6.3}$$

where $G(1,T) = G(q,q^{-1}T)\cdot A(q,q^{-1}T)^{-1}\in M_n(\mathcal{H}_K(X))$ (cf. (2.2.7)).

Proof. In terms of modules, the columns of the matrix $Y_c(T)$ correspond to $\mathcal{H}_K(X)-$linear maps $\alpha : \mathrm{M}\to\mathcal{A}_\Omega(c,R)$, verifying $\sigma_q\circ\alpha = \alpha\circ\sigma_q^{\mathrm{M}}$, for all $q\in U$ (cf. Section 3.2.1). We must show that such an α also commutes with δ_q. This follows immediately by the continuity of α. Indeed, the inclusion $\mathcal{H}_K(X)\to\mathcal{A}_\Omega(c,R)$ is continuous, and hence every $\mathcal{H}_K(X)-$linear map $\mathcal{H}_K(X)^n\to\mathcal{A}_\Omega(c,R)$ is continuous. □

5.7 Twisted Taylor formula for $(\sigma,\delta)-$modules, and rough estimate of radius

Let X be a q-invariant affinoid. Let $D_q := \sigma_q\circ\frac{d}{dT} = \lim_{q'\to q}\frac{\sigma_{q'}-\sigma_q}{T(q'-q)} = \frac{1}{qT}\cdot\delta_q$. For all $q\in\mathcal{Q}(X)$ and all $f(T)\in\mathcal{H}_K(X)$, one has

$$\mathrm{D}_q(f\cdot g) = \sigma_q(f)\cdot\mathrm{D}_q(g) + \mathrm{D}_q(f)\cdot\sigma_q(g)\ , \tag{5.7.1}$$

$$(d/dT\circ\sigma_q) = q\cdot(\sigma_q\circ d/dT)\ , \tag{5.7.2}$$

$$\mathrm{D}_q^n = q^{n(n-1)/2}\cdot\sigma_q^n\circ(d/dT)^n\ , \tag{5.7.3}$$

$$\|\mathrm{D}_q^n(f(T))\|_X\leqslant\frac{|n!|}{r_X^n}\cdot\|f(T)\|_X \qquad \text{(cf. Lemma 1.9)}\ . \tag{5.7.4}$$

Hence, for all $c \in K$, $\mathrm{D}_q^n(T-c)^i = \frac{i!}{(i-n)!} \cdot q^{n(n-1)/2} \cdot (q^nT-c)^{i-n}$ if $n \leqslant i$, and $\mathrm{D}_q^n(T-c)^i = 0$ if $n > i$. This shows that if $f(T) := \sum_{i\geqslant 0} a_i \cdot \frac{(T-c)^i}{(i!)\cdot q^{i(i-1)/2}} \in \mathcal{A}_\Omega(c,R)$ is a formal series, with $|q-1||c| < R \leqslant \rho_{c,X}$, then $a_n = \mathrm{D}_q^n(f)(c/q^n)$, and the usual Taylor formula can be written as

$$f(T) = \sum_{n\geqslant 0} \mathrm{D}_q^n(f)(c/q^n) \cdot \frac{(T-c)^n}{(n!)\cdot q^{n(n-1)/2}} \quad . \tag{5.7.5}$$

The following proposition gives the analogue of the classical rough estimate for differential and $q-$difference equations (cf. [Chr83, 4.1.2], [DV04, 4.3]).

Proposition 5.20. *Let $c \in X(\Omega)$. Assume that the system (5.6.1) has a Taylor solution $Y_c \in M_n(\mathcal{A}_\Omega(c,R_c))$, with $|q-1||c| < R_c \leqslant \rho_{c,X}$. For all q-invariant sub-affinoid $X' \subseteq X$, containing $\mathrm{D}^+(c,|q-1||c|)$, one has*

$$R_c \geqslant \frac{|p|^{\frac{1}{p-1}}}{\max(\ r_{X'}^{-1}\|A(q,T)\|_{X'}\ ,\ \|G(q,T)/qT\|_{X'}\ )} \ . \tag{5.7.6}$$

In particular if X' is a disk $\mathrm{D}^+(c,\rho)$, with $|q-1||c| \leqslant \rho \leqslant \rho_{c,X}$, then

$$R_c \geqslant \frac{|p|^{\frac{1}{p-1}}\cdot\rho}{\max(\ |A(q,T)|_{(c,\rho)}\ ,\ \frac{|G(q,T)|_{(c,\rho)}}{\max(1,|c|/\rho)}\ )} \ . \tag{5.7.7}$$

Proof. The matrix $Y_c(T)$ satisfies $\sigma_q^n(Y_c(T)) = A_{[n]}(q,T)\cdot Y_c(T)$, and $D_q^n(Y_c(T)) = F_{[n]}(q,T)\cdot Y_c(T)$, where $F_{[0]} = \mathrm{Id} = A_{[0]}$, $A_{[1]} := A(q,T)$, $F_{[1]} := \frac{1}{qT}G(q,T)$, and

$$A_{[n]} := \sigma_q^{n-1}(A_{[1]})\cdots\sigma_q(A_{[1]})\cdot A_{[1]} \ , \tag{5.7.8}$$
$$F_{[n+1]} := \sigma_q(F_{[n]})\cdot F_{[1]} + D_q(F_{[n]})\cdot A_{[1]} \ . \tag{5.7.9}$$

Hence one has

$$Y_c(T) := \sum_{i\geqslant 0} F_{[n]}(c/q^n)\frac{(T-c)^n}{(n!)\cdot q^{n(n-1)/2}} \quad , \tag{5.7.10}$$

which is a hybrid between the usual Taylor formula and the Taylor formula for $q-$difference equations. Inequalities 5.7.6 then follow from the inequality

$$|F_{[n]}(c/q^n)|_\Omega \leqslant \|F_{[n]}\|_{X'} \leqslant \max\left(\|F_{[1]}\|_{X'}\ ,\ \frac{1}{r_{X'}}\cdot\|A_{[1]}\|_{X'}\right)^n \tag{5.7.11}$$

If $X' = \mathrm{D}^+(c,\rho)$, then the last term is equal to $\frac{1}{\rho^n}\cdot\max\left(\frac{|G(q,T)|_{(c,\rho)}}{\max(1,|c|/\rho)}\ ,\ |A(q,T)|_{(c,\rho)}\right)^n$. Indeed $r_{\mathrm{D}^+(c,\rho)} = \rho$, $F_{[1]} = \frac{1}{qT}G(q,T)$, and $|T|_{(c,\rho)} = |(T-c)+c|_{(c,\rho)} = \max(\rho,|c|)$, hence $|F_{[1]}|_{(c,\rho)} = \frac{1}{|q|\max(|c|,\rho)}\cdot|G(q,T)|_{(c,\rho)}$, and $|q| = 1$. □

6. Generic radius of convergence and solvability

Definition 6.1 (Generic radius of convergence). Let $q \in \mathcal{Q}(X)$ (resp. $q \in \mathcal{Q}(X) - \boldsymbol{\mu}(\mathcal{Q})$), let $c \in X(K^{\mathrm{alg}})$, and let $\mathrm{D}^+(c,\rho)$, $|q-1||c| < \rho \leqslant \rho_{c,X}$, be a q-invariant disk. Let M be the $(\sigma_q,\delta_q)-$module (resp. σ_q-module) defined by the system (5.6.1) (resp. (5.3.8)). Let $R_{t_{c,\rho}} := Ray(Y(x,y),t_{c,\rho}) =$

$Ray(Y(x,y),|.|_{(c,\rho)})$ be the radius of convergence[5] of $Y_{A(q,T)}(T,t_{c,\rho})$. Assume that

$$|q-1||t_{c,\rho}|<R_{t_{c,\rho}}\ .^{6} \tag{6.0.12}$$

We define the $(c,\rho)-$*generic radius of convergence of* M to be the real number

$$Ray(\mathrm{M},|.|_{c,\rho})\ :=\ \min\big(\ R_{t_{c,\rho}}\ ,\ \rho_{c,X}\ \big)\ >\ |q-1||c|\ . \tag{6.0.13}$$

6.0.1 The assumption (6.0.12) ensures that the disk of convergence of $Y(x,y)$ at $y=t_{c,\rho}$ is q-invariant. While the bound $Ray(\mathrm{M},|.|_{c,\rho})\leqslant\rho_{c,X}$ ensures that $Y(x,y)$ is invertible in the disk $\mathrm{D}^-(t_{c,\rho},R)$, for all $0<R\leqslant Ray(\mathrm{M},|.|_{c,\rho})$ (cf. Lemma 5.15). We recall that $|t_{c,\rho}|=\min(|c|,\rho)$, and that $\|\cdot\|_{\mathrm{D}^+(c,\rho)}=\max_{y_0\in\mathrm{D}^+_{K^{\mathrm{alg}}}(c,\rho)}|\cdot|_{y_0}$. Hence, by the transfer principle (cf. Section 5.4), one has:

$$R_{t_{c,\rho}}:=Ray(Y(x,y),t_{c,\rho})=Ray(Y(x,y),\mathrm{D}^+(c,\rho))=\min_{y_0\in\mathrm{D}^+_{K^{\mathrm{alg}}}(c,\rho)}Ray(Y(x,y),y_0)\ . \tag{6.0.14}$$

The number $Ray(\mathrm{M},|.|_{(c,\rho)})$ is invariant under change of basis in M, while the number $R_{t_{c,\rho}}=Ray(Y(x,y),|.|_{(c,\rho)})$ depends on the choice of basis. Observe that $Ray(\mathrm{M},|.|_{(c,\rho)})$ depends on the affinoid X, and on the semi-norm $|.|_{(c,\rho)}$ defined by $t_{c,\rho}$, but not on the particular choice of $t_{c,\rho}$ (cf. Section 1.4.1).

Definition 6.2 (Solvability). Let M be a σ_q-module (resp. a $(\sigma_q,\delta_q)-$module) on $\mathcal{H}_K(X)$. We will say that M is solvable at $t_{c,\rho}$ if

$$Ray(\mathrm{M},|.|_{(c,\rho)})=\rho_{c,X}\ . \tag{6.0.15}$$

6.0.2 *Continuity and* log-*concavity of the Radius.* Notice that every point $|.|_*$ in the Berkovich space associated to X is of the form $|.|_{c,\rho}$, for a suitable $\rho\geqslant0$, and for a point c in $X(L)$, where $(L,|.|)/(K,|.|)$ is a sufficiently large extension of complete valued fields. One may verify that $|.|_*\mapsto Ray(\mathrm{M},|.|_{c,\rho})$ is a well defined function on the Berkovich space (i.e. the Radius does not depends on the chosen c, but only on $|.|_*$). In a recent pre-print (cf. [BV07]) it have been proved that the function $|.|_*\mapsto Ray(\mathrm{M},|.|_*)$ is continuous on the Berkovich Space. We refer to [BV07] for a very inspiring treatment to this subject.

We notice that this generalizes a previous statement (cf. [CD94]) proving, for all $c\in X(L)$, the continuity of the function $\rho\mapsto Ray(\mathrm{M},|.|_{c,\rho})$.

Let now $(L,|.|)/(K,|.|)$ be any extension of complete valued fields. Let $c\in X(L)$. The function $\rho\mapsto Ray(\mathrm{M},|.|_{c,\rho})$ defined on $[0,\rho_{c,X}]$ is log-concave (cf. Def. 1.4), and it can be proved that it is piecewise log-affine. This follows essentially by the definition of the Radius (cf. (5.3.10)), and by Lemma 1.6.

6.1 Solvability over an annulus and over the Robba ring

Let $\mathrm{B}:=\mathcal{A}_K(I)$, with $I=]r_1,r_2[$, and let M be a σ_q-module (resp. a $(\sigma_q,\delta_q)-$module) on $\mathcal{A}_K(I)$. For all $c\in K$, $|c|\in I$, one has $t_{c,|c|}=t_{0,|c|}$. For all affinoid $X\subseteq\mathcal{C}(I)$ containing the disk $\mathrm{D}^-(c,|c|)$ one has $\rho_{c,X}=|c|$. Then the norm $|.|_{c,|c|}:\mathcal{A}_K(I)\longrightarrow\mathbb{R}_{\geqslant}$ and the generic radius $Ray(\mathrm{M},|.|_{(c,|c|)})$, do not depend on the choice of c or the affinoid X, but only on $|c|$. Hence, for all $\rho\in I$, we chose an arbitrary $c\in\Omega$, with $|c|=\rho\in I$, and we set

$$t_\rho:=t_{c,\rho}\ ,\quad\text{and}\qquad Ray(\mathrm{M},\rho):=Ray(\mathrm{M},|.|_{(c,\rho)})\ . \tag{6.1.1}$$

[5]In the case of the q-difference equation (5.3.8), the radius $R_{t_{c,\rho}}$ is given by definition (5.3.10). In the case of the system (5.6.1) the radius $R_{t_{c,\rho}}$ is given indifferently by definition (5.3.2) or by definition (5.3.10), indeed under our assumptions these two definitions are equal since $Y_{A(q,T)}(x,y)=Y_{G(1,T)}(x,y)$. However observe that the definition (5.3.10) exists only if $q\in\mathcal{Q}-\boldsymbol{\mu}(\mathcal{Q})$, while definition (5.3.2) preserves its meaning on the root of unity.

[6]Observe that $\rho_{c,X}=\rho_{t_{c,\rho},X}$, indeed $\mathrm{D}^+(c,r)=\mathrm{D}^+(t_{c,\rho},r)$, for all $r\geqslant\rho$.

To define the radius we need the assumption $|q-1||t_\rho| < \rho_{t_\rho,X} = \rho$ (cf. Definition 6.1). Since $|t_\rho| = \rho$, this assumption is equivalent to

$$|q-1| < 1\ . \tag{6.1.2}$$

Definition 6.3 (solvability at ρ). Let $q \in \mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$ (cf. Definition (2.0.4)). Let M be a σ_q−module on $\mathcal{A}_K(I)$. We will say that M is solvable at $\rho \in I$ if

$$Ray(\mathrm{M},\rho) = \rho\ . \tag{6.1.3}$$

6.1.1 *Solvability over $\mathcal{R}_K$ or $\mathcal{H}_K^\dagger$.* Let $q \in \mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$. Let M be a σ_q−module over $\mathcal{R}_K$. By definition M comes, by scalar extension, from a module $\mathrm{M}_{\varepsilon_1}$ defined on an annulus $\mathcal{C}(]1-\varepsilon_1,1[)$. If $\varepsilon_2 > 0$, and if $\mathrm{M}_{\varepsilon_2}$ is another module on $\mathcal{C}(]1-\varepsilon_2,1[)$ satisfying $\mathrm{M}_{\varepsilon_2} \otimes_{\mathcal{A}_K(]1-\varepsilon_2,1[)} \mathcal{R}_K \xrightarrow{\sim} \mathrm{M}$, then there exists a $\varepsilon_3 \leqslant \min(\varepsilon_1,\varepsilon_2)$ such that

$$\mathrm{M}_{\varepsilon_1} \otimes \mathcal{A}_K(]1-\varepsilon_3,1[) \xrightarrow{\ \sim\ } \mathrm{M}_{\varepsilon_2} \otimes \mathcal{A}_K(]1-\varepsilon_3,1[)\ . \tag{6.1.4}$$

Hence the limit $\lim_{\rho\to 1^-} Ray(\mathrm{M}_\varepsilon,\rho)$ is independent of the choice of the module M_ε.

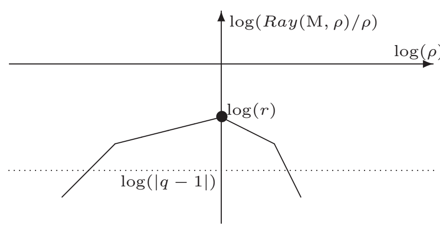
Definition 6.4. Let $q \in \mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$, and let $|q-1| < r \leqslant 1$. We define

$$\sigma_q - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{[r]}\ , \tag{6.1.5}$$

as the full sub category of $\sigma_q - \mathrm{Mod}(\mathcal{H}_K^\dagger)$ whose objects satisfy

$$Ray(\mathrm{M},1) \geqslant r\ , \qquad (r > |q-1|)\ , \tag{6.1.6}$$

as illustrated below in the log-graphic of the function $\log(\rho) \mapsto \log(Ray(\mathrm{M},\rho)/\rho)$ (cf. Def. 1.4):

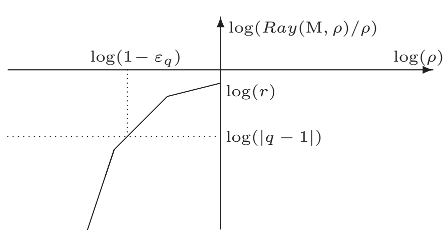


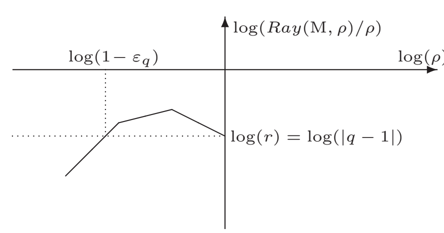
Objects in $\sigma_q - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{[1]}$ will be called *solvable*.

Definition 6.5. Let $q \in \mathcal{Q}_1 - \boldsymbol{\mu}(\mathcal{Q}_1)$, and let $|q-1| \leqslant r \leqslant 1$. We define

$$\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{[r]}\ , \tag{6.1.7}$$

as the full sub category of $\sigma_q - \mathrm{Mod}(\mathcal{R}_K)$ formed by objects M satisfying $\lim_{\rho\to 1^-} Ray(\mathrm{M},\rho) \geqslant r$, and there exists $\varepsilon_q > 0$ such that $Ray(\mathrm{M},\rho) > |q-1|$, for all $\rho \in]1-\varepsilon_q,1[$. There are two possible cases $r > |q-1|$, and $r = |q-1|$, as illustrated in the following pictures:





Objects in $\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{[1]}$ will be called *solvable*.

Remark 6.6. Notice that in definition 6.4 the existence of $\varepsilon_q > 0$ such that $Ray(\mathrm{M},\rho) > |q-1|$, for all $\rho \in]1-\varepsilon_q, 1+\varepsilon_q[$ is automatically verified since one assumes $r > |q-1|$.

6.1.2 *Analogous definitions for (σ_q,δ_q)-modules.* In the case of (σ_q,δ_q)-modules, the generic radius of convergence is defined even if q is a root of unity. We give then analogous definitions of $(\sigma_q,\delta_q)-\mathrm{Mod}(\mathrm{B})^{[r]}$, for $\mathrm{B}:=\mathcal{R}_K$ or $\mathrm{B}:=\mathcal{H}_K^\dagger$, without any restrictions on q.

6.2 Generic radius for discrete and analytic objects over $\mathcal{R}_K$ and $\mathcal{H}_K^\dagger$

In this section $\mathrm{B}=\mathcal{R}_K$ or $\mathrm{B}=\mathcal{H}_K^\dagger$.

Definition 6.7. For all $\varepsilon>0$ let

$$I_\varepsilon := \begin{cases}]\,1-\varepsilon\,,\;1\,[\,, & \text{if}\quad \mathrm{B}=\mathcal{R}_K \\ \\ ]\,1-\varepsilon\,,\;1+\varepsilon\,[\,, & \text{if}\quad \mathrm{B}=\mathcal{H}_K^\dagger \end{cases}\;. \tag{6.2.1}$$

Definition 6.8. For all subset $S\subseteq \mathrm{D}^-(1,1)=\mathcal{Q}_1$, for all $0<\tau<1$, we set

$$S_\tau := S\cap \mathrm{D}^-(1,\tau)\;. \tag{6.2.2}$$

Definition 6.9. Let $0<r\leqslant 1$. Let $S\subseteq \mathrm{D}^-(1,1)$, $S^\circ\neq\emptyset$. We denote by

$$\sigma-\mathrm{Mod}(\mathrm{B})_S^{[r]}\;, \tag{6.2.3}$$

the full subcategory of $\sigma-\mathrm{Mod}(\mathrm{B})_S$ whose objects M have the following properties:

i) The restriction of M to every $q\in S$ belongs to $\sigma_q-\mathrm{Mod}(\mathrm{B})^{[r]}$;

ii) For all τ such that $0<\tau<r$, there exists $\varepsilon_\tau>0$ such that the restriction $\mathrm{Res}^{\langle S\rangle}_{\langle S_\tau\rangle}(\mathrm{M})$ comes, by scalar extension, from an object

$$\mathrm{M}_{\varepsilon_\tau}\in\;\; \sigma-\mathrm{Mod}(\mathcal{A}_K(I_{\varepsilon_\tau}))^{\mathrm{disc}}_{S_\tau} \tag{6.2.4}$$

such that, for all $\rho\in I_{\varepsilon_\tau}$, and for all $q,q'\in S_\tau$ one has (cf. (5.3.9))

$$Y_{A(q,T)}(T,t_\rho)=Y_{A(q',T)}(T,t_\rho)\;. \tag{6.2.5}$$

Objects in $\sigma_q-\mathrm{Mod}(\mathrm{B})^{[1]}_S$ will be called *solvable*.

Example 6.10. This example justifies the condition i) given in the preceding definition. Let $r:=\omega:=|p|^{\frac{1}{p-1}}$, and let $S=\mathrm{D}^-(1,\omega)$. Let M be the discrete σ-module over the Robba ring defined by the family of equations $\{\ \sigma_q-A(q,T)\ \}_{q\in S}$, where $A(q,T):=\exp((q^{-1}-1)T^{-1})$. Then $Y(x,y):=\exp(x^{-1}-y^{-1})$ is the simultaneous solution of every equation of this family. Observe that $A(q,T)\in\mathcal{R}_K$ if and only if $|q^{-1}-1|<\omega$, but if $|q-1|$ tends to ω^-, then *the matrices $A(q,T)$ do not all belong to the same annulus*. Indeed $A(q,T)\in\mathcal{A}_K(I_\varepsilon)$ if and only if $|q^{-1}-1|<\omega(1-\varepsilon)$.

Remark 6.11. Condition i) implicitly implies that $S\subseteq\mathrm{D}^-(1,r)$ if $\mathrm{B}=\mathcal{H}_K^\dagger$ (cf. Def. 6.4), and $S\subseteq\mathrm{D}^+(1,r)$ if $\mathrm{B}=\mathcal{R}_K$ (cf Def. 6.5).

6.2.1 *Analogous definitions for (σ_q,δ_q)-modules.* One defines analogously $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})^{[r]}_S$, *but without restrictions on $S\subseteq\mathrm{D}^-(1,r)$*, as the subcategory of $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S$, whose objects verify conditions i) and ii), in which equation (6.2.5) is replaced by (cf. Definitions (5.3.1) and (5.3.9))

$$Y_{G(1,T)}(T,t_\rho)=Y_{A(q,T)}(T,t_\rho)\;, \tag{6.2.6}$$

for all $\rho\in I_{\varepsilon_\tau}$, and all $q\in S_\tau$.

7. The Propagation Theorem

7.1 Taylor admissible modules

Definition 7.1 (Taylor admissible discrete modules on S). Let $X := \mathrm{D}^+(c_0,R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i,R_i)$ be an affinoid, and let $S \subseteq \mathcal{Q}_1(X)$, be a subset with $S^\circ \neq \emptyset$ (cf. (2.0.5)). Let $(\mathrm{M},\sigma^{\mathrm{M}})$ be a discrete σ-module defined by the family of equations

$$\{\sigma_q - A(q,T)\}_{q\in S}\ ,\quad A(q,T)\in GL_n(\mathcal{H}_K(X))\ ,\ \forall\, q\in S\ . \tag{7.1.1}$$

We will say that $(\mathrm{M},\sigma^{\mathrm{M}})$ is *Taylor admissible on X, with generic radius greater than r*, if :

(1) One has $S \subseteq \mathrm{D}^-(1, r/\max(|c_0|,R_0))$;

(2) there exists a matrix $Y(x,y)$, convergent in $\mathcal{U}_R$ (cf. (5.3.3)), with $R\geqslant r$ satisfying, for all $q\in S$, the condition (5.5.1), that is

$$r\ \leqslant\ R\ \leqslant\ r_X\ ; \tag{7.1.2}$$

(3) $Y(x,y)$ is simultaneous solution of every equation of the family (7.1.1).

The full subcategory of $\sigma - \mathrm{Mod}(\mathcal{H}_K(X))_S^{\mathrm{disc}}$ whose objects are Taylor admissible, with generic radius greater than r, will be denoted by

$$\sigma - \mathrm{Mod}(\mathcal{H}_K(X))_S^{[r]}\ . \tag{7.1.3}$$

Moreover we set

$$\sigma - \mathrm{Mod}(\mathcal{H}_K(X))_S^{\mathrm{adm}}\ :=\ \bigcup_r \sigma - \mathrm{Mod}(\mathcal{H}_K(X))_S^{[r]}\ \ . \tag{7.1.4}$$

where $r\leqslant r_X$ runs in the set of real numbers such that $S\subseteq \mathrm{D}^-(1,r/\max(|c_0|,R_0))$. We define analogously the categories $(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K(X))_S^{[r]}$ and $(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K(X))_S^{\mathrm{adm}}$ of *admissible $(\sigma,\delta)-$modules on S*. Namely the condition $S^\circ\neq\emptyset$ is suppressed, and if $(\mathrm{M},\sigma^{\mathrm{M}},\delta_1^{\mathrm{M}})$ is a discrete (σ,δ)-modules on S defined by a system of equations (cf. (3.2.4)), then the Taylor solution $Y_{G(1,T)}(x,y)$ (cf. (5.3.1)) of the differential equation defined by δ_1^{M} satisfies (7.1.2), and moreover is simultaneously solution of every equation defined by σ_q^{M}, for all $q\in S$.

7.1.1 *Taylor Admissibility over $\mathcal{H}_K^\dagger(X)$.* We define

$$\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger(X))_S^{[r]}\ ,\quad (\text{resp. } (\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K^\dagger(X))_S^{[r]}) \tag{7.1.5}$$

as the full subcategory of $\sigma-\mathrm{Mod}(\mathcal{H}_K^\dagger(X))_S$ (resp. $(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K^\dagger(X))_S$) formed by objects whose restriction belongs to $\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_S^{[r]}$ (resp. $(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K(X))_S^{[r]}$).

Remark 7.2. If $X=\{|T|=1\}$, $\mathcal{H}_K^\dagger(X)=\mathcal{H}_K^\dagger$ (cf. (1.2.4)), this definition is equivalent to Def. 6.9.

7.1.2 *Taylor admissibility over $\mathcal{R}_K$.* We preserve the notations of section 6.2.

Definition 7.3. We will say that an object is Taylor admissible over an annulus $\mathcal{C}(I)$ if its restriction to every sub-annulus $\mathcal{C}(J)$, with J compact, $J\subseteq I$, is Taylor admissible (cf. Definition 7.1).

One defines Taylor admissibility over $\mathcal{R}_K$ by reducing to the case of modules over a single annulus $\mathcal{C}(I_\varepsilon)$, for some $\varepsilon>0$ sufficiently close to 0. One finds in this way exactly the Definition 6.9:

Definition 7.4. Let $S\subseteq \mathrm{D}^-(1,1)$, with $S^\circ\neq\emptyset$. Let $\tau_S := \sup_{q\in S}|q-1|$. We set

$$\sigma - \mathrm{Mod}(\mathcal{R}_K)_S^{\mathrm{adm}}\ :=\ \sigma - \mathrm{Mod}(\mathcal{R}_K)_S^{[\tau_S]}\ . \tag{7.1.6}$$

We give the same definition for (σ,δ)-modules, without assuming that “ $S^\circ\neq\emptyset$ ” : $(\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_K)_S^{\mathrm{adm}} := (\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_K)_S^{[\tau_S]}$.

7.2 Propagation Theorem

Remark 7.5. We preserve notations of Definition 7.1. If M is Taylor admissible on X, then, in particular, M is trivialized by $\mathcal{A}_K(c,R)$, for all $c\in X(K)$. Hence we can apply C-Deformation and C-Confluence to M, with $\mathrm{C}=\mathcal{A}_K(c,R)$ (cf. section 4.2). It will follows from the proof of Theorem 7.7, that this confluence does not depend on the chosen point $c\in X(K)$.

Theorem 7.6 (Propagation Theorem first form). *Let X be an affinoid. Then, if $q\in\mathcal{Q}_1(X)-\boldsymbol{\mu}(\mathcal{Q}_1(X))$, the natural restriction functor*

$$\bigcup_U \mathrm{Res}_q^U \;:\; \bigcup_U \sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}} \longrightarrow \sigma_q-\mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}} \tag{7.2.1}$$

is an equivalence, where U runs over the set of all open neighborhood of q. The analogous fact is true for (σ,δ)-modules without supposing $q\notin\boldsymbol{\mu}(\mathcal{Q})$.

Proof. By Lemma 4.3, $\cup_U\mathrm{Res}_{\{q\}}^U$ is fully faithful. Indeed for all modules M, N over U, by admissibility, there exists a number R, with $|q-1|\max(|c_0|,R_0)<R\leqslant r_X$, such that, for all $c\in X(K)$, the algebra $\mathrm{C}:=\mathcal{A}_K(c,R)$ trivializes both M and N. The essential surjectivity of $\cup_U\mathrm{Res}_{\{q\}}^U$ will follow from Theorem 7.7 below. □

Theorem 7.7 (Propagation Theorem second form). *Let $X=\mathrm{D}^+(c_0,R_0)-\cup_{i=1}^n\mathrm{D}^-(c_i,R_i)$. Let $q\in\mathcal{Q}_1(X)-\boldsymbol{\mu}(\mathcal{Q}_1(X))$. Let*

$$Y(q\cdot T)=A(T)\cdot Y(T)\;,\quad A(T)\in GL_n(\mathcal{H}_K(X)) \tag{7.2.2}$$

be a Taylor admissible $q-$difference equation (cf. Def. 7.1). Then there exists a matrix $A(Q,T)$ uniquely determined by the following properties:

i) *$A(Q,T)$ is analytic and invertible in the domain*

$$\mathrm{D}^-\Big(1\;,\;\frac{R}{\max(|c_0|,R_0)}\Big)\times X\quad\subset\;\mathbb{A}_K^2\;, \tag{7.2.3}$$

ii) *The matrix $A(Q,T)$ specialized at (q,T) is equal to $A(T)$,*

iii) *For all $q'\in\mathrm{D}^-(1,R/\max(|c_0|,R_0))$, the Taylor solution matrix $Y_A(x,y)$ of the equation (7.2.2) (cf. (5.3.9)) simultaneously satisfies*

$$Y_A(q'\cdot T,y)\;=\;A(q',T)\cdot Y_A(T,y)\;\;. \tag{7.2.4}$$

Moreover the matrix $A(Q,T)$ is independent of the choice of solution $Y_A(x,y)$.

Proof. By equation (7.2.4), the matrix $A(Q,T)$ must be equal to

$$A(Q,T)=Y_A(Q\cdot T,y)\cdot Y_A(T,y)^{-1}=Y_A(Q\cdot T,y)\cdot Y_A(y,T)=Y_A(Q\cdot T,T)\;. \tag{7.2.5}$$

This makes sense since $Y_{A(q,T)}(x,y)$ is invertible in its domain of convergence (cf. Lemma 5.16). Hence $A(Q,T)$ converges in the domain of convergence of $Y_A(QT,T)$ and is invertible in that domain, since $Y_A(x,y)$ is. By admissibility, there exists $|q-1|\max(|c_0|,R_0)<R\leqslant r_X$ such that $Y_A(x,y)$ converges for all $(x,y)\in\mathcal{U}_R$, i.e. for all (x,y) such that $|x-y|<R$ (cf. (5.3.3)). Then $Y_A(QT,T)$ converges for $|Q-1||T|<R$. Since $|T|\leqslant\sup_{c\in A}|c|=\max(|c_0|,R_0)$, it follows that $Y(QT,T)$ converges for $|Q-1|<R/\max(|c_0|,R_0)$. □

Remark 7.8. By the propagation Theorem, every object of $\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}}$ and of $(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}}$ is automatically analytic.

Corollary 7.9. *Let $\max(|c_0|,R_0)<r\leqslant r_X$, and let $S\subseteq\mathrm{D}^-(1,r/\max(|c_0|,R_0))$, such that $S^\circ\neq\emptyset$.*

For all $q \in S^\circ$ one has the following diagram in which all functors are equivalences by Remark 4.2.2:

$$\begin{array}{ccc} \sigma - \mathrm{Mod}(\mathcal{H}_K(X))_S^{[r]} & \overset{(2.4.3)}{=\!=\!=} & (\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K(X))_S^{[r]} \\ \mathrm{Res}^S_{\{q\}} \downarrow \wr & \odot & \wr \downarrow \mathrm{Res}^S_{\{q\}} \\ \sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} & \underset{\text{Forget } \delta_q}{\overset{\sim}{\longleftarrow}} & (\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} . \end{array} \tag{7.2.6}$$

By considering the union for all r (cf. Equation (7.1.4)) one has the following statement. If $\tau_q := |q-1|\max(|c_0|, R_0)$, one then has the equivalences:

$$\begin{array}{ccc} \bigcup_{r>\tau_q} \sigma - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_{\mathrm{D}^-(1,r)} & \overset{(2.4.3)}{=\!=\!=} & \bigcup_{r>\tau_q} (\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_{\mathrm{D}^-(1,r)} \\ \bigcup_{r>\tau_q} \mathrm{Res}^{\mathrm{D}^-(1,r)}_{\{q\}} \downarrow \wr & \odot & \wr \downarrow \bigcup_{r>\tau_q} \mathrm{Res}^{\mathrm{D}^-(1,r)}_{\{q\}} \\ \sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}} & \underset{\text{Forget } \delta_q}{\overset{\sim}{\longleftarrow}} & (\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}} . \end{array} \tag{7.2.7}$$

In particular, if $q, q' \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$ verify $\max(|q-1|, |q'-1|)\max(|c_0|, R_0) < r$, then, by the formalism introduced in Section 4.2, if $\mathrm{D} := \mathrm{D}^-(1, r/\max(|c_0|, R_0))$, one has an equivalence:

$$\mathrm{Res}^{\mathrm{D}}_{q'} \circ (\mathrm{Res}^{\mathrm{D}}_q)^{-1} \;:\; \sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\sim} \sigma_{q'} - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} . \tag{7.2.8}$$

The same statement holds for (σ,δ)-modules without assuming $q, q' \notin \boldsymbol{\mu}_{p^\infty}$. □

Definition 7.10. In the notation of Corollary 7.9 (cf. Equation (7.2.8)), if $q, q' \notin \boldsymbol{\mu}_{p^\infty}$, we set

$$\mathrm{Def}^{\mathrm{Tay}}_{q,q'} := \mathrm{Res}^{\mathrm{D}}_{q'} \circ (\mathrm{Res}^{\mathrm{D}}_q)^{-1} \;:\; \sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\sim} \sigma_{q'} - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} . \tag{7.2.9}$$

We denote again by $\mathrm{Def}^{\mathrm{Tay}}_{q,q'}$, without assuming $q, q' \in \boldsymbol{\mu}_{p^\infty}$, the analogous functor for (σ,δ)-modules. Moreover, if $q \notin \mathcal{Q}(X) - \boldsymbol{\mu}_{p^\infty}$, then we set

$$\mathrm{Conf}^{\mathrm{Tay}}_q := \mathrm{Def}^{\mathrm{Tay}}_{q,1} \circ (\text{Forget } \delta_q)^{-1} \;:\; \sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\sim} \delta_1 - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} . \tag{7.2.10}$$

By remark 7.5, the functor $\mathrm{Conf}^{\mathrm{Tay}}_q : (\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\sim} \sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]}$ of diagram (7.2.6) coincides with $\mathrm{Conf}^{\mathrm{C}}_q$ (cf. Definition 4.6), where C is equal to $\mathcal{A}_K(c,r)$, where r is as in the corollary 7.9, and where $c \in X(K)$ is arbitrarily chosen.

7.2.1 *Root of unity.* If $q \in \boldsymbol{\mu}_{p^\infty}$, then the categories $\sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]}_S$ and $\sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}_S$ are not defined. In this case we cannot expect any equivalence between $(\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}$ with a full subcategory of $\sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))$ because the first category is K-linear and the second is not. In this case we will see in Proposition 8.6 that the functor "Forget δ_q" is not very interesting since it sends every $(\sigma_q,\delta_q)-$module with Frobenius structure into the trivial σ_q-module (i.e., a direct sum of the copies of the unit object).

7.2.2 Starting from a Taylor *admissible* σ_q-module M over B, one can *compute* the differential equation $\mathrm{Conf}^{\mathrm{Tay}}_q(\mathrm{M}) \in \delta_1 - \mathrm{Mod}(\mathrm{B})$ by the relation

$$G(1,T) = \lim_{q\to 1} \frac{A(q,T) - \mathrm{Id}}{q-1} = \lim_{n\to+\infty} \frac{A(q^{p^n},T) - \mathrm{Id}}{q^{p^n} - 1} , \tag{7.2.11}$$

where $A(q^{p^n},T) = A(q, q^{p^n-1}T)A(q, q^{p^n-2}T)\cdots A(q,T)$. The propagation theorem provides the convergence of this limit in $M_n(\mathrm{B})$; The reader may have the feeling that this limit should be easy to compute, but (without introducing the Taylor solution) the convergence of this limit and its explicit computation are *highly non trivial facts*. It is surprising to see that the admissibility condition, which is not a strong assumption, actually implies such a deep fact.

Remark 7.11. It should be possible to generalize the main theorem to other kind of operators, different from σ_q. In other words it should be possible to "deform" differential equations into "$\sigma-$difference equations", where σ in an automorphism different from σ_q, but sufficiently close to the identity. In a future work we will study the action of a $p-$adic Lie group on differential equations.

7.3 Extending the Confluence Functor to the case $|q-1|=|q|=1$

Let $q\in\mathcal{Q}(X)-\boldsymbol{\mu}(\mathcal{Q}(X))$ be such that $q^{k_0}\in\mathcal{Q}_1(X)$, for some $k_0\geqslant 1$.[7] By composing with the evident functor

$$\sigma_q-\mathrm{Mod}(\mathcal{H}_K(X))\longrightarrow\sigma_{q^{k_0}}-\mathrm{Mod}(\mathcal{H}_K(X))\;, \tag{7.3.1}$$

one defines "k_0-*Taylor admissible objects*" of $\sigma_q-\mathrm{Mod}(\mathcal{H}_K(X))$ as objects whose image is Taylor admissible in $\sigma_{q^{k_0}}-\mathrm{Mod}(\mathcal{H}_K(X))$. Since the sequence $\{q^{k_0p^n}\}_{n\geqslant 0}$ tends to 1, then, for k_0 sufficiently large, q^{k_0} satisfies the condition of section 5.2, in order that d_{k^0} verifies equality (5.2.1). We obtain then a Confluence Functor:

$$\sigma_q-\mathrm{Mod}(\mathcal{H}_K(X))^{k_0-\mathrm{adm}}\longrightarrow\delta_1-\mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}\;. \tag{7.3.2}$$

The converse of this fact (i.e. the deformation of a differential equation into a $q-$difference equation with $|q|=1$ and $|q-1|$ large) remains an open problem.

Remark 7.12. Notice that there exist equations in $\sigma_q-\mathrm{Mod}(\mathcal{H}_K(X))$ which are not k_0-Taylor admissible, for all $k_0\geqslant 1$. For example consider the rank one equation σ_q-a, with $a\in K$, $|a|>1$. Suppose also that $|q-1|<|p|^{\frac{1}{p-1}}$, in order that $\liminf_n|[n]_q^!|^{1/n}=|p|^{\frac{1}{p-1}}$. Then the radius is small and one can compute it explicitly by applying [DV04, Prop.4.6]. One has $Ray((\mathrm{M},\sigma_q^{\mathrm{M}}),\rho)=|a|^{-1}|p|^{\frac{1}{p-1}}|q-1|\rho<|q-1|\rho$, and $Ray((\mathrm{M},\sigma_{q^{k_0}}^{\mathrm{M}}),\rho)=|a|^{-k_0}|p|^{\frac{1}{p-1}}|q^{k_0}-1|\rho<|q^{k_0}-1|\rho$.

7.4 Propagation Theorem over $\mathcal{H}_K^\dagger$ and $\mathcal{R}_K$

The Propagation Theorem is true over every base ring B appearing in this paper, up to a correct definition for the notion of "Taylor admissible". We state here the results for $\mathcal{H}_K^\dagger$ and $\mathcal{R}_K$.

Proposition 7.13. *Let again* $\mathrm{B}:=\mathcal{H}_K^\dagger$, *or* $\mathrm{B}:=\mathcal{R}_K$, *let* $0<r\leqslant 1$, *and let* $S\subseteq\mathrm{D}^-(1,r)$, *be a subset, with* $S^\circ\neq\emptyset$. *Let* $\mathrm{M}\in\sigma-\mathrm{Mod}(\mathrm{B})_S^{[r]}$ *(i.e. in particular* M *is admissible). Then* M *is the restriction to* S *of an* analytically C-constant *module over all the disk* $\mathrm{D}^-(1,r)$. *Moreover, the restriction functor is an equivalence:*

$$\sigma-\mathrm{Mod}(\mathrm{B})_{\mathrm{D}^-(1,r)}^{[r]}\xrightarrow[\sim]{\mathrm{Res}_S^{\mathrm{D}^-(1,r)}}\sigma-\mathrm{Mod}(\mathrm{B})_S^{[r]}\;. \tag{7.4.1}$$

In particular solvable modules extend to the whole disk $\mathrm{D}^-(1,1)$. *The analogous assertion holds for* $(\sigma,\delta)-$*modules, without supposing that* $S^\circ\neq\emptyset$*:*

$$(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_{\mathrm{D}^-(1,r)}^{[r]}\xrightarrow[\sim]{\mathrm{Res}_S^{\mathrm{D}^-(1,r)}}(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S^{[r]}\;. \tag{7.4.2}$$

Proof. By Lemma 4.3, it suffices to prove the essential surjectivity of $\mathrm{Res}_S^{\mathrm{D}^-(1,r)}$. The proof is straightforward and essentially the same as the proof of the Propagation Theorem 7.6. □

[7] For an annulus centered at 0, the condition $q^{k_0}\in\mathcal{Q}_1(A)=\mathrm{D}^-(1,1)$ is equivalent to $\bar{q}\in\mathbb{F}_p^{\mathrm{alg}}$.

COROLLARY 7.14. *Let $q, q' \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$. Let $r \in \mathbb{R}$ satisfy*

$$\max(|q-1|, |q'-1|) < r \leqslant 1 \ . \tag{7.4.3}$$

Then one has an equivalence

$$\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \xrightarrow[\sim]{\mathrm{Def}^{\mathrm{Tay}}_{q,q'}} \sigma_{q'} - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \ . \tag{7.4.4}$$

The same equivalence holds between $(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{R}_K)^{[r]}$ and $(\sigma_{q'}, \delta_{q'}) - \mathrm{Mod}(\mathcal{R}_K)^{[r]}$, without assuming $q \notin \boldsymbol{\mu}_{p^\infty}$. Moreover, if $q \notin \boldsymbol{\mu}_{p^\infty}$, and if $|q-1| < r$, then we have an equivalence

$$(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{R}_K)^{[r]} \xrightarrow[\sim]{\text{“Forget } \delta_q\text{”}} \sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{[r]}. \tag{7.4.5}$$

As usual we set $\mathrm{Conf}^{\mathrm{Tay}}_q := \mathrm{Def}^{\mathrm{Tay}} \circ (\text{Forget } \delta_q)^{-1}$. The analogous statement holds for $\mathcal{H}^\dagger_K$. □

7.4.1 *Unipotent equations* We shall compute the deformation $\mathrm{Def}^{\mathrm{Tay}}_{1,q}$ of the differential module U_m defined by the equation

$$\delta_1(Y_{U_m}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \cdot Y_{U_m} \ , \qquad Y_{U_m}(x,y) = \begin{pmatrix} 1 & \ell_1 & \cdots & \ell_{m-2} & \ell_{m-1} \\ 0 & 1 & \ell_1 & \cdots & \ell_{m-2} \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & 1 & \ell_1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \ , \tag{7.4.6}$$

where $\ell_n := [\log(x) - \log(y)]^n/n!$. One has $\sigma_q^x(\ell_n(x,y)) = [\log(qx) - \log(y)]^n/n! = (\log(q) + \log(x) - \log(y))^n/n! = \sum_{i=0}^n \frac{\log(q)^{n-k}}{(n-k)!} \cdot \ell_k$. The matrix of $\sigma_q^{U_m}$ is then

$$A(q,T) = \begin{pmatrix} 1 & \log(q) & \frac{\log(q)^2}{2} & \cdots & \frac{\log(q)^{m-1}}{(m-1)!} \\ 0 & 1 & \log(q) & \cdots & \frac{\log(q)^{m-2}}{(m-2)!} \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & \log(q) \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} \ . \tag{7.4.7}$$

7.5 Classification of solvable rank one $q-$difference equations over $\mathcal{R}_{K_\infty}$

In this section we classify rank one solvable $q-$difference equations over $\mathcal{R}_{K_\infty}$ by applying the deformation $\mathrm{Def}^{\mathrm{Tay}}_{1,q}$ to the classification of the differential equations obtained in [Pul07]. We recall the classification of the rank one solvable differential equations over $\mathcal{R}_{K_\infty} := \cup_{s\geqslant 0}\mathcal{R}_{K_s}$ (see below).

We fix a Lubin-Tate group $\mathfrak{G}_P$ isomorphic to $\widehat{\mathbb{G}}_m$ over $\mathbb{Z}_p$. We recall that $\mathfrak{G}_P$ is defined by an uniformizer w of $\mathbb{Z}_p$, and by a series $P(X) \in X\mathbb{Z}_p[[X]]$, satisfying $P(X) \equiv \mathrm{w} \cdot X \pmod{X^2\mathbb{Z}_p[[X]]}$ and $P(X) \equiv X^p \pmod{p\mathbb{Z}_p[[X]]}$. Such a formal series is called a *Lubin-Tate series*. We fix now a sequence $\boldsymbol{\pi} := (\pi_m)_{m\geqslant 0}$, $\pi_m \in \mathbb{Q}_p^{\mathrm{alg}}$, such that $P(\pi_0) = 0$, $\pi_0 \neq 0$ and $P(\pi_{m+1}) = \pi_m$, for all $m \geqslant 0$. The element $(\pi_m)_{m\geqslant 0}$ is a generator of the Tate module of $\mathfrak{G}_P$ which is a free rank one $\mathbb{Z}_p-$module. We set $K_s := K(\pi_s)$ and $K_\infty := \cup_{s\geqslant 0} K_s$. We denote by k_s and k_∞ the respective residual fields. The tower $K \subseteq K_0 \subseteq K_1 \subseteq \ldots$ does not depend n the choice of $\boldsymbol{\pi}$, nor on $\mathfrak{G}_P \cong \widehat{\mathbb{G}}_m$. One has $K_s = K(\xi_s)$, where ξ_s is a primitive p^{s+1}-th root of unity. For example, one can choose $\mathfrak{G}_P = \mathbb{G}_m$, hence $P(X) = (X+1)^p - 1$, and $\pi_m = \xi_m - 1$, where ξ_m is a compatible sequence of primitive $p^{m+1}-$th root of 1, i.e. $\xi_0^p = 1$ and $\xi_m^p = \xi_{m-1}$, for all $m \geqslant 0$. One has the following facts:

i) Every rank one *solvable* differential module over $\mathcal{R}_K$ has a basis in which the associated operator is

$$\mathrm{L}(a_0, \boldsymbol{f}^-(T)) \ := \ \delta_1 - \Big(a_0 - \sum_{j=0}^{s} \pi_{s-j} \sum_{i=0}^{j} f_i^-(T)^{p^{j-i}} \partial_{T,\log}(f_i^-(T))\Big) \ , \tag{7.5.1}$$

where $a_0 \in \mathbb{Z}_p$, and $\boldsymbol{f}^-(T) := (f_0^-(T),\dots,f_s^-(T))$ is a Witt vector in $\mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$, with $K_s := K(\pi_s)$. Notice that even if π_j does not belong to K, the resulting polynomial $\sum_{j=0}^{s}\pi_{s-j}\sum_{i=0}^{j} f_i^-(T)^{p^{j-i}}\partial_{T,\log}(f_i^-(T))$ has, by assumption, coefficients in K.

ii) The Taylor solution at ∞ of the differential module in this basis is given by the so called $\boldsymbol{\pi}$-exponential attached to $\boldsymbol{f}^-(T)$:

$$T^{a_0}\cdot \mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1) \ := \ T^{a_0}\cdot\exp\Bigl(\sum_{j=0}^{s}\pi_{s-j}\frac{\phi_j^-(T)}{p^j}\Bigr)\ , \tag{7.5.2}$$

where $\langle\phi_0^-(T),\dots,\phi_s^-(T)\rangle \in (T^{-1}\mathcal{O}_{K_s}[T^{-1}])^{s+1}$ is the phantom vector of $\boldsymbol{f}^-(T)$, namely one has $\phi_j^-(T) = \sum_{i=0}^{j} p^i f_i^-(T)^{p^{j-i}}$.

iii) The correspondence $\boldsymbol{f}^-(T)\mapsto \mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ is a group morphism

$$\mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}]) \xrightarrow{\mathrm{e}_{p^s}(-,1)} 1+\pi_s T^{-1}\mathcal{O}_{K_s}[[T^{-1}]]\ . \tag{7.5.3}$$

Notice that if $L(0,\boldsymbol{f}^-(T))$ has its coefficients in $\mathcal{R}_K$ $(\subset \mathcal{R}_{K_s})$ then also $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ lies in $1+T^{-1}\mathcal{O}_K[[T^{-1}]]$ (because it is its Taylor solution at ∞).

iv) Conversely, $\mathrm{L}(a_0,\boldsymbol{f}^-(T))$ is solvable for all pairs $(a_0,\boldsymbol{f}^-(T)) \in \mathbb{Z}_p\times\mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$.

v) The operator $\mathrm{L}(a_0,\boldsymbol{f}^-(T))$ has a (strong) Frobenius structure (cf. Def. 8.5) if and only if $a_0 \in \mathbb{Z}_{(p)} := \mathbb{Z}_p\cap\mathbb{Q}$.

vi) The operators $\mathrm{L}(a_0,\boldsymbol{f}_1^-(T))$ and $\mathrm{L}(b_0,\boldsymbol{f}_2^-(T))$ (with coefficients in $\mathcal{R}_K(\subset\mathcal{R}_{K_s})$) define isomorphic differential modules (over $\mathcal{R}_K$) if and only if $a_0-b_0\in\mathbb{Z}$ and the Artin-Schreier equation

$$\bar{\mathrm{F}}(\overline{\boldsymbol{g}^-(T)}) - \overline{\boldsymbol{g}^-(T)} = \overline{\boldsymbol{f}_1^-(T)-\boldsymbol{f}_2^-(T)} \tag{7.5.4}$$

has a solution $\overline{\boldsymbol{g}^-(T)}$ in $\mathbf{W}_s(k^{\mathrm{alg}}((t)))$, where t is the reduction of T, and $\bar{\mathrm{F}}$ is the Frobenius of $\mathbf{W}_s(k^{\mathrm{alg}}((t)))$ (sending $(\bar{g}_0,\dots,\bar{g}_s)$ into $(\bar{g}_0^p,\dots,\bar{g}_s^p)$). This happens if and only if the equation $L(0,\boldsymbol{f}_1^-(T)-\boldsymbol{f}_2^-(T))$ is trivial over $\mathcal{R}_K$, and also if and only if $\mathrm{e}_{p^s}(\boldsymbol{f}_1^-(T)-\boldsymbol{f}_2^-(T),1)$ is overconvergent.[8]

By deformation, every solvable q-difference equation, with $|q-1|<1$, has a solution at ∞ of the form $T^{a_0}\cdot\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$. Its matrix in this basis is then

$$A(q,T) = \mathrm{e}_{p^s}(\boldsymbol{f}^-(qT),1)/\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1) = \mathrm{e}_{p^s}(\boldsymbol{f}^-(qT)-\boldsymbol{f}^-(T),1)\ .$$

The deformation guarantees that $A(q,T)\in\mathcal{R}_K$. This is confirmed by the fact that $\boldsymbol{f}^-(qT)$ and $\boldsymbol{f}^-(T)$ have the same reduction in $\mathbf{W}_s(k^{\mathrm{alg}}((t)))$, and hence $\mathrm{e}_{p^s}(\boldsymbol{f}^-(qT)-\boldsymbol{f}^-(T),1)\in\mathcal{R}_K$ by point vi) of the previous classification.

8. Quasi unipotence and $p-$adic local monodromy theorem

In this section we show how to deduce the q-analogue of the p-adic local monodromy theorem (cf. [And02], [Ked04], [Meb02]) by deformation.

Let K be a complete discrete valued field with perfect residue field (this hypothesis is necessary to have the p-adic local monodromy theorem). Let $\mathcal{E}_K^\dagger\subset\mathcal{R}_K$ be the so called *bounded Robba ring*, $\mathcal{E}_K^\dagger := \{\sum_{i\in\mathbb{Z}} a_iT^i\in\mathcal{R}_K \mid \sup|a_i|<+\infty, \lim_{i\to-\infty}|a_i|=0\}$. Then, since K is discrete valued, $(\mathcal{E}_K^\dagger,|\cdot|_{(0,1)})$ is a *Henselian* valued field, with residue field $k((t))$. It has two topologies arising from

[8] Indeed the overconvergence of $\mathrm{e}_{p^s}(\boldsymbol{f}_1^-(T)-\boldsymbol{f}_2^-(T),1)$ is independent on the residual field, for this reason we can look for solution of the Artin-Schreier-Witt equation (7.5.4) with coefficients in the more general field k^{alg} instead of k.

$|\cdot|_{(0,1)}$, and from the inclusion in $\mathcal{R}_K$. It is not complete with respect to none of these two topologies, but $\mathcal{E}_K^\dagger$ is dense in $\mathcal{R}_K$. One has the inclusions

$$\mathcal{H}_K^\dagger \;\subset\; \mathcal{E}_K^\dagger \;\subset\; \mathcal{R}_K \;. \tag{8.0.5}$$

8.1 Frobenius Functor and Frobenius Structure

Let $\varphi : K \to K$ be an absolute Frobenius (i.e. a ring morphism lifting of the $p-$th power map of k). Since $\mathcal{R}_K$ is not a local ring, and does not have a residue ring, we need a special definition:

Definition 8.1. An absolute *Frobenius* on $\mathcal{R}_K$ (resp. $\mathcal{H}_K^\dagger$, $\mathcal{E}_K^\dagger$) is a continuous ring morphism, again denoted by $\varphi : \mathcal{R}_K \to \mathcal{R}_K$, extending φ on K and such that $\varphi(\sum a_iT^i) = \sum \varphi(a_i)\varphi(T)^i$, where $\varphi(T) = \sum_{i\in\mathbb{Z}} b_iT^i \in \mathcal{R}_K$ (resp. $\varphi(T) \in \mathcal{H}_K^\dagger$, $\varphi(T) \in \mathcal{E}_K^\dagger$) verifies $|b_i| < 1$, for all $i \neq p$, and $|b_p - 1| < 1$.

Definition 8.2. We denote by ϕ the particular absolute Frobenius on $\mathcal{R}_K$ given by the choice

$$\phi(T) := T^p \;, \qquad \phi(f(T)) := f^{\varphi}(T^p) \;. \tag{8.1.1}$$

where $f^{\varphi}(T)$ is the series obtained from $f(T)$ by applying $\varphi : K \to K$ on the coefficients.

Let B be one of the rings $\mathcal{H}_K^\dagger$, $\mathcal{E}_K^\dagger$, or $\mathcal{R}_K$. For all $q \in \mathrm{D}^-(1,1)$, the following diagrams are commutative

$$\begin{array}{ccc} \mathrm{B} & \xrightarrow{\phi} & \mathrm{B} \\ {\scriptstyle\sigma_{q^p}}\downarrow & \odot & \downarrow{\scriptstyle\sigma_q} \\ \mathrm{B} & \xrightarrow[\phi]{} & \mathrm{B} \end{array} \quad ; \quad \begin{array}{ccc} \mathrm{B} & \xrightarrow{\phi} & \mathrm{B} \\ {\scriptstyle p\cdot\delta_1}\downarrow & \odot & \downarrow{\scriptstyle\delta_1} \\ \mathrm{B} & \xrightarrow[\phi]{} & \mathrm{B} \end{array} \quad . \tag{8.1.2}$$

Definition 8.3 (Frobenius functor). Let $S \subseteq \mathrm{D}^-(1,r)$, $0 < r \leqslant 1$. Let

$$r' := \min(\; r^{1/p} \;,\; r\cdot|p|^{-1} \;) \;. \tag{8.1.3}$$

The Frobenius functor (cf. def. 6.9)

$$\phi^* :(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[r]} \longrightarrow (\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[r']} \;, \tag{8.1.4}$$

$$(\text{resp. } \phi^* : \quad \sigma - \mathrm{Mod}(\mathrm{B})_S^{[r]} \longrightarrow \quad \sigma - \mathrm{Mod}(\mathrm{B})_S^{[r']} \;) \tag{8.1.5}$$

is defined as $\phi^*(\mathrm{M},\sigma^{\mathrm{M}},\delta_1^{\mathrm{M}}) = (\;\phi^*(\mathrm{M})\;,\;\sigma^{\phi^*(\mathrm{M})}\;,\;\delta_1^{\phi^*(\mathrm{M})}\;)$, where

i) $\phi^*(\mathrm{M}) := \mathrm{M}\otimes_{\mathrm{B},\phi}\mathrm{B}$ is the scalar extension of M via ϕ,

ii) the morphism $\sigma^{\phi^*(\mathrm{M})}$ is given by $\sigma_q^{\phi^*(\mathrm{M})} = \sigma_{q^p}^{\mathrm{M}} \otimes \sigma_q^{\mathrm{B}}$:

$$q \longmapsto \sigma_{q^p}^{\mathrm{M}} \otimes \sigma_q \quad : \quad S \xrightarrow{\sigma^{\phi^*(\mathrm{M})}} \mathrm{Aut}_K^{\mathrm{cont}}(\phi^*(\mathrm{M})) \;, \tag{8.1.6}$$

iii) the derivation is given by

$$\delta_1^{\phi^*(\mathrm{M})} = (p\cdot\delta_1^{\mathrm{M}})\otimes\mathrm{Id}_{\mathrm{B}} + \mathrm{Id}_{\mathrm{M}}\otimes\delta_1^{\mathrm{B}} \;, \tag{8.1.7}$$

iv) a morphism $\alpha : \mathrm{M}\to\mathrm{N}$ is sent into $\alpha\otimes 1 : \phi^*(\mathrm{M}) \to \phi^*(\mathrm{N})$.

Remark 8.4. The fact that the functor ϕ^* sends $(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[r]}$ into $(\sigma,\delta) - \mathrm{Mod}(\mathrm{B})_S^{[r']}$ with this particular value of r' (cf. Equation (8.1.3)) results from the fact that this result is true for *differential* equations (cf. [Pul05, Appendix], and [CM02b, Prop.7.2]), and from the confluence.

8.1.1 We observe that the pull-back $\varphi^*(\mathrm{M})$ is actually a σ-module over $S^{1/p} := \{q \in K \mid q^p \in S\}$. Indeed $\phi^*(\mathrm{M})$ is canonically endowed with the action of $\sigma_{q^{1/p}}^{\phi^*(\mathrm{M})} := \sigma_q^{\mathrm{M}} \otimes \sigma_{q^{1/p}} : \phi^*(\mathrm{M}) \to \phi^*(\mathrm{M})$, for all roots $q^{1/p}$ of q. This fact was used in [ADV04] to define the so called confluent weak Frobenius structure (cf. Definition 8.27).

If $\mathrm{M} \in (\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{[r]}$, then we can consider its Taylor solution at 1: $Y(T,1) = \sum_{i\geqslant 0} Y_i(T-1)^i \in GL_n(\mathcal{A}_K(1,1))$, $Y_i \in M_n(K)$. Then the Taylor solution of $\phi^*(\mathrm{M})$ is given by

$$Y^\phi(T^p,1) \;:=\; \sum_{i\geqslant 0} \varphi(Y_i)(T^p-1)^i \;. \tag{8.1.8}$$

The matrices of $\phi^*(\sigma_q)$ and $\phi^*(\delta_1)$ are the following. Let $\mathbf{e} = \{\mathrm{e}_1,\ldots,\mathrm{e}_n\}$ be a basis of M. Let $\sigma_q - A(q,T)$ and $\delta_1 - G(1,T)$ be the operators associated to σ_q^{M} and δ_1^{M} in this basis. Then the operators associated to $\phi^*(\mathrm{M})$ in the basis $\mathbf{e}\otimes 1$ are

$$\sigma_q - A^\varphi(q^p,T^p)\;, \qquad \delta_1 - p\cdot G^\varphi(1,T^p)\;, \tag{8.1.9}$$

where, according with (2.1.7), one has $A(q^p,T) = A(q,q^{p-1}T)\cdots A(q,qT)A(q,T)$.

8.1.2 *Frobenius Structure.* The functor $\phi^* : \delta_1-\mathrm{Mod}(\mathcal{R}_K)^{[1]} \xrightarrow{\sim} \delta_1-\mathrm{Mod}(\mathcal{R}_K)^{[1]}$ is an equivalence (cf. [CM02b, Cor.8.14]). By deformation ϕ^* is hence an auto-equivalence of $\sigma-\mathrm{Mod}(\mathcal{R}_K)_S^{[1]}$ (if $S^\circ \neq \emptyset$) and $(\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_K)_S^{[1]}$ (without assuming $S^\circ\neq\emptyset$).

Definition 8.5 (Frobenius structure). Let B be one of the rings $\mathcal{H}_K^\dagger$, $\mathcal{E}_K^\dagger$, or $\mathcal{R}_K$. Let $S \subseteq \mathrm{D}^-(1,1)$ be a subset. Let M be a discrete $\sigma-$module (resp. $(\sigma,\delta)-$module) over S. We will say that M has a *Frobenius structure of order* $h\geqslant 1$, if there exists an B-isomorphism $(\phi^*)^h(\mathrm{M}) \xrightarrow{\sim} \mathrm{M}$ of σ-modules over S (cf. Section 8.1.1), where $(\phi^*)^h := \phi^*\circ\cdots\circ\phi^*$, $h-$times. We denote by

$$\sigma-\mathrm{Mod}(\mathrm{B})_S^{(\phi)}\;, \qquad (\text{resp.}\quad (\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S^{(\phi)}\;) \tag{8.1.10}$$

the full subcategory of $\sigma-\mathrm{Mod}(\mathrm{B})_S^{[1]}$ (resp. $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S^{[1]}$) whose objects have a Frobenius structure of some order.

If M has a Frobenius structure, then $r = r'$ (cf. (8.1.3)) and hence M is *solvable*:

$$\sigma-\mathrm{Mod}(\mathrm{B})_S^{(\phi)} \;\subset\; \sigma-\mathrm{Mod}(\mathrm{B})_S^{[1]}\;. \tag{8.1.11}$$

Hence objects in $\sigma-\mathrm{Mod}(\mathrm{B})_S^{(\phi)}$ and $(\sigma,\delta)-\mathrm{Mod}(\mathrm{B})_S^{(\phi)}$ are, in particular, *admissible*.

If $Y(T,1)$ is the Taylor solution of $\mathrm{M}\in(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{[1]}$ at 1, then the fact that M has a Frobenius structure of some order $h\geqslant 1$, is equivalent to the existence of a matrix $H(T)\in GL_n(\mathcal{H}_K^\dagger)$ such that

$$Y^{\varphi^h}(T^{p^h},1) = H(T)\cdot Y(T,1)\;. \tag{8.1.12}$$

Indeed $\mathcal{A}_K(1,1)$ is a $\mathcal{H}_K^\dagger$-discrete σ-algebra over $\mathrm{D}^-(1,1)$ trivializing M (cf. Def.3.2). In particular the equivalences $\mathrm{Def}_{q,q'}^{\mathrm{Tay}}$ and $\mathrm{Conf}_q^{\mathrm{Tay}}$ send objects with Frobenius structure into objects with Frobenius structure.

Proposition 8.6. *Let ξ be a p^n-th root of unity, and let $q\in\mathcal{Q}_1-\boldsymbol{\mu}(\mathcal{Q}_1)$. Let $\mathrm{M}\in\sigma_q-\mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$. Then $\mathrm{Def}_{q,\xi}^{\mathrm{Tay}}(\mathrm{M})$ is trivial (i.e. isomorphic to a direct sum of copies of the unit object).*

Proof. Let $Y(T,1)\in GL_n(\mathcal{H}_K^\dagger)$ be the Taylor solution at 1 of M in some basis $\mathbf{e}$. Then, by (8.1.12), there exists $H(T)$ such that $Y^{\varphi^h}(T^{p^h},1) = H(T)\cdot Y(T,1)$. Hence, one also has $Y^{\varphi^{nh}}(T^{p^{nh}},1) = H_n(T)\cdot Y(T,1)$, for some $H_n(T)\in GL_n(\mathcal{H}_K^\dagger)$. Since $\sigma_\xi(Y^{\varphi^{nh}}(T^{p^{nh}})) = Y^{\varphi^{nh}}(T^{p^{nh}})$, it follows that in the basis $H_n(T)\cdot\mathbf{e}$ the matrix of σ_ξ is trivial: $A(\xi,T)=\mathrm{Id}$ (cf. Section 3.2.1). □

8.2 Special coverings of $\mathcal{H}_K^\dagger$

We recall briefly the notions of *special coverings*. The residue field of $\mathcal{E}_K^\dagger$ is $k(\!(t)\!)$ (with respect to the norm $|.|_{(0,1)}$). On the other hand, the residue ring of $\mathcal{H}_K^\dagger$ (with respect to the Gauss norm $|.|_{(0,1)}$) is $k[t,t^{-1}]$. One has

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{H}_K^\dagger} & \subseteq & \mathcal{O}_{\mathcal{E}_K^\dagger} \\ \downarrow & \odot & \downarrow \\ k[t,t^{-1}] & \subseteq & k(\!(t)\!). \end{array} \tag{8.2.1}$$

We denote by $\mathcal{O}_K[T,T^{-1}]^\dagger$ the weak completion of $\mathcal{O}_K[T,T^{-1}]$, in the sense of Monsky and Washnitzer (cf. [MW68]). One has

$$\mathcal{H}_K^\dagger = \mathcal{O}_K[T,T^{-1}]^\dagger \otimes_{\mathcal{O}_K} K\ . \tag{8.2.2}$$

Let us look at the residual situation. The morphism

$$\widehat{\eta} := \mathrm{Spec}(k(\!(t)\!)) \quad \hookrightarrow \quad \mathbb{G}_{m,k} = \mathrm{Spec}(k[t,t^{-1}]) \tag{8.2.3}$$

gives rise, by pull-back, to a map

$$\left\{ \begin{array}{l} \text{Finite Étale} \\ \text{coverings of } \widehat{\eta} \end{array} \right\} \xleftarrow{\text{Pull-back}} \left\{ \begin{array}{l} \text{Finite Étales} \\ \text{coverings of } \mathbb{G}_{m,k} \end{array} \right\}\ . \tag{8.2.4}$$

It is known (cf. [Kat86, 2.4.9]) that this map is surjective, and moreover that there exists a full sub-category of the right hand category, called *special coverings of* $\mathbb{G}_{m,k}$, which is equivalent, via pull-back, to the category on the left hand side. Special coverings are defined by the property that they are tamely ramified at ∞, and that their geometric Galois group has a unique $p-$Sylow subgroup (cf. [Kat86, 1.3.1]).

On the other hand, if $\pi \in \mathcal{O}_K$ is a uniformizing element, then both $(\mathcal{O}_{\mathcal{E}_K^\dagger}, (\pi))$ and $(\mathcal{O}_K[T,T^{-1}]^\dagger, (\pi))$ are Henselian couples in the sense of [Ray70, Ch.II] (cf. [Mat02, 5.1]). One can show that the preceding situation lifts to characteristic 0. One has the following equivalences:

$$\begin{array}{ccccc} \left\{ \begin{array}{l} \text{Special} \\ \text{extensions of } \mathcal{H}_K^\dagger \end{array} \right\} & \xrightarrow[\sim]{-\otimes \mathcal{E}_K^\dagger} & \left\{ \begin{array}{l} \text{Finite unramified} \\ \text{extensions of } \mathcal{E}_K^\dagger \end{array} \right\} & \xrightarrow[\sim]{-\otimes \mathcal{R}_K} & \left\{ \begin{array}{l} \text{Special} \\ \text{extensions of } \mathcal{R}_K \end{array} \right\} \\ {\scriptstyle -\otimes K}\uparrow\wr & \odot & \wr\uparrow{\scriptstyle -\otimes K} & & \\ \left\{ \begin{array}{l} \text{Special extensions} \\ \text{of } \mathcal{O}_K[T,T^{-1}]^\dagger \end{array} \right\} & \xrightarrow[\sim]{-\otimes \mathcal{O}_{\mathcal{E}_K^\dagger}} & \left\{ \begin{array}{l} \text{Finite onramified} \\ \text{extensions of } \mathcal{O}_{\mathcal{E}_K^\dagger} \end{array} \right\} & & \\ {\scriptstyle -\otimes k}\downarrow\wr & \odot & \wr\downarrow{\scriptstyle -\otimes k} & & \\ \left\{ \begin{array}{l} \text{Special} \\ \text{coverings of } \mathbb{G}_{m,k} \end{array} \right\} & \xrightarrow[\sim]{\text{Pull-back}} & \left\{ \begin{array}{l} \text{Finite étale} \\ \text{coverings of } \hat{\eta} \end{array} \right\} & & \end{array} \tag{8.2.5}$$

where, by special extension of $\mathcal{O}_K[T,T^{-1}]^\dagger$ (resp. $\mathcal{H}_K^\dagger$, $\mathcal{R}_K$) we mean a finite étale Galois extension of $\mathcal{O}_K[T,T^{-1}]^\dagger$ (resp. $\mathcal{H}_K^\dagger$, $\mathcal{R}_K$) coming, by Henselianity, from a special cover of $\mathbb{G}_{m,k}$.

Lemma 8.7. *Let* $\mathrm{F}/k(\!(t)\!)$ *be a finite Galois extension with Galois group* G*. Let* $\mathcal{S}^\dagger(\mathrm{F})/\mathcal{H}_K^\dagger$ *be the corresponding Special extension of* $\mathcal{H}_K^\dagger$*. Then* $(\mathcal{S}^\dagger(\mathrm{F}))^{\mathrm{G}} = \mathcal{H}_K^\dagger$*.*

Proof. By [SGA03, Exposé V, Cor.3.4], $(\mathcal{S}^\dagger(\mathrm{F}))^{\mathrm{G}}/\mathcal{H}_K^\dagger$ is a Special extension. The assertion is then easy since, by the above equivalence there is bijection between Special sub-algebras of $\mathcal{S}^\dagger(\mathrm{F})$ over $\mathcal{H}_K^\dagger$, and sub-extensions of $\mathrm{F}/k(\!(t)\!)$. $\square$

8.2.1 *Extension of σ_q to Special extensions.*

Lemma 8.8 [ADV04, Section 11.3]. *Let $F/k(\!(t)\!)$ be a finite separable extension. Let $\mathcal{F}^\dagger/\mathcal{O}_K[T,T^{-1}]^\dagger$ be the corresponding special extensions. The automorphism σ_q of $\mathcal{O}_K[T,T^{-1}]^\dagger$ extends to an automorphism $\mathcal{F}^\dagger$. The extension is unique up to $\mathcal{O}_K[T,T^{-1}]^\dagger$-automorphisms of $\mathcal{F}^\dagger$. The same statement holds for the extensions $(\mathcal{H}_K^\dagger)'/\mathcal{H}_K^\dagger$, $(\mathcal{E}_K^\dagger)'/\mathcal{E}_K^\dagger$, $(\mathcal{R}_K)'/\mathcal{R}_K$ corresponding to $F/k(\!(t)\!)$. In particular there exists a unique extension of σ_q to $\mathcal{F}^\dagger$, $(\mathcal{H}_K^\dagger)'$, $(\mathcal{E}_K^\dagger)'$, $(\mathcal{R}_K)'$ inducing the identity on F.*

Proof. The proof results from the formal properties of Henselian couples (cf. [Ray70]). □

By uniqueness the extension of σ_q commutes with the action of $\mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!))$.

Remark 8.9. Every finite extensions of $\mathbb{C}(\!(T)\!)$ is of the form $\mathbb{C}(\!(T^{m/n})\!)$. Up to change the variable we have an isomorphism $\mathbb{C}(\!(T^{m/n})\!) \cong \mathbb{C}(\!(Z)\!)$. Analogously it can be seen that a finite unramified extension of $\mathcal{E}_K^\dagger$ is (non canonically) isomorphic to $\mathcal{E}_{K'}^\dagger$ for some finite K'/K. In this case the link between the variable Z and the variable T is rather complicate and essentially unknown. *The great problem of the theory is that the extended automorphism does not send Z into qZ.* The general "Confluence" theory introduced in section 4 will be crucial in solving this problem.

8.3 Quasi unipotence of differential equations and canonical extension

In this section we recall some known facts on p-adic *differential* equations.

Definition 8.10. We denote by $\widetilde{\mathcal{H}_K^\dagger}$ (resp. $\widetilde{\mathcal{E}_K^\dagger}$, $\widetilde{\mathcal{R}_K}$) the union of all finite special (resp. unramified, special) extensions of $\mathcal{H}_K^\dagger$ (resp. $\mathcal{E}_K^\dagger$, $\mathcal{R}_K$) in an algebraically closure of the field of fractions of $\mathcal{R}_K$.

Definition 8.11. Let $S \subseteq \mathrm{D}^-(1,1)$ be a subset (resp. $S \subseteq \mathrm{D}^-(1,1)$, with $S^\circ \neq \emptyset$). A discrete $(\sigma,\delta)-$module on S (resp. discrete $\sigma-$module on S) is called *quasi-unipotent* if it is trivialized by the discrete $(\sigma,\delta)-$algebra

$$\widetilde{\mathcal{H}_K^\dagger}[\log(T)] \qquad (\text{resp. } \widetilde{\mathcal{E}_K^\dagger}[\log(T)]\ ,\ \widetilde{\mathcal{R}_K}[\log(T)]\ )\ . \tag{8.3.1}$$

Let $\mathrm{B} := \mathcal{H}_K^\dagger$, or $\mathcal{E}_K^\dagger$, $\mathcal{R}_K$. We observe that M is trivialized by $\widetilde{\mathrm{B}}[\log(T)]$, if and only if M is trivialized by $\mathrm{B}'[\log(T)]$, where B' is a (*finite*) special extension of B. Indeed the entries of a fundamental matrix of solutions of M in $\widetilde{\mathrm{B}}[\log(T)]$ all lie in a finite extension.

Theorem 8.12 ($p-$adic local monodromy theorem, cf. [And02],[Ked04],[Meb02]). *Objects in $\delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$ become quasi-unipotent possibly after a suitable extension of the field of constants K. In other words, if $\mathrm{M} \in \delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$, then there exists a finite extension K'/K such that $\mathrm{M} \otimes_K K'$ is quasi unipotent (i.e. trivialized by $\widetilde{\mathcal{H}_{K'}^\dagger}[\log(T)]$).* □

Theorem 8.13 ([Mat02, 7.10,7.15]). *If a differential equation $\mathrm{M} \in \delta_1 - \mathrm{Mod}(\mathcal{R}_K)$ is quasi-unipotent, then it has a Frobenius structure. Moreover, the scalar extension functor*

$$-\otimes \mathcal{R}_K\ :\ \delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)} \ \longrightarrow\ \delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)} \tag{8.3.2}$$

is essentially surjective. □

Theorem 8.14 ([Mat02, 7.15]). *There exists a full sub-category of $\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$, denoted by $\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$, which is equivalent to $\delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$ via the scalar extension functor (8.3.2). Objects in $\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$ category are trivialized by $\widetilde{\mathcal{H}_K^\dagger}[\log(T)]$.* □

Definition 8.15 (Canonical extension). Objects in $\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$ will be called *special objects.* We will denote by

$$\delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)} \xrightarrow[\sim]{\mathrm{Can}} \delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}} \ \subset\ \delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)} \tag{8.3.3}$$

the section of the functor (8.3.2), whose image is the category of special objects (cf. Theorem 8.14). We will call it the *canonical extension functor*.

Corollary 8.16 ([And02, 7.1.6]). *Let $\mathrm{M} \in \delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$, then, up to replacing K by a finite extension K'/K, M decomposes in a direct sum of submodules of the form $\mathrm{N} \otimes U_m$, where N is a module trivialized by a special extension of $\mathcal{R}_K$, and U_m is the $m-$dimensional object defined by the operator (cf. section 7.4.1)*

$$\delta_1 - \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} . \qquad \square \tag{8.3.4}$$

Remark 8.17. The $\log(T)$ appearing in (8.3.1), is added uniquely to trivialize the module of the form U_m, for $m \geqslant 2$ (cf. section 7.4.1).

Lemma 8.18. *Let $\mathrm{N} \in \delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$ be a special object trivialized by $\widetilde{\mathcal{H}_K^\dagger}$. Let $\widetilde{Y} = (\widetilde{y}_{i,j}) \in GL_n(\widetilde{\mathcal{H}_K^\dagger})$ be a fundamental matrix solution of N. Let $(\mathcal{E}^\dagger)'$ (resp. $\mathcal{R}'$) be the smallest special extension of $\mathcal{E}_K^\dagger$ (resp. $\mathcal{R}_K$), such that $\mathrm{N} \otimes \mathcal{E}_K^\dagger$ is trivialized by $(\mathcal{E}^\dagger)'$ ($\mathrm{N} \otimes \mathcal{R}_K$ is trivialized by $\mathcal{R}'$). Then one has*

$$(\mathcal{E}^\dagger)' = \mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}] , \qquad \mathcal{R}' = \mathcal{R}_K[\{\widetilde{y}_{i,j}\}_{i,j}] . \tag{8.3.5}$$

In other words, the smallest special extension of $\mathcal{E}_K^\dagger$ (resp. $\mathcal{R}_K$) trivializing M is generated by the solutions of M.

Proof. Since M is trivialized by $(\mathcal{E}^\dagger)'$, one has $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}] \subseteq (\mathcal{E}^\dagger)'$. Hence the differential field $\mathcal{E}_{K'}^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$ is an unramified extension, and is then a special extension. Since $(\mathcal{E}^\dagger)'$ is minimal, $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}] = (\mathcal{E}^\dagger)'$. The case over $\mathcal{R}_K$ follows from the case over $\mathcal{E}_K^\dagger$. $\square$

Corollary 8.19. *We preserve the notations of Lemma 8.18. There exists a unique $K-$linear ring automorphism σ_q of $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$, which induces the identity on the residue field.*

Proof. By Lemma 8.18, $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$ is a special extension (i.e. Henselian). Hence, by Section 8.2.1, the extension of σ_q to $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$ is unique. $\square$

Corollary 8.20. *Let $S \subseteq \mathrm{D}_K^-(1,1)$ The scalar extension functor*

$$- \otimes \mathcal{R}_K \;:\; (\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{(\phi)} \;\longrightarrow\; (\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)} \tag{8.3.6}$$

is essentially surjective. Moreover there exists a full sub-category of $(\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{(\phi)}$, which we call $(\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{\mathrm{Sp}}$, equivalent via $- \otimes \mathcal{R}_K$ to $(\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)}$. The same statement is true for σ-modules under the assumption $S^\circ \neq \emptyset$.

Proof. By Proposition 7.13, we can assume that $S = \mathrm{D}^-(1,1)$. By Theorem 8.13 there exists a basis of M in which the matrix $G(1,T)$ of δ_1^{M} lies in $M_n(\mathcal{H}_K^\dagger)$. Moreover, $\mathrm{Can}(\mathrm{M},\delta_1^{\mathrm{M}})$ is Taylor admissible, since all solvable differential equations are Taylor admissible. By Proposition 7.13, for all $q \in \mathrm{D}^-(1,1)$, the matrix $A(q,T) := Y_G(qT,T)$ belongs also to $GL_n(\mathcal{H}_K^\dagger)$. This proves the essential surjectivity. The fully faithfulness follows by deformation of Thm. 8.14 (cf. Cor. 7.9). $\square$

8.3.1 It is not clear to us if the smallest special extension of $\mathcal{H}_K^\dagger$ trivializing a given $\mathrm{M} \in \delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$ is generated (over $\mathcal{H}_K^\dagger$) by the entries of a fundamental matrix of solution of M. So we are obliged to give the following definition.

Definition 8.21. We denote by $\widetilde{\mathrm{C}^{\text{ét}}_K}$ the sub-algebra of $\widetilde{\mathcal{H}^\dagger_K}$ generated, over $\mathcal{H}^\dagger_K$, by the entries of every fundamental solutions matrix of each object in $\delta_1 - \mathrm{Mod}(\mathcal{H}^\dagger_K)^{\mathrm{Sp}}$ which is trivialized by $\widetilde{\mathcal{H}^\dagger_K}$.

With the notation of Corollary 8.20, the inclusions $(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}^\dagger_K)^{\mathrm{Sp}}_S \subset (\sigma,\delta)-\mathrm{Mod}(\mathcal{H}^\dagger_K)^{(\phi)}_S \subset (\sigma,\delta)-\mathrm{Mod}(\mathcal{H}^\dagger_K)^{[1]}_S$ are strict (the same holds for (σ,δ)-modules). For example the equation $\delta_1(y) = a_0 y$, with $a_0 \in \mathbb{Z}_p - \mathbb{Z}_{(p)}$, is solvable, but without Frobenius structure (cf. section 7.5). On the other hand an object with Frobenius structure could have non zero p-adic slope at 1^+ (hence irregular at ∞), hence it is not special. Unfortunately we have no examples of *non special* equations with Frobenius structure, but trivialized by $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$.

8.4 Quasi unipotence of $\sigma-$modules and $(\sigma,\delta)-$modules with Frobenius structure

This section is devoted to prove the following

Theorem 8.22 ($p-$adic local monodromy theorem (generalized form)). *Let $S \subset \mathrm{D}^-(1,1)$, be a subset (resp. $S^\circ \neq \emptyset$). Then every object $\mathrm{M} \in (\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}_S$ (resp. $\mathrm{M} \in \sigma - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}_S$) is quasi unipotent, after replacing K, if necessary, by a finite extension K'/K depending on M.*

This result simplifies, and generalizes the analogous result of [ADV04]. The proof is obtained by deformation of the p-adic local monodromy theorem of *differential* equation (cf. Theorem 8.12).

The proof is essentially the following. Assume that $S = \{q\}$, with $q \notin \boldsymbol{\mu}_{p^\infty}$. By canonical extension (cf. Cor. 8.20) M is trivialized by $\widetilde{\mathcal{R}_K}[\log(T)]$ if and only if $\mathrm{Can}(\mathrm{M})$ is trivialized by $\widetilde{\mathcal{H}^\dagger_K}[\log(T)]$ (or equivalently by $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$). Hence we can assume that $\mathrm{M} \in \sigma_q - \mathrm{Mod}(\mathcal{H}^\dagger_K)^{\mathrm{Sp}}$. Firstly apply the confluence functor to obtain a differential equation $\mathrm{Conf}^{\mathrm{Tay}}_q(\mathrm{M},\sigma^{\mathrm{M}}_q)$. We prove then in Lemma 8.23 below that $\mathrm{Conf}^{\mathrm{Tay}}_q(\mathrm{M},\sigma^{\mathrm{M}}_q)$ is $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$-extensible to $\mathrm{D}^-(1,1)$ (cf. Def. 4.4). Hence we obtain, by deformation, another q-difference module $\mathrm{Def}^{\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]}_{1,q}(\mathrm{Conf}^{\mathrm{Tay}}_q(\mathrm{M},\sigma^{\mathrm{M}}_q))$ over $\mathcal{H}^\dagger_K$ (cf. section 4). This q-difference module is quasi unipotent since, by definition, it has the same solutions in $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$ of the quasi unipotent differential equation $\mathrm{Conf}^{\mathrm{Tay}}_q(\mathrm{M},\sigma^{\mathrm{M}}_q)$. We shows then that there is an embedding $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)] \subseteq \mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ commuting with δ_1, φ, and with σ_q, for all $q \in \mathrm{D}^-(1,1)$ (cf. Lemma 8.24). This proves that the restriction of $\mathrm{Def}^{\mathrm{Tay}}_{1,q}$ to the category of objects trivialized by $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$ coincides with $\mathrm{Def}^{\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]}_{1,q}$ (cf. Sections 4.2.3,4.2.1), because $\mathrm{Def}^{\mathrm{Tay}}_{1,q} = \mathrm{Def}^{\mathrm{C}}_{1,q}$, with $\mathrm{C} = \mathcal{A}_K(1,1)$ (cf. Remark 7.5) or equivalently $\mathrm{C} = \mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ (cf. Corollary 8.26). Hence

$$\mathrm{Def}^{\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]}_{1,q}(\mathrm{Conf}^{\mathrm{Tay}}_q(\mathrm{M},\sigma^{\mathrm{M}}_q)) \;=\; \mathrm{Def}^{\mathrm{Tay}}_{1,q}(\mathrm{Conf}^{\mathrm{Tay}}_q(\mathrm{M},\sigma^{\mathrm{M}}_q)) \;=\; (\mathrm{M},\sigma^{\mathrm{M}}_q)\,. \tag{8.4.1}$$

In particular $(\mathrm{M},\sigma^{\mathrm{M}}_q)$ is trivialized by $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$ and is hence quasi unipotent.

Lemma 8.23. *Let $\mathrm{M} \in \delta_1 - \mathrm{Mod}(\mathcal{H}^\dagger_K)^{\mathrm{Sp}}$. Assume that K is sufficiently large so that M is quasi unipotent. Let $(\mathcal{H}^\dagger_K)'$ be the smallest special extension of $\mathcal{H}^\dagger_K$ such that M is trivialized by $(\mathcal{H}^\dagger_K)'[\log(T)]$. Let $\widetilde{Y} \in GL_n(\widetilde{\mathcal{H}^\dagger_K}[\log(T)])$ be a fundamental matrix solution of the differential equation M. Then there exists a finite extension K'/K such that the matrix*

$$\widetilde{A}(q,T) := \sigma_q(\widetilde{Y}) \cdot \widetilde{Y}^{-1} \tag{8.4.2}$$

belongs to $GL_n(\mathcal{H}^\dagger_{K'})$, for all $q \in \mathrm{D}^-_{K'}(1,1)$. In particular the operator σ_q acting on $\widetilde{\mathcal{E}^\dagger_K}$ stabilizes both $\widetilde{\mathcal{H}^\dagger_K}$, and $\widetilde{\mathrm{C}^{\text{ét}}_K}$, and hence M is $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$-extensible to the whole disc $\mathrm{D}^-(1,1)$ (cf. Def. 4.4).

Proof. We can suppose that $K = K'$. By Corollary 8.16, and by canonical extension (cf. definition 8.15), one can assume that $\mathrm{M} = N$, or $\mathrm{M} = U_m$, where N is trivialized by an Galois étale extension $(\mathcal{H}_K^\dagger)'$ of $\mathcal{H}_K^\dagger$, and where U_m is defined over $\mathcal{H}_K^\dagger$ as in Corollary 8.16. The case "$\mathrm{M} = U_m$" is trivial, since both the matrices of $\delta_1^{U_m}$ and of $\sigma_q^{U_m}$ can be described explicitly as in section 7.4.1. Let now $\mathrm{M} = \mathrm{N}$ (i.e., M is trivialized by $\widetilde{\mathcal{H}_K^\dagger}$). In this case the solution matrix $\widetilde{Y}$ lies in $GL_n((\mathcal{H}_K^\dagger)')$. The special extension $(\mathcal{H}_K^\dagger)'/\mathcal{H}_K^\dagger$ corresponds via the equivalence of section 8.2 to a finite Galois extension $\mathrm{F}/k((t))$. Let $\mathrm{G} := \mathrm{Gal}(\mathrm{F}/k((t)))$, then G acts on $(\mathcal{H}_K^\dagger)'$ by $\mathcal{H}_K^\dagger$-automorphisms, and moreover the fixed points under this action are exactly the elements of $\mathcal{H}_K^\dagger$ (cf. Lemma 8.7). After enlarging K, if necessary, for all $\gamma \in \mathrm{G}$, one has

$$\gamma(\widetilde{Y}) = \widetilde{Y} \cdot H_\gamma \;, \quad \text{with} \quad H_\gamma \in GL_n(K) \;. \tag{8.4.3}$$

Indeed by Lemma 8.18 the corresponding Galois extension $(\mathcal{E}_K^\dagger)'/\mathcal{E}_K^\dagger$ is generated by the entries of $\widetilde{Y}$. Hence $(\mathcal{E}_K^\dagger)'/\mathcal{E}_K^\dagger$ is a Picard-Vessiot extension of $\mathcal{E}_K^\dagger$ with *differential* Galois group G. It follows then by Picard-Vessiot theory that $H_\gamma \in GL_n(K)$ (cf. [vdPS03, Obs.1.26]). Since σ_q commutes with every $\gamma \in \mathrm{G}$ (cf. Section 8.2.1), one finds

$$\gamma(\widetilde{A}(q,T)) = \gamma(\sigma_q(\widetilde{Y}) \cdot \widetilde{Y}^{-1}) = \sigma_q(\widetilde{Y}) \cdot H_\gamma \cdot (\widetilde{Y} \cdot H_\gamma)^{-1} = \widetilde{A}(q,T) \;. \tag{8.4.4}$$

Hence $\widetilde{A}(q,T)$ belongs to $\mathcal{H}_K^\dagger$, for all $|q-1| < 1$. □

Lemma 8.24. *Let $\mathcal{A}_{K^{\mathrm{alg}}}(1,1) := \bigcup_{K'/K=\mathrm{finite}} \mathcal{A}_{K'}(1,1)$. There exists an embedding $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)] \subseteq \mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ commuting with the actions of δ_1, of φ, and of σ_q, for all $q \in \mathrm{D}^-_{K^{\mathrm{alg}}}(1,1)$. In other words $\mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ is a $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]$-$(\sigma,\delta)$-algebra over the disc $\mathrm{D}^-_{K^{\mathrm{alg}}}(1,1)$, and one has the following diagram of discrete $\mathcal{H}_K^\dagger-(\sigma,\delta)-$algebras over $\mathrm{D}^-(1,1)$:*

$$\begin{array}{ccccccccc} \mathcal{A}_{K^{\mathrm{alg}}}(1,1) & \supset & \widetilde{\mathrm{C}_K^{\text{ét}}} & \subset & \widetilde{\mathcal{H}_K^\dagger} & \subset & \widetilde{\mathcal{E}_K^\dagger} & \subset & \widetilde{\mathcal{R}_K} \\ \cup & & \cup & & & & \cup & & \cup \\ \mathcal{A}_K(1,1) & \supset & \mathcal{H}_K^\dagger & & \subset & & \mathcal{E}_K^\dagger & \subset & \mathcal{R}_K \;. \end{array} \tag{8.4.5}$$

Proof. In the following we assume K to be sufficiently large in order that every Special objects appearing in the proof is quasi unipotent. Let $\mathrm{M} \in \delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{\mathrm{Sp}}$, be a Special differential equation trivialized by $\widetilde{\mathcal{H}_K^\dagger}$. Let $(\mathrm{C}_K^{\text{ét}})'$ be the smallest sub-$\mathcal{H}_K^\dagger$-algebra of $\widetilde{\mathcal{H}_K^\dagger}$ trivializing M. By definition $(\mathrm{C}_K^{\text{ét}})'$ is generated over $\mathcal{H}_K^\dagger$ by the entries $\{\widetilde{y}_{i,j}\}_{i,j}$ of a matrix solution $\widetilde{Y}$ of M in $\widetilde{\mathcal{H}_K^\dagger}$. We consider $\mathcal{H}_K^\dagger[\log(T)]$, $(\mathrm{C}_K^{\text{ét}})'[\log(T)]$, $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]$, $\mathcal{A}_K(1,1)$ as differential algebras (we forget the actions of σ_q in a first time). We have an embedding $\mathcal{H}_K^\dagger[\log(T)] \subset \mathcal{A}_K(1,1)$ commuting with δ_1 sending the symbol $\log(T)$ into the power series $\sum_{n\geqslant 1}(-1)^{n-1}(T-1)^n/n \in \mathcal{A}_K(1,1)$. We extends this embedding to $(\mathrm{C}_K^{\text{ét}})'[\log(T)]$ as follows. Since the differential equation M has its coefficients in $\mathcal{H}_K^\dagger$ we can consider its Taylor solutions $Y(T,1)$ at the point 1. Since M is solvable, then $Y(T,1) \in GL_n(\mathcal{A}_K(1,1))$. Let now $\mathcal{F}_K := \mathrm{Frac}(\mathcal{H}_K^\dagger)$ be the field of fractions of $\mathcal{H}_K^\dagger$. Since $\mathcal{F}_K$ is a field, then (up to enlarge K) we can apply the Picard-Vessiot theory to obtain an isomorphism $\mathcal{F}_K[\{\widetilde{y}_{i,j}\}_{i,j}] \xrightarrow{\sim} \mathcal{F}_K[\{y_{i,j}(T,1)\}_{i,j}]$, sending $\widetilde{y}_{i,j}$ into $y_{i,j}(T,1)$, and commuting with δ_1. Clearly this isomorphism identifies $(\mathrm{C}_K^{\text{ét}})' = \mathcal{H}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$ with $\mathcal{H}_K^\dagger[\{y_{i,j}(T,1)\}_{i,j}] \subset \mathcal{A}_K(1,1)$. If M' is another differential equation, and if $\mathcal{H}_K^\dagger[\{\widetilde{y}'_{i,j}\}_{i,j}]$ is the corresponding Picard-Vessiot extension identified with $\mathcal{H}_K^\dagger[\{y'_{i,j}(T,1)\}_{i,j}] \subset \mathcal{A}_K(1,1)$, then the embedding corresponding to $\mathrm{M}\oplus\mathrm{M}'$ extends these two embedding since the entries of a solutions of $\mathrm{M}\oplus\mathrm{M}'$ are the families $\{\widetilde{y}_{i,j}, \widetilde{y}'_{h,k}\}_{i,j,h,k}$

and $\{y_{i,j}(T,1), y'_{h,k}(T,1)\}_{i,j,h,k}$ respectively. It is hence clear that this family of embedding are compatible, so that we obtain an embedding $\widetilde{\mathrm{C}_K^{\text{ét}}} \subseteq \mathcal{A}_K(1,1)$ commuting with δ_1, and consequently $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)] \subseteq \mathcal{A}_K(1,1)$ also commutes with δ_1. Notice that $\log(T)$ is algebraically free over $\mathcal{H}_K^\dagger$ and hence over $\widetilde{\mathrm{C}_K^{\text{ét}}}$ which is union of finite algebras over $\mathcal{H}_K^\dagger$ (cf. Lemma 8.7). We can now check that this embedding commutes with σ_q (resp. φ), by looking to its action on the entries $\{\widetilde{y}_{i,j}\}_{i,j}$, and $\{y_{i,j}(T,1)\}_{i,j}$. Hence it is enough to prove that the isomorphism $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}] \xrightarrow{\sim} \mathcal{E}_K^\dagger[\{y_{i,j}(T,1)\}_{i,j}]$ commutes with σ_q (resp. φ). Observe that, if we fix an embedding of $\mathcal{F}_K[\{y_{i,j}(T,1)\}_{i,j}]$ in an fixed algebraically closure of $\mathcal{E}_K^\dagger$, then $\mathcal{E}_K^\dagger[\{y_{i,j}(T,1)\}_{i,j}]$ is, by definition, the smallest field containing $\mathcal{E}_K^\dagger$ and $\{y_{i,j}(T,1)\}_{i,j}$. The action of $\delta_1, \sigma_q, \varphi$ are defined on $\mathcal{E}_K^\dagger[\{y_{i,j}(T,1)\}_{i,j}]$ as the extensions of $\delta_1, \sigma_q, \varphi$ on $\mathcal{F}_K[\{y_{i,j}(T,1)\}_{i,j}]$. This extension exists since $\mathcal{H}_K^\dagger[\{y_{i,j}(T,1)\}_{i,j}] \cap \mathcal{E}_K^\dagger = \mathcal{H}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}] \cap \mathcal{E}_K^\dagger = \mathcal{H}_K^\dagger$ (cf. Lemma 8.7), and since $\delta_1, \sigma_q, \varphi$ act on $Y(T,1)$ by multiplication by matrices with coefficients in $\mathcal{H}_K^\dagger$, (cf. [Bou59, Sect.6,no.1,Prop.1]). By Lemma 8.19, there exists a unique extension of σ_q to $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]$, and of course a unique extension of φ since $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}]/\mathcal{E}_K^\dagger$ is unramified. Hence the isomorphism $\mathcal{E}_K^\dagger[\{\widetilde{y}_{i,j}\}_{i,j}] \xrightarrow{\sim} \mathcal{E}_K^\dagger[\{y_{i,j}(T,1)\}_{i,j}]$ commutes with σ_q and φ. □

Remark 8.25. The same statement holds for $\mathcal{A}_K(c,1)$ instead of $\mathcal{A}_K(1,1)$, providing that $|c| = 1$, $c \in K$, and $\varphi(c) = c$.

Proof of Theorem 8.22. By Proposition 7.13, one has $(\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)} = (\sigma,\delta)-\mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{(\phi)}$, (resp. $\sigma - \mathrm{Mod}(\mathcal{R}_K)_S^{(\phi)} = \sigma - \mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{(\phi)}$). On the other hand, $(\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{(\phi)} = \sigma - \mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{(\phi)}$ (cf. (2.4.3)). Moreover, if $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$, then $(\sigma,\delta) - \mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{(\phi)} = (\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$ (resp. $\sigma - \mathrm{Mod}(\mathcal{R}_K)_{\mathrm{D}^-(1,1)}^{(\phi)} = \sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$). Hence, without loss of generality, we can assume that M is a Taylor admissible $(\sigma_q,\delta_q)-$module, with Frobenius structure. The proof follows now by the discussion after Theorem 8.22. □

Corollary 8.26. *Let $S \subset \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$ (resp. $S \subset \mathrm{D}^-(1,1)$). We have the following equalities*

$$\mathrm{Conf}_q^{\mathrm{Tay}} \overset{Rem. 7.5}{=} \mathrm{Conf}_q^{\mathcal{A}_K(1,1)} = \mathrm{Conf}_q^{\mathcal{A}_{K^{\mathrm{alg}}}(1,1)} \overset{Lemma\ 8.24}{=} \mathrm{Conf}_q^{\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]}, \tag{8.4.6}$$

where the first three equalities holds for these functors on $\sigma-\mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{[1]}$ (resp. $(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{[1]}$), while, in the last equality, one considers the restrictions of these functors to the full subcategory of $\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{[1]}$ (resp. $(\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{[1]}$) of objects trivialized by $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]$. In particular the last equality holds on $\sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{\mathrm{Sp}}$ (resp. $(\sigma,\delta) - \mathrm{Mod}(\mathcal{H}_K^\dagger)_S^{\mathrm{Sp}}$). The same relation holds for deformation functors.

Proof. By Remark 7.5 the restriction of $\mathrm{Conf}_q^{\mathrm{Tay}}$ to the category of *solvable* objects coincides with $\mathrm{Conf}_q^{\mathcal{A}_K(1,1)}$. A *solvable* object over $\mathcal{H}_K^\dagger$ is trivialized by $\mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ if and only if it is trivialized by $\mathcal{A}_K(1,1)$. Indeed both these conditions are verified if and only if its Taylor solution at 1 converges on $\mathrm{D}^-(1,1)$. Hence $\mathrm{Conf}_q^{\mathcal{A}_K(1,1)} = \mathrm{Conf}_q^{\mathcal{A}_{K^{\mathrm{alg}}}(1,1)}$ on solvable objects. Now, by Theorem 8.22, Special objects are trivialized by $\widetilde{\mathrm{C}_K^{\text{ét}}}[\log(T)]$ hence by $\mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ (cf. Lemma 8.24). □

8.5 The confluence of André-Di Vizio

In this last section we prove that the restriction of $\mathrm{Conf}_q^{\mathrm{Tay}}$ to $\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$ is isomorphic to the functor "Conf" defined in [ADV04, Section 15.1]. In all this last section $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$.

We recall that an *antecedent* of a σ_q-module M over $\mathcal{R}_K$ is a $\sigma_{q^p}-$module M_1 such that $\phi^*(\mathrm{M}_1)$ is isomorphic to M as σ_q-module. The antecedent is unique up to isomorphisms, because this fact is true for differential equations (cf. Remark 8.4). In order to preserve the notations of [ADV04], we fix a $s \geqslant 1$, and we call M_1 the s-th. antecedent of M, i.e. $\Phi : (\phi^*)^s(\mathrm{M}_1) \xrightarrow{\sim} \mathrm{M}$.

The following definition was given in [ADV04] under the assumption $|q-1| < |p|^{1/(p-1)}$. The same definition holds for $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$.

Definition 8.27 ([ADV04, 12.11]). Let $s \in \mathbb{N}_{>0}$. A *Confluent Weak Frobenius Structure* (CWFS) on a σ_q-module $\mathrm{M}_0 := (\mathrm{M}_0, \sigma_q^{\mathrm{M}_0}) \in \sigma_q - \mathrm{Mod}(\mathcal{R}_K)$ is a sequence $\{\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}\}_{m \geqslant 0}$ of $q^{p^{sm}}$-difference operators on M_0, together with a family of isomorphisms

$$\Phi_m : \big(\; (\phi^*)^s(\mathrm{M}_0) \; , \; (\phi^*)^s(\sigma_{q^{p^{s(m+1)}}}^{\mathrm{M}_{m+1}}) \; \big) \xrightarrow{\sim} (\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}_m}) \; , \tag{8.5.1}$$

of $q^{p^{sm}}-$difference modules (identifying $(M_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}_m})$ to the s-th antecedent of $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$), such that:

i) The operators $\Delta_{q^{p^{sm}}}^{\mathrm{M}_m} := (\sigma_{q^{p^{sm}}}^{\mathrm{M}_m} - \mathrm{Id}^{\mathrm{M}_0})/(q^{p^{sm}} - 1)$ converge to a derivation $\Delta^{\mathrm{M}_\infty}$ on M_0.

ii) If $\mathrm{M}_\infty := (\mathrm{M}_0, \Delta^{\mathrm{M}_\infty})$ is this differential module, then the sequence of isomorphisms (8.5.1) converges to a Frobenius isomorphism $\Phi_\infty : \phi^*(\mathrm{M}_\infty) \xrightarrow{\sim} \mathrm{M}_\infty$.

We denote by

$$\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{\mathrm{conf}(\phi)} \tag{8.5.2}$$

the category whose objects are families of operators $(\mathrm{M}_0, \{\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}\}_{m \geqslant 0})$ on M_0 admitting the existence of a family $\{\Phi_m\}_{m \geqslant 0}$ making it on a confluent weak Frobenius structure on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$. A morphism $\alpha : (\mathrm{M}_0, \{\sigma_{q^{p^m}}^{\mathrm{M}_m}\}_{m \geqslant 0}) \longrightarrow (\mathrm{N}_0, \{\sigma_{q^{p^m}}^{\mathrm{N}_m}\}_{m \geqslant 0})$ is a $\mathcal{R}_K$-linear morphisms $\alpha : \mathrm{M}_0 \to \mathrm{N}_0$ verifying simultaneously $\alpha \circ \sigma_{q^{p^{sm}}}^{\mathrm{M}_m} = \sigma_{q^{p^{sm}}}^{\mathrm{N}_m} \circ \alpha$, for all $m \geqslant 0$.

8.5.1 *Construction of CWFSs.* A q-difference module $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ admits infinitely many Confluent Weak Frobenius Structures (CWFS), even if $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ admits a (strong) Frobenius structure. Indeed if a CWFS $(\mathrm{M}_0, \{\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}\}_{m \geqslant 0}, \{\Phi_m\}_{m \geqslant 0})$ on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ is given, we give now an algorithm to produce infinitely many CWFS on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$. Let $\{\psi_m : \mathrm{M}_0 \xrightarrow{\sim} \mathrm{M}_0\}_{m \geqslant 0}$ be a sequence of $\mathcal{R}_K$-linear automorphisms of M_0 such that $\lim_m \psi_m = \mathrm{Id}^{\mathrm{M}_0}$. Define

$$\sigma_{q^{p^{sm}}}^{\mathrm{M}'_m} := \psi_m \circ \sigma_{q^{p^{sm}}}^{\mathrm{M}_m} \circ \psi_m^{-1} \; , \qquad \Phi'_m := \psi_m \circ \Phi_m \circ [\phi^*(\psi_{m+1})]^{-1} \; . \tag{8.5.3}$$

One easily checks that $(\mathrm{M}_0, \{\sigma_{q^{p^{sm}}}^{\mathrm{M}'_m}\}_m, \{\Phi'_m\}_m)$ is again a CWFS on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$. Notice that this new CWFS is not always isomorphic to the first one (even if $\psi_0 = \mathrm{Id}^{\mathrm{M}_0}$). Indeed, by definition, an isomorphism is a single arrow $\alpha : \mathrm{M}_0 \to \mathrm{M}_0$ satisfying *simultaneously* $\alpha \circ \sigma_{q^{p^{sm}}}^{\mathrm{M}_m} = \sigma_{q^{p^{sm}}}^{\mathrm{M}'_m} \circ \alpha$, for all $m \geqslant 0$. Nevertheless, since $\lim_m \psi_m = \mathrm{Id}^{\mathrm{M}_0}$, the limit differential equation is the same for all CWFS defined in this way (cf. Remark 8.29). We observe moreover that ψ_m defines an isomorphism of $q^{p^{sm}}$-difference modules between $(\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}_m})$ and $(\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}'_m})$, this agrees with the uniqueness of the antecedent by Frobenius. If $\widetilde{Y}_m$ is the solution of $(\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}_m})$ in $GL_n(\widetilde{\mathcal{R}_K}[\log(T)])$, and if $B_m(T) \in GL_n(\mathcal{R}_K)$ is the matrix of ψ_m, then the solution of $(\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}'_m})$ is given by $B_m(T)\widetilde{Y}_m$.

Remark 8.28. Assume that M_0 admits a (strong) Frobenius structure. The constancy of the solution does not follows from the preview definition. Indeed a solution of $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$, with values in C is a morphism $\alpha : \mathrm{M}_0 \to \mathrm{C}$ satisfying $\alpha \circ \sigma_q^{\mathrm{M}_0} = \sigma_q^{\mathrm{C}} \circ \alpha$ (cf. section 3). The fact that α is a solution of $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ does not implies that α commutes also with $\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}$. Indeed the data $(\mathrm{M}_0, \{\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}\}_{m \geqslant 0})$ is *not necessarily* a discrete σ-module over $S = \{q^{p^{sm}}\}_{m \geqslant 0}$, because the map $S \to \mathrm{Aut}^{\mathrm{cont}}(\mathrm{M}_0)$ sending q into $\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}$ is not supposed to have any coherency (cf. Remark 2.5,(iii)). To obtain the constancy of the solutions we need to "*rigidify*" these constructions by introducing the notion of C-*constant* σ-module (cf. Remarks 0.2, 0.1, and Example 2.6).

8.5.2 We have an evident fully faithful functor

$$\chi_q^{(\phi)} \ :\ \sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)} \ \longrightarrow\ \sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{\mathrm{conf}(\phi)} \tag{8.5.4}$$

defined by

$$\chi_q^{(\phi)}(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0}) \ :=\ (\mathrm{M}_0, \{(\sigma_q^{\mathrm{M}_0})^{p^{sm}}\}_{m\geqslant 0})\ , \tag{8.5.5}$$

where s is sufficiently large to have an isomorphism $\Phi : (\phi^*)^s(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0}) \xrightarrow{\sim} (\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$, and $\Phi_m := \Phi$, for all $m \geqslant 0$. On the other hand we have another functor (cf. [ADV04, Sect.12.3])

$$Lim_\infty^{(\phi)} \ :\ \sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{\mathrm{conf}(\phi)} \ \longrightarrow\ \delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)} \tag{8.5.6}$$

sending $(\mathrm{M}_0, \{\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}\}_{m\geqslant 0}, \{\Phi_m\}_{m\geqslant 0})$ into its limit differential equation $(\mathrm{M}_0, \Delta^{\mathrm{M}_\infty})$. We have actually

$$Lim_\infty^{(\phi)} \circ \chi_q^{(\phi)} \ =\ \mathrm{Conf}_q^{\mathrm{Tay}}\ . \tag{8.5.7}$$

Indeed if $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ has a (strong) Frobenius Structure, then $(\mathrm{M}_0, \{(\sigma_q^{\mathrm{M}_0})^{p^{sm}}\}_{m\geqslant 0})$ is a (solvable) Taylor admissible σ-module over $S := \{q^{p^{sm}}\}_{m\geqslant 0}$ (cf. Def. 7.4). Hence, by Section 7.2.2, the differential equation $\mathrm{Conf}_q^{\mathrm{Tay}}(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ is given by the limit $\Delta^{\mathrm{M}_\infty} := \lim_{m\to\infty} \Delta_{q^{p^{sm}}}^{\mathrm{M}_m}$ of definition 8.27. Moreover since the operator $\sigma_{q^{p^{sm}}}^{\mathrm{M}_m}$ is determined by the knowledge of the solutions of $(\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}_m})$ in $GL_n(\widetilde{\mathcal{R}_K}[\log(T)])$, then $\chi_q^{(\phi)}(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ is the *unique* CWFS on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ such that the fundamental matrix solution of $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ in $GL_n(\widetilde{\mathcal{R}_K}[\log(T)])$ (or equivalently its Taylor solution in $\mathcal{A}_K(1,1)$, cf. Lemma 8.24) is simultaneously solution of every $(\mathrm{M}_0, \sigma_{q^{p^{sm}}}^{\mathrm{M}_m})$.

Remark 8.29. It is not clear whether the limit differential equation $(\mathrm{M}_0, \Delta^{\mathrm{M}_\infty})$ depends on the particular CWFS on $(\mathrm{M}_0, \sigma_q^{\mathrm{M}_0})$ or, analogously, if there exists two non isomorphic q-difference modules endowed with CWFS giving the same limit differential equation. Indeed both these phenomena arise in the category $\sigma_q - \mathrm{Mod}(\mathcal{R}_K)^{\mathrm{conf}}$ defined below.

Lemma 8.30. *If K is algebraically closed, then the functor $\chi_q^{(\phi)}$ is isomorphic to the functor $D_{\sigma_q}^{\mathrm{conf}(\phi)} \circ V_{\sigma_q}^{(\phi)}$ of [ADV04, Cor. 14.8]. Hence the functor $\mathrm{Conf}_q^{\mathrm{Tay}} (\overset{(8.5.7)}{=} Lim_\infty^{(\phi)} \circ \chi_q^{(\phi)})$ is isomorphic to the confluence functor $\mathrm{Conf} := Lim_\infty^{(\phi)} \circ D_{\sigma_q}^{\mathrm{conf}(\phi)} \circ V_{\sigma_q}^{(\phi)}$ as it was defined in [ADV04, Section 15.1].*

Proof. As explained in the introduction, $V_{\sigma_q}^{(\phi)}(\mathrm{M}, \sigma_q^{\mathrm{M}})$ (resp. $V_d^{(\phi)}(\mathrm{M}, \delta_1^{\mathrm{M}})$) is the (dual of the) space of solutions of $(\mathrm{M}, \sigma_q^{\mathrm{M}})$ (resp. $(\mathrm{M}, \delta_1^{\mathrm{M}})$) in $\widetilde{\mathcal{R}_K}[\log(T)]$. By definition $D_d^{(\phi)} \circ V_d^{(\phi)} \cong \mathrm{Id}$, and $D_{\sigma_q}^{(\phi)} \circ V_{\sigma_q}^{(\phi)} \cong \mathrm{Id}$. Then $D_d^{(\phi)} \circ V_{\sigma_q}^{(\phi)} = \mathrm{Conf}_q^{\widetilde{\mathcal{R}_K}[\log(T)]}$ is the functor sending $(\mathrm{M}, \sigma_q^{\mathrm{M}})$ into the differential equation having the same solutions in $\widetilde{\mathcal{R}_K}[\log(T)]$. By definition (cf. [ADV04, Prop. 12.17]) one has $D_{\sigma_q}^{\mathrm{conf}(\phi)} \cong \chi_q^{(\phi)} \circ D_{\sigma_q}^{(\phi)}$. This proves that the functor Conf of [ADV04] is equal to $\mathrm{Conf}_q^{\widetilde{\mathcal{R}_K}[\log(T)]}$. By Corollary 8.26 we conclude. □

8.5.3 Lemma 8.30 clarifies the nature of the functor Conf of [ADV04] (cf. Corollary 8.26). Indeed Conf is equal to $\mathrm{Conf}_q^{\mathrm{Tay}}$, and sends a q-difference equation into the differential equation having the same Taylor solutions (or equivalently having the same "étale" solutions in $\widetilde{\mathcal{R}_K}[\log(T)]$, cf. Lemma 8.24 and Corollary 8.26). This functor actually does not depend on the existence of a Frobenius Structure and exists in the more general context of *admissible* modules. This generalizes the constructions of [ADV04] to all $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$, removing also the assumption "$K = K^{\mathrm{alg}}$". Notice that the equivalence provided by the Propagation Theorem requires only the definition and the formal properties of the Taylor solution $Y(x,y)$. For this reason the equivalences $\mathrm{Conf}_q^{\mathrm{Tay}}$ and $\mathrm{Def}_{q,q'}^{\mathrm{Tay}}$ are not a consequence of the heretofore developed theory. Conversely our Confluence implies the main results of [ADV04] and also of [DV04].

8.5.4 *A conjecture of [ADV04].* Section 8.5.1 proves that the fully faithful functor $\chi_q^{(\phi)}$ is not an equivalence. This answer to a question asked in [ADV04, Corollary 14.8, and after]. Nevertheless observe that the existence of a CWFS on $(\mathrm{M}_0,\sigma_q^{\mathrm{M}_0})$ is equivalent to the existence of a *strong* Frobenius structure on it. This have been firstly proved for rank one equations (cf. [ADV04, Prop.7.3], the case with rational coefficient follows actually from section 7.5, indeed every rank one equation with rational exponent has a (strong) Frobenius Structure). The general case is proved as follows.

Definition 8.31. We define

$$\sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{\mathrm{conf}} \tag{8.5.8}$$

as the category whose objects are $\mathcal{R}_K$-modules M together with a family of σ_q-semi-linear automorphisms $\{\sigma_{q^{p^{sm}}}^{\mathrm{M}}:\mathrm{M}\xrightarrow{\sim}\mathrm{M}\}_{m\geqslant 0}$ (without any condition of compatibility) such that the limit

$$\delta_1^{\mathrm{M}}:=\lim_{m\to\infty}\frac{\sigma_{q^{p^{sm}}}^{\mathrm{M}}-\mathrm{Id}}{q^{p^{sm}}-1} \tag{8.5.9}$$

converges to a connection δ_1^{M} on M. Morphisms between $(\mathrm{M},\{\sigma_{q^{p^{sm}}}^{\mathrm{M}}\}_{m\geqslant 0})$ and $(\mathrm{N},\{\sigma_{q^{p^{sm}}}^{\mathrm{N}}\}_{m\geqslant 0})$ are $\mathcal{R}_K$-linear morphisms $\alpha:\mathrm{M}\to\mathrm{N}$ satisfying simultaneously $\alpha\circ\sigma_{q^{p^{sm}}}^{\mathrm{M}}=\sigma_{q^{p^{sm}}}^{\mathrm{N}}\circ\alpha$, for all $m\geqslant 0$.

Remark 8.32. We have a functor

$$\mathit{Lim}_\infty\ :\ \sigma_q-\mathrm{Mod}(\mathcal{R}_K)\ \longrightarrow\ \delta_1-\mathrm{Mod}(\mathcal{R}_K) \tag{8.5.10}$$

sending $(\mathrm{M},\{\sigma_{q^{p^{sm}}}^{\mathrm{M}}\}_{m\geqslant 0})$ into its limit differential equation. Indeed if $\alpha:\mathrm{M}\to\mathrm{N}$ satisfies simultaneously $\alpha\circ\sigma_{q^{p^{sm}}}^{\mathrm{M}}=\sigma_{q^{p^{sm}}}^{\mathrm{N}}\circ\alpha$, for all $m\geqslant 0$, then, by passing to the limit, one has $\alpha\circ\delta_1^{\mathrm{M}}=\delta_1^{\mathrm{N}}\circ\alpha$. We have then the following commutative diagram of categories:

$$\begin{array}{ccccc}
\sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{[r]} & \xrightarrow{\chi_q} & \sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{\mathrm{conf}} & \xrightarrow{\mathit{Lim}_\infty} & \delta_1-\mathrm{Mod}(\mathcal{R}_K) \\
\cup & \odot & \cup & \odot & \cup \\
\sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{(\phi)} & \xrightarrow[\chi_q^{(\phi)}]{} & \sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{\mathrm{conf}(\phi)} & \xrightarrow[\mathit{Lim}_\infty^{(\phi)}]{} & \delta_1-\mathrm{Mod}(\mathcal{R}_K)^{(\phi)},
\end{array} \tag{8.5.11}$$

where $r\geqslant|q-1|$, and where χ_q sends $(\mathrm{M},\sigma_q^{\mathrm{M}})$ into $(\mathrm{M},\{(\sigma_q^{\mathrm{M}})^{p^{sm}}\}_{m\geqslant 0})$. By Section 7.2.2, as above

$$\mathit{Lim}_\infty\circ\chi_q=\mathrm{Conf}_q^{\mathrm{Tay}}\ :\ \sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{[r]}\ \xrightarrow{\sim}\ \delta_1-\mathrm{Mod}(\mathcal{R}_K)^{[r]}\ \subset\ \delta_1-\mathrm{Mod}(\mathcal{R}_K)\ . \tag{8.5.12}$$

Corollary 8.33. *Let $q\in\mathrm{D}^-(1,1)-\boldsymbol{\mu}_{p^\infty}$. Let $(\mathrm{M},\sigma_q^{\mathrm{M}})\in\sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{[r]}$, with $r\geqslant|q-1|$. Then $(\mathrm{M},\sigma_q^{\mathrm{M}})$ admits a CWFS if and only if it admits a (strong) Frobenius structure.*

Proof. Assume that $\mathit{Lim}_\infty\circ\chi_q(\mathrm{M},\sigma_q^{\mathrm{M}})$ lies in $\delta_1-\mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$. By (8.1.12), $\mathrm{Def}_{1,q}^{\mathrm{Tay}}\circ\mathit{Lim}_\infty\circ\chi_q(\mathrm{M},\sigma_q^{\mathrm{M}})$ lies in $\sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$. Now since $\mathrm{Def}_{1,q}^{\mathrm{Tay}}\circ\mathit{Lim}_\infty\circ\chi_q=\mathrm{Def}_{1,q}^{\mathrm{Tay}}\circ\mathrm{Conf}_q^{\mathrm{Tay}}=\mathrm{Id}$, then $\mathrm{Def}_{1,q}^{\mathrm{Tay}}\circ\mathit{Lim}_\infty\circ\chi_q(\mathrm{M},\sigma_q^{\mathrm{M}})$ is isomorphic to $(\mathrm{M},\sigma_q^{\mathrm{M}})$, and has hence (strong) Frobenius structure. □

8.6 The theory of slopes

In a sequence of papers, G.Christol and Z.Mebkhout developed a theory of slopes for p-adic differential equations over the Robba ring. We summarize the main properties in the following theorem.

Theorem 8.34 (cf. [CM02a]). *Let* M *be a solvable differential module over* $\mathcal{R}_K$. *There exists a unique decomposition of* M, *called* break decomposition

$$\mathrm{M} = \oplus_{x\in\mathbb{R}_{\geqslant 0}}\mathrm{M}(x)\ , \tag{8.6.1}$$

satisfying the following properties. Let t_ρ *be a generic point for the norm* $|\cdot|_\rho$ *(cf. (6.1.1)), then there exists* $\varepsilon > 0$ *such that*

i) *For all* $\rho \in]1-\varepsilon, 1[$, $\mathrm{M}(x)$ *is (the biggest submodule of* M*) trivialized by* $\mathcal{A}_K(t_\rho, \rho^{x+1})$,

ii) *For all* $\rho \in]1-\varepsilon, 1[$, *and for all* $y < x$, $\mathrm{M}(x)$ *has no solutions in* $\mathcal{A}_K(t_\rho, \rho^{y+1})$.

The number $\mathrm{Irr}(\mathrm{M}) := \sum_{x\geqslant 0} x \cdot \mathrm{rank}_{\mathcal{R}_K}(\mathrm{M}(x))$ *is called* p-adic irregularity *of* M, *and it lies in* $\mathbb{N}$.

The fact that Irr(M) is integer is known as the *Hasse-Arf property.* This theorem has an analogous in the theory of representations of the Galois group of a local field:

Proposition 8.35 (cf. [Kat88]). *Let* $\mathcal{I}$, $\mathcal{P}$ *be the inertia and the wild inertia subgroups of* $\mathrm{G} := \mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t)))$. *Denote by* $\{\mathcal{I}^{(x)}\}_{x\geqslant 0}$ *the "upper numbering filtration" of* $\mathcal{I}$. *Let* V *be a* $\mathbb{Z}[1/p]$-*representation of* G, *such that* $\mathcal{P}$ *acts through a finite discrete quotient. Then* V *admits a* break decomposition $\mathrm{V} = \oplus_{x\geqslant 0}\mathrm{V}(x)$ *of* G-*submodules* $\mathrm{V}(x)$ *such that* $\mathrm{V}(0) = \mathrm{V}^{\mathcal{P}}$, *and for all* $x > 0$*:*

i) $(\mathrm{V}(x))^{\mathcal{I}^{(x)}} = 0$*;*

ii) *For all* $y > x$, $(\mathrm{V}(x))^{\mathcal{I}^{(y)}} = \mathrm{V}(x)$.

The number Swan$(\mathrm{V}) := \sum_{x\geqslant 0} \mathrm{rank}_{\mathbb{Z}[1/p]}\mathrm{V}(x)$ *is called* Swan conductor *of* V, *and it lies in* $\mathbb{N}$.

For an very inspiring overview about this analogy we refer to [And04].

Different authors (cf. [Tsu98b], [Mat02], [Cre00]) proved that the equivalence functor introduced by J.-M.Fontaine (cf. [Fon90], [Tsu98a]), associating to a finite representation of G, a (φ, ∇)-module over $\mathcal{E}_K^\dagger$ (and hence a differential module over $\mathcal{R}_K$) preserves the break decompositions. *The Swan conductor of a representation equals the Irregularity of the corresponding differential equation.*

In [And02] the author state a family of axiomatic conditions in a general Tannakian category in order to have a "*theory of slopes*". The previous two cases respect the formalism of [And02].

In a second time he conjectured (cf. [And04, Conjecture 4.2]) that a similar theory of slopes, should exists also for $\sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$ and ask if this "new" theory of slopes is compatible with that of Christol-Mebkhout on $\delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$ (via the confluence), and hence with the ramification theory on $\underline{\mathrm{Rep}}_{K^{\mathrm{alg}}}(\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a)$ (via the Fontaine's functor T_1 of the introduction). He suggested to proceed in analogy with the theory of Christol-Mebkhout (cf. [CM02a]), reproducing their proofs in the context of q-difference equations in order to obtain a statement analogous to theorem 8.34. Finally he asked whether this "new" theory of slopes on $\sigma_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is compatible or not with the theory of slopes of Christol-Mebkhout in $\delta_1 - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ via the equivalence Conf that he obtained in [ADV04] for $|q-1| < |p|^{\frac{1}{p-1}}$.

Afterwards, at the end of 2005, he actually obtained such a theory of slopes for $\sigma_q-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$, with $|q-1| < |p|^{\frac{1}{p-1}}$, and established the two corollaries below in this case. These verifications will be included in a fort coming paper of Y.André. This part have been exposed by Y.André at the 24-*th Nordic and* 1-*st Franco-Nordic Congress of Mathematicians* (6 to 9 January 2006, Rejkyavik, Iceland).

The next corollaries prove the above conjecture in the more general context of σ-modules. We prove it for all $|q-1| < 1$, without any assumptions about the Frobenius structure, and without assuming $K = K^{\mathrm{alg}}$. The equivalence established by Corollary 7.14 gives in fact the following analogous of Theorem 8.34 for $\sigma-$modules and (σ, δ)-modules. Thank to Proposition 7.13, without loss of generality, we can reduce this statement to the case $S = \{q\}$:

Corollary 8.36. *Let* $|q-1|<1$, $q\in K$, *(resp.* $q\notin\boldsymbol{\mu}_{p^\infty}$*). Let* $\mathrm{M}\in(\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{R}_K)^{[1]}$ *(resp.* $\mathrm{M}\in\sigma_q-\mathrm{Mod}(\mathcal{R}_K)^{[1]}$*). Then* M *admits a break decomposition* $\mathrm{M}=\oplus_{x\geqslant 0}\mathrm{M}(x)$*, where* $\mathrm{M}(x)$ *is characterized by the following properties (analogues to i) and ii) of Theorem 8.34). There exists* $\varepsilon>0$ *such that*

i) *For all* $\rho\in]1-\varepsilon,1[$, $\mathrm{M}(x)$ *is (the biggest submodule of* M*) trivialized by* $\mathcal{A}_K(t_\rho,\rho^{x+1})$*,*
ii) *For all* $\rho\in]1-\varepsilon,1[$*, and for all* $y<x$, $\mathrm{M}(x)$ *has no solutions in* $\mathcal{A}_K(t_\rho,\rho^{y+1})$*.*

This decomposition is compatible with the confluence i.e. $\mathrm{M}(x)=\mathrm{Def}^{\mathrm{Tay}}_{1,q}(\mathrm{Conf}^{\mathrm{Tay}}_q(\mathrm{M})(x))$*. In particular the irregularity* $\mathrm{Irr}_{\sigma_q}(\mathrm{M}):=\sum_{x\geqslant 0}x\cdot\mathrm{rank}_{\mathcal{R}_K}\mathrm{M}(x)$ *is a natural number.*

Proof. The "*slopes*" and the "*Irregularity*" are defined, by Christol and Mebkhout (cf. [CM02b]), by means of the generic *radius of the Taylor solutions*. The K-linear equivalences $\mathrm{Conf}^{\mathrm{Tay}}_q$ and $\mathrm{Def}^{\mathrm{Tay}}_{1,q}$ preserve, by definition, the generic Taylor solution. It follows immediately that the q-difference equation inherits then, via the equivalence $\mathrm{Conf}^{\mathrm{Tay}}_q$, the *slopes* of the attached differential equation, together with their formal properties (break decomposition, Hasse-Arf property, ...). □

Corollary 8.37. *With the notations of [ADV04] and [And04], if* $K=K^{\mathrm{alg}}$ *is algebraically closed, the functor* $D^{(\phi)}_{\sigma_q}:\sigma_q-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}\longrightarrow\underline{\mathrm{Rep}}_{K^{\mathrm{alg}}}(\mathcal{I}_{k^{\mathrm{alg}}((t))}\times\mathbb{G}_a)$ *preserves the slopes (by corollary 8.36 on the left hand side, and by the Swan conductor on the right hand side).*

Proof. One has $D^{(\phi)}_{\sigma_q}=D^{(\phi)}_d\circ\mathrm{Conf}^{\mathrm{Tay}}_q$. Since $D^{(\phi)}_d$ and $\mathrm{Conf}^{\mathrm{Tay}}$ preserve the slopes, so does $D^{(\phi)}_{\sigma_q}$. □

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