Small connections are cyclic

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Abstract

The main local invariants of a (one variable) differential module over the complex numbers are given by means of a cyclic basis. In the p-adic setting the existence of a cyclic vector is often unknown. It results then interesting to obtain definitions that does not involve cyclic vectors. We investigate the existence of such a cyclic vector in a Banach algebra or an algebra defined by a family of semi-norms. We follow the explicit method of Katz [Kat87], and we prove the existence of such a cyclic vector under the assumption that the matrix of the derivation is small enough in norm.

Contents

1. Katz's simple algorithm for cyclic vectors

Let $(\mathscr{B}, d)$ be a commutative ring $\mathscr{B}$ with unit, together with a derivation[1] $d : \mathscr{B} \to \mathscr{B}$. We denote by $\mathscr{B}^{d=0} := \{b \in \mathscr{B}$ such that $d(b) = 0\}$ the sub-ring of constants. A differential module M is a free $\mathscr{B}$-module of finite rank together with an action of the derivation

$$\nabla \ : \ \mathrm{M} \to \mathrm{M} \tag{1.1}$$

i.e. a Z-linear map satisfying $\nabla(bm) = d(b)m + b\nabla(m)$ for all $b \in \mathscr{B}$, $m \in \mathrm{M}$. A cyclic vector for M is an element $m \in \mathrm{M}$ such that the family $\{m, \nabla(m), \nabla^2(m), \dots, \nabla^{n-1}(m)\}$ is a basis of M over $\mathscr{B}$. Such a vector does not always exists. Namely if $d = 0$ is the trivial derivation, then ∇ is merely a $\mathscr{B}$-linear map and (M, ∇) is a torsion module over the ring of polynomials $\mathscr{B}[X]$ where the action of X on M is given by ∇. There is another counterexample in the case in which $\mathscr{B} = \mathbb{F}_p(X)$ is a functions field in characteristic $p > 0$: let $\mathrm{M} := \mathbb{F}_q[X]^n$, with $n > q = p^r$, together with the trivial connection $\nabla(f_1, \dots, f_n) = (f_1', \dots, f_n')$, then, since $d^q = 0$, one has $\nabla^q = 0$ so M does not have any cyclic vector. The same happens replacing $\mathbb{F}_q$ by a ring A having a maximal ideal $\mathfrak{m}$ such that $A/\mathfrak{m} \cong \mathbb{F}_q$. The trivial connection of $A[X]^n$ (with respect to d/dx) can not admit a cyclic vector, since otherwise its reduction to $\mathbb{F}_q[X]$ would be cyclic too.

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[1] i.e. a $\mathbb{Z}$-linear map satisfying the Leibniz rule $d(ab) = ad(b) + d(a)b$

1.1 Three cyclic vector theorems.

P.Deligne provided the existence of such a cyclic vector for all differential modules over a field of characteristic 0 with non trivial derivation (cf. [Del70, Ch.II,Lemme 1.3]).

Theorem 1.1 [Del70, Ch.II, Lemme 1.3]. *Let $\mathscr{B}$ be a field of characteristic 0, then all differential modules over $\mathscr{B}$ admit a cyclic vector.*

Subsequently N.Katz generalized the result of Deligne providing the following simple explicit algorithm:

Theorem 1.2 ([Kat87]). *Assume that there exists an element $t \in \mathscr{B}$ such that $d(t) = 1$. Assume moreover that $(n-1)!$ is invertible in $\mathscr{B}$, and that $\mathscr{B}^{d=0}$ contains a field k such that [2] $\#k > n(n-1)$. Let $a_0, a_1, \ldots, a_{n(n-1)}$ be $n(n-1)+1$ distinct elements of k, and let $\mathbf{e} := \{e_0, \ldots, e_{n-1}\} \subset \mathrm{M}$ be a basis of M over $\mathscr{B}$. Then Zarisky locally on Spec($\mathscr{B}$) one of the vectors*

$$c(\mathbf{e}, t-a_i) \ := \ \sum_{j=0}^{n-1} \frac{(t-a_i)^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \nabla^k(e_{j-k}) \tag{1.2}$$

is a cyclic vector of M.

Theorem 1.3 ([Kat87]). *If $\mathscr{B}$ is a local $\mathbb{Z}[1/(n-1)!]$-algebra, and if $a \in \mathscr{B}^{d=0}$ is such that the maximal ideal of $\mathscr{B}$ contains $t-a$, then $c(\mathbf{e}, t-a)$ is a cyclic vector for* M.

The arguments of the Katz's proofs are the following. We consider the polynomial ring $\mathscr{B}[X]$ and we extend the derivation of $\mathscr{B}$ by $d(X) = 1$. We denote again by ∇ the action of d on $\mathrm{M} \otimes_{\mathscr{B}} \mathscr{B}[X]$ given by $\nabla \otimes \mathrm{Id}_{\mathscr{B}[X]} + \mathrm{Id}_{\mathrm{M}} \otimes d$. Each element c_0 in $\mathrm{M} \otimes_{\mathscr{B}} \mathscr{B}[X]$ can be uniquely represented as $c_0 := \sum_{j \geqslant 0} c_{0,j} X^j$, with $c_{0,j} \in \mathrm{M}$ for $j = 0, 1, \ldots$. The derivatives $\nabla^i(c_0)$ of c_0 then have the same form $c_i := \nabla^i(c_0) = \sum_{j \geqslant 0} c_{i,j} X^j$, with $c_{i,j} = \sum_{k=0}^{i} k! \binom{j+k}{j} \binom{i}{k} \nabla^{i-k}(c_{0,j+k})$.

$$\begin{array}{rclcccccccccc}
c_0 & = & c_{0,0} & + & c_{0,1} \cdot X & + & c_{0,2} \cdot X^2 & + & \cdots & + & c_{0,n-1} \cdot X^{n-1} & + & \cdots \\
\nabla(c_0) & = & c_{1,0} & + & c_{1,1} \cdot X & + & c_{1,2} \cdot X^2 & + & \cdots & + & c_{1,n-1} \cdot X^{n-1} & + & \cdots \\
\nabla^2(c_0) & = & c_{2,0} & + & c_{2,1} \cdot X & + & c_{2,2} \cdot X^2 & + & \cdots & + & c_{2,n-1} \cdot X^{n-1} & + & \cdots \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
\nabla^{n-1}(c_0) & = & c_{n-1,0} & + & c_{n-1,1} \cdot X & + & c_{n-1,2} \cdot X^2 & + & \cdots & + & c_{n-1,n-1} \cdot X^{n-1} & + & \cdots \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array} \tag{1.3}$$

The main point is now that, if $(n-1)!$ is invertible in $\mathscr{B}$, and if the degree (with respect to X) of c_0 is less or equal to $n-1$, then the 0-components $\{c_{0,0}, c_{1,0}, \ldots, c_{n-1,0}\}$ of $\{c_0, \nabla(c_0), \ldots, \nabla^{n-1}(c_0)\}$ *uniquely determine* c_0. In fact we have the inversion formula

$$c_{0,j} \ := \ \frac{1}{j!} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} \nabla^{j-k}(c_{k,0}) \ , \quad j = 0, \ldots, n-1 \ . \tag{1.4}$$

The idea is then to choose the 0-components equal to the basis of M: $c_{k,0} := e_k$. We then obtain the vector (1.2):

$$c(\mathbf{e}, X) \ := \ \sum_{j=0}^{n-1} \frac{X^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \nabla^k(e_{j-k}) \ . \tag{1.5}$$

This choice implies that the determinant of the base change is a polynomial $P(X) \in \mathscr{B}[X]$ verifying $P(0) = 1$, because the matrix $H(X) \in M_n(\mathscr{B}[X])$ expressing $\{c_0, \nabla(c_0), \ldots, \nabla^{n-1}(c_0)\}$ in the basis $\mathbf{e}$ verifies $H(0) = \mathrm{Id}$. In other words $P(X)$ is invertible as a formal power series in $\mathscr{B}[[X]]$, so that c_0 is a cyclic vector for $\mathrm{M} \otimes_{\mathscr{B}} \mathscr{B}[[X]]$.

[2] The symbol $\#k$ means the number of elements of k, or, if k is infinite, its cardinality.

We now specialize X into an element $t-a$ verifying $d(t-a)=1$, this guarantee that the specialization commutes with the action of the derivation. Let us come to the proof of the above results. If $\mathscr{B}$ is local, and if $t-a$ belongs to the maximal ideal, then $P(t-a)$ is clearly invertible since it is of the form $P(t-a)=P(0)+(t-a)Q(t-a)=1+y$, with y in the maximal ideal. This proves theorem 1.3. Notice that if $\mathscr{B}$ is a field of characteristic 0, then $\mathscr{B}^{d=0}$ is an infinite field, hence there exists at least a constant $a\in\mathscr{B}^{d=0}$ such that $P(t-a)\neq 0$, this is enough to prove Deligne's Theorem 1.1.[3] Now we come to the proof of Theorem 1.2. Katz proves that the ideal $\mathscr{I}$ of $\mathscr{B}$ generated by the values $\{P(t-a_i)\}_{i=0,\ldots,n(n-1)}$ is the unit ideal. He argues as follows. We observe that the polynomial $P(X)$ has degree $\leqslant n(n-1)$, since

$$c_0\wedge\nabla(c_0)\wedge\ldots\wedge\nabla^{n-1}(c_0)\;=\;P(X)\cdot e_0\wedge e_1\wedge\cdots\wedge e_{n-1} \tag{1.6}$$

and the n vectors $c_0,\nabla(c_0),\ldots,\nabla^{n-1}(c_0)$ have all degree $\leqslant(n-1)$. So we write $P=\sum_{s=0}^{n(n-1)}r_sX^s$ and $P(t-a_i)=\sum_{s=0}^{n(n-1)}r_s(t-a_i)^s$. Now for $i\neq j$ one has $(t-a_i)-(t-a_j)=a_j-a_i\neq 0$ in k, so $(t-a_i)-(t-a_j)$ is invertible in $\mathscr{B}$. Hence the Van Der Monde matrix $V:=((t-a_i)^j)_{0\leqslant i,j\leqslant n(n-1)}$ is invertible because its determinant is $\prod_{0\leqslant i<j\leqslant n(n-1)}(a_j-a_i)$. This implies that the ideal $\mathscr{I}$ is equal to the ideal generated by the coefficients $r_0,\ldots,r_{n(n-1)}$. [4] Since $r_0=1$, then $\mathscr{I}=\mathscr{B}$. This concludes the Katz's proofs.

1.1.1 *About the assumptions of Katz's Theorems.* The assumption about the existence of t such that $d(t)=1$ is not completely constrictive. Indeed it is enough to assume the existence of an element $\widetilde{t}\in\mathscr{B}$ such that $d(\widetilde{t})=f$ is invertible in $\mathscr{B}$. Then we replace the derivation d by $\widetilde{d}:=f^{-1}\cdot d$ in order to have $\widetilde{d}(\widetilde{t})=1$. We then consider the connection $\widetilde{\nabla}:=f^{-1}\cdot\nabla$ on M, and we form the Katz's cyclic vector (1.2) constructed from the data of $(\widetilde{d},\widetilde{t},\widetilde{\nabla})$. Then

Lemma 1.4. *The vector c is a cyclic vector for the differential module (M,∇) over $(\mathscr{B},d)$ if and only if c is a cyclic vector for $(\mathrm{M},f\cdot\nabla)$ over $(\mathscr{B},f\cdot d)$, for an arbitrary invertible element $f\in\mathscr{B}$.*

Proof. It is enough to prove that if c is cyclic with respect to (M,∇) then it is a cyclic vector with respect to $(\mathrm{M},f\nabla)$. We have to prove that the base change matrix from the basis $\{c,\nabla(c),\ldots,\nabla^{n-1}(c)\}$ to the family $\{c,(f\nabla)(c),(f\nabla)^2(c),\ldots,(f\nabla)^{n-1}(c)\}$ is invertible. The Leibnitz rule of ∇ gives the relation $\nabla\circ f=f\circ\nabla+d(f)$ where f and $d(f)$ denote respectively the multiplication in M by $f\in\mathscr{B}$ and $d(f)\in\mathscr{B}$. One sees then that $(f\nabla)^k=f^k\nabla^k+\sum_{0\leqslant i\leqslant k-1}\alpha_i(f)\nabla^i$, for convenient elements $\alpha_i(f)\in\mathscr{B}$. This implies that the base change matrix is triangular with $(1,f,f^2,\ldots,f^{n-1})$ in the diagonal. □

Remark 1.5. *The Katz's algorithm is not invariant under the above change of derivation. In other words the Katz's vector c_0 obtained from (d,t,∇) does not coincide with the Katz's vector $\widetilde{c}_0$ constructed from $(\widetilde{d},\widetilde{t},\widetilde{\nabla})$.*[5] *If one of them is a cyclic vector, then it is simultaneously cyclic for ∇ and $\widetilde{\nabla}$, thanks to the above lemma. But actually, in our knowledge, the fact that one of them is cyclic does not imply necessarily that the other is cyclic too.*

1.2 The Katz's base change matrix.

We now investigate the explicit form of the base change matrix $H(X)$. For this we need to introduce some notation. If a basis $\mathbf{e}$ of M is fixed then we can associate to the n-times iterated connection

[3] Notice that Deligne does not ask for the existence of $t\in\mathscr{B}$ satisfying $d(t)=1$. But it is easy to reduce the general case to this one by replacing the non trivial derivation d by $\widetilde{d}:=f\cdot d$, with $f:=d(t)^{-1}$, and then using Lemma 1.4.

[4] The ideal $\mathscr{I}$ is the set of linear combinations $\sum_{i=0}^{n(n-1)}b_iP(t-a_i)={}^tw\cdot(P(t-a_i))_i$ with coefficients b_i in $\mathscr{B}$. Since V is invertible, for all vector v with coefficients in $\mathscr{B}$, there exists w such that ${}^tw\cdot V={}^tv$ and reciprocally. So that any linear combination ${}^tw\cdot(P(t-a_i))_i$ of the family $\{P(t-a_i)\}_i$ is in fact a linear combination of the family $\{r_i\}_i$, because ${}^tw\cdot(P(t-a_i))_i={}^tw\cdot V\cdot(r_i)_i={}^tv\cdot(r_i)_i$, and reciprocally. So $\mathscr{I}$ is the ideal generated by the family $\{r_i\}_i$.

[5] Notice that $d(t)=\widetilde{d}(\widetilde{t})=1$. Once we change d we also have to change t in order to preserve this relation.

$\nabla^n := \nabla \circ \nabla \circ \cdots \circ \nabla$ a matrix $G_n = (g_{n;i,j})_{i,j=0,\ldots,n-1} \in M_n(\mathscr{B})$ whose rows are the image of the basis $\mathbf{e}$ by ∇^n:

$$\nabla^n(e_i) := \sum_{j=0}^{n-1} g_{n;i,j} \cdot e_j \ . \tag{1.7}$$

Proposition 1.6. *The Katz's base change matrix $H(X)$ verifying $(\nabla^i(c_0(\mathbf{e},X)))_i = H(X)(e_i)_i$ has the form*

$$H(X) \ := \ H_0(X) + H_1(X) \cdot G_1 + \cdots + H_{2n-2}(X) \cdot G_{2n-2} \ , \tag{1.8}$$

where the matrices $H_s(X)$, $s = 0, \ldots 2n-2$, all belong to $\mathbb{Z}[\frac{1}{(n-1)!}][X]$ and satisfy the following properties:

i) *One has*

$$H_0(X) \ = \ \begin{pmatrix} 1 & X & \frac{X^2}{2} & \frac{X^3}{3!} & \cdots & \cdots & \frac{X^{n-1}}{(n-1)!} \\ 0 & 1 & X & \frac{X^2}{2} & \frac{X^3}{3!} & \cdots & \frac{X^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & X & \frac{X^2}{2} & \cdots & \frac{X^{n-3}}{(n-3)!} \\ 0 & 0 & 0 & 1 & X & \cdots & \frac{X^{n-4}}{(n-4)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & X \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} \tag{1.9}$$

ii) *If $H_s(X) = (h_{s;i,j}(X))_{i,j}$ then*

$$h_{s;i,j}(X) \ = \ \alpha(s;i,j)\frac{X^{s+j-i}}{(s+j-i)!} \tag{1.10}$$

with

$$\alpha(s;i,j) \ = \ \epsilon_{s;i,j} \cdot \left[\sum_{k=\max(0,s+j-(n-1))}^{\min(i,s)} (-1)^{s+k} \binom{s-k+j}{j}\binom{i}{k} \right] \in \mathbb{Z} \tag{1.11}$$

where

$$\epsilon_{s;i,j} \ = \ \begin{cases} 1 & \textit{if} \quad (s,j) \in [0,n-1+i] \times [\max(0,i-s), \min(n-1,n-1+i-s)] \\ 0 & \textit{if} \quad (s,j) \notin [0,n-1+i] \times [\max(0,i-s), \min(n-1,n-1+i-s)] \end{cases} \tag{1.12}$$

iii) *In particular one has $h_{s;i,j} = 0$ if $j-i$ does not belong to the interval $[\max(1-s,1-n), n-1-s]$.*

Proof. Applying ∇^i to the vector $c_0(\mathbf{e},X)$, and re-summing by setting $s := m+k-j$ one obtains

$$\nabla^i(c_0(\mathbf{e},X)) \ = \ \sum_{m=0}^{n-1}\sum_{j=0}^{m}\sum_{k=0}^{i}(-1)^{m-j}\binom{m}{j}\binom{i}{k}d^{i-k}(\frac{X^m}{m!})\nabla^{m-j+k}(e_j) \tag{1.13}$$

$$= \ \sum_{s=0}^{n-1+i} \ \sum_{j=\max(0,i-s)}^{\min(n-1,n-1+i-s)} \alpha(s;i,j)\frac{X^{s+j-i}}{(s+j-i)!}\nabla^s(e_j) \ , \tag{1.14}$$

where $\alpha(s;i,j)$ is

$$\alpha(s;i,j) \ := \ \left[\sum_{k=\max(0,s+j-(n-1))}^{\min(i,s)} (-1)^{s-k} \binom{s-k+j}{j}\binom{i}{k} \right] \in \mathbb{Z} \ . \tag{1.15}$$

In matrix form, if $H_s(X) = (h_{s;i,j}(X))_{i,j=0,\dots,n-1}$, then

$$
\begin{aligned}
(\nabla^i(c_0(\mathbf{e},X)))_i &= \Big(\sum_{s=0}^{2n-2} H_s(X)G_s\Big)\cdot(e_i)_i \;=\; \sum_{s=0}^{2n-2}\Big(H_s(X)G_s\Big)\cdot(e_i)_i && (1.16)\\
&= \sum_{s=0}^{2n-2}\Big(\sum_{j=0}^{n-1} h_{s;i,j}(X)g_{s;j,k}\Big)_{i,k}\cdot(e_i)_i \;=\; \sum_{s=0}^{2n-2}\Big(\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} h_{s;i,j}(X)g_{s;j,k}e_k\Big)_i && (1.17)\\
&= \sum_{s=0}^{2n-2}\Big(\sum_{j=0}^{n-1} h_{s;i,j}(X)(\sum_{k=0}^{n-1} g_{s;j,k}e_k)\Big)_i \;=\; \sum_{s=0}^{2n-2}\Big(\sum_{j=0}^{n-1} h_{s;i,j}(X)\nabla^s(e_j)\Big)_i\;. && (1.18)
\end{aligned}
$$

So that $\nabla^i(c_0(\mathbf{e},X)) = \sum_{s=0}^{2n-2}\sum_{j=0}^{n-1} h_{s;i,j}(X)\nabla^s(e_j)$. This means that

$$h_{s;i,j}(X) \;=\; \alpha(s;i,j)\cdot\frac{X^{s+j-i}}{(s+j-i)!}\;. \tag{1.19}$$

□

Below we write the first examples of $H(X)$ for $n = 2, 3, 4, 5$.

$n = 2$:

$$H(X) = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -X & 0 \\ 0 & X \end{pmatrix} G_1 + \begin{pmatrix} 0 & 0 \\ -X & 0 \end{pmatrix} G_2$$

$n = 3$:

$$H(X) = \begin{pmatrix} 1 & X & \frac{X^2}{2!} \\ 0 & 1 & X \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -X & -2\frac{X^2}{2} & 0 \\ 0 & -X & -\frac{X^2}{2} \\ 0 & 0 & 2X \end{pmatrix} G_1 + \begin{pmatrix} \frac{X^2}{2} & 0 & 0 \\ 0 & -2\frac{X^2}{2} & 0 \\ 0 & -3X & \frac{X^2}{2} \end{pmatrix} G_2 + \begin{pmatrix} 0 & 0 & 0 \\ \frac{X^2}{2} & 0 & 0 \\ X & -2\frac{X^2}{2} & 0 \end{pmatrix} G_3 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{X^2}{2} & 0 & 0 \end{pmatrix} G_4$$

$n = 4$:

$$H(X) = \begin{pmatrix} 1 & X & \frac{X^2}{2!} & \frac{X^3}{3!} \\ 0 & 1 & X & \frac{X^2}{2!} \\ 0 & 0 & 1 & X \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -X & -2\frac{X^2}{2} & -3\frac{X^3}{3!} & 0 \\ 0 & -X & -2\frac{X^2}{2} & \frac{X^3}{3!} \\ 0 & 0 & -X & 2\frac{X^2}{2} \\ 0 & 0 & 0 & 3X \end{pmatrix} G_1 + \begin{pmatrix} \frac{X^2}{2} & 3\frac{X^3}{3!} & 0 & 0 \\ 0 & \frac{X^2}{2} & -3\frac{X^3}{3!} & 0 \\ 0 & 0 & -5\frac{X^2}{2} & \frac{X^3}{3!} \\ 0 & 0 & -6X & 3\frac{X^2}{2} \end{pmatrix} G_2 +$$

$$\begin{pmatrix} -\frac{X^3}{3!} & 0 & 0 & 0 \\ 0 & 3\frac{X^3}{3!} & 0 & 0 \\ 0 & 4\frac{X^2}{2} & -3\frac{X^3}{3!} & 0 \\ 0 & 4X & -8\frac{X^2}{2} & \frac{X^3}{3!} \end{pmatrix} G_3 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{X^3}{3!} & 0 & 0 & 0 \\ -\frac{X^2}{2} & 3\frac{X^3}{3!} & 0 & 0 \\ -X & 7\frac{X^2}{2} & -3\frac{X^3}{3!} & 0 \end{pmatrix} G_4 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{X^3}{3!} & 0 & 0 & 0 \\ -2\frac{X^2}{2} & 3\frac{X^3}{3!} & 0 & 0 \end{pmatrix} G_5 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{X^3}{3!} & 0 & 0 & 0 \end{pmatrix} G_6$$

$n = 5$:

$$H(X) = \begin{pmatrix} 1 & X & \frac{X^2}{2!} & \frac{X^3}{3!} & \frac{X^4}{4!} \\ 0 & 1 & X & \frac{X^2}{2!} & \frac{X^3}{3!} \\ 0 & 0 & 1 & X & \frac{X^2}{2} \\ 0 & 0 & 0 & 1 & X \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -X & -2\frac{X^2}{2!} & -3\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 \\ 0 & -X & -2\frac{X^2}{2!} & -3\frac{X^3}{3!} & \frac{X^4}{4!} \\ 0 & 0 & -X & -2\frac{X^2}{2!} & 2\frac{X^3}{3!} \\ 0 & 0 & 0 & -X & 3\frac{X^2}{2!} \\ 0 & 0 & 0 & 0 & 4X \end{pmatrix} G_1 + \begin{pmatrix} \frac{X^2}{2!} & 3\frac{X^3}{3!} & 6\frac{X^4}{4!} & 0 & 0 \\ 0 & \frac{X^2}{2!} & 3\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 \\ 0 & 0 & \frac{X^2}{2!} & -7\frac{X^3}{3!} & \frac{X^4}{4!} \\ 0 & 0 & 0 & -9\frac{X^2}{2!} & 3\frac{X^3}{3!} \\ 0 & 0 & 0 & -10X & 6\frac{X^2}{2!} \end{pmatrix} G_2 +$$

$$\begin{pmatrix} -\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 & 0 & 0 \\ 0 & -\frac{X^3}{3!} & 6\frac{X^4}{4!} & 0 & 0 \\ 0 & 0 & 9\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 \\ 0 & 0 & 10\frac{X^2}{2!} & -11\frac{X^3}{3!} & \frac{X^4}{4!} \\ 0 & 0 & 10X & -20\frac{X^2}{2!} & 4\frac{X^3}{3!} \end{pmatrix} G_3 + \begin{pmatrix} \frac{X^4}{4!} & 0 & 0 & 0 & 0 \\ 0 & -4\frac{X^4}{4!} & 0 & 0 & 0 \\ 0 & -5\frac{X^3}{3!} & 6\frac{X^4}{4!} & 0 & 0 \\ 0 & -5\frac{X^2}{2!} & 15\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 \\ 0 & -5X & 25\frac{X^2}{2!} & -13\frac{X^3}{3!} & \frac{X^4}{4!} \end{pmatrix} G_4 +$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{X^4}{4!} & 0 & 0 & 0 & 0 \\ \frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 & 0 & 0 \\ \frac{X^2}{2!} & -9\frac{X^3}{3!} & 6\frac{X^4}{4!} & 0 & 0 \\ X & -14\frac{X^2}{2!} & 21\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 \end{pmatrix} G_5 + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{X^4}{4!} & 0 & 0 & 0 & 0 \\ 2\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 & 0 & 0 \\ 3\frac{X^2}{2!} & -13\frac{X^3}{3!} & 6\frac{X^4}{4!} & 0 & 0 \end{pmatrix} G_6 + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{X^4}{4!} & 0 & 0 & 0 & 0 \\ 3\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 & 0 & 0 \end{pmatrix} G_7 +$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{X^4}{4!} & 0 & 0 & 0 & 0 \end{pmatrix} G_8$$

2. Small connections are cyclic over an ultrametric Banach algebra.

In this section we provide a sufficient condition for differential modules over an ultrametric Banach algebras, in order to guarantee that the Katz's vector (1.2) is a cyclic vector.

2.1 Norms and matrices

We recall that an ultrametric norm on a commutative ring with unit $\mathscr{B}$ is a map $|.| : \mathscr{B} \to \mathbb{R}_{\geqslant 0}$ satisfying $|0| = 0$, $|1| = 1$, $|a+b| \leqslant \max(|a|,|b|)$, $|ab| \leqslant |a| \cdot |b|$, for all $a, b \in \mathscr{B}$. We require moreover $|na| = |n||a|$ for all $n \in \mathbb{Z}$, $a \in \mathscr{B}$. Hence, in particular, the norm on $\mathbb{Z}$ induced by $|.|$ is ultrametric and so $|n| \leqslant 1$ for all $n \in \mathbb{Z}$. If $\mathscr{B}$ is complete an separated[6] with respect to $|.|$ then we say that $\mathscr{B}$ is an *ultrametric Banach algebra*. A norm on $M_n(\mathscr{B})$ is a map $\|.\| : M_n(\mathscr{B}) \to \mathbb{R}_{\geqslant 0}$ satisfying $\|0\| = 0$, $\|1\| = 1$, $\|A+B\| \leqslant \max(\|A\|,\|B\|)$, $\|AB\| = \|A\| \cdot \|B\|$, $\|bA\| = |b|\|A\|$ for all $b \in \mathscr{B}$, $A, B \in M_n(\mathscr{B})$. In the sequel we will consider on $M_n(\mathscr{B})$ two norms

$$\text{sup-norm:} \qquad |(a_{i,j})| \quad := \sup_{i,j} |a_{i,j}| \;, \tag{2.1}$$

$$\rho\text{-sup-norm:} \qquad |(a_{i,j})|^{(\rho)} := \sup_{i,j} |a_{i,j}| \rho^{j-i} \;, \qquad \rho > 0 \;. \tag{2.2}$$

Notice that if $\mathscr{C}$ is a $\mathscr{B}$-algebra together with a norm $|.|_{\mathscr{C}}$ extending[7] that of $\mathscr{B}$, and if $c \in \mathscr{C}$ is an element with norm $|c| = \rho^{-1}$, then $|A|^{(\rho)} = |\Lambda_c^{-1} A \Lambda_c|$, for all $A \in M_n(\mathscr{B})$, where Λ_c is the diagonal matrix with diagonal equal to $(1, c, c^2, \ldots, c^{n-1})$.

2.1.1 *Norm of derivation* Let $(\mathrm{B}, |.|)$ be an ultrametric Banach algebra, and let $\|.\| : \mathrm{M}_n(\mathscr{B}) \to \mathbb{R}_{\geqslant 0}$ be a fixed norm. Let now $d : \mathscr{B} \to \mathscr{B}$ be a continuous derivation. We extend d to $M_n(\mathscr{B})$ by $d((a_{i,j})_{i,j}) := (d(a_{i,j}))_{i,j}$. Let $|d|$ denotes the norm operator of d acting on $\mathscr{B}$:

$$|d| \;:=\; \sup_{b \neq 0, b \in \mathscr{B}} \frac{|d(b)|}{|b|} \;. \tag{2.3}$$

We will always assume that the norm $\|.\|$ verifies

$$\|d(A)\| \;\leqslant\; |d| \cdot \|A\| \tag{2.4}$$

for all $A \in M_n(\mathscr{B})$. This holds for the sup-norm and the ρ-sup-norm.

2.2 Norm of the matrix of the connection and cyclic vectors

We consider as above an ultrametric Banach algebra $(\mathrm{B}, |.|)$, together with a continuous derivation $d : \mathscr{B} \to \mathscr{B}$. Let $\|.\| : \mathrm{M}_n(\mathscr{B}) \to \mathbb{R}_{\geqslant 0}$ be a fixed norm satisfying (2.4), for which $M_n(\mathscr{B})$ is complete and separated. Let (M, ∇) be a differential module. We assume that there is a element $t \in \mathscr{B}$ such that $d(t) = 1$. In order to consider the Katz's base change matrix (1.8) we assume that $(n-1)!$ is invertible in $\mathscr{B}$. As in the above sections we denote by G_n the matrix of the n-th iterated connection $\nabla^n : \mathrm{M} \to \mathrm{M}$ with respect to a basis $\mathbf{e}$. The simple idea of this section is the following.

Lemma 2.1. *If the matrices $G_1, \ldots, G_{2n-2}$ are small enough in norm, in order to verify*

$$\|H_0(-t) H_s(t) G_s\| < 1 \tag{2.5}$$

for all $s = 1, \ldots, 2n-2$, then the Katz's base change matrix

$$H(t) \;:=\; H_0(t) + H_1(t) G_1 + \cdots + H_{2n-2} G_{2n-2} \tag{2.6}$$

is invertible.

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[6] $\mathscr{B}$ is separated if and only if $|a| = 0$ implies $a = 0$.

[7] i.e. in order that the structural morphism $\mathscr{B} \to \mathscr{C}$ is an isometry

Proof. Indeed $H_0(t)$ is always invertible with inverse

$$H_0(t)^{-1} \;=\; H_0(-t)\;. \tag{2.7}$$

So that $H(t)$ is invertible if and only if $H_0(t)^{-1}H(t) = 1+\sum_{s=1}^{2n-2} H_0(-t)H_s(t)G_s$ is invertible. □

Of course a sufficient condition to have (2.5) is

$$\|G_s\| < (\|H_s(t)\|\cdot\|H_0(-t)\|)^{-1}\;. \tag{2.8}$$

In the following subsection we provide an explicit upper bound on the sup-norm and on the ρ-sup-norm of $G := G_1$ sufficient to guarantee (2.8) for all $s = 1,\ldots,2n-2$. In order to do that we relate the norm of G_s with that of G_1 by the following

LEMMA 2.2. *For all $s \geqslant 1$ one has*

$$\|G_s\| \;\leqslant\; \|G_1\|\cdot\sup(\|G_1\|,|d|)^{s-1}\;. \tag{2.9}$$

Proof. We have the recursive relation $G_{s+1} = d(G_s)+G_sG_1$. Since we assume $\|d(G_s)\| \leqslant |d|\|G_s\|$, then one easily has $\|G_{s+1}\| \leqslant \|G_s\|\cdot\max(|d|,\|G_1\|)$. By induction the lemma is proved. □

2.3 Upper bound for the sup-norm.

Let now the chosen norm $\|.\| = |.|$ be the sup-norm (2.1). We are looking for a condition on $|G_1|$ that guarantee

$$|G_s| < (|H_s(t)|\cdot|H_0(-t)|)^{-1}\;, \tag{2.10}$$

for all $s = 1,\ldots,2n-2$. Thanks to Proposition 1.6 one has[8]

$$|H_0(t)| \;=\; |H_0(-t)| = \sup_{i=0,\ldots,n-1} |t^i|/|i!| \tag{2.11}$$

$$|H_s(t)| \;\leqslant\; |H_0(-t)|\;,\ \text{for all } s = 1,\ldots,2n-2\;. \tag{2.12}$$

Indeed since $\alpha(s;i,j)$ is an integer, and since the norm $|.|$ is ultrametric, one has $|\alpha(s;i,j)| \leqslant 1$. From this we have

$$(|H_0(-t)||H_s(t)|)^{-1} \;\geqslant\; |H_0(t)|^{-2}\;. \tag{2.13}$$

On the other hand by Lemma 2.2 one has

$$|G_s| \;\leqslant\; |G_1|\cdot\max(|G_1|,|d|)^{s-1}\;. \tag{2.14}$$

Hence it is enough to prove that

$$|G_1|\cdot\max(|G_1|,|d|)^{s-1} \;<\; |H_0(t)|^{-2}\;, \tag{2.15}$$

for all $s = 1,\ldots,2n-2$.

PROPOSITION 2.3. *Assume that*

$$|G_1| \;<\; |H_0(t)|^{-2}\cdot\min\Big(1,\frac{1}{|d|^{2n-3}}\Big) \;=\; \min\Big(1,\frac{1}{|t|},\frac{|2|}{|t^2|},\ldots,\frac{|(n-1)!|}{|t^{n-1}|}\Big)^2\cdot\min\Big(1,\frac{1}{|d|^{2n-3}}\Big)\;. \tag{2.16}$$

Then (M,∇) is cyclic and the Katz's vector $c_0(\mathbf{e},t)$ is a cyclic vector for M.

Proof. We observe that both minimums are $\leqslant 1$, moreover $\min(1,1/|t|,|2|/|t^2|,\ldots,|(n-1)!|/|t^{n-1}|) \leqslant 1/|t|$. Since $d(t) = 1$, then $|d||t| \geqslant 1$, and hence $1/|t| \leqslant |d|$. Our assumption then implies $|G_1| < |d|$. Hence (2.15) becomes $|G_1|\cdot|d|^{s-1} < |H_0(t)|^{-2}$ for all $s = 1,\ldots,2n-2$. This inequality is fulfilled if and only if $|G_1| < \min_{s=1,\ldots,2n-2}|H_0(t)|^{-2}/|d|^{s-1} = |H_0(t)|^{-2}\cdot\min(1,1/|d|^{2n-3})$ which is our assumption. □

[8]Notice that $|.|$ is not assumed to be multiplicative, hence $|t^i| \leqslant |t|^i$.

2.4 Upper bound for the ρ-sup-norm with $\rho = |t|^{-1}$.

We assume that

$$\rho \;:=\; |t|^{-1}\,, \qquad |(a_{i,j})_{i,j}|^{(|t|^{-1})} \;=\; \sup_{i,j} |a_{i,j}||t|^{i-j}\;. \tag{2.17}$$

As above we shall provide a condition on G_1 to guarantee

$$|G_s|^{(|t|^{-1})} < (|H_s(t)|^{(|t|^{-1})}\cdot|H_0(-t)|^{(|t|^{-1})})^{-1}\;, \tag{2.18}$$

for all $s = 1,\ldots,2n-2$. Since $|.|$ is not assumed to be multiplicative, hence $|t^i| \leqslant |t|^i$. This implies

$$|H_0(-t)|^{(|t|^{-1})} = |H_0(t)|^{(|t|^{-1})} = \sup_{i,j=0,\ldots,n-1}\frac{|t^i||t|^{-i}}{|i!|} \;\leqslant\; \sup_{i=0,\ldots,n-1}\frac{1}{|i!|} \;=\; \frac{1}{|(n-1)!|}\;. \tag{2.19}$$

Of course if $|.|$ is power multiplicative[9], the above inequality is actually an equality. Notice that since $|.|$ is ultrametric on $\mathbb{Z}$, then $|(n-1)!| \leqslant 1$.

Lemma 2.4. *One has*

$$|H_s(t)|^{(|t|^{-1})} \;\leqslant\; \frac{|t^s|}{|(n-1)!|}\;. \tag{2.20}$$

Proof. Thanks to proposition 1.6, for all $s = 1,\ldots,2n-2$ one has

$$|H_s(t)|^{(|t|^{-1})} \;=\; \max_{i,j=0,\ldots,n-1}|h_{s;i,j}||t|^{i-j} = \max_{i,j=0,\ldots,n-1}|\alpha(s;i,j)|\frac{|t^{s+j-i}|}{|(s+j-i)!|}|t|^{i-j}\;. \tag{2.21}$$

Now $|\alpha(s,i,j)| \leqslant 1$, and it is equal to 0 for $j-i \notin [\max(1-s,1-n),n-1-s]$. So, since $|t^{s+j-i}| \leqslant |t^s||t|^{j-i}$, then we obtain $|H_s(t)|^{(|t|^{-1})} \leqslant \max_{j-i\in[\max(1-s,1-n),n-1-s]}\frac{|t^{s+j-i}|}{|(s+j-i)!|}|t|^{i-j} \leqslant \max_{r\in[\max(1-s,1-n),n-1-s]}\frac{|t^s|}{|(s+r)!|} = \frac{|t^s|}{|(n-1)!|}$. □

Then one has

$$(|H_0(-t)|^{(|t|^{-1})}\cdot|H_s(t)|^{(|t|^{-1})})^{-1} \;\geqslant\; \frac{|(n-1)!|^2}{|t|^s}\;. \tag{2.22}$$

On the other hand by Lemma 2.2 one has

$$|G_s|^{(|t|^{-1})} \;\leqslant\; |G_1|^{(|t|^{-1})}\cdot\max(|G_1|^{(|t|^{-1})},|d|)^{s-1}\;. \tag{2.23}$$

So condition (2.18) is fulfilled if

$$|G_1|^{(|t|^{-1})}\cdot\max(|G_1|^{(|t|^{-1})},|d|)^{s-1} \;<\; \frac{|(n-1)!|^2}{|t|^s} \tag{2.24}$$

for all $s = 1,\ldots,2n-2$.

Proposition 2.5. *Assume that*

$$|G_1|^{(|t|^{-1})} \;<\; \frac{|(n-1)!|^2|d|}{(|d||t|)^{2n-2}}\;. \tag{2.25}$$

Then (M,∇) *is cyclic and the Katz's vector* $c_0(\mathbf{e},t)$ *is a cyclic vector for* M.

Proof. Since $d(t)=1$, then $|d||t| \geqslant 1$. Our assumption then implies $|G_1|^{(|t|^{-1})} < |(n-1)!|^2|d| \leqslant |d|$. Hence (2.24) becomes $|G_1|^{(|t|^{-1})}\cdot|d|^{s-1} < \frac{|(n-1)!|^2}{|t|^s}$ for all $s = 1,\ldots,2n-2$. This inequality is fulfilled if and only if $|G_1|^{(|t|^{-1})} < \min_{s=1,\ldots,2n-2}\frac{|(n-1)!|^2|d|}{(|t||d|)^s} = \frac{|(n-1)!|^2|d|}{(|t||d|)^{2n-2}}$ which is our assumption. □

[9]The norm $|.|$ is power multiplicative if it verifies $|b^n| = |b|^n$ for all $b \in \mathscr{B}$, and all integer $n \geqslant 0$

2.5 Upper bound for the ρ-sup-norm with $\rho = |d|$.

We now set

$$\rho \ := \ |d| \ , \qquad |(a_{i,j})_{i,j}|^{(|d|)} \ = \ \sup_{i,j} |a_{i,j}||d|^{j-i} \ . \tag{2.26}$$

We quickly reproduce the computations of section 2.4. As usual we have to prove that $|G_s|^{(|d|)} < (|H_0(-t)|^{(|d|)} \cdot |H_s(t)|^{(|d|)})^{-1}$. One has

$$|H_0(-t)|^{(|d|)} \ = \ |H_0(t)|^{(|d|)} \ = \ \max_{i=0,\ldots,n-1} \frac{|t^i||d|^i}{|i!|} \ \leqslant \ \max_{i=0,\ldots,n-1} \frac{(|d||t|)^i}{|i!|} \tag{2.27}$$

As usual this becomes an equality if $|.|$ is power multiplicative.

Lemma 2.6. *Let $\rho \geqslant 1$ be a real number, and let $s \geqslant 0$ be an integer. The sequence of real numbers $i \mapsto \rho^i/|(s+i)!|$ is increasing.*

Proof. One has $\rho^{i+1}/|(s+i+1)!| \geqslant \rho^i/|(s+i)!|$ if and only if $\rho/|s+i+1| \geqslant 1$. This last is true since the norm of a integer is $\leqslant 1$, because the norm is ultrametric. □

Since $d(t) = 1$, then $|d||t| \geqslant 1$, so we then have

$$|H_0(-t)|^{(|d|)} \ \leqslant \ \frac{(|d||t|)^{n-1}}{|(n-1)!|} \ . \tag{2.28}$$

Lemma 2.7. *One has*

$$|H_s(t)|^{(|d|)} \ \leqslant \ |t^s| \cdot \frac{(|d||t|)^{n-1-s}}{|(n-1)!|} \tag{2.29}$$

Proof. As in lemma 2.4 one has

$$|H_s(t)|^{(|d|)} = \max_{i,j} |\alpha(s;i,j)| \frac{|t^{s+j-i}||d|^{j-i}}{|(s+j-i)!|} \leqslant \max_{i,j} \frac{|t^s|(|d||t|)^{j-i}}{|(s+j-i)!|} = |t^s| \cdot \max_r \frac{(|d||t|)^r}{|(s+r)!|} \ , \tag{2.30}$$

where i, j runs in $[0, n-1]$, and $r \in [\max(1-s, 1-n), n-1-s]$. By Lemma 2.6 the last maximum is equal to $(|d||t|)^{n-1-s}/|(n-1)!|$. □

Then one has

$$(|H_0(-t)|^{(|d|)} \cdot |H_s(t)|^{(|d|)})^{-1} \ \geqslant \ \frac{|(n-1)!|^2}{|t^s| \cdot (|d||t|)^{2n-2-s}} \ . \tag{2.31}$$

As usual one also has $|G_s|^{(|d|)} \leqslant |G_1|^{(|d|)} \cdot \max(|G_1|^{(|d|)}, |d|)^{s-1}$, so what we need is

$$|G_1|^{(|d|)} \cdot \max(|G_1|^{(|d|)}, |d|)^{s-1} \ < \ \frac{|(n-1)!|^2}{|t|^s(|d||t|)^{2n-2-s}} \tag{2.32}$$

for all $s = 1, \ldots, 2n-2$.

Proposition 2.8. *Assume that*

$$|G_1|^{(|d|)} \ < \ \frac{|(n-1)!|^2|d|}{(|d||t|)^{2n-2}} \ . \tag{2.33}$$

Then (M, ∇) is cyclic and the Katz's vector $c_0(\mathbf{e}, t)$ is a cyclic vector for M.

Proof. Since $d(t) = 1$, then $|d||t| \geqslant 1$. Our assumption then implies $|G_1|^{(|t|^{-1})} < |(n-1)!|^2|d| \leqslant |d|$. Hence (2.32) becomes $|G_1|^{(|d|)} \cdot |d|^{s-1} < \frac{|(n-1)!|^2}{|t|^s(|d||t|)^{n-1-s}}$, for all $s = 1, \ldots, 2n-2$. But this is actually our assumption. □

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