Introduction to: p-Adic Confluence of q-Difference Equations

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From the Introduction of the paper

The main aim of this paper is to provide a *theory of confluence* for q-difference equations in the p-adic framework.

A motivation : the rough idea of the confluence

Heuristically we say that a family of $q-$difference equations $\{\ \sigma_q(Y_q) = A(q,T)\cdot Y_q\ \}_{q\in D^-(1,\epsilon)-\{1\}}$ (where σ_q is the automorphism $f(T)\mapsto f(qT)$), is confluent to the differential equation $\delta_1(Y_q) = G(1,T)\cdot Y_q$, with $\delta_1 := T\frac{d}{dT}$, if one has $\lim_{q\to 1}\frac{A(q,T)-1}{q-1} = G(1,T)$ and, in some suitable meaning

$$\lim_{q\to 1} Y_q = Y_1\ . \tag{0.0.1}$$

Roughly speaking, in this paper we show that in the p-adic framework, if a differential equation is given, then, for ϵ sufficiently small, one may choose the family $\{G(q,T)\}_q$ in order to have $Y_q = Y_1$, for all $q\in \mathrm{D}^-(1,\epsilon)$. Conversely if q_0 is not a root of unity, and if a single equation $\sigma_{q_0}(Y_{q_0}) = A(q_0,T)\cdot Y_{q_0}$ is given, then, under some assumptions on the radius of convergence of its *generic* Taylor solution Y_{q_0}, one can find a differential equation, and family as above with the property that $Y_q = Y_{q_0} = Y_1$, for all $q\in\mathrm{D}^+(1,|q_0-1|)$. In this sense, in the p-adic context, the solutions of q-difference equations are not simply a "*discretization*" of the solutions of differential equations, but they are actually equal. We want now to state these facts more precisely.

The work of Y.André and L.Di Vizio

In [ADV04] the authors initiated the study of the phenomena of confluence in a $p-$adic setting. For K a complete discrete valuation field of mixed characteristic, they found an equivalence between the category of $q-$difference equations with Frobenius structure over the Robba ring $\mathcal{R}_{K^{\mathrm{alg}}}$ (here called $\sigma_q-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$), and the category of differential equations with Frobenius structure over the Robba ring $\mathcal{R}_{K^{\mathrm{alg}}}$ (here called $\delta_1-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$).

One of the restrictions of [ADV04] is that the number q is required to satisfy $|q-1| < |p|^{\frac{1}{p-1}}$. Indeed, in the annulus $|q-1| = |p|^{\frac{1}{p-1}}$ one encounters the $p-$th root of unity and, if $\xi^p = 1$, then the category $\sigma_\xi-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is different in nature from the category of differential equation, since it is not $K^{\mathrm{alg}}-$linear.

The equivalence of [ADV04] is obtained as follows. In [And02] one proves that the Tannakian group of $\delta_1-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is $\mathcal{I}_{k^{\mathrm{alg}}((t))}\times\mathbb{G}_a$, where k is the perfect residue field of K, and $\mathcal{I}_{k^{\mathrm{alg}}((t))}$ is the absolute Galois group of $k^{\mathrm{alg}}((t))$. On the other hand in [ADV04] one shows that $\sigma_q-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ has the same Tannakian group $\mathcal{I}_{k^{\mathrm{alg}}((t))}\times\mathbb{G}_a$. By composition with the respec-

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tive Tannakian equivalences (T_q and T_1 below), one obtains then the so called the *confluence functor* " Conf_q " (in the notations of [ADV04] one has $T_1 = \mathrm{V}_d^{(\phi)}$ and $T_q = \mathrm{V}_{\sigma_q}^{(\phi)}$):

$$\begin{array}{ccc} \sigma_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} & \overset{\mathrm{Conf}_q}{\underset{\cong}{\dashrightarrow}} & \delta_1 - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} \\ & {\scriptstyle T_q}\searrow^{\cong} \quad {}^{\cong}\swarrow{\scriptstyle T_1} & \\ & \underline{\mathrm{Rep}}_{K^{\mathrm{alg}}}(\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a) & \end{array} \tag{0.0.2}$$

The strategy of [ADV04] consists in showing that, as in the case of differential equations (cf. [And02]), every object M in $\sigma_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)}$ is quasi-unipotent, i.e. becomes unipotent after a special extension of $\mathcal{R}_K$ (cf. section 8.3 of the paper). Once a basis of M is fixed, this means that M admits a complete basis of solutions $\widetilde{Y} \in \mathrm{GL}_n(\widetilde{\mathcal{R}_K}[\log(T)])$, where $\widetilde{\mathcal{R}_K}$ is the union of all special extensions of $\mathcal{R}_K$ (it is a sort of lifting of $k((t))^{\mathrm{alg}}$). We will call "*étale*" solutions the solutions of M in $\widetilde{\mathcal{R}_K}[\log(T)]$. The proof of this relevant result needs a substantial effort, and is actually not less complicated than the classical $p-$adic local monodromy theorem for differential equations itself (i.e. the fact that T_1 is an equivalence). Thanks to the fact that this important, but also very peculiar, class of q-difference and differential equations are trivialized by $\widetilde{\mathcal{R}_K}[\log(T)]$, one can define the functor T_1 (resp. T_q) associating to a differential (resp. q-difference) equation $(\mathrm{M}, \delta_1^{\mathrm{M}})$ (resp. $(\mathrm{M}, \sigma_q^{\mathrm{M}})$) the K^{alg}-vector space $T_1(\mathrm{M}, \delta_1^{\mathrm{M}})$ (resp. $T_q(\mathrm{M}, \sigma_q^{\mathrm{M}})$) of its "étale" solutions in $\widetilde{\mathcal{R}_K}[\log(T)]$.[1] The action of $\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a$ on the space of the "étale" solutions arises from its action on $\widetilde{\mathcal{R}_K}[\log(T)]$ by $\mathcal{R}_K$-linear automorphisms commuting with δ_1 and σ_q on $\widetilde{\mathcal{R}_K}[\log(T)]$.

Hence one sees for the first time in [ADV04] the fact that the "étale" solutions of a q-difference equation with Frobenius structure, are also the "étale" solutions of a differential equation. Moreover the functor Conf_q is nothing but the functor sending a q-difference equation (with (strong) Frobenius structure) into the differential equation having the same solutions.

In the present paper we prove that this "permanence" of the solutions holds also for *Taylor solutions* (see below). We develop then a p-adic theory of Confluence using, as a unique tool, this fact, here called *propagation principle*. We prove indeed that this principle is sufficient to define the Confluence and Deformation equivalences, over almost all p-adic ring of functions, with very basic assumptions on the equations. This theory requires only the definition and the formal properties of the *generic Taylor solution* $Y(x, y)$. For this reason it is not a consequence of the heretofore developed theory. Conversely we deduce, as a special case, the confluence of [ADV04] by comparing Taylor solutions and "étale" solutions (cf. the end of the introduction).

The generic q-Taylor solution

Let now K be an arbitrary ultrametric complete valued field of mixed characteristic $(0, p)$. Let $X = \mathrm{D}^+(c_0, R_0) - \cup_{i=1,\ldots,n} \mathrm{D}^-(c_i, R_i)$ be an affinoid, where $\mathrm{D}^-(c, R)$ denotes the open disc centered at c of radius R. Let $\mathcal{H}_K(X)$ be the ring of analytic elements on X. Consider a q-difference equation

$$\sigma_q(Y) = A(q, T) \cdot Y \ , \quad A(q, T) \in GL_n(\mathcal{H}_K(X)) \tag{0.0.3}$$

on X. Denote by $(\mathrm{M}, \sigma_q^{\mathrm{M}})$ the q-difference module over X defined by this equation.

A major difference between the complex and the p-adic settings is that in the latter there are disks (not centered at 0) which are q-invariant. A disk $\mathrm{D}^-(c, R) \subset X(K)$ is q-invariant (i.e. the map

[1] Following the definition section 3.2 of the paper, $\mathrm{V}_d^{(\phi)}(\mathrm{M}) := (\mathrm{M} \otimes_{\mathcal{R}_K} \widetilde{\mathcal{R}_K}[\log(T)])^{\delta_1 = 0}$ is actually the dual of the space of solutions $\mathrm{Hom}_{\mathcal{R}_K}^{\delta_1}(\mathrm{M}, \widetilde{\mathcal{R}_K}[\log(T)])$ (resp. same remark for $\mathrm{V}_{\sigma_q}^{(\phi)}(\mathrm{M}) := (\mathrm{M} \otimes_{\mathcal{R}_K} \widetilde{\mathcal{R}_K}[\log(T)])^{\sigma_q = \mathrm{Id}}$ and $\mathrm{Hom}_{\mathcal{R}_K}^{\sigma_q}(\mathrm{M}, \widetilde{\mathcal{R}_K}[\log(T)])$).

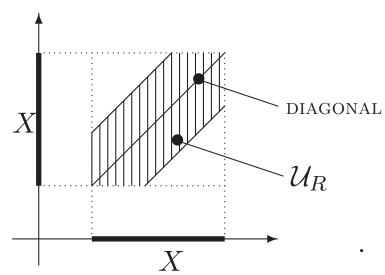
$x \mapsto qx$ is a bijection of $\mathrm{D}^-(c,R)$) if and only if $|q-1||c| < R$, and $|q|=1$ (cf. Lemma 5.1 of the paper). Starting from this consideration, in [DV04] the author define, for q-difference equations, the q-analogue of the generic Taylor solution of a differential equation (cf. Definition 5.11 of the paper):

$$Y(x,y) := \sum_{n\geqslant 0} H_n(q,T)\frac{(x-y)_{q,n}}{[n]_q^!}\;, \tag{0.0.4}$$

where $H_n(q,T)$ is obtained by iterating the equation (0.0.3): $d_q^n(Y) = H_n(q,T)\cdot Y$, where $d_q := \frac{\sigma_q - 1}{(q-1)T}$. For a large class of equations it happens that, for all $c \in X(K)$, the series $Y(x,c)$ represents a function which converges on a disk $\mathrm{D}^-(c,R)$, with $|q-1||c| < R$. More precisely $Y(x,y)$ converges in a neighborhood of the diagonal of the type $\mathcal{U}_R := \{(x,y)\in X\times X \mid |x-y| < R\}$, with

$$|q-1|\cdot \mathfrak{s}_X < R\,, \tag{0.0.5}$$

where $\mathfrak{s}_X := \sup_{c\in X}|c|$ as shown in the following picture (one easily sees that $\mathfrak{s}_X = \max(|c_0|, R_0)$):



We call such equations *Taylor admissible*. The matrix function $Y(x,y) : \mathcal{U}_R \to GL_n(K)$ is invertible and satisfies the cocycle conditions: $Y(x,y)\cdot Y(y,z) = Y(x,z)$ and $Y(x,y)^{-1} = Y(y,x)$, for all $(x,y),(y,z),(x,z)\in\mathcal{U}_R$. Moreover $Y(qx,y) = A(q,x)Y(x,y)$ and, for all $c\in X(K)$, the matrix $Y(x,c)\in GL_n(\mathcal{A}_K(c,R))$ is a fundamental basis of solutions of the equation (0.0.3). In particular the q-difference algebra $\mathcal{A}_K(c,R)$ of analytic functions over the disk $\mathrm{D}^-(c,R)$, trivializes $(\mathrm{M},\sigma_q^{\mathrm{M}})$.

The following fact is the main point of this paper (cf. Theorem 7.7 of the paper). If now $q' \neq q$ belongs to the disk $\mathrm{D}^-(q,R/\mathfrak{s}_X) = \mathrm{D}^-(1,R/\mathfrak{s}_X)$, then the matrix

$$A(q',x) := Y(q'x,y)\cdot Y(x,y)^{-1} = Y(q'x,y)\cdot Y(y,x) = Y(q'x,x) \tag{0.0.6}$$

is an analytic function of x on all of X. Indeed $(q'x,x)\in\mathcal{U}_R$, for all $x\in X$, and hence the matrix $A(q',x)$ maps $x\mapsto(q'x,x)\mapsto Y(q'x,x) = A(q',x)$. One shows easily that $A(q',x)\in GL_n(\mathcal{H}_K(X))$, for all $q'\in\mathrm{D}^-(1,R/\mathfrak{s}_X)$, since $Y(x,y)$ is invertible. This fact implies that $Y(x,y)$ *is simultaneously the Taylor solution of every equation of the family $\{\sigma_{q'}(Y) = A(q',T)Y\}_{q'}$, for all $q'\in\mathrm{D}^-(1,R/\mathfrak{s}_X)$*. Equivalently, this means that the q-difference module $(\mathrm{M},\sigma_q^{\mathrm{M}})$ is canonically endowed with an action of $\sigma_{q'}$, for all $q'\in\mathrm{D}^-(1,R/\mathfrak{s}_X)$. This remarkable fact will be called *Propagation Principle*. As one can see, this happens actually under the following weak assumptions on $(\mathrm{M},\sigma_q^{\mathrm{M}})$:

i) q is not a root of unity; (0.0.7)

ii) $Y(x,y)$ converges on some $\mathcal{U}_R$ with $|q-1|\cdot\mathfrak{s}_X < R \leqslant r_X$; (0.0.8)

where $r_X = \min(R_0,R_1,\ldots,R_n)$ is a number depending on the geometry of X. The category of q-difference modules $(\mathrm{M},\sigma_q^{\mathrm{M}})$ satisfying these two properties for a suitable unspecified R satisfying $|q-1|\mathfrak{s}_X < r \leqslant R \leqslant r_X$ will be denoted by $\sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]}$.

The assumption $|q-1|\mathfrak{s}_X < R$ assures that the image of the map $x\mapsto(qx,x) : X\mapsto X\times X$ is contained in $\mathcal{U}_R$. The bound $R\leqslant r_X$ assures that the function $Y(x,y)$ does not converge outside X. Indeed the properties of $Y(x,y)$ outside X are not invariant under $\mathcal{H}_K(X)$-base changes in M. Finally condition ii) also assures that the map $x\mapsto qx$ is a bijection of X globally fixing each individual hole of X (cf. section 5.2 of the paper). Since $r_X \leqslant \mathfrak{s}_X$, we are assuming implicitly that $|q-1|<1$. But no restrictive assumptions on X or on K are made.

Obviously this process works just as well if the initial function $Y(x,y)$ is the generic Taylor solution of a differential equation. The category of *differential* equations whose Taylor solution converges on $\mathcal{U}_R$, for an unspecified R satisfying $r \leqslant R \leqslant r_X$, will be denoted by $\delta_1-\mathrm{Mod}(\mathcal{H}_K(X))^{[r]}$.

Discrete and analytic σ-modules

Let $\mathcal{Q}(X)$ be the set of $q \in K$ for which $x \mapsto qx$ is a bijection of X. Then $\mathcal{Q}(X)$ is a topological subgroup of $K^\times$, and the disk $\mathrm{D}^-(1,R/\mathfrak{s}_X)$, with $R \leqslant r_X$, is an open subgroup of $\mathcal{Q}(X)$. The group $\mathcal{Q}(X)$ acts continuously on $\mathcal{H}_K(X)$ via $q \mapsto \sigma_q$. The data of M, together with the simultaneous σ_q-semi-linear action of σ_q^{M}, for all $q \in \mathrm{D}^-(1,R/\mathfrak{s}_X)$, is then a *semi-linear representation of the sub-group* $\mathrm{D}^-(1,R/\mathfrak{s}_X) \subseteq \mathcal{Q}(X)$. This representation has three remarkable properties:

(a) The map $(q',x) \mapsto A(q',x)$ is analytic in (q',x). In particular the representation is continuous;

(b) The group $\mathrm{D}^-(1,R/\mathfrak{s}_X)$ depends on R, and hence on M;

(c) The matrix $Y(x,y)$ is simultaneously the generic Taylor solution of the q-difference module $(\mathrm{M},\sigma_q^{\mathrm{M}})$, for all $q \in \mathrm{D}^-(1,R/\mathfrak{s}_X)$.

Inspired by the first two properties we define a new class of objects called *discrete or analytic σ-modules* as follows. Consider a subset $S \subset \mathcal{Q}(X)$. A *discrete σ-module* on S is nothing but a $\mathcal{H}_K(X)$ semi-linear representation of the group $\langle S\rangle$ generated by S. If $S=U$ is an open subset of $\mathcal{Q}(X)$, we define *analytic σ-modules on U* to be a discrete σ-modules over U together with a certain condition of analyticity of σ_q^{M} with respect to q. These categories are denoted by $\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_S^{\mathrm{disc}}$ and $\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{an}}$ respectively. In this paper the words "*discrete*" or "*analytic*" will be referred to the discreteness or analyticity of σ_q^{M} with respect to q. We heuristically imagine the analytic σ-modules as *semi-linear representations of the (co-variant) sheaf of groups $U \mapsto \langle U\rangle$.*

Remark 0.1. It is important to notice that morphisms between analytic σ-modules over U are morphisms of representations. More precisely once a basis of M (resp. N) is fixed, we have a family of operators $\{\sigma_q(Y)=A(q,T)Y\}_{q\in\langle U\rangle}$ (resp. $\{\sigma_q(Y)=\widetilde{A}(q,T)Y\}_{q\in\langle U\rangle}$) such that $A(q,T)$ (resp. $\widetilde{A}(q,T)$) depends analytically on (q,T).[2] A morphism $\alpha:\mathrm{M}\to\mathrm{N}$ then must simultaneously commute with σ_q^{M} and σ_q^{N}, for all $q \in \langle U\rangle$. In other words the matrix B of α must simultaneously verify $A(q,T)B=\sigma_q(B)\widetilde{A}(q,T)$, for all $q\in\langle U\rangle$. Actually there are *non isomorphic* analytic σ-modules over U defining isomorphic q-difference equations at every $q\in\langle U\rangle$ (see example 2.6 of the paper). This is analogous to have *non isomorphic* sheaves having isomorphic stalks at every point.

Taylor admissible σ-modules

We now want to analyse property (c): the constancy of the solutions. If $S \not\subseteq \boldsymbol{\mu}_{p^\infty}$, we call *Taylor admissible σ-modules over S* those σ-modules for which the q-Taylor solution $Y(x,y)$ is the same for all $q\in\langle S\rangle$, and satisfy the condition ii), for all $q\in S$ (cf. (0.0.8)). If $S=U$ is open, by the Propagation Principle, Taylor admissible σ-modules are *automatically* analytic on U (cf. Remark 7.8 of the paper). This category is denoted by $\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}} \subseteq \sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{an}}$. We heuristically imagine Taylor admissible σ-modules as *semi-linear representations of the (co-variant) sheaf of groups $U\mapsto\langle U\rangle$*, which are *locally constant.*

Taylor admissibility is a particular case of a more classical notion. If $\mathrm{C}/\mathcal{H}_K(X)$ is an algebra admitting an action of the group $\langle S\rangle$ extending that on $\mathcal{H}_K(X)$, then a semi-linear representation

[2] The data of an analytic σ-module is actually nothing but "*a family of q-difference equations depending analytically on q*".

of $\langle S\rangle$ over $\mathcal{H}_K(X)$ is called C-*admissible* if it is trivialized by C. For a discrete σ-module M over S to be trivialized by C means exactly that there exists $Y\in GL_n(\mathrm{C})$ which is a simultaneous solution of all operators defined by M. If M is trivialized by C we will say that M is C-*constant*. We observe that if $S=q^{\mathbb{Z}}$, then C is nothing but a q-difference algebra over $\mathcal{H}_K(X)$. So the constancy of the solutions does not depend on the analyticity of M, rather it is a *discrete* fact.

In section 3 of the paper we define *discrete σ-algebras*, and we develop a basic differential/difference Galois theory for discrete σ-algebras. The analogue of the Picard-Vessiot theorem providing the existence of a discrete σ-algebra trivializing a given discrete σ-module *is missing*. We are thus obliged to work with the category of discrete σ-modules trivialized by a fixed discrete σ-algebra C. In section 4 of the paper we develop formally the theory of C-Confluence and C-Deformation, which will also depend on the chosen discrete σ-algebra C.

Remark 0.2. Notice that solutions will be defined formally as morphisms $\mathrm{M}\to\mathrm{C}$ commuting simultaneously with the actions of σ_q for all $q\in S$ (cf. Section 3.2 of the paper). This fact, together with Remark 2.4.1 of the paper explains why the notion of C-*constant* σ-module implies the constancy of the solutions (with respect to q).

The Confluence functor

Let $(\mathrm{M},\sigma^{\mathrm{M}})$ be an analytic σ-module over U. By analyticity we also have an action of the *Lie algebra* of $\langle U\rangle$ (here systematically identified with $K\cdot\delta_1$). In other words the following limit converges to a connection $\delta_1^{\mathrm{M}}:\mathrm{M}\to\mathrm{M}$ (cf. section 2.4 of the paper):

$$\delta_1^{\mathrm{M}} \;:=\; \lim_{q\in\langle U\rangle,q\to1}\frac{\sigma_q^{\mathrm{M}}-1}{q-1}\;\in\;\mathrm{End}_K^{\mathrm{cont}}(\mathrm{M})\;, \tag{0.0.9}$$

where q runs over the (open) group $\langle U\rangle$ generated by U. In terms of matrices, the matrix $G(1,T)$ of δ_1^{M} is $G(1,T)=q\frac{\partial}{\partial q}\Big(A(q,T)\Big)_{|_{q=1}}$ (cf. equation (2.4.5) of the paper). By continuity, morphisms of analytic σ-modules also commute with the connection (cf. remark 2.4.1 of the paper). Hence we obtain a functor called $\mathrm{Conf}_U:\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{an}}\longrightarrow\delta_1-\mathrm{Mod}(\mathcal{H}_K(X))$, sending $(\mathrm{M},\sigma^{\mathrm{M}})$ into $(\mathrm{M},\delta_1^{\mathrm{M}})$ (cf. Remark 2.13 of the paper). This functor is not an equivalence, but it does induce an equivalence:

$$\mathrm{Conf}_U^{\mathrm{Tay}}\;:\;\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{[r]}\;\xrightarrow{\;\sim\;}\;\delta_1-\mathrm{Mod}(\mathcal{H}_K(X))^{[r]}\;, \tag{0.0.10}$$

where $\mathrm{Conf}_U^{\mathrm{Tay}}$ simply denotes the restriction of Conf_U to the category $\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{[r]}\subseteq\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{\mathrm{adm}}$ of Taylor admissible σ-modules verifying condition ii) with $r\leqslant R\leqslant r_X$ (cf. (0.0.8)), where $r>0$ is large enough to have $U\subset\mathrm{D}^-(1,r/\mathfrak{s}_X)$ (cf. Corollary 7.9 of the paper). The Propagation Principle gives a quasi inverse functor (cf. Remark 2.13 of the paper for a formal presentation).

On the other hand let $q\in U-\boldsymbol{\mu}_{p^\infty}$. An analytic σ-module over U defines a q-difference module by forgetting the action of $\sigma_{q'}^{\mathrm{M}}$, for all $q'\neq q$. Again the Propagation Principle provides an equivalence

$$\mathrm{Res}_q^U\;:\;\sigma-\mathrm{Mod}(\mathcal{H}_K(X))_U^{[r]}\;\xrightarrow{\;\sim\;}\;\sigma_q-\mathrm{Mod}(\mathcal{H}_K(X))^{[r]}\;, \tag{0.0.11}$$

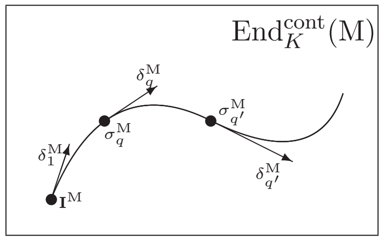
where $r\leqslant r_X$ is sufficiently large to have $U\subseteq\mathrm{D}^-(1,r/\mathfrak{s}_X)$ (cf. Corollary 7.9 of the paper). We call the composite equivalence $\mathrm{Conf}_q^{\mathrm{Tay}}$. Thus we have

$$\mathrm{Conf}_q^{\mathrm{Tay}}:=\mathrm{Conf}_U^{\mathrm{Tay}}\circ(\mathrm{Res}_q^U)^{-1}\;:\;\sigma_q-\mathrm{Mod}(\mathcal{H}_K(X))^{[r]}\xrightarrow{\;\sim\;}\delta_1-\mathrm{Mod}(\mathcal{H}_K(X))^{[r]}\;. \tag{0.0.12}$$

The equivalence $\mathrm{Conf}_q^{\mathrm{Tay}}$ sends a q-difference equation satisfying conditions i) and ii) (cf. (0.0.7),(0.0.8)), into the differential equation having the same generic Taylor solution.

Roots of unity and q-tangent operators

In this last equivalence the number q must not belong to $\boldsymbol{\mu}_{p^\infty}$. If $q' = \xi$, with $\xi^{p^n} = 1$, the category of σ_ξ-difference equations is not K-linear and cannot be equivalent to the category of differential equations. Nevertheless, if, for $q \notin \boldsymbol{\mu}_{p^\infty}$, the radius R of the q-Taylor solution is large, the Propagation Principle gives an operator $\sigma_\xi^{\mathrm{M}} : \mathrm{M} \to \mathrm{M}$ acting on M. The idea is to replace the category $\sigma_\xi - \mathrm{Mod}(\mathcal{H}_K(X))$ with another category. The expected object "at ξ" should also be endowed with an action of the Lie algebra, *as we have just done in the case* $\xi = 1$. For all $q \in \langle U \rangle$ the action of the Lie algebra of $\langle U \rangle$ is given by the limit $\delta_q^{\mathrm{M}} := \lim_{q' \to q} \frac{\sigma_{q'}^{\mathrm{M}} - \sigma_q^{\mathrm{M}}}{q'-q} \in \mathrm{End}_K^{\mathrm{cont}}(\mathrm{M})$, for $q, q' \in \langle U \rangle$, as shown in the picture:



Clearly $\delta_q^{\mathrm{M}} = \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}}$, so to give δ_q^{M} is equivalent to give δ_1^{M}. In a root of unity the "limit object" is a mixed data $(\mathrm{M}, \sigma_\xi^{\mathrm{M}}, \delta_1^{\mathrm{M}})$, i.e. a connection δ_1^{M} on M together with an action of σ_ξ^{M} on M. We call these new objects (σ_ξ, δ_ξ)*-modules*. In the sequel every terminology is given simultaneously for σ-modules and (σ, δ)-modules. The additional data of δ_ξ^{M} makes the category of (σ_ξ, δ_ξ)-modules K-linear. Moreover δ_ξ^{M} preserve the "*information*" in a neighborhood of ξ, indeed we find equivalences

$$\mathrm{Conf}_\xi^{\mathrm{Tay}} := \mathrm{Conf}_U^{\mathrm{Tay}} \circ (\mathrm{Res}_\xi^U)^{-1} \;:\; (\sigma_\xi, \delta_\xi) - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\sim} \delta_1 - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \,, \qquad (0.0.13)$$

$$\mathrm{Def}_{\xi,q}^{\mathrm{Tay}} := \mathrm{Res}_q^U \circ (\mathrm{Res}_\xi^U)^{-1} \;:\; (\sigma_\xi, \delta_\xi) - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \xrightarrow{\sim} (\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{H}_K(X))^{[r]} \quad (0.0.14)$$

If q is not a root of unity, then the data of δ_1^{M} is superfluous, indeed if the module is Taylor admissible the Propagation Principle allows one to re-construct δ_1^{M} from σ_q^{M}.

In the classical setting over the complex numbers $\mathbb{C}$, understanding of the case $q = \xi \in \boldsymbol{\mu}_{p^\infty}$ remains an open problem.

Quasi unipotence and comparison with André-Di Vizio's Confluence

Up to a correct definition for the notion of Taylor admissibility, the previous theory can be generalized to more general rings of functions. From section 7.4 of the paper on we obtain the theory over $\mathcal{R}_K$. We prove that every q-difference equations with Frobenius Structure over $\mathcal{R}_K$, is quasi unipotent (i.e. is trivialized by $\widetilde{\mathcal{R}_K}[\log(T)]$), for all $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$, generalizing the main result of [ADV04]. We actually prove this theorem in the more general context of σ-modules, and (σ, δ)-modules. We deduce it by the quasi unipotence of p-adic *differential* equations with Frobenius Structure over $\mathcal{R}_K$, and by deformation. The idea is the following. As already mentioned, we are obliged to work with σ-modules trivialized by a fixed discrete σ-algebra C, and the C-Confluence and C-Deformations functors depend on C. In the "quasi unipotent" context this algebra is $\mathrm{C} := \widetilde{\mathcal{R}_K}[\log(T)]$, while in the context of the propagation theorem $\mathrm{C} := \mathcal{A}_K(c, R)$, for an arbitrary point $c \in X$, and suitable $R > 0$. To compare Taylor solutions to the "étale solutions" in $GL_n(\widetilde{\mathcal{R}_K}[\log(T)])$, the idea is to find a discrete σ-algebra of *functions over a disk* containing $\widetilde{\mathcal{R}_K}[\log(T)]$. Actually such an algebra does not exist. Thus we use a theorem of S.Matsuda (cf. Theorem 8.13 of the paper) providing an equivalence between $\delta_1 - \mathrm{Mod}(\mathcal{R}_K)^{(\phi)}$ with the sub-category of $\delta_1 - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\phi)}$ formed by Special objects. Special objects are trivialized by a special extension of $\mathcal{H}_K^\dagger$ (cf. Section 8.3 of the paper). The ring $\mathcal{A}_K(1,1)$ is a discrete σ-algebra over $\mathcal{H}_K^\dagger$. We prove then that the algebra $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)]$ generated over $\mathcal{H}^{\dagger}_K$ by all the "étale solutions" of Special objects admits an embedding $\widetilde{\mathrm{C}^{\text{ét}}_K}[\log(T)] \subset \mathcal{A}_{K^{\mathrm{alg}}}(1,1)$ commuting with δ_1, with the Frobenius, and with σ_q^{M}, for all $q \in \mathrm{D}^-(1,1) - \boldsymbol{\mu}_{p^\infty}$ (cf. Lemma 8.24 of the paper). This will prove that the C-Confluence and the C-Deformation functors defined by using $\mathrm{C} = \mathcal{A}_K(1,1)$, or $\mathrm{C} = \widetilde{\mathcal{R}_K}[\log(T)]$ are actually the same (cf. Corollary 8.26 of the paper). Moreover it proves also that the confluence of André-Di Vizio coincides with our $\mathrm{Conf}_q^{\mathrm{Tay}}$ (cf. Section 8.5 of the paper), thus it is independent on the Frobenius.

Structure of the paper

Section 1 is devoted to notation. In section 2, we give definitions and basic facts on *discrete/analytic* $\sigma-$*modules, and* $(\sigma,\delta)-$*modules*. In section 3 we define *discrete* σ*-algebras and* (σ,δ)*-algebras*, and we give the abstract definition of *solutions*. In section 4 we give the formal notion of *confluence*. In section 5 we introduce *generic Taylor solutions* and *generic radius of convergence*. In section 6 we define *Taylor admissible objects* and obtain the main *Propagation Theorem* 7.7 of the paper. In the last section 8 we apply the previous theory to the Robba ring, and to the $p-$*adic local monodromy theorem*.

REFERENCES

ADV04 Yves André and Lucia Di Vizio, *q-difference equations and p-adic local monodromy*, Astérisque (2004), no. 296, 55–111. MR MR2135685

And02 Y. André, *Filtrations de type Hasse-Arf et monodromie p-adique*, Invent. Math. **148** (2002), no. 2, 285–317.

DV04 Lucia Di Vizio, *Introduction to p-adic q-difference equations*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 615–675.

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