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# Rank one solvable $p$-adic differential equations and finite Abelian characters via Lubin–Tate groups

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**Abstract** We introduce a new class of exponentials of Artin–Hasse type, called $\boldsymbol{\pi}$-exponentials. These exponentials depend on the choice of a generator $\boldsymbol{\pi}$ of the Tate module of a Lubin–Tate group $\mathfrak{G}$ over $\mathbb{Z}_p$. They arise naturally as solutions of solvable differential modules over the Robba ring. If $\mathfrak{G}$ is isomorphic to $\widehat{\mathbb{G}}_m$ over $\mathbb{Z}_p$, we develop methods to test their over-convergence, and get in this way a stronger version of the Frobenius structure theorem for differential equations. We define a natural transformation of the Artin–Schreier complex into the Kummer complex. This provides an explicit generator of the Kummer unramified extension of $\mathcal{E}^{\dagger}_{K_\infty}$, whose residue field is a given Artin–Schreier extension of $k((t))$, where $k$ is the residue field of $K$. We then compute explicitly the group, under tensor product, of isomorphism classes of rank one solvable differential equations. Moreover, we get a canonical way to compute the rank one $\varphi$-module over $\mathcal{E}^{\dagger}_{K_\infty}$ attached to a rank one representation of $\mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t)))$, defined by an Artin–Schreier character.

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## 0 Introduction

The aim of this paper is to make the theory of rank one solvable differential equations over the Robba ring $\mathcal{R}_K$ (cf. 1.1) as explicit as possible, where $(K, |.|)$ is a complete ultrametric field with residue field $k$. It is known (cf. [1,16,20]) that, under some restrictions on $K$ and $k$, a $p$-adic differential module with Frobenius structure over $\mathcal{R}_K$ becomes unipotent, after pull back on a covering of $\mathcal{R}_K$, coming from a separable extension of $\mathrm{E} := k((t))$. In particular, in [20] the aim is to express this module via extension of rank one modules, and get a $p$-adic analogue of Turritin's classical Theorem for $K((T))$-differential modules.

Let $\partial_T := T\frac{\mathrm{d}}{\mathrm{d}T}$. We shall answer to the following questions:

1. When is a given differential equation
$$L = \partial_T - g(T), \quad g(T) = \sum a_i T^i \in \mathcal{R}_K \tag{0.1}$$
solvable? Can we read the solvability of $L$ from the coefficients $a_i$ of $g(T)$?
2. What is the irregularity of $L$?
3. Can we explicitly describe the group (under tensor product) $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K)$ of isomorphism classes of rank one solvable differential equations over $\mathcal{R}_K$?
4. How does this differential equation change under Artin–Schreier extensions? In particular what is the family of rank one solvable modules becoming trivial after a given separable extension of $\mathrm{E} = k((t))$?
5. Let $\mathrm{E}^{\mathrm{sep}}$ be the separable closure of E. What is explicitly the rank one $\varphi$-module attached to an Artin–Schreier character (or rank one representation) of $\mathrm{G}_{\mathrm{E}} := \mathrm{Gal}(\mathrm{E}^{\mathrm{sep}}/\mathrm{E})$ via the theory of Fontaine–Katz? In particular what is the solvable equation attached to this $\varphi$-module?

## 0.1 Robba exponentials

The first example of irregular solvable differential equation was given by Dwork with the function $\exp(\pi T^{-1})$ which is the Taylor solution at $\infty$ of the irregular operator $\partial_T + \pi T^{-1}$, where $\pi$ is a solution of the equation $X^{p-1} = -p$. Dwork shows that the exponential $\vartheta(T^{-1}) := \exp(\pi(T^{-p} - T^{-1}))$ is over-convergent (i.e. converges for $|T| > 1 - \varepsilon$, for some $\varepsilon > 0$). This provides the so called "Frobenius Structure" isomorphism between $\partial_T + \pi T^{-1}$, and its pull back by Frobenius $\partial_T + p\pi T^{-p}$.

In [22], Robba generalizes the example of Dwork by producing a class of exponentials, here called $E_m(T)$, commonly known as Robba's exponentials. Namely Robba shows that, for all number $\pi_0$ such that $|\pi_0| = |p|^{\frac{1}{p-1}}$, there exists a sequence $\alpha_1, \alpha_2, \dots$ such that, for all $m \geq 1$, the exponential

$$E_m(T^{-1}) = \exp\left(\pi_0\left(\frac{T^{-p^m}}{p^m} + \alpha_1\frac{T^{-p^{m-1}}}{p^{m-1}} + \cdots + \alpha_m T^{-1}\right)\right) \qquad (0.2)$$

converges in the disk $|T| > 1$, and hence the operator $L = \partial_T + \pi_0(T^{-p^m} + \alpha_1 T^{-p^{m-1}} + \cdots + \alpha_m T^{-1})$, with $E_m(T^{-1})$ as solution, is solvable at $\rho = 1$. Moreover Robba shows the necessity of the condition $|\pi_0\alpha_i| = |\pi_0|^{1/p^i}$, for all $i \geq 0$. This construction leads Robba to define the $p$-adic irregularity of a solvable differential equation as the slope at $1^-$ of the radius of convergence (cf. Definition 1.8).

But Robba's construction is not sufficient for two reasons. The first one is that the numbers $\alpha_i$ are obtained as intersection of a decreasing sequences of disks, and then the field $K$ must be algebraically closed and spherically complete. The second reason is that Robba was not able to prove the over-convergence of $E_m(T^{-p})/E_m(T^{-1})$, since the $\alpha_i$'s are essentially unknown.

These problems are solved by Matsuda in [18]. He simplifies remarkably the proof of Robba by using the Artin–Hasse exponential. The idea of introducing the Artin–Hasse exponential is due to Dwork (cf. [13, 21.1]), and Robba (cf. [22, 10.12]). Matsuda shows that, if $\xi_{m+1}$ is a primitive $p^{m+1}$th root of 1, and if $\xi_{m+1-j} := \xi_{m+1}^{p^j}$, then we can choose $\alpha_i = (\xi_i - 1)/(\xi_0 - 1)$. Then

$$E_m(T^{-1}) = \exp\left((\xi_1 - 1)\frac{T^{-p^m}}{p^m} + (\xi_2 - 1)\frac{T^{-p^{m-1}}}{p^{m-1}} + \cdots + (\xi_{m+1} - 1)T^{-1}\right). \qquad (0.3)$$

Matsuda proves also that, if $p \neq 2$, then the exponential $E_m(T^{-p})/E_m(T^{-1})$ is over-convergent. He obtains these results by a quite complicates, but elementary, explicit estimation of the valuation of the coefficients of this exponential.

For the first time we see, in the paper of Matsuda, the algebraic nature of these analytic exponentials. Indeed, if $\alpha : \mathrm{G_E} \to \Lambda^\times$ is a character of $\mathrm{G_E}$ into a finite extension $\Lambda/\mathbb{Q}_p$, such that $\alpha(\mathrm{G_E})$ is finite, then Matsuda shows that the irregularity of the differential equation, attached to the $\varphi$-$\nabla$-module over $\mathcal{E}_K^\dagger$ defined by $\alpha$, is equal to the Swan conductor of $\alpha$.

Independently from Matsuda, Chinellato, under the direction of Dwork, obtains a new algorithm showing the existence of the $\alpha_i$s (cf. [8]).

Even with the great progress given by Matsuda, André, Kedlaya, Crew, Mebkhout, Tsuzuki and others, the questions given in 0.1 are still open, and are the object of this paper.

We generalize, and improve, the techniques of Matsuda and Chinellato, by invoking the Lubin–Tate theory. We recall that the Artin–Hasse exponential $E(-,T)$ is the group morphism $E(-,T) : \mathbf{W}(\mathrm{B}) \to 1 + T\mathrm{B}[[T]]$, functorial on the ring B, sending the Witt vector $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots) \in \mathbf{W}(\mathrm{B})$ into the series

$$E(\boldsymbol{\lambda}, T) = \exp\left(\phi_0 T + \phi_1 \frac{T^p}{p} + \phi_2 \frac{T^{p^2}}{p^2} + \cdots\right), \tag{0.4}$$

where $\langle \phi_0, \phi_1, \ldots\rangle \in \mathrm{B}^{\mathbb{N}}$ is the phantom vector of $\boldsymbol{\lambda}$ (cf. Eq. (1.17)). If $\mathrm{B} = \mathcal{O}_{K^{\mathrm{alg}}}$, then this exponential has bounded coefficients, and hence converges at least for $|T| < 1$. Given a Frobenius automorphism of $\mathbb{Z}_p[[X]]$, that is a series $P(X) \in X\mathbb{Z}_p[[X]]$ lifting $X^p \in \mathbb{F}_p[[X]]$, we consider a sequence $\{\pi_j\}_{j\geq 0}$ in $\mathcal{O}_{K^{\mathrm{alg}}}$, such that $P(\pi_0) = 0$, and $P(\pi_{j+1}) = \pi_j$, for all $j \geq 0$. Then we provide, for all $m \geq 0$, a Witt vector $[\pi_m] \in \mathbf{W}(\mathcal{O}_{K^{\mathrm{alg}}})$, whose phantom vector is $\langle \pi_m, \ldots, \pi_0, 0, 0, \ldots\rangle$. In this way, we obtain a large class of exponentials of "Robba" type:

$$E_m(T) := E([\pi_m], T) = \exp\left(\pi_m T + \pi_{m-1}\frac{T^p}{p} + \cdots + \pi_0 \frac{T^{p^m}}{p^m}\right). \tag{0.5}$$

We show then that *the radius of convergence of these exponentials is* 1 *if and only if* $P(X)$ *is a Lubin–Tate series* (*cf. Eq.* (1.24)), *which thus defines a Lubin–Tate group* $\mathfrak{G}_P$. In this case $\boldsymbol{\pi} := (\pi_j)_{j\geq 0}$ is a generator of the Tate module of $\mathfrak{G}_P$ (cf. Proposition 2.2). If $\mathfrak{G}_P$ is the formal multiplicative group $\widehat{\mathbb{G}}_m$, that is if $P(X) = (X+1)^p - 1$, then we recover Matsuda's exponentials (0.3). On the other hand, if $P(X) = pX + X^p$, we recover, for $m = 0$, Dwork's exponential. Observe that, in the case considered by Dwork, the formal group $\mathfrak{G}_P$ is isomorphic, but not equal, to $\widehat{\mathbb{G}}_m$.

Furthermore, we show that $E_m(T^p)/E_m(T)$ *is over-convergent, for all* $m \geq 0$, *if and only if* $\mathfrak{G}_P$ *is isomorphic* (*but not necessary equal*) *to* $\widehat{\mathbb{G}}_m$. This is the reason of the over-convergence of the exponentials $E_m(T^p)/E_m(T)$ of Matsuda and Dwork.

From this starting point we develop the explicit link between abelian characters of $\mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t)))$ and rank one solvable differential equations over $\mathcal{R}_K$, and examine various applications.

## 0.2 Organization of the paper

In Sects. 1.1, 1.2, 1.3, 1.4, and 1.5 we give the definitions and recall some facts used in the sequel.

In Sect. 2.1 we define some canonical Witt vectors with coefficients in $\mathbb{Z}_p[[X]]$, and show their properties with respect to the Artin–Hasse exponential. In Sect. 2.2 we introduce a new class of exponentials called $\boldsymbol{\pi}$-exponentials (cf. Definition 2.4), and show their main properties with respect to the convergence/over-convergence.

In Sect. 2.3, we give the first application. Fix a Lubin–Tate group $\mathfrak{G}_P$ isomorphic to $\widehat{\mathbb{G}}_m$, and a generator $\boldsymbol{\pi} = (\pi_j)_{j\geq 0}$ of the Tate module. Let $L$ be a complete discrete valued field, with residue field $k_L$, and let $\varphi : L \to L$ be a lifting of the Frobenius $x \mapsto x^p$ of $k_L$. Let $L_m := L(\xi_m)$, where $\xi_m$ is a primitive $p^{m+1}$th root of unity. It is well known that we have the Henselian bijection

$$\{\text{Finite unramified extensions of } L\} \xrightarrow{\sim} \{\text{Finite separable extensions of } k_L\}.$$

We shall describe an inverse of this map. Let $k'/k_L$ be a finite cyclic abelian extension of degree $d$, and let $L'/L$ be the corresponding unramified extension. If $(d,p) = 1$, and if $k_L$ contains the $d$th roots of 1, then $k'/k_L$ is of Kummer type, and hence $L' = L(\theta)$, where $\theta$ is the Teichmüller representative of a Kummer generator $\bar{\theta} \in k'$.

On the other hand, if $d = p^m$, then $k'$ is of Artin–Schreier type (cf. Remark 1.10), and it is generated, over $k_L$, by (the entries of) a Witt vector $\bar{\boldsymbol{\nu}} \in \mathbf{W}_m(k_L^{\mathrm{sep}})$, which is solution of an equation of the type $\bar{\mathrm{F}}(\bar{\boldsymbol{\nu}}) - \bar{\boldsymbol{\nu}} = \bar{\boldsymbol{\lambda}}$, where $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_L)$ is a so called Witt vector "defining" $k'$. In this case $L'_m/L_m$ is again a Kummer extension, since all cyclic extensions of $L_m$ whose degree is $p^m$ are Kummer. Now choose an arbitrary lifting $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_L)$ of $\bar{\boldsymbol{\lambda}}$, and solve the equation $\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{\lambda}$, $\boldsymbol{\nu} \in \mathbf{W}_m(\widehat{L}^{\mathrm{unr}})$. Then a Kummer generator $\theta$ of $L'_m$ is given by the value at $T = 1$ of a certain $\boldsymbol{\pi}$-exponential, called $\theta_{p^m}(\boldsymbol{\nu}, T)$, (cf. Theorem 2.9).

The Artin–Schreier theory and Kummer theory are given by some *complexes* computing the Galois cohomology. Roughly speaking, we shall obtain a natural transformation of functors which "deforms" the Artin–Schreier complex into the Kummer complex and induces a quasi isomorphism:

$$\begin{array}{ccccccc} 1 & \longrightarrow & (L_m)^\times & \xrightarrow{x\mapsto x^{p^{m+1}}} & (L_m)^\times & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \theta} & & \uparrow{\scriptstyle e^{p^m}} & & \\ 0 & \longrightarrow & \mathbf{W}_m(k_L) & \xrightarrow[\bar{\mathrm{F}}-1]{} & \mathbf{W}_m(k_L) & \longrightarrow & 0\,. \end{array} \qquad (0.6)$$

Actually, such a natural transformation can not exist, because the Artin–Schreier complex is in characteristic $p$, and the Kummer complex is in characteristic 0. As a matter of fact, we lift the Artin–Schreier complex to characteristic 0 and deform it into the Kummer complex, by using *the value at* $T = 1$ *of some* *over-convergent* $\boldsymbol{\pi}$*-exponentials called* $\theta_{p^m}(-,T)$ *and* $\mathrm{e}_{p^m}(-,T)^{p^{m+1}}$ (see diagram (2.44)). This provides a well defined morphism between the cohomologies.

Under some assumptions on $K$ (cf. Eq. (2.53)), even if the field $L=\mathcal{E}_K^\dagger$ is not complete, we show that this diagram exists for $\mathcal{E}_K^\dagger$ and its finite unramified extensions (cf. Corollary 2.4). The commutative diagram is then:

$$\begin{array}{ccccccccc}
1 \longrightarrow & \boldsymbol{\mu}_{p^{m+1}} & \longrightarrow & (\mathcal{E}^\dagger_{K_m})^\times & \xrightarrow{f\mapsto f^{p^{m+1}}} & (\mathcal{E}^\dagger_{K_m})^\times & \xrightarrow{\delta_{\mathrm{Kum}}} & \mathrm{H}^1(\mathrm{G}_{\mathcal{E}^\dagger_{K_m}},\boldsymbol{\mu}_{p^{m+1}}) & \longrightarrow 1 \\
 & \uparrow & & \uparrow{\scriptstyle \theta_{p^m}(-,1)} & & \uparrow{\scriptstyle \mathrm{e}_{p^m}(-,1)^{p^{m+1}}} & & \uparrow{\scriptstyle \overline{\mathrm{e}}:=\overline{\mathrm{e}_{p^m}(-,1)^{p^{m+1}}}} & \\
 & \mathbf{W}_m(\mathcal{O}_K^{\sigma=1}) & \hookrightarrow & \mathbf{W}_m(\mathcal{O}_K^\dagger) & \xrightarrow[\varphi-1]{} & \mathbf{W}_m(\mathcal{O}_K^\dagger) & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
0 \longrightarrow & \mathbb{Z}/p^{m+1}\mathbb{Z} & \longrightarrow & \mathbf{W}_m(\mathrm{E}) & \xrightarrow[\bar{\mathrm{F}}-1]{} & \mathbf{W}_m(\mathrm{E}) & \xrightarrow[\delta]{} & \mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Z}/p^{m+1}\mathbb{Z}) & \longrightarrow 0
\end{array}$$

(0.7)

where $\mathrm{G_E}=\mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t)))$ and $\mathrm{G}_{\mathcal{E}^\dagger_{K_m}}=\mathrm{Gal}(\mathcal{E}^{\dagger,\mathrm{alg}}_{K_m}/\mathcal{E}^\dagger_{K_m})$. We specify the kernel and the image of the morphism $\overline{\mathrm{e}}$ between the cohomologies. If $\overline{\boldsymbol{f}}(t)\in\mathbf{W}_m(k((t)))$ is a Witt vector defining an Artin–Schreier separable extension of $k((t))$, then (up to add the $p^{m+1}$th roots of 1) a generator of the corresponding unramified extension of $\mathcal{E}^\dagger_{K_m}$ is given by $\theta_{p^m}(\boldsymbol{\nu},1)$, where $\boldsymbol{\nu}\in\mathbf{W}_m(\widehat{\mathcal{E}}_K^{\mathrm{unr}})$ is a solution of the equation $\varphi(\boldsymbol{\nu})-\boldsymbol{\nu}=\boldsymbol{f}(T)$, and $\boldsymbol{f}(T)\in\mathbf{W}_m(\mathcal{O}_K^\dagger)$ is an arbitrary lifting of $\overline{\boldsymbol{f}}(t)$.

Let $K_m:=K(\pi_m)=K(\xi_m)$, $K_\infty:=\cup_m K_m$, and let $k_m$ be the residue field of $K_m$. In Sects. 3, 3.1, and 3.2 we classify all solvable rank one differential equations over $\mathcal{R}_{K_\infty}$. The key point is the following equality, arising from the diagram (0.7), and useful for describing the Kummer generator $\theta_{p^m}(\boldsymbol{\nu},1)$:

$$\theta_{p^m}(\boldsymbol{\nu},1)^{p^{m+1}}=\mathrm{e}_{p^m}(\boldsymbol{f}(T),1)^{p^{m+1}}. \tag{0.8}$$

The expression $\mathrm{e}_{p^m}(\boldsymbol{f}(T),1)$ has no meaning, because $\mathrm{e}_{p^m}(-,Z)$ is not over-convergent as a function of $Z$. We make sense of this symbol in some cases: in Sect. 3.1 we define a class of exponentials of the form

$$\mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1)=\exp\left(\pi_m\phi_0^-(T)+\pi_{m-1}\frac{\phi_1^-(T)}{p}+\cdots+\pi_0\frac{\phi_m^-(T)}{p^m}\right), \tag{0.9}$$

where $\boldsymbol{f}^-(T)\in\mathbf{W}_m(T^{-1}\mathcal{O}_K[T^{-1}])$, and $\langle\phi_0^-(T),\ldots,\phi_m^-(T)\rangle\in(T^{-1}\mathcal{O}_K[T^{-1}])^{m+1}$ is its phantom vector. This exponential is $T^{-1}$-adically convergent and defines a series in $1+T^{-1}\mathcal{O}_{K_m}[[T^{-1}]]$, whose $p^{m+1}$th power lies in $\mathcal{R}_{K_m}$. These Witt vectors correspond to totally ramified Artin–Schreier extensions of $\mathrm{E}:=k((t))$. The exponential (0.9) is then the desired Kummer generator $\theta_{p^m}(\boldsymbol{\nu},1)$.

We state then the explicit bijection between the abelian Galois theory for $\mathrm{E} = k((t))$, and the theory of rank one differential equations over $\mathcal{R}_{K_\infty}$. Matsuda, in [18], has pointed out, under some restrictions, that such a correspondence should exist. We go further by removing any restrictions, improving his methods, and by making the correspondence more explicit (cf. Theorems 3.1 and 3.2). Namely we introduce the fundamental exponential $\mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1)$. We show that every rank one differential module $M$ over $\mathcal{R}_{K_\infty}$ comes, by scalar extension, from a module $M_{]0,\infty]}$, over $K_\infty[T^{-1}]$, whose Taylor solution at $\infty$ is of the form

$$T^{a_0} \cdot \mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1), \tag{0.10}$$

for some $m \geq 0$, $a_0 \in \mathbb{Z}_p$, and $\boldsymbol{f}^-(T) \in \mathbf{W}_m(T^{-1}\mathcal{O}_{K_m}[T^{-1}])$. Moreover, the isomorphism class of $M$ depends only on the class of $a_0$ in $\mathbb{Z}_p/\mathbb{Z}$, and on the Artin–Schreier character $\alpha$ defined by the reduction of $\boldsymbol{f}^-(T)$ in $\mathbf{W}_m(k_m((t)))$. Suppose that $a_0$ belongs to $\mathbb{Z}_{(p)} := \mathbb{Q} \cap \mathbb{Z}_p$. Then $a_0$ corresponds to the moderate extensions of $\mathrm{E} = k((t))$, generated by $t^{a_0}$. On the other hand, $\boldsymbol{f}^-(T)$ corresponds to the Artin–Schreier extension given by (the kernel of) the Artin–Schreier character defined by the reduction $\boldsymbol{f}^-(T)$. We recover in this way the well known bijection

$$\left\{\begin{array}{c}\text{Rank one}\\ \text{characters of } \mathcal{I}_{k_\infty((t))}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{Isomorphism classes of rank one}\\ \text{solvable differential equations}\\ \text{over } \mathcal{R}_{K_\infty} \text{with rational residue}\end{array}\right\}, \tag{0.11}$$

where $\mathcal{I}_{k_\infty((t))}$ is the inertia subgroup of $\mathrm{Gal}(k_\infty((t))^{\mathrm{sep}}/k_\infty((t)))$.

The central point is that the following $\boldsymbol{\pi}$-exponential is over-convergent

$$\frac{\mathrm{e}_{p^m}(\boldsymbol{f}^-_{(\bar{\mathrm{F}})}(T), 1)}{\mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1)} = \mathrm{e}_{p^m}(\boldsymbol{f}^-_{(\bar{\mathrm{F}})}(T) - \boldsymbol{f}^-(T), 1), \tag{0.12}$$

where $\boldsymbol{f}^-_{(\bar{\mathrm{F}})}(T)$ is an arbitrary lifting of the reduction $\bar{\mathrm{F}}(\overline{\boldsymbol{f}^-}(t)) \in \mathbf{W}_m(k_m((t)))$. If a lifting of the $p$-th power map $\varphi : \mathcal{R}_K \to \mathcal{R}_K$ is given, then this result implies the usual Frobenius Structure Theorem. Observe that we do not need the existence of $\varphi$ (cf. Remark 2.9), because actually the isomorphism class of a given module $M$ depends only on the reduction of $\boldsymbol{f}^-(T)$ in characteristic $p$. This represents a progress in two directions, with respect to the analogous Theorem of [6]: firstly we do not suppose $k$ perfect, and secondly we get a precise description of the isomorphism class of $M$. In particular, we find that, if $a_0 = 0$, then "the order" (cf. Definition 1.11) of the Frobenius structure is 1 (cf. Remark 4.1).

In Sect. 3.2 we prove these Theorems essentially by reducing the study to the "elementary" $\boldsymbol{\pi}$-exponentials corresponding to simpler Witt vector called $s$-co-monomials. These exponentials are studied in detail in Sect. 2.2.

We give then some complements (Sects. 4.1, 4.2, 4.3, 4.4, 4.5). In particular, in the Sect. 4.2, we compute the group of rank one solvable equations killed by a given Artin–Schreier extension and then answer the question (4) of 0.1.

In Sect. 4.3 we extend the definition of $\boldsymbol{\pi}$-exponentials to a larger class of differential equations, and we provide an algorithm (see Proof of Lemma 4.5), which gives a *criterion of solvability*, (cf. Corollary 4.6). In particular we show that *there is no irregular rank one equations if $K/\mathbb{Q}_p$ is unramified* (cf. Corollary 4.7). This answers the question (1) of 0.1. Then we *compute the irregularity* in some classical cases (cf. Sect. 4.4). We describe the Tannakian group of the category whose objects are successive extensions of rank one solvable modules. We remove the hypothesis "$K$ is spherically complete" present in the literature. In Sect. 4.6 we compute the $\phi - \nabla$-module attached to a character with finite image of $\mathrm{G_E}$. This answers question (5) of 0.1.

## 1 Definitions and notations

### 1.1 General notations

Let $p > 0$ be a fixed prime number. Let $(K, |.|)$ be a complete valued field containing $(\mathbb{Q}_p, |.|)$. For every valued extension field $L/K$, we denote by $\mathcal{O}_L = \{x \in L \mid |x| \leq 1\}$ the ring of integers of $L$, by $\mathfrak{p}_L = \{x \in L \mid |x| < 1\}$ its maximal ideal and by $k_L = \mathcal{O}_L/\mathfrak{p}_L$ its residue field. We set $k := k_K$, and take $K^{\mathrm{alg}}$ to be a fixed algebraic closure of $K$, and $k^{\mathrm{alg}} = k_{K^{\mathrm{alg}}}$ will be its residue field. $\Omega/K$ will be a spherically complete extension field containing $K^{\mathrm{alg}}$, satisfying $|\Omega| = \mathbb{R}_{\geq 0}$, and whose residue field $k_\Omega/k$ is not algebraic. We set

$$\omega := |p|^{\frac{1}{p-1}}.$$

We denote by $\partial_T := T\frac{\mathrm{d}}{\mathrm{d}T}$ the usual derivation. For all rings $R$ we denote by $R^\times$ the group of invertible elements in $R$.

#### *1.1.1 Analytic functions and the Robba ring*

For every (non vacuous) interval $I \subseteq [0, \infty[ \subset \mathbb{R}$ we set $\mathcal{A}_K(I) := \big\{f(T) = \sum_{i\in\mathbb{Z}} a_i T^i \mid \sup_i(|a_i|\rho^i) < \infty, \forall \rho \in I\big\}$, The topology of $\mathcal{A}_K(I)$ is defined by the family of absolute values

$$|f(T)|_\rho := \max_{i\in\mathbb{Z}} |a_i|\rho^i, \quad \forall \rho \in I. \tag{1.1}$$

Let $\mathcal{R}_K := \cup_\varepsilon \mathcal{A}_K(]1-\varepsilon, 1[)$ be the Robba ring. $\mathcal{R}_K$ is complete with respect to the limit topology. Let $\mathcal{E}_K := \{\sum_{i\in\mathbb{Z}} a_i T^i \mid \sup_i |a_i| < \infty \text{ }, \lim_{i\to-\infty} a_i = 0\}$ be the Amice ring. $\mathcal{E}_K$ is endowed with the Gauss norm $|.|_1$ and is complete. We denote by $\mathcal{O}_{\mathcal{E}_K} := \{f \in \mathcal{E} \mid |f|_1 \leq 1\}$ its ring of integers. If the valuation on $K$ is not discrete we may have $|a_i| < \sup_i |a_i|$, for all $i \in \mathbb{Z}$.

**Definition 1.1** *For all algebraic extensions $H/K$ we set*

$$\mathcal{A}_H(I) := \mathcal{A}_K(I) \otimes_K H, \quad \mathcal{R}_H := \mathcal{R}_K \otimes_K H, \quad \mathcal{E}_H := \mathcal{E}_K \otimes_K H. \tag{1.2}$$

Since $K$ is algebraically closed in $\mathcal{A}_K(I)$ (resp. $\mathcal{R}_K$, $\mathcal{E}_K$) then $\mathcal{A}_H(I)$ (resp. $\mathcal{R}_H$, $\mathcal{E}_H$) is a domain. All $p$-adic differential equations over $\mathcal{A}_H(I)$ (resp. $\mathcal{R}_H$, $\mathcal{E}_H$) come, by scalar extension, from an equation over $\mathcal{A}_L(I)$ (resp. $\mathcal{R}_L$, $\mathcal{E}_L$) with $L/K$ finite. This will justify the Definition 1.9.

**Definition 1.2** *For all formal series $f(T) = \sum_{i\in\mathbb{Z}} a_iT^i$ we define*

$$f^-(T) := \sum_{i\leq -1} a_iT^i, \quad f^+(T) := \sum_{i\geq 1} a_iT^i, \tag{1.3}$$

*we have $f(T) = f^-(T) + a_0 + f^+(T)$.*

**Definition 1.3** *For all algebraic extension $H/K$, let $\mathcal{E}^\dagger_{H,T} := \mathcal{R}_H \cap \mathcal{E}_H$. We denote by $\mathcal{O}^\dagger_{H,T} := \mathcal{O}_{\mathcal{E}_H} \cap \mathcal{R}_H$. If no confusion is possible we will write $\mathcal{E}^\dagger_H$ (respectively $\mathcal{O}^\dagger_H$) instead of $\mathcal{E}^\dagger_{H,T}$ (respectively $\mathcal{O}^\dagger_{H,T}$).*

*Remark* 1.1 The quotients $\mathcal{O}_{\mathcal{E}_K}/\{f \in \mathcal{O}_{\mathcal{E}_K} : |f|_1 < 1\}$ or $\mathcal{O}^\dagger_K/\{f \in \mathcal{O}^\dagger_K : |f|_1 < 1\}$ are equals to $k((t))$ if and only if the valuation on $K$ is discrete. Nevertheless, if the valuation is not discrete, the rings $\mathcal{O}_{\mathcal{E}_K}$ and $\mathcal{O}^\dagger_K$ are always local, their maximal ideals $\mathfrak{p}_{\mathcal{O}_{\mathcal{E}_K}}$ and $\mathfrak{p}_{\mathcal{O}^\dagger_K}$ are formed by series $f = \sum_i a_iT^i$ such that $|a_i| < 1$, for all $i \in \mathbb{Z}$, observe that, since the valuation is not discrete, this condition do not implies that $|f|_1 < 1$. The residue fields $\mathcal{O}^\dagger_K/\mathfrak{p}_{\mathcal{O}^\dagger_K}$, and $\mathcal{O}_{\mathcal{E}_K}/\mathfrak{p}_{\mathcal{O}_{\mathcal{E}_K}}$ are actually always equals to $k((t))$.

### 1.2 Generalities on rank one differential equations

Let B be one of the rings $\mathcal{A}_K(I)$, $\mathcal{R}_K$, $\mathcal{E}^\dagger_K$, $\mathcal{E}_K$. Let $\partial_T - g(T)$, $g(T) \in$ B be a first order linear differential operator. The differential module defined by $\partial_T - g(T)$ is the free rank one module $M$ over B, endowed with the action of the derivation $\partial_T : M \to M$ given, in the chosen basis $\mathbf{e}$, by $\partial_T(\mathbf{e}) = g(T) \cdot \mathbf{e}$. We will say that $g(T)$ is the matrix of the derivation $\partial_T$ in the basis $\mathbf{e}$. In the sequel we will work with both derivations $\partial_T$ and $\mathrm{d}/\mathrm{d}T$. We set

$$g_s(T) = \text{the matrix of } \partial_T^s; \quad g_{[s]}(T) = \text{the matrix of } \left(\mathrm{d}/\mathrm{d}T\right)^s; \tag{1.4}$$

Then one has

$$g_{[s+1]} = \frac{\mathrm{d}}{\mathrm{d}T}(g_{[s]}(T)) + g_{[s]}(T)g_{[1]}(T), \quad g_{[0]}(T) := 1. \tag{1.5}$$

Let C be a B-differential algebra. A solution of $\partial_T - g(T)$, $g(T) \in$ B, with values in C is, by definition, an element $y \in$ C satisfying $d(y) = g(T) \cdot y$. If $M$ is the rank one module defined by $\partial_T - g(T)$, then the solution $y$ define a morphism of B-modules $\mathbf{e} \mapsto y : M \to$ C commuting with the derivation. The operator corresponding to the basis $f \cdot \mathbf{e}$, $f \in$ B$^\times$, is $\partial_T - (g(T) + \frac{\partial_T(f)}{f})$. On the other hand, the tensor product of the modules defined by $\partial_T - g(T)$ and $\partial_T - \tilde{g}(T)$ is the module defined by the operator $\partial_T - (g(T) + \tilde{g}(T))$. Then we will identify the group, under tensor product, of isomorphism classes of (free) rank one differential modules (here called Pic(B)) with the group

$$\mathrm{B}/\partial_{T,\log}(\mathrm{B}^{\times}),$$

where $\partial_{T,\log} : \mathrm{B}^{\times} \to \mathrm{B}$ is the morphism of groups $f \mapsto \partial_T(f)/f$.

*1.2.1 Taylor solution and radius of convergence*

Let $I \subseteq \mathbb{R}_{\geq 0}$ be some interval. In this subsection, $M$ will be a *rank one* $\mathcal{A}_K(I)$-differential module defined by the operator $\partial_T - g(T)$.

Let $x \in \Omega$, $|x| \in I$. We regard $\Omega[[T-x]]$ as an $\mathcal{A}_K(I)$-differential algebra by the Taylor map $f(T) \mapsto \sum_{k\geq 0}(\frac{d}{dT})^k(f)(x)\frac{(T-x)^k}{k!} : \mathcal{A}_K(I) \to \Omega[[T-x]]$. The Taylor solution of $\partial_T - g(T)$ at $x$ is (recall that $g(T) = Tg_{[1]}(T)$)

$$s_x(T) := \sum_{k\geq 0} g_{[k]}(x)\frac{(T-x)^k}{k!}. \tag{1.6}$$

Indeed $\partial_T(s_x(T)) = g(T)s_x(T)$. The radius of convergence of $s_x(T)$ at $x$ is, by the usual definition, $\mathrm{Ray}(M,x) = \liminf_s(|g_{[k]}(x)|/|k!|)^{-\frac{1}{k}}$.

**Definition 1.4** *The radius of convergence of $M$ at $\rho \in I$ is*

$$\begin{aligned}\mathrm{Ray}(M,\rho) &:= \min\Big(\rho,\ \liminf_k(|g_{[k]}|_\rho/|k!|)^{-1/k}\Big)\\ &= \min\Big(\rho,\ \omega\big[\limsup_k(|g_{[k]}|_\rho)^{1/k}\big]^{-1}\Big).\end{aligned}$$

The second equality follows from the fact that the sequence $|k!|^{1/k}$ is convergent to $\omega$, and $|g_{[k]}|_\rho^{1/k}$ is bounded by $\max(|g_{[1]}|_\rho, \rho^{-1})$. The presence of $\rho$ in the minimum makes this definition invariant under change of basis in $M$.

**Theorem 1.1** (Transfer) *For all $\rho \in I$ we have*

$$\mathrm{Ray}(M,\rho) = \min\left(\rho,\ \inf_{x\in\Omega,|x|=\rho}\mathrm{Ray}(M,x)\right). \tag{1.7}$$

*Assume now that $I = [0,\rho]$. Then*

$$\mathrm{Ray}(M,\rho) = \min\left(\rho,\ \min_{x\in\Omega,|x|\leq\rho}\mathrm{Ray}(M,x)\right). \tag{1.8}$$

*In particular* $\mathrm{Ray}(M,\rho) \leq \min(\rho, \mathrm{Ray}(M,0))$.

*Proof* Since for $\rho = |x|$ we have $|g_{[s]}(T)|_\rho \geq |g_{[s]}(x)|$, hence by Definition 1.4, $\mathrm{Ray}(M,\rho) \leq \min(\rho, \mathrm{Ray}(M,x))$. Let $t_\rho \in \Omega$ be such that $\{x \in \Omega \mid |x - t_\rho| < \rho\} \cap K = \emptyset$, then $|g_{[s]}|_\rho = |g_{[s]}(t_\rho)|$, for all $s \geq 0$ ([10, 9.1]), hence $\mathrm{Ray}(M,\rho) = \min(\rho, \mathrm{Ray}(M,t_\rho))$. The last assertion follows similarly. □

**Lemma 1.1** (Small Radius) *Let $\rho \in I$. Then*

$$\mathrm{Ray}(M,\rho) \geq \omega\rho \cdot \min\left(1, |g(T)|_\rho^{-1}\right). \tag{1.9}$$

*Moreover* $\mathrm{Ray}(M,\rho) < \omega\rho$ *if and only if* $|g(T)|_\rho > 1$, *and in this case we have*

$$\mathrm{Ray}(M,\rho) = \omega\rho \cdot |g(T)|_\rho^{-1}. \tag{1.10}$$

*Proof* By induction on Eq. (1.5), $|g_{[s]}|_\rho \leq \max(\rho^{-1}, |g_{[1]}|_\rho)^s = \rho^{-s} \max(1, |g|_\rho)^s$ (cf. Eq. (1.4)), and equality holds if $|g_{[1]}|_\rho > \rho^{-1}$. Then apply Definition 1.4. □

**Definition 1.5** *$M$ is called* solvable *at $\rho \in I$, if* $\mathrm{Ray}(M,\rho) = \rho$.

**Theorem 1.2** [7] *The map $\rho \mapsto \mathrm{Ray}(M,\rho) : I \to \mathbb{R}_{\geq}$ is continuous and locally of the form $r \cdot \rho^{\beta+1}$, for suitable $r \in \mathbb{R}_{\geq}$, and $\beta \in \mathbb{N}$. More precisely there exist a partition $I = \cup_{n\in\mathbb{Z}} I_n$, $\sup I_n = \inf I_{n+1}$, and two sequences $\{r_n\}_{n\in\mathbb{Z}}$, $\{\beta_n\}_{n\in\mathbb{Z}}$, such that $\beta_n \in \mathbb{Z}$, $\mathrm{Ray}(M,\rho) = r_n\rho^{(\beta_n+1)}$, $\forall\ \rho \in I_n$, and (cf. Lemma 1.3)*

$$\beta_n \geq \beta_{n+1}. \tag{1.11}$$

*Proof* The existence of the partition follows from the Small Radius Lemma 1.1 and Theorem 1.3. For more details see [9, 8.6] and [7, 2.5]. □

**Definition 1.6** *We will call the property* (1.11) *the* log-concavity *of the function* $\rho \mapsto \mathrm{Ray}(M,\rho)$. *We will call $\beta_n$ the slope of $M$ in the interior of $I_n$. More generally if $\rho = \sup I_n = \inf I_{n+1}$, we set* $\mathrm{sl}^-(M,\rho) := \beta_n$ *and* $\mathrm{sl}^+(M,\rho) := \beta_{n+1}$.

*Remark* 1.2 The Taylor solution of $M \otimes N$ is the product of the Taylor solutions of $M$ and $N$. Hence, by the transfer Theorem 1.1, $\mathrm{Ray}(M \otimes N,\rho) \geq \min(\mathrm{Ray}(M,\rho), \mathrm{Ray}(N,\rho))$. If $\mathrm{Ray}(M,\rho) \neq \mathrm{Ray}(N,\rho)$, then we have $\mathrm{Ray}(M \otimes N,\rho) = \min(\mathrm{Ray}(M,\rho), \mathrm{Ray}(N,\rho))$.

### *1.2.2 Solvability, slopes and irregularities*

In this subsection, $M$ is the rank one module over $\mathcal{R}_K$, defined by $\partial_T + g(T)$, $g(T) := \sum_{i\in\mathbb{Z}} a_i T^i \in \mathcal{R}_K$.

**Lemma 1.2** *There exists $d > 0$ such that $M$ is isomorphic to the module defined by $\partial_T + \sum_{i\geq -d} a_i T^i$. In other words there exists $f(T) \in \mathcal{R}_K^\times$ such that $\partial_{T,\log}(f) = \sum_{i<-d} a_i T^i$.*

*Proof* By hypothesis $g(T) \in \mathcal{A}_K(]1-\varepsilon,1[)$, for some $\varepsilon > 0$. Then $\sum_{i\neq 0} a_i T^i/i \in \mathcal{A}_K(]1-\varepsilon,1[)$. In particular $\lim_{i\to-\infty} |a_i/i|\rho^i = 0$, for all $\rho \in ]1-\varepsilon,+\infty[$. Let $d > 0$ be such that $\sup_{i<-d}(|a_i/i|\tilde{\rho}^i) < \omega$ for a fixed $\tilde{\rho} \in ]1-\varepsilon,1[$. Then $\sup_{i<-d}(|a_i/i|\rho^i) < \omega$, for all $\rho \geq \tilde{\rho}$. Then $f(T) = \exp(-\sum_{i<-d} a_i T^i/i)$ lies in $\mathcal{R}_K$. □

**Definition 1.7** *Let $M$ be a differential module over $\mathcal{R}_K$. The module $M$ is called* solvable *if and only if* $\lim_{\rho\to 1^-} \mathrm{Ray}(M,\rho) = 1$. *We will denote the category of solvable differential modules over $\mathcal{R}_K$ by* $\mathrm{MLS}(\mathcal{R}_K)$.

**Lemma 1.3** *Let $M \in \mathrm{MLS}(\mathcal{R}_K)$ be defined in some basis by the operator $\partial_T - g(T)$, $g(T) \in \mathcal{R}_K$. Then*

1. *There exist $0 < \varepsilon < 1$ and a last slope $\beta := \mathrm{sl}^-(M,1) \geq 0$ such that*

$$\mathrm{Ray}(M,\rho) = \rho^{\beta+1}, \quad \text{for all } \rho \in ]1-\varepsilon,1[\,. \tag{1.12}$$

2. *There exists $\varepsilon'$ such that $|g(T)|_\rho \leq 1$, for all $\rho \in ]1-\varepsilon',1[$.*
3. *If $g(T) = \sum_{-d}^{\infty} a_i T^i$, $d > 0$, then $|a_{-d}| \leq \omega$ and, for $\rho$ close to $0$,*

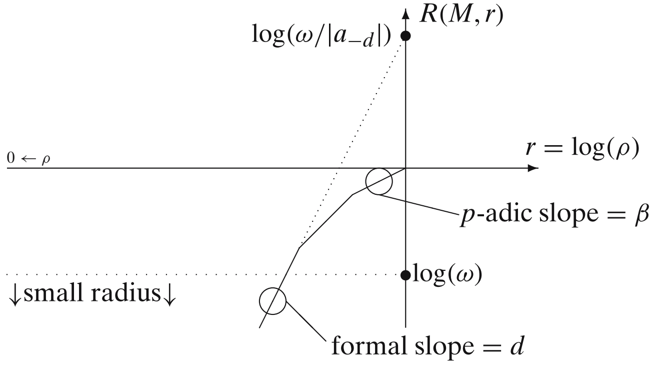
$$\mathrm{Ray}(M,\rho) = \omega|a_{-d}|^{-1}\rho^{d+1}. \tag{1.13}$$

4. *Moreover, if $d > 0$, and if $|a_{-d}| = \omega$, then $\beta = d$.*
5. *If $d \leq 0$, then $\mathrm{Ray}(M,\rho) = \rho$, for all $\rho \in ]0,1[$ and $\beta = 0$.*

*Proof* The slopes are positive natural numbers, hence the decreasing sequence $\{\beta_n\}_n$ becomes constant for $n \to \infty$. Then $\beta = \min_{n\in\mathbb{Z}}\{\beta_n\}$. The second assertion follows from the Small Radius Lemma 1.1. Let now $g(T) = \sum_{i\geq -d} a_i T^i$, with $d > 0$. We study the function $\rho \mapsto \mathrm{Ray}(M,\rho)/\rho$. Let

$$R(M,r) := \log(\mathrm{Ray}(M,\rho)) - \log(\rho), \quad r := \log(\rho). \tag{1.14}$$

Then $R(M,r) \leq 0$, for all $r \leq 0$, and the function $r \mapsto R(M,r) : ]-\infty,0[ \longrightarrow ]-\infty,0]$ is of the following form



Since $d > 0$, one has $|g(T)|_\rho = |a_{-d}|\rho^{-d} > 1$, for $\rho$ close to 0. Hence, near 0, we can apply the Small Radius Lemma (cf. Remark 1.3): we have $\mathrm{Ray}(M,\rho) = \omega|a_{-d}|^{-1}\rho^{d+1}$. Since $\lim_{\rho\to 1^-}\mathrm{Ray}(M,\rho) = 1$, hence by log-concavity and continuity, we must have $\omega|a_{-d}|^{-1} \geq 1$ (or equivalently $\log(\omega/|a_{-d}|) \geq 0$ as in the picture) and if $|a_{-d}| = \omega$, then, again by continuity and log-concavity, this graph is a line, and $\beta = d$. If $d \leq 0$, then $|g(T)|_\rho \leq 1$, for all $\rho < 1$, hence the Small Radius Lemma gives $R(M,r) \geq \log(\omega)$ for all $r \leq 0$. Since $R(M,r) \to 0$ for $r \to 0$ (solvability), then by log-concavity and continuity this implies $R(M,r) = 0, \forall r \leq 0$. □

*Remark* 1.3 We maintain the notation of Lemma 1.3 part (3). We recall that $\mathrm{sl}^+(M,0) = \min(0,d)$ is equal to the classical formal slope $\mathrm{Irr}_F(M)$ of $M$ as $K((T))$-differential module (cf. [27]). This is actually true in all ranks.

**Definition 1.8** *Let $M$ be a solvable rank one differential module over $\mathcal{R}$. The $p$-adic* irregularity *of $M$ is the natural number* $\mathrm{Irr}(M) := \mathrm{sl}^-(M,1)$.

*Remark* 1.4 If $M$ is defined by an operator $\partial_T + g(T)$, $g(T) = \sum_{-d}^{\infty} a_i T^i \in \mathcal{R}_K$, then by log-concavity and continuity we have $\mathrm{Irr}_F(M) \geq \mathrm{Irr}(M)$.

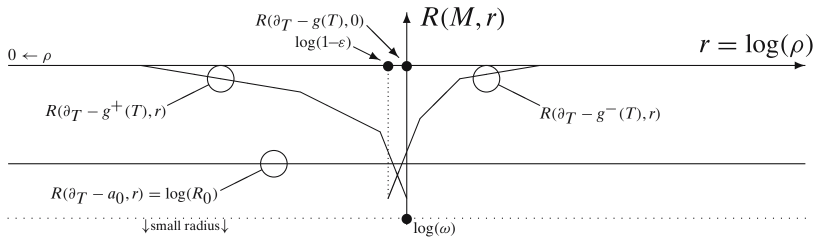
**Definition 1.9** *If $K'/K$ is a finite extension, then we denote by* $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K'})$ *the group, under tensor product, of isomorphism classes of solvable rank one differential modules over $\mathcal{R}_{K'}$. For all algebraic extensions $H/K$, we set*

$$\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_H) := \bigcup_{K\subset K'\subset H\ ,\ K'/K \text{finite}} \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K'}). \tag{1.15}$$

**Corollary 1.1** *We have* $\mathrm{Irr}(M\otimes N) \leq \max(\mathrm{Irr}(M), \mathrm{Irr}(N))$, *for all $M, N \in$* MLS *$(\mathcal{R}_K)$. Moreover the equality holds if* $\mathrm{Irr}(M) \neq \mathrm{Irr}(N)$.

**Proposition 1.1** *Let $\partial_T - g(T)$, $g(T) = \sum_{i\in\mathbb{Z}} a_i T^i \in \mathcal{R}_K$, be a solvable differential equation. Then $\partial_T - g^-(T)$, $\partial_T - a_0$, $\partial_T - g^+(T)$ are all solvable (cf. Definition 1.2).*

*Proof* Let us call $M_{]1-\varepsilon,\infty]}$, $M_0$, $M_{[0,1[}$ the differential modules defined by $\partial_T - g^-(T)$, $\partial_T - a_0$, $\partial_T - g^+(T)$, respectively. Then $M = M_{]1-\varepsilon,\infty]} \otimes M_0 \otimes M_{[0,1[}$. By the Small Radius Lemma 1.1, the equation $\partial_T - g^-(T)$ (respectively $\partial_T - g^+(T)$) has a convergent solution at $\infty$ (respectively at 0), hence $\mathrm{Ray}(M_{]1-\varepsilon,\infty]},\rho) = \rho$, for large values of $\rho$, and $\mathrm{Ray}(M_{[0,1[},\rho) = \rho$, for $\rho$ close to 0. While $\mathrm{Ray}(M_0,\rho) = R_0 \cdot \rho$, for all $\rho$ (cf. Lemma 1.4). Hence the slopes of $M_{]1-\varepsilon,\infty]}$ (respectively $M_{[0,1[}$, $M_0$) in the interval $]1-\varepsilon, 1[$ are strictly positive (respectively strictly negative, respectively equal to 0) as in the picture (cf. (1.14)).



We have $\mathrm{Ray}(M,\rho) = \inf(\mathrm{Ray}(M_{]1-\varepsilon,\infty]},\rho), \mathrm{Ray}(M_{[0,1[},\rho), \mathrm{Ray}(M_0,\rho))$, for all $1-\varepsilon < \rho < 1$, with the exception of a finite numbers of $\rho$, this follows by Remark 1.2. By continuity of the radius, we have equality even for these isolated values of $\rho$. Since $\lim_{\rho\to 1^-} \mathrm{Ray}(M,\rho) = 1$, this implies $\mathrm{Ray}(M_{[0,1[},\rho) = \rho$ for all $\rho < 1$, $\mathrm{Ray}(M_0,\rho) = \rho$ for all $\rho$, and $\mathrm{Ray}(M_{]1-\varepsilon,\infty]},\rho) = \rho$ for all $\rho \geq 1$. □

The classification of the equations of the type $\partial_T - a_0$, $a_0 \in K$, is well known (see Sect. 1.2.5), while the solvable equations of the form $\partial_T - g^+(T)$ are always trivial:

**Proposition 1.2** *Let $\partial_T - g^+(T)$, $g^+(T) = \sum_{i\geq 1} a_i T^i \in \mathcal{A}_H([0,1[) \subset \mathcal{R}_H$ be solvable at* $1^-$ *(cf. Definition 1.7). Let M be the module attached to $\partial_T - g^+(T)$, then*

1. *We have $g^+(T) \in T\mathcal{O}_H[[T]]$. Hence M comes, by scalar extension, from a differential module $M_{[0,1[}$ over $\mathcal{O}_H[[T]]$;*
2. *We have* $\mathrm{Ray}(M_{[0,1[}\ ,\ \rho) = \rho$*, for all $\rho < 1$;*
3. *$M_{[0,1[}$ is trivial as $\mathcal{O}_H[[T]]$-module;*
4. *The exponential* $\exp(\sum_{i\geq 1} a_i T^i/i)$ *lies in* $1 + T\mathcal{O}_H[[T]]$.

*Proof* We have $|a_i| \leq 1$, because the Small Radius Lemma 1.1. Since $\partial_T - g^+(T)$ has a convergent solution at 0 (namely this Taylor solution is $\exp(\sum_{i\geq 1} a_i T^i/i)$), then $\mathrm{Ray}(M_{[0,1[},\rho) = \rho$ for all $\rho$ close to 0. Since $\lim_{\rho\to 1^-} \mathrm{Ray}(M_{[0,1[},\rho) = 1$, then by log-concavity we must have $\mathrm{Ray}(M_{[0,1[},\rho) = \rho$, for all $\rho < 1$. By the transfer Theorem 1.1 the Taylor solution $\exp(\sum_{i\geq 1} a_i T^i/i)$ converges in the disk $|T| < 1$ and belongs to $\mathcal{O}_K[[T]]$ (because a non trivial solution of a differential equation has no zeros in its disk of convergence). □

**Corollary 1.2** *Every rank one solvable differential module over $\mathcal{R}_K$ has a basis in which the matrix lies in $\mathcal{O}_K[T^{-1}]$.*

*Proof* By Proposition 1.2 there exists a basis in which the matrix lies in $\mathcal{R}_K \cap \mathcal{O}_K[[T^{-1}]]$. The base change matrix to obtain this basis is an exponential convergent in $[0,1[$. Now, by Lemma 1.2 we recover the good basis. This last base change matrix is again an exponential convergent in $]1-\varepsilon,\infty[$. □

*1.2.3 Frobenius structure and p-th ramification*

**Definition 1.10** *An* absolute Frobenius on *K is a $\mathbb{Q}_p$-endomorphism $\sigma : K \to K$ such that $|\sigma(x) - x^p| < 1$, for all $x \in \mathcal{O}_K$.*
*If an absolute Frobenius $\sigma : K \to K$ is given, an* absolute Frobenius on $\mathcal{R}_K$ *is then a continuous endomorphism of rings $\varphi : \mathcal{R}_K \to \mathcal{R}_K$ extending $\sigma$, and such that*

$$\varphi(T) - T^p = \sum a_i(\varphi)T^i, \quad \text{with } |a_i(\varphi)| < 1 \quad \text{for all } i \in \mathbb{Z},\ a_i(\varphi) \in K. \quad (1.16)$$

By continuity $\varphi$ is given by $\sigma$ and by the choice of $\varphi(T)$. Namely if we set $(\sum a_i T^i)^\sigma := \sum \sigma(a_i) T^i$, then $\varphi(f(T)) = f^\sigma(\varphi(T))$, for all $f \in \mathcal{R}_K$. The simplest absolute Frobenius is given by the choice $\varphi(T) = T^p$ and we denote it by $\varphi_\sigma$.

Let $\varphi : \mathcal{R} \to \mathcal{R}$ be an absolute Frobenius. By scalar extension (and change of derivation), we have a functor: $\varphi^* : \mathrm{MLC}(\mathcal{R}) \rightsquigarrow \mathrm{MLC}(\mathcal{R})$. If $M \in \mathrm{MLC}(\mathcal{R})$ is defined by the operator $\partial_T - g(T)$, then $\varphi^*(M)$ is defined by the operator $\partial_T - (\partial_{T,\log}(\varphi(T)) \cdot g^\sigma(\varphi(T)))$. The isomorphism class of $\varphi^*(M)$ does not depend on the choice of $\varphi$ (cf. [9, 7.1]).

### *1.2.4 p-th ramification*

Let $\sigma$ be an absolute Frobenius on $K$. For all analytic functions $f(T) := \sum_i a_i T^i \in \mathcal{A}(I)$, we set $\varphi_p(f(T)) := f(T^p)$. Observe that $\varphi_p$ is not an absolute Frobenius. We set $\varphi_\sigma(f(T)) := f^\sigma(T^p)$. The $p$-th ramification map $\varphi_p : \mathcal{A}(I^p) \to \mathcal{A}(I)$ defines, as before, a functor denoted by $\varphi_p^* : \mathrm{MLC}(\mathcal{A}_K(I^p)) \rightsquigarrow \mathrm{MLC}(\mathcal{A}_K(I))$.

**Theorem 1.3** [9] *Let* $M \in \mathrm{MLC}(\mathcal{A}_K(I^p))$. *Then for all* $\rho \in I$ *one has* $\mathrm{Ray}(\varphi_\sigma^*(M), \rho) = \mathrm{Ray}(\varphi_p^*(M), \rho)$, *and*

$$\mathrm{Ray}(\varphi_\sigma^*(M), \rho) \geq \rho \min\left(\left(\frac{\mathrm{Ray}(M, \rho^p)}{\rho^p}\right)^{1/p}, |p|^{-1} \frac{\mathrm{Ray}(M, \rho^p)}{\rho^p}\right),$$

*and equality holds if* $\mathrm{Ray}(M, \rho) \neq \omega^p \rho$.

*Proof* Since $f(T) \mapsto f^\sigma(T)$ is an isometry, we have the first equality. The second one follows from a quite complex, but elementary, explicit computation. □

*Example* 1.1 The radius of the operator $\partial_T - 1/p$ is equal to $\omega|p|\rho = \omega^p \rho$ (cf. Lemma 1.1), but its image by Frobenius is the trivial module.

**Corollary 1.3** *Let* $M \in \mathrm{MLS}(\mathcal{R}_K)$, *let* $\varphi : \mathcal{R}_K \to \mathcal{R}_K$ *be an absolute Frobenius, then* $\mathrm{sl}^-(\varphi^*(M), 1) = \mathrm{sl}^-(M, 1)$.

**Definition 1.11** (Frobenius structure) *Let M be a module over* $\mathcal{R}_K$. *We will say that M has a* Frobenius structure of order *h, if M is isomorphic to* $(\varphi^*)^h(M)$.

*Remark* 1.5 If $M$ has a Frobenius structure, then it is solvable by Theorem 1.3 applied to "antecedents" of $M$. (see [9, 8.6 and 7.7 infra]).

*Remark* 1.6 By Theorem 1.3 we have $\mathrm{Irr}(\varphi^*(M)) = \mathrm{Irr}(\varphi_p^*(M)) = \mathrm{Irr}(M)$.

### *1.2.5 Moderate characters*

Let $a_0 \in K$. We denote by $\mathrm{M}(a_0, 0)$ the module defined by the constant operator $\partial_T - a_0$ (cf. Sect. 1.2). We will call *moderate* every *solvable* differential module (over $\mathcal{R}_K$) of the form $\mathrm{M}(a_0, 0)$. By [22, 5.4], $\mathrm{M}(a_0, 0)$ is solvable if and only if

$a_0 \in \mathbb{Z}_p$. Moreover the equation $\partial_{T,\log}(f(T)) = a_0$ has a solution $f(T) \in \mathcal{R}_K^\times$ if and only if $a_0 \in \mathbb{Z}$, and in this case $f(T) = T^{a_0}$. This shows that the group under tensor product of moderate differential modules is isomorphic to $\mathbb{Z}_p/\mathbb{Z}$. On the other hand it is well known that an $\mathrm{M}(a_0, 0)$ has a Frobenius structure if and only if $a_0 \in \mathbb{Z}_{(p)}$ (cf. Lemma 4.2).

**Lemma 1.4** *Let* $\alpha(a_0) := \limsup_s(|a_0(a_0-1)(a_0-2)\cdots(a_0-s+1)|^{\frac{1}{s}})$. *Then* $\mathrm{Ray}(M(a_0,0),\rho) = \rho \cdot R_0 \leq \rho$, *for all* $\rho > 0$, *with* $R_0 := \min(1, \omega \cdot \alpha(a_0)^{-1})$.

*Proof* A direct computation gives $g_{[s]}(T) = \alpha_s(a_0)T^{-s}$, with $\alpha_s(a_0) := a_0(a_0 - 1)\cdots(a_0 - s + 1)$ (cf. Eq. (1.4)). Then apply Definition 1.4. □

## 1.3 Notations on Witt vectors and covectors

Let $R$ be a ring. Notations concerning the ring $\mathbf{W}(R)$ of Witt vectors will follow [3], except for the indexation "$m$" of the ring $\mathbf{W}_m(R)$ of Witt vector of finite length. We set $\mathbf{W}_m(R) := \mathbf{W}(R)/\mathrm{V}^{m+1}\mathbf{W}(R)$ (see Sect. 1.3.1). We denote by

$$\phi_n := \phi_n(X_0,\ldots,X_n) := X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n \tag{1.17}$$

the Witt polynomial. Vectors in $R^{\mathbb{N}}$ and in $R^{m+1}$ will be distinguished from Witt vectors by the notation $\langle\phi_0,\phi_1,\ldots\rangle$ instead of $(\phi_0,\phi_2,\ldots)$. For all Witt vector $\boldsymbol{r} = (r_0, r_1, \ldots) \in \mathbf{W}(R)$, the vector $\phi(\boldsymbol{r}) = \langle\phi_0(r_0), \phi_1(r_0, r_1), \ldots\rangle$ is called the phantom vector of $\boldsymbol{r}$. The map $\boldsymbol{r} \mapsto \phi(\boldsymbol{r}) : \mathbf{W}(R) \to R^{\mathbb{N}}$ is a ring morphism.

**Lemma 1.5** ([3, Lemme 3 Sect. 1, $N^0$2]) *Let* $\boldsymbol{\lambda} \mapsto \phi(\boldsymbol{\lambda}) : \mathbf{W}(R) \xrightarrow{\phi} R^{\mathbb{N}}$ *be the phantom component map. If* $p \in R$ *is not a zero divisor, then* $\phi$ *is injective. If* $p \in R$ *is invertible then* $\phi$ *is bijective.*

**Lemma 1.6** ([3, Lemme 2 Sect. 1, $N^0$2]) *Let* $\sigma : R \to R$ *be a ring morphism satisfying* $\sigma(a) \equiv a^p \pmod{pR}$, *for all* $a \in R$. *Then a vector* $\langle\phi_0,\ldots,\phi_m\rangle \in R^{m+1}$ *is the phantom vector of a Witt vector if and only if*

$$\phi_i \equiv \sigma(\phi_{i-1}) \qquad \mathrm{mod}\ p^i R, \quad \text{for all } i = 1,\ldots,m. \tag{1.18}$$

*Remark* 1.7 All assertions concerning relations between Witt vectors or properties of $\boldsymbol{\pi}$-exponentials (see below) will be proved by translating these relations or properties in terms of phantom components.

### *1.3.1 Frobenius and Verschiebung*

We denote by $\mathrm{F} : \mathbf{W}(R) \to \mathbf{W}(R)$ and $\mathrm{V} : \mathbf{W}(R) \to \mathbf{W}(R)$ the usual Frobenius and Verschiebung morphisms. We denote again by $\mathrm{F} : \mathbf{W}_{m+1}(R) \to \mathbf{W}_m(R)$, $\mathrm{V} : \mathbf{W}_m(R) \to \mathbf{W}_{m+1}(R)$ the reduction of F and V to $\mathbf{W}_m(R)$. We have again $\mathrm{FV}(\boldsymbol{r}) = p \cdot \boldsymbol{r}$ in $\mathbf{W}_m(R)$. We recall that $\phi(\mathrm{V}(r_0, r_1, \ldots)) = \langle 0, p\phi_0, p\phi_1, \ldots\rangle$ and $\phi(\mathrm{F}(r_0, r_1, \ldots)) = \langle\phi_1, \phi_2, \ldots\rangle$.

If $R$ has characteristic $p$, then $\mathrm{F}(r_0, r_1, \dots) = (r_0^p, r_1^p, \dots)$. Hence it is possible to reduce the morphism F of $\mathbf{W}(R)$ to a morphism of $\mathbf{W}_m(R)$ into itself, by setting $\bar{\mathrm{F}}(r_0, \dots, r_m) = (r_0^p, \dots, r_m^p)$. We denote this morphism by $\bar{\mathrm{F}} : \mathbf{W}_m(R) \to \mathbf{W}_m(R)$.

*1.3.2 Completeness*

Let $R$ be a topological ring. We identify topologically $\mathbf{W}_m(R)$ with $R^{m+1}$, via the function $(r_0, \dots, r_m) \mapsto \langle r_0, \dots, r_m \rangle$. The operations on $\mathbf{W}_m(R)$ are continuous, because defined by polynomials.

**Lemma 1.7** *If $R$ has a basis $\mathcal{U}_R$ of neighborhood of* 0 *formed by ideals, then $R$ is complete if and only if $\mathbf{W}_m(R)$ is complete for all $m \geq 0$.*

*Proof* It is evident for $m = 0$. Let $m \geq 1$ and $\{\boldsymbol{r}_n\}_n$, $\boldsymbol{r}_n := (r_{n,0}, \dots, r_{n,m})$, be a Cauchy sequence in $\mathbf{W}_m(R)$. The sequence $r_{0,n}$ is Cauchy in $R$ and we denote $r_0 := \lim_n r_{0,n}$. The translate sequence $\boldsymbol{r}_n^1 := \boldsymbol{r}_n - (r_0, 0, \dots, 0)$ is Cauchy, so we can suppose $r_0 = 0$. For every ideal $I \in \mathcal{U}_R$ there exists $n_I$ such that $\boldsymbol{r}_{n_1}^1 - \boldsymbol{r}_{n_2}^1 = (S_{0,n_1,n_2}, \dots, S_{m,n_1,n_2}) \in \mathbf{W}_m(I)$, for all $n_1, n_2 \geq n_I$. Let us write $S_{1,n_1,n_2} = r_{n_1,1}^1 - r_{n_2,1}^1 + P(r_{n_1,0}^1, r_{n_2,0}^1)$. By [3, Sect. 1 $n^0 3$ *a*)] the polynomial $S_{k,n_1,n_2}$ is isobaric without constant term. Since $r_{n,0}^1 \in I$, for $n \geq n_I'$, sufficiently large and since $I$ is an ideal, hence $r_{n_1,1}^1 - r_{n_2,1}^1 \in I$, for all $n_1, n_2 \geq n_I'$. So the sequence $r_{n,1}^1$ is Cauchy and converges to $r_1 \in R$. Moreover the sequence $\boldsymbol{r}_n^2 := \boldsymbol{r}_n^1 - (0, r_1, 0, \dots, 0)$ is such that both $r_{n,0}^2$ and $r_{n,1}^2$ go to 0. This process can be iterated indefinitely. □

**Corollary 1.4** *If $(R, |.|)$ is an ultrametric valued ring, then $R$ is complete if and only if $\mathbf{W}_m(R)$ is complete for all $m \geq 0$.*

*1.3.3 Length*

Let $R$ be a ring of characteristic $p$. If the vector $\boldsymbol{r} \in \mathbf{W}_m(R)$ is such that $r_0 = \cdots = r_{k-1} = 0$ and $r_k \neq 0$, then we define the length of $\boldsymbol{r}$ as $\ell(\boldsymbol{r}) := m - k$, and $\ell(\mathbf{0}) := -\infty$. If $R$ is not of characteristic $p$, then we will define $\ell(\boldsymbol{r})$ as the length of the image of $\boldsymbol{r}$ in $\mathbf{W}_m(R/pR)$.

*1.3.4 Covectors*

We recall that the covectors module $\mathbf{CW}(R)$ is the additive group defined by the following inductive limit ([3, Sect. 1 ex. 23 page 47]): $\mathbf{CW}(R) := \varinjlim(\mathbf{W}_m(R) \stackrel{\mathrm{V}}{\to} \mathbf{W}_{m+1}(R) \stackrel{\mathrm{V}}{\to} \cdots)$. In the sequel we must work with a slightly different sequence. Let $R$ be a ring of characteristic $p$. Then $\mathrm{V}\bar{\mathrm{F}} = \bar{\mathrm{F}}\mathrm{V}$ and $\mathrm{V}\bar{\mathrm{F}}(r_0, \dots, r_m) = (0, r_0^p, \dots, r_m^p)$. Let $\widetilde{\mathbf{CW}}(R)$ be the following inductive limit:

$$\widetilde{\mathbf{CW}}(R) := \varinjlim(\mathbf{W}_m(R) \stackrel{\mathrm{V}\bar{\mathrm{F}}}{\to} \mathbf{W}_{m+1}(R) \stackrel{\mathrm{V}\bar{\mathrm{F}}}{\to} \cdots). \tag{1.19}$$

If $R$ is a perfect field of characteristic $p$, then $\mathbf{CW}(R)$ is isomorphic to $\widetilde{\mathbf{CW}}(R)$. This results from the following commutative diagram:

$$\begin{array}{ccccccccc} R & \xrightarrow{\mathrm{V}} & \mathbf{W}_1(R) & \xrightarrow{\mathrm{V}} & \mathbf{W}_2(R) & \xrightarrow{\mathrm{V}} & \cdots & \longrightarrow & \mathbf{CW}(R) \\ \| & \odot & \wr\downarrow \bar{\mathrm{F}} & \odot & \wr\downarrow \bar{\mathrm{F}}^2 & & & & \wr\downarrow \\ R & \xrightarrow{\mathrm{V}\bar{\mathrm{F}}} & \mathbf{W}_1(R) & \xrightarrow{\mathrm{V}\bar{\mathrm{F}}} & \mathbf{W}_2(R) & \xrightarrow{\mathrm{V}\bar{\mathrm{F}}} & \cdots & \longrightarrow & \widetilde{\mathbf{CW}}(R)\ . \end{array} \tag{1.20}$$

*Remark* 1.8 If $R$ is a field of characteristic $p$, then $\widetilde{\mathbf{CW}}(R) = \widetilde{\mathbf{CW}}(R^p)$.

### 1.4 Notations in Artin–Schreier theory

**Definition 1.12** *Let $R$ be a field of characteristic $p > 0$ and let $R^{\mathrm{sep}}/R$ be a fixed separable closure of $R$. We denote by $\mathrm{G}_R = \mathrm{Gal}(R^{\mathrm{sep}}/R)$. If $R$ is a complete discrete valuation field, we denote by $\mathcal{I}_R$ the inertia group and by $\mathcal{P}_R$ the pro-$p$-sylow subgroup of $\mathcal{I}_R$.*

We have $\mathrm{H}^1(\mathrm{G}_R, \mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_R, \mathbb{Z}/p^m\mathbb{Z})$ (cf. [23, Ch.$X$, Sect. 3]). The situation is then expressed by the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{Z}/p^{m+1}\mathbb{Z} & \longrightarrow & \mathbf{W}_m(R) & \xrightarrow{\bar{\mathrm{F}}-1} & \mathbf{W}_m(R) & \xrightarrow{\delta} & \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_R, \mathbb{Z}/p^{m+1}\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow \imath & \odot & \downarrow \mathrm{V} & \odot & \downarrow \mathrm{V} & \odot & \downarrow \jmath & & \\ 0 & \longrightarrow & \mathbb{Z}/p^{m+2}\mathbb{Z} & \longrightarrow & \mathbf{W}_{m+1}(R) & \xrightarrow{\bar{\mathrm{F}}-1} & \mathbf{W}_{m+1}(R) & \xrightarrow{\delta} & \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_R, \mathbb{Z}/p^{m+2}\mathbb{Z}) & \longrightarrow & 0 \end{array} \tag{1.21}$$

where $\imath : 1 \mapsto p$ is the usual inclusion, and $\jmath$ is the composition with $\imath$. For $\boldsymbol{\lambda} \in \mathbf{W}_m(R)$, the character $\alpha = \delta(\boldsymbol{\lambda})$ sends the automorphism $\gamma$ to the element $\alpha(\gamma) := \gamma(\boldsymbol{\nu}) - \boldsymbol{\nu} \in \mathbb{Z}/p^{m+1}\mathbb{Z}$, where $\boldsymbol{\nu} \in R^{\mathrm{sep}}$ is a solution of the equation $\bar{\mathrm{F}}(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{\lambda}$. Taking the inductive limit, we get the following exact sequence:

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbf{CW}(R) \xrightarrow{\bar{\mathrm{F}}-1} \mathbf{CW}(R) \to \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_R, \mathbb{Q}_p/\mathbb{Z}_p) \to 0\,, \tag{1.22}$$

where the word "cont" means that all characters $\mathrm{G}_R \to \mathbb{Q}_p/\mathbb{Z}_p$ factorize on a finite quotient of $\mathrm{G}_R$. Indeed $\varinjlim_m \mathrm{Hom}(\mathrm{G}_R, \mathbb{Z}/p^m\mathbb{Z})$ can be seen as the subset of $\mathrm{Hom}(\mathrm{G}_R, \mathbb{Q}_p/\mathbb{Z}_p)$ formed by the elements killed by a power of $p$.

*Remark* 1.9 If the vertical arrows V are replaced by $\mathrm{V}\bar{\mathrm{F}}$ in the diagram (1.21), then the morphisms $\imath$ and $\jmath$ remain the same. Indeed $\delta(\boldsymbol{\lambda}) = \delta(\bar{\mathrm{F}}(\boldsymbol{\lambda}))$, because $\bar{\mathrm{F}}(\boldsymbol{\lambda}) = \boldsymbol{\lambda} + (\bar{\mathrm{F}} - 1)(\boldsymbol{\lambda})$, for all $\boldsymbol{\lambda} \in \mathbf{W}_s(R)$. Hence we have also

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \to \widetilde{\mathbf{CW}}(R) \xrightarrow{\bar{\mathrm{F}}-1} \widetilde{\mathbf{CW}}(R) \to \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_R, \mathbb{Q}_p/\mathbb{Z}_p) \to 0. \tag{1.23}$$

*Remark* 1.10 Let $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_m) \in \mathbf{W}_m(R)$. The kernel of $\alpha := \delta(\boldsymbol{\lambda})$ is the subgroup of $\mathrm{G}_R$ whose corresponding extension field is $R(\{\nu_0, \dots, \nu_m\})$, (i.e. the smallest field containing the set $\{\nu_0, \dots, \nu_m\}$), where $\boldsymbol{\nu} = (\nu_0, \dots, \nu_m) \in \mathbf{W}_m(R^{\mathrm{sep}})$ is solution of $\bar{\mathrm{F}}(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{\lambda}$. All cyclic separable extensions of $R$, whose degree is a power of $p$, are of this form for a suitable $m \geq 0$, and $\boldsymbol{\lambda}$.

*1.4.1*

Let $\kappa$ be a field of characteristic $p > 0$, and let $R := \kappa((t))$. The Galois group of an abelian extension of $\kappa((t))$ is the product of its $p$-torsion part (controlled by the Artin–Schreier theory) and its moderate part (controlled by Kummer theory).

**Definition 1.13** *We set* $\mathbf{P}(\kappa) := \mathrm{Hom}^{\mathrm{cont}}(\mathcal{P}_R, \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}^{\mathrm{cont}}(\mathcal{I}_R, \mathbb{Q}_p/\mathbb{Z}_p)$.

*Remark* 1.11 We will see that $\mathbf{P}(\kappa) \cong \frac{\mathbf{CW}(t^{-1}\kappa[t^{-1}])}{(\bar{\mathrm{F}}-1)\mathbf{CW}(t^{-1}\kappa[t^{-1}])}$. This group describes the abelianization of the pro-$p$-Sylow of the quotient $\mathcal{P}_R$. On the other hand $\mathbf{CW}(\kappa)/(\bar{\mathrm{F}} - 1)\mathbf{CW}(\kappa) = \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_R/\mathcal{I}_R, \mathbb{Q}_p/\mathbb{Z}_p)$.

## 1.5 Notations in Lubin–Tate theory

For notations and results on Lubin–Tate theory we refer to [17]. In this paper we will treat only Lubin–Tate groups over the field $\mathbb{Q}_p$. We recall briefly only the facts used in this paper. Let $\mathrm{w} := p \cdot u \in p\mathbb{Z}_p$, $u \in \mathbb{Z}_p^\times$, be a uniformizing element. Let $\mathfrak{F}_{\mathrm{w}}$ be the family of formal power series $P(X) \in \mathbb{Z}_p[[X]]$ satisfying

$$P(X) \equiv \mathrm{w}X \qquad (\mathrm{mod}\ X^2\mathbb{Z}_p[[X]]), \quad P(X) \equiv X^p \qquad (\mathrm{mod}\ \mathrm{w}\mathbb{Z}_p[[X]]). \quad (1.24)$$

A series in $\mathfrak{F}_{\mathrm{w}}$ will be called a Lubin–Tate series. For all $P \in \mathfrak{F}_{\mathrm{w}}$, there exists a unique formal group law $\mathfrak{G}_P(X, Y) \in \mathbb{Z}_p[[X, Y]]$ such that $P(\mathfrak{G}_P(X, Y)) = \mathfrak{G}_P(P(X), P(Y))$ (i.e. $P(X)$ is an endomorphism of $\mathfrak{G}_P(X, Y)$).

**Lemma 1.8** *Let* $P, \widetilde{P} \in \mathfrak{F}_{\mathrm{w}}$. *For all* $a \in \mathbb{Z}_p$ *there exists a unique formal series* $[a]_{P,\widetilde{P}}(X) \in \mathbb{Z}_p[[X]]$ *such that*

1. $[a]_{P,\widetilde{P}}(X) \equiv aX \ (\mathrm{mod}\ X^2\mathbb{Z}_p[[X]])$,
2. $[a]_{P,\widetilde{P}}(\mathfrak{G}_P(X, Y)) = \mathfrak{G}_{\widetilde{P}}([a]_{P,\widetilde{P}}(X), [a]_{P,\widetilde{P}}(Y))$.

*In other words* $[a]_{P,\widetilde{P}}(X)$ *is a morphism of group laws.*

We set $[a]_P(X) := [a]_{P,P}(X)$. By the uniqueness, we have that $P(X) = [\mathrm{w}]_P(X)$. The setting $x \times y := \mathfrak{G}_P(x, y)$ defines a new group law on $\mathfrak{p}_K$, denoted by $\mathfrak{G}_P(\mathfrak{p}_K)$. Let $P^{(k)}$ denote the series $P \circ P \circ \cdots \circ P$, $k$-times. Following [17] let

$$\Lambda_{P,m} = \mathrm{Ker}(P^{(m)}) = \mathrm{Ker}([\mathrm{w}^m]_P) = \{x \in \mathbb{C}_p \mid P^{(m)}(x) = 0 \quad \text{and} \quad |x| < 1\} \quad (1.25)$$

be the set of $[\mathrm{w}]^m$-torsion points of $\mathfrak{G}_P(\mathfrak{p}_{\mathbb{C}_p})$, and $\Lambda_P := \cup_m \Lambda_{P,m}$. We have $\Lambda_P \subset \mathbb{Q}_p^{\mathrm{alg}}$. Moreover $\mathbb{Q}_p(\Lambda_{P,m})/\mathbb{Q}_p$ is Galois and depend only on w. The formal group law $\mathfrak{G}_P$ makes $\Lambda_{P,m}$ a group.

**Theorem 1.4** ([17, Theorem 2]) *We have the following properties*:

1. *We have* $\Lambda_{P,m} \cong \mathbb{Z}/p^m\mathbb{Z}$, *for all* $m \geq 1$, *and then* $\Lambda_P \cong \mathbb{Q}_p/\mathbb{Z}_p$.
2. *Let* $\gamma \in \mathrm{Gal}(\mathbb{Q}_p(\Lambda_P)/\mathbb{Q}_p)$. *There exists an unique unit* $u_\gamma \in \mathbb{Z}_p^\times$ *such that*

$$\gamma(x) = [u_\gamma]_P(x), \quad \forall x \in \Lambda_P.$$

3. *The map* $\gamma \mapsto u_\gamma$ *is an isomorphism of* $\mathrm{Gal}(\mathbb{Q}_p(\Lambda_P)/\mathbb{Q}_p)$ *onto the group* $\mathbb{Z}_p^\times$. *The same map gives an isomorphism*

$$\mathrm{Gal}(\mathbb{Q}_p(\Lambda_P)/\mathbb{Q}_p(\Lambda_{P,m})) \xrightarrow{\sim} 1 + \mathrm{w}^m\mathbb{Z}_p, \quad \forall m \geq 1.$$

4. *Let* $u \in \mathbb{Z}_p^\times$, *then* $[u]_P(x) = (u^{-1},\mathbb{Q}_p(\Lambda_{P,m})/\mathbb{Q}_p)(x)$, *for all* $x \in \Lambda_{P,m}$, *where* $(u^{-1},\mathbb{Q}_p(\Lambda_{P,m})/\mathbb{Q}_p) \in \mathrm{Gal}(\mathbb{Q}_p(\Lambda_P)/\mathbb{Q}_p)$ *is the norm residue symbol.*

*Remark* 1.12 The simplest Lubin–Tate series is $P(X) = \mathrm{w}X + X^p$. If $\mathrm{w} = p$, then a non trivial zero $\pi_0$ of $P$ is the "$\pi$" of Dwork. If again $\mathrm{w} = p$ and $P(X) = (X+1)^p - 1$, then $\mathfrak{G}_P \cong \widehat{\mathbb{G}}_m$, and all torsion points are of the form $\xi - 1$, with $\xi^{p^k} = 1$, for some $k \geq 0$. This was the choice made by Matsuda [18].

**Theorem 1.5** ([15, Proposition 8.3.22]) *Let* $\mathfrak{G}$ *and* $\widetilde{\mathfrak{G}}$ *be two Lubin–Tate groups relative to the uniformizers* w *and* $\tilde{\mathrm{w}}$, *respectively. Then* $\mathfrak{G}$ *is isomorphic to* $\widetilde{\mathfrak{G}}$ (*as formal groups over* $\mathbb{Z}_p$) *if and only if* $\mathrm{w} = \tilde{\mathrm{w}}$.

### *1.5.1 Tate module*

The multiplication by $[\mathrm{w}]_P$ sends $\Lambda_m$ into $\Lambda_{m-1}$. The Tate module of $\mathfrak{G}_P$ is, by definition, $\mathrm{T}(\mathfrak{G}_P) := \lim_{\leftarrow m} \Lambda_{P,m}$. A generator $\boldsymbol{\pi} = (\pi_{P,j})_{j\geq 0}$ of the Tate module $\mathrm{T}(\mathfrak{G}_P)$ is a sequence $(\pi_{P,j})_{j\geq 0}$, $\pi_j \in \Lambda_P$, such that $P(\pi_{P,0}) = 0$, $\pi_{P,0} \neq 0$ and $P(\pi_{P,j+1}) = \pi_{P,j}$, for all $j \geq 0$. If no confusion is possible, we will write $\pi_j$ instead of $\pi_{P,j}$. The Newton polygon of $P$ shows that $P$ has exactly $p-1$ non trivial zeros of value $\omega = |p|^{\frac{1}{p-1}}$, and inductively $P(X) - \pi_{j-1}$ has $p$ zeros of valuation $\omega^{\frac{1}{p^j}}$. Hence $|\pi_j| = \omega^{1/p^j}$, for all $j \geq 0$, and the Galois extension $\mathbb{Q}_p(\Lambda_{P,m}) = \mathbb{Q}_p(\pi_{m-1})$ is totally ramified. On the other hand the field $K(\pi_{m-1})$ is not always totally ramified.

**Definition 1.14** *We set* $K_m := K(\pi_m)$ (*respectively* $K(\Lambda_P)$), *and denote by* $k_m$ (*respectively* $k_{\mathrm{w}}$) *its residue field. Moreover, if* $\mathrm{w} = p$, *we put* $K_\infty := K(\Lambda_P)$ *and* $k_\infty := k_p$. *For all algebraic extensions* $L/K$, $L_\infty$ *will be the smallest field containing* $L$ *and* $K_\infty$.

*Example* 1.2 If $P(X) = (X+1)^p - 1$, then $\mathfrak{G}_P = \widehat{\mathbb{G}}_m$, and $\Lambda_m = \{\xi_m - 1 \mid \xi_m^{p^m} = 1\}$ is the set of $p^m$th root of unity minus 1. A generator of $\mathrm{T}(\widehat{\mathbb{G}}_m)$ is a family $(\xi_j - 1)_{j\geq 0}$ satisfying $\xi_j^{p^i} = \xi_{j-i}$, for all $0 \leq i \leq j$.

**Definition 1.15** *Let $P, \widetilde{P} \in \mathfrak{F}_{\mathrm{w}}$ be two Lubin–Tate series. We will say that $x \in \Lambda_P$ and $y \in \Lambda_{\widetilde{P}}$ are equivalent if $y = [1]_{P,\widetilde{P}}(x)$ (cf. Lemma 1.8).*

*Remark* 1.13 Since $[1]_{P,\widetilde{P}}(x) = x +$ (things divisible by $x^2$), it follows that $|x - [1]_{P,\widetilde{P}}(x)| \leq |x|^2$. In particular, if $\mathrm{w} = p$ and if $\pi_m$ is fixed, then there exists a unique $p^{m+1}$th root of 1, say $\xi_m$, such that $|(\xi_m - 1) - \pi_m| \leq \omega^{\frac{2}{p^m}}$ (cf. Remark 1.12).

## 2 $\pi$-exponentials and applications

### 2.1 Construction of Witt vectors

Let $P(X) \in \mathbb{Z}_p[[X]]$ be a series, with $P(0) = 0$, satisfying

$$P(X) \equiv X^p \mod p\mathbb{Z}_p[[X]]. \tag{2.1}$$

We consider the Frobenius $\sigma_P : \mathbb{Z}_p[[X]] \to \mathbb{Z}_p[[X]]$ given by $\sigma_P(h(X)) := h(P(X))$.

**Lemma 2.1** ([3, Ch. IX, Sect. 1, ex. 14, a]) *There is a unique ring morphism*

$$[-] : \mathbb{Z}_p[[X]] \xrightarrow[h(X)\mapsto[h(X)]]{} \mathbf{W}(\mathbb{Z}_p[[X]]) \tag{2.2}$$

*such that $\phi_j \circ [-] = \sigma_P^j$. In other words, for all $h(X) \in \mathbb{Z}_p[[X]]$, the Witt vector $[h(X)]$ is the unique one whose phantom vector is equal to*

$$\langle h(X),\ h(P(X)),\ h(P(P(X))),\ \ldots\ \rangle. \tag{2.3}$$

*Moreover $[-]$ is also the unique ring morphism satisfying the relation*

$$\mathrm{F}([h(X)]) = [h(P(X))]. \tag{2.4}$$

*Proof* By Lemma 1.6, the ring morphism $h(X) \mapsto \langle h(X), h(P(X)), \ldots\rangle$ : $\mathbb{Z}_p[[X]] \to (\mathbb{Z}_p[[X]])^{\mathbb{N}}$ has its values in the image of the phantom component map $\phi : \mathbf{W}(\mathbb{Z}_p[[X]]) \hookrightarrow (\mathbb{Z}_p[[X]])^{\mathbb{N}}$. Since, by Lemma 1.5, $\phi$ is injective, the Lemma is proved. □

**Definition 2.1** *Let* B *be a complete topologized $\mathbb{Z}_p$-ring, and let $b \in$* B *be a topologically nilpotent element. The specialization $X \mapsto b : \mathbb{Z}_p[[X]] \to$* B *provides,* *by functoriality, a morphism* $\mathbf{W}(\mathbb{Z}_p[[X]]) \to \mathbf{W}(\mathrm{B})$*. For brevity, we denote by* $[h(b)]$ *the image of* $h(X)$ *via the morphism*

$$\mathbb{Z}_p[[X]] \xrightarrow{[-]} \mathbf{W}(\mathbb{Z}_p[[X]]) \xrightarrow{X \mapsto b} \mathbf{W}(\mathrm{B}). \tag{2.5}$$

*We will denote again by* $[h(b)]$ *its image in* $\mathbf{W}_m(\mathrm{B})$*.*

*Remark* 2.1 The phantom vector of $[h(b)]$ is

$$\langle h(b),\ h(P(b)),\ h(P(P(b))),\ \ldots \rangle. \tag{2.6}$$

In general there is no morphism $\mathbb{Z}_p[b] \to \mathbf{W}(\mathrm{B})$ sending $h(b)$ into $[h(b)]$, the notation $[h(b)]$ is imprecise, but more handy.

**Lemma 2.2** (key lemma) *Let* $(\mathrm{B}, |.|)$ *be a* $\mathbb{Z}_p$*-ring, complete with respect to an absolute value* $|.|$*, extending the absolute value of* $\mathbb{Z}_p$*. Let* $h(X) = \sum_{i\geq 0} a_i X^i \in \mathbb{Z}_p[[X]]$*, and let* $[h(b)] = (\lambda_0, \lambda_1, \ldots) \in \mathbf{W}(\mathrm{B})$*, with* $|b| < 1$*. Then the following statements are equivalent:*

1. $|a_0| = |p|^r$*,*
2. $|\lambda_0|, \ldots, |\lambda_{r-1}| < 1$*, and* $|\lambda_r| = 1$*.*

*Proof* Let $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots) = [h(b)]$. We denote by $\overline{\mathrm{B}}$ the residue ring. The condition (2) is equivalent to $\bar{\lambda}_r \neq 0$, and $\bar{\lambda}_i = 0$, for all $i < r$, or, if $k \geq 0$ is given, it is equivalent to $\bar{\lambda}_r^{p^k} \neq 0$, and $\bar{\lambda}_i^{p^k} = 0$, for all $i < r$. This last condition is equivalent to the condition (2) for the vector $\mathrm{F}^k(\boldsymbol{\lambda})$. Now the phantom vector of $\mathrm{F}^k(\boldsymbol{\lambda})$ is $\langle h(P^{(k)}(b)), h(P^{(k+1)}(b)), \ldots \rangle$ (cf. Sect. 1.3.1). Moreover $|P(b)| \leq \sup(|b|^p, |p||b|)$, hence, for all $\varepsilon > 0$, there exists $k \geq 0$ such that $|P^{(i)}(b)| < \varepsilon$, for all $i \geq k$. If $\varepsilon$ is small enough, then $|h(P^{(i)}(b))| = |a_0|$, for all $i \geq k$. Let $(\nu_0, \nu_1, \ldots) := \mathrm{F}^k(\boldsymbol{\lambda})$, then, since $p^j \nu_j = h(P^{(j)}(b)) - (\nu_0^{p^j} + \cdots + p^{j-1}\nu_{j-1}^p)$, we see, by induction, that $|a_0| = |p|^r$ if and only if $|\nu_r| = 1$ and $|\nu_j| = |p|^{r-j}$, for all $j \leq r-1$. □

**Definition 2.2** *We fix now a sequence* $\boldsymbol{\varpi} := \{\varpi_j\}_{j\geq 0}$ *in* $\mathbb{Q}_p^{\mathrm{alg}}$ *satisfying* $|\varpi_0| < 1$*,* $P(\varpi_0) = 0$*, and* $P(\varpi_{j+1}) = \varpi_j$*, for all* $j \geq 1$*.*

*Remark* 2.2 The ring $\mathbb{Z}_p[\varpi_m]$ is complete, for all $m \geq 0$. Indeed $\varpi_m$ is algebraic and integral over $\mathbb{Z}_p$, hence $\mathbb{Z}_p[\varpi_m]$ is a free module over $\mathbb{Z}_p$.

*Remark* 2.3 If $P$ is a Lubin–Tate series, and if $\varpi_0 \neq 0$, then $\boldsymbol{\varpi}$ is a generator of the Tate module $\mathrm{T}(\mathfrak{G}_P)$, while if $P(X) \equiv X^p \mod X^{p+1}\mathbb{Z}_p[[X]]$, then $\varpi_j = 0$ for all $j \geq 0$. Observe that, taking $h(T) := T$ and $b := \varpi_m$ in the lemma 2.1, then $[\varpi_m] \in \mathbf{W}(\mathbb{Z}_p[\varpi_m])$ is the unique Witt vector whose phantom vector is $\langle \varpi_m, \varpi_{m-1}, \ldots, \varpi_0, 0, \ldots \rangle$. The uniqueness follows from the injectivity of the phantom map $\phi : \mathbf{W}(\mathbb{Z}_p[\varpi_m]) \hookrightarrow (\mathbb{Z}_p[\varpi_m])^{\mathbb{N}}$.

**Proposition 2.1** *For all $\mathbb{Z}_p[\varpi_m]$-algebra* B *of characteristic* 0

$$[\varpi_j]\mathbf{W}(\mathrm{B}) \subset [\varpi_{j+1}]\mathbf{W}(\mathrm{B}), \quad j = 0, \dots, m-1. \tag{2.7}$$

*Moreover, for all $\boldsymbol{\lambda} \in \mathbf{W}(\mathrm{B})$, and all $j = 0, \dots, m-1$*

$$\mathrm{F}([\varpi_{j+1}]) = [\varpi_j]; \quad \mathrm{V}([\varpi_j] \cdot \boldsymbol{\lambda}) = [\varpi_{j+1}] \cdot \mathrm{V}(\boldsymbol{\lambda}). \tag{2.8}$$

*Hence* $\mathrm{F}([\varpi_{j+1}]\mathbf{W}(\mathrm{B})) \subset [\varpi_j]\mathbf{W}(\mathrm{B})$ *and* $\mathrm{V}([\varpi_j]\mathbf{W}(\mathrm{B})) \subset [\varpi_{j+1}]\mathbf{W}(\mathrm{B})$.

*If now $\varpi_0 \neq 0$, then the kernel of the morphism $\boldsymbol{\lambda} \mapsto [\varpi_m]\boldsymbol{\lambda}$ is the ideal* $\mathrm{V}^{m+1}\mathbf{W}(\mathrm{B})$. *The induced morphism* $\mathbf{W}_m(\mathrm{B}) \to \mathbf{W}(\mathrm{B})$ *is a functorial isomorphism of* $\mathbf{W}_m(\mathrm{B})$ *into the ideal* $[\varpi_m]\mathbf{W}(\mathrm{B})$ *(as* $\mathbf{W}(\mathrm{B})$*-modules), which commutes with* $\mathrm{V} : \mathbf{W}_m(\mathrm{B}) \to \mathbf{W}_{m+1}(\mathrm{B})$ *and* $\mathrm{F} : \mathbf{W}_{m+1}(\mathrm{B}) \to \mathbf{W}_m(\mathrm{B})$

$$\begin{array}{ccc} \mathbf{W}(\mathrm{B}) & \xrightarrow{\boldsymbol{\lambda} \mapsto [\varpi_m]\boldsymbol{\lambda}} & \mathbf{W}(\mathrm{B}) \\ \downarrow & & \uparrow \\ \mathbf{W}_m(\mathrm{B}) & \xrightarrow{\sim} & [\varpi_m] \cdot \mathbf{W}(\mathrm{B}) \end{array} . \tag{2.9}$$

*Proof* Let $h(X) := P(X)/X$, then $[\varpi_j]\boldsymbol{\lambda} = [P(\varpi_{j+1})]\boldsymbol{\lambda} = [\varpi_{j+1} \cdot h(\varpi_{j+1})]\boldsymbol{\lambda} = [\varpi_{j+1}] \cdot [h(\varpi_{j+1})]\boldsymbol{\lambda}$. This shows the inclusion (2.7). All other assertions are easily verified on the phantom components. □

**Corollary 2.1** *Let* $\langle \phi_0, \dots, \phi_m \rangle \in \mathbb{Z}_p[\varpi_m]^{m+1}$. *If there exists a formal series* $h(X) = \sum_{i \geq 0} a_i X^i \in \mathbb{Z}_p[[X]]$ *satisfying*

$$h(\varpi_{m-j}) = \phi_j, \quad \text{for all } 0 \leq j \leq m, \tag{2.10}$$

*then* $\langle \phi_0, \dots, \phi_m \rangle$ *is the phantom vector of* $[h(\varpi_m)] := (\nu_0, \dots, \nu_m) \in \mathbf{W}_m(\mathbb{Z}_p[\varpi_m])$. *Moreover,* $|a_0| = |p|^r$, *for some* $r \geq 0$, *if and only if* $|\nu_0|, \dots, |\nu_{r-1}| < 1$ *and* $|\nu_r| = 1$.

### *2.1.1 Artin–Hasse exponential and Robba exponentials*

**Definition 2.3** ([3, ex. 58]) *Let* B *be a* $\mathbb{Z}_{(p)}$*-ring, and let*

$$E(T) := \exp\left(T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots\right) \in 1 + T\mathbb{Z}_{(p)}[[T]]. \tag{2.11}$$

*For all* $\boldsymbol{\lambda} := (\lambda_0, \lambda_1, \dots) \in \mathbf{W}(\mathrm{B})$, *the Artin–Hasse exponential relative to* $\boldsymbol{\lambda}$ *is*

$$E(\boldsymbol{\lambda}, T) := \prod_{j \geq 0} E(\lambda_j \cdot T^{p^j}) = \exp\left(\phi_0 T + \phi_1 \frac{T^p}{p} + \phi_2 \frac{T^{p^2}}{p^2} + \cdots\right) \in 1 + TB[[T]], \tag{2.12}$$

*where* $\langle\phi_0,\phi_1,\dots\rangle$ *is the phantom vector of* $\boldsymbol{\lambda}$.

*Remark* 2.4 The Artin–Hasse exponential is then a group morphism

$$E(-,T):\mathbf{W}(\mathrm{B})\to 1+T\mathrm{B}[[T]], \tag{2.13}$$

functorial on the $\mathbb{Z}_{(p)}$-ring B.

**Proposition 2.2** *Let* $[\varpi_m]\in\mathbf{W}(\mathbb{Z}_p[\varpi_m])$ *be as in Definition* 2.2. *The exponential*

$$E_m(T):=E([\varpi_m],T)=\exp\left(\varpi_m T+\varpi_{m-1}\frac{T^p}{p}+\cdots+\varpi_0\frac{T^{p^m}}{p^m}\right) \tag{2.14}$$

*converges exactly in the disk* $|T|<1$, *for all* $m\geq 0$, *if and only if* $P(X)$ *is a Lubin–Tate series, and* $\boldsymbol{\varpi}:=(\varpi_j)_{j\geq 0}$ *is a generator of the Tate module* $\mathrm{T}(\mathfrak{G}_P)$.

*Proof* Assume that the radius of convergence of $E([\varpi_m],T)$ is equal to 1, for all $m\geq 0$. Then, for $m=0$, the radius of convergence of $\exp(\varpi_0 T)$ is 1, hence $|\varpi_0|=\omega$. The Newton polygon of $P(X)$ implies that $P(X)\equiv \mathrm{w}X \mod X\mathbb{Z}_p[[X]]$, for some w, with $|\mathrm{w}|=|p|$, hence $P(X)$ is a Lubin–Tate series. Conversely, assume that $P(X)$ is a Lubin–Tate series, and that $\boldsymbol{\varpi}:=(\varpi_j)_{j\geq 0}$ is a generator of $\mathrm{T}(\mathfrak{G}_P)$. Consider the differential operator $L:=\partial_T+\varpi_m T^{-1}+\varpi_{m-1}T^{-p}+\cdots+\varpi_0 T^{-p^m}$. Then $E_m(T^{-1})$ is the Taylor solution at $+\infty$ of $L$. Since $|\varpi_0|=\omega$, by Lemma 1.3, we have $\mathrm{Ray}(L,\rho)=\rho^{p^m+1}$, for all $\rho<1$. In particular, the irregularity of $L$ is $p^m$. Then $E_m(T^{-1})$ is not convergent for $|T|<1$, because otherwise, by transfer at $\infty$, $E_m(T^{-1})\in\mathcal{R}$, and $L$ will be trivial. □

**Theorem 2.1** *Let* $P(X)=\mathrm{w}X+\cdots$ *be a Lubin–Tate series, and let* $\boldsymbol{\varpi}:=(\varpi_j)_{j\geq 0}$ *be a generator of* $\mathrm{T}(\mathfrak{G})$. *Then the formal series* $E_m(T^p)/E_m(T)$ *is over-convergent (i.e. convergent for* $|T|<1+\varepsilon$, *for some* $\varepsilon>0$) *if and only if*

$$|\mathrm{w}-p|\leq|p|^{m+2}.$$

*In particular,* $E_m(T^p)/E_m(T)$ *is over-convergent for all* $m\geq 0$ *if and only if* $\mathfrak{G}_P$ *is isomorphic to the formal multiplicative group* $\widehat{\mathbb{G}}_m$ (*cf. Theorem* 1.5).

*Proof* This Theorem will follow easily from the theory of $\boldsymbol{\pi}$-exponentials (cf. the proof of Theorem 2.5 infra), and is placed here for expository reasons. □

## 2.2 $\boldsymbol{\pi}$-exponentials

We maintain the notations of Sect. 2.1. In this section we fix a uniformizing element w of $\mathbb{Z}_p$, a $\mathbb{Q}_p$-Lubin–Tate series $P\in\mathbb{Z}_p[[X]]$, $P\in\mathfrak{F}_{\mathrm{w}}$, and a generator $\boldsymbol{\pi}=(\pi_j)_{j\geq 0}$ of the Tate module. We fix three natural numbers $n,m,d$ such that

$$d=n\cdot p^m>0, \quad \text{and} \quad (n,p)=1. \tag{2.15}$$

**Definition 2.4** *Let* B *be a* $\mathbb{Z}_p[\pi_m]$*-algebra. Let* $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(\mathrm{B})$*, and let* $\langle \phi_0, \ldots, \phi_m \rangle \in \mathrm{B}^{m+1}$ *be its phantom vector. We set*

$$\mathrm{e}_d(\boldsymbol{\lambda}, T) := E([\pi_m]\boldsymbol{\lambda}, T^n) = \exp\left(\pi_m\phi_0 T^n + \pi_{m-1}\phi_1 \frac{T^{np}}{p} + \cdots + \pi_0\phi_m \frac{T^d}{p^m}\right). \tag{2.16}$$

*We will call* $\mathrm{e}_d(\boldsymbol{\lambda}, T) \in 1 + \pi_m T\mathrm{B}[[T]]$ *the* $\boldsymbol{\pi}$-exponential *attached to* $\boldsymbol{\lambda}$.

**Proposition 2.3** *The map* $\boldsymbol{\lambda} \mapsto \mathrm{e}_d(\boldsymbol{\lambda}, T)$ *defines a group morphism*

$$\mathbf{W}_m(\mathrm{B}) \longrightarrow 1 + \pi_m T\mathrm{B}[[T]]\,. \tag{2.17}$$

*Moreover, for all* $\boldsymbol{\lambda}, \boldsymbol{\nu} \in \mathbf{W}_m(\mathrm{B})$*, we have*

$$\mathrm{e}_d(\boldsymbol{\lambda}, T) = \prod_{j=0}^{m} E_{m-j}\left(\lambda_j T^{np^j}\right), \tag{2.18}$$

$$E_m(T) = \mathrm{e}_{p^m}((1,0,\ldots,0), T), \quad \mathrm{e}_d(\boldsymbol{\lambda}, T) = \mathrm{e}_{p^m}(\boldsymbol{\lambda}, T^n),$$
$$\mathrm{e}_d(\boldsymbol{\lambda}, T^p) = \mathrm{e}_{p\cdot d}(\mathrm{V}(\boldsymbol{\lambda}), T), \qquad \mathrm{e}_d(\boldsymbol{\lambda} + \boldsymbol{\nu}, T) = \mathrm{e}_d(\boldsymbol{\lambda}, T) \cdot \mathrm{e}_d(\boldsymbol{\nu}, T).$$

*Furthermore, if* $\mathrm{B} = \mathcal{O}_L$ *is the ring of integers of some finite extension* $L/K$*, and if, for some* $r \geq 1$*, there exists a Frobenius* $\sigma$ *on* $\mathcal{O}_L$ *lifting the* $p^r$*-th power map* $x \mapsto x^{p^r}$ *of* $k_L$*, and satisfying* $\sigma(\pi_j) = \pi_j$*,* $\forall\, 0 \leq j \leq m$*, then we have*

$$\mathrm{e}_d^{\sigma}(\boldsymbol{\lambda}, T) = \mathrm{e}_d(\sigma(\boldsymbol{\lambda}), T), \tag{2.19}$$

*where* $\sigma(\lambda_0, \ldots, \lambda_m) = (\sigma(\lambda_0), \ldots, \sigma(\lambda_m))$ *and, for all* $f(T) = \sum a_i T^i$*, we set* $f^{\sigma}(T) = \sum \sigma(a_i) T^i$ *(cf. Sect.* 1.2.4*).*

*Proof* All the assertions are easily verified on the phantom components. □

### *2.2.1 Study of the differential module attached to a* $\boldsymbol{\pi}$*-exponential*

We maintain the notations of Sect. 2.2. As usual $d = np^m > 0$, with $(n,p) = 1$. In this subsection $H/K$ is an algebraic extension (not necessary complete) and

$$H_m := H(\pi_m). \tag{2.20}$$

*Remark* 2.5 The Witt vectors we are considering have a finite number of entries. Hence the exponential $\mathrm{e}_d(\boldsymbol{\lambda}, T)$ has its coefficients in a *finite* (thus complete) extension of $K$. This will solve all problems concerning convergence.

**Definition 2.5** *Let* $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(\mathcal{O}_H)$*, and let* $\langle \phi_0, \ldots, \phi_m \rangle \in \mathcal{O}_H^{m+1}$ *be its phantom vector. We define*

$$\mathrm{L}_d(\boldsymbol{\lambda}) := \partial_T - \partial_{T,\log}(\mathrm{e}_d(\boldsymbol{\lambda}, T^{-1})) = \partial_T + n \cdot \left( \sum_{j=0}^{m} \pi_{m-j} \cdot \phi_j \cdot T^{-np^j} \right).$$

*We denote by* $\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})$ *the differential module over* $\mathcal{R}_{H_m}$ *defined by* $\mathrm{L}_d(\boldsymbol{\lambda})$.

**Lemma 2.3** $\mathrm{L}_d(\boldsymbol{\lambda})$ *is solvable at* $\rho = 1$*, and hence* $\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda}) \in \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_m})$.

*Proof* The Taylor solution at $+\infty$ of $\mathrm{L}_d(\boldsymbol{\lambda})$ is $\mathrm{e}_d(\boldsymbol{\lambda}, T^{-1}) \in 1 + \pi_m T^{-1} \mathcal{O}_{H_m}[[T^{-1}]]$, which has bounded coefficients and so converges for $|T| > 1$. By transfer (cf. Theorem 1.1), $\mathrm{Ray}(\mathrm{L}_d(\boldsymbol{\lambda}), \rho) = \rho$, for all $\rho > 1$. By continuity of the radius, $\mathrm{Ray}(\mathrm{L}_d(\boldsymbol{\lambda}), 1) = 1$. □

**Proposition 2.4** *The map* $\boldsymbol{\lambda} \mapsto \mathrm{e}_d(\boldsymbol{\lambda}, T^{-1})$ *defines a group morphism*

$$\mathbf{W}_m(\mathcal{O}_H) \longrightarrow 1 + \pi_m T^{-1} \mathcal{O}_{H_m}[[T^{-1}]] . \tag{2.21}$$

*More precisely, for all* $\boldsymbol{\lambda}, \boldsymbol{\nu} \in \mathbf{W}_m(\mathcal{O}_H)$*, one has:*

$$\varphi_p^*(\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})) = \widetilde{\mathrm{M}}_{pd}(\mathrm{V}(\boldsymbol{\lambda})) \ , \ \widetilde{\mathrm{M}}_d(\boldsymbol{\lambda} + \boldsymbol{\nu}) = \widetilde{\mathrm{M}}_d(\boldsymbol{\lambda}) \otimes \widetilde{\mathrm{M}}_d(\boldsymbol{\nu}), \tag{2.22}$$

*where* $\varphi_p(f(T)) = f(T^p)$ *(cf.* Sect. 1.2.4). *Moreover, if there exists an absolute Frobenius* $\sigma$ *on* $H_m$ *(cf. Definition 1.10) such that* $\pi_j^\sigma = \pi_j$*, for all* $0 \leq j \leq m$*, then*

$$\varphi_\sigma(\mathrm{e}_d(\boldsymbol{\lambda}, T)) = \mathrm{e}_d(\varphi_\sigma(\boldsymbol{\lambda}), T^p), \quad \varphi_\sigma^*(\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})) = \widetilde{\mathrm{M}}_{pd}(\mathrm{V}(\sigma(\boldsymbol{\lambda}))),$$

*where* $\varphi_\sigma(f(T)) = f^\sigma(T^p)$*, (cf.* Sect. 1.2.3), *and* $\sigma(\lambda_0, \ldots, \lambda_m) := (\sigma(\lambda_0), \ldots, \sigma(\lambda_m))$.

*Proof* The first part is a direct consequence of Proposition 2.3. The last assertion is a consequence of Corollary 2.3 and is placed here for expository reasons. Observe that, in the sequel, we do not suppose the existence of $\sigma$ on $H$. Indeed, our "Frobenius structure Theorem" does not require the existence of $\varphi$ (cf. Remark 2.10). □

*Remark* 2.6 In particular $\widetilde{\mathrm{M}}_d$ defines a morphism of groups

$$\widetilde{\mathrm{M}}_d : \mathbf{W}_m(\mathcal{O}_H) \longrightarrow \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_m}). \tag{2.23}$$

**Theorem 2.2** *Let* $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(\mathcal{O}_H)$ *and let* $\langle \phi_0, \ldots, \phi_m \rangle \in \mathcal{O}_H^{m+1}$ *be its phantom vector. The following assertions are equivalent:*

1. $\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})$ *is trivial (i.e. isomorphic to* $\mathcal{R}_{H_m}$*).*
2. *The exponential* $\mathrm{e}_d(\boldsymbol{\lambda}, T)$ *is over-convergent (i.e. convergent in some disk* $|T| < 1 + \varepsilon$*, with* $\varepsilon > 0$*).*
3. $|\lambda_0|, \ldots, |\lambda_m| < 1$.

*Moreover, if* $|\lambda_0|, \ldots, |\lambda_{r-1}| < 1$ *and* $|\lambda_r| = 1, r \leq m$*, then we have* (*cf. Sect.* 1.3.3)

$$\mathrm{Irr}(\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})) = n \cdot p^{\ell(\boldsymbol{\lambda})} = d/p^r. \tag{2.24}$$

*Proof* The equivalence (1) $\Leftrightarrow$ (2) is evident. By Eq. (2.18) and Proposition 2.2, condition (3) implies that $\mathrm{e}_d(\boldsymbol{\lambda}, T^{-1}) \in \mathcal{R}_{H_m}$, so $\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})$ is trivial. The converse follows from the last assertion below. Let then $|\lambda_0|, \ldots, |\lambda_{r-1}| < 1$, and $|\lambda_r| = 1$, $r \leq m$. Clearly $|\phi_m(\boldsymbol{\lambda})| = 1$ if and only if $|\lambda_0| = 1$ (cf. Eq. (1.17)). Then, if $r = 0$, we can apply Lemma 1.3, and hence $\mathrm{Irr}(M) = \mathrm{Irr}_F(M) = d$. Let now $0 < r \leq m$, then $E_{m-j}(\lambda_j T^{-p^j})$ belongs to $\mathcal{R}_{H_m}^{\times}$, for all $j = 0, \ldots, r-1$. Then we change basis by the function $f(T) := \prod_{j=0}^{r-1} E_{m-j}(\lambda_j T^{-p^j})^{-1} \in \mathcal{R}_{H_m}^{\times}$. By Proposition 2.3, the new solution is

$$f(T) \cdot \mathrm{e}_d(\boldsymbol{\lambda}, T) = \mathrm{e}_d((0, \ldots, 0, \lambda_r, \ldots, \lambda_m), T) = \mathrm{e}_{d/p^r}\left((\lambda_r, \ldots, \lambda_m), T^{p^r}\right). \tag{2.25}$$

In other words, we have $\widetilde{\mathrm{M}}_d(\lambda_0, \ldots, \lambda_m) \xrightarrow{\sim} \varphi_p^*(\widetilde{\mathrm{M}}_{d/p^r}(\lambda_r, \ldots, \lambda_m))$ (cf. Eq. (2.22) and Sect. 1.2.4). By Remark 1.6, the Theorem is proved by induction, since $|\lambda_r| = 1$. □

*Remark* 2.7 In particular, $\widetilde{\mathrm{M}}_d$ passes to the quotient $\mathbf{W}_m(k_H)$, and induces an injective additive map called $\mathrm{M}_d$:

$$\begin{array}{ccc} \mathbf{W}_m(\mathcal{O}_H) & \xrightarrow{\widetilde{\mathrm{M}}_d} & \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_m}) \\ \downarrow & \nearrow_{\mathrm{M}_d} & \\ \mathbf{W}_m(k_H) & & \end{array} \tag{2.26}$$

**Corollary 2.2** *Consider the morphism of groups*

$$\mathbb{Z}_p[[X]] \xrightarrow{[-]} \mathbf{W}_m(\mathbb{Z}_p[\pi_m]) \subset \mathbf{W}_m(\mathcal{O}_{H_m}) \xrightarrow{\widetilde{\mathrm{M}}_d} \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_m}). \tag{2.27}$$

*Let* $h(X) := \sum_{i\geq 0} a_i X^i \in \mathbb{Z}_p[[X]]$ *be such that* $|a_0| = |p|^r$ $(v_p(a_0) = r)$*. Then* $\widetilde{\mathrm{M}}_d([h(\pi_m)])$ *has irregularity* $d/p^r$*, and is trivial if and only if* $r \geq m+1$*. In other words, the kernel of the composite map is the ideal* $p^{m+1}\mathbb{Z}_p[[X]] + X\mathbb{Z}_p[[X]]$.

*Proof* Combine Corollary 2.1 and the reduction Theorem 2.2. □

### 2.2.2 Dependence on the Lubin–Tate group and on $\boldsymbol{\pi}$

We maintain the notations of the previous sections. As usual $d = np^m > 0$, $(n,p) = 1$.

**Theorem 2.3** (Dependence on the choice of $\boldsymbol{\pi}$) *Let* $\boldsymbol{\pi} = (\pi_j)_{j\geq 0}$*,* $\boldsymbol{\pi}' = (\pi'_j)_{j\geq 0}$ *be two generators of* $\mathrm{T}(\mathfrak{G}_P)$*. Denote by* $\mathrm{M}'_d(-)$*,* $E'_j(T)$ *and* $\mathrm{e}'_d(-,T)$ *the constructions attached to* $\boldsymbol{\pi}'$*. Then* $\mathrm{M}_d(1,0,\dots,0)$ *and* $\mathrm{M}'_d(1,0,\dots,0)$ *are isomorphic over* $\mathcal{R}_{H_m}$ *if and only if* $\pi_m = \pi'_m$*. Moreover, in this case,* $\mathrm{M}_d(\boldsymbol{\lambda})$ *and* $\mathrm{M}'_d(\boldsymbol{\lambda})$ *are isomorphic over* $\mathcal{R}_{H_m}$*, for all* $\boldsymbol{\lambda} \in \mathbf{W}_m(k_H)$*.*

*Proof* The solution at $\infty$ of $\mathrm{M}_d\,(1,0,\dots,0)$ is $\mathrm{e}_d((1,0,\dots,0),T^{-1}) = E_m(T^{-n})$. We shall show that $E_m(T^{-n})/E'_m(T^{-n}) \in \mathcal{R}^\times$, that is $E_m(T^{-1})/E'_m(T^{-1}) \in \mathcal{R}^\times$, if and only if $\pi_m = \pi'_m$. We have

$$E_m(T^{-1})/E'_m(T^{-1}) = \exp\left(\sum_{j=0}^{m} \pi_{m-j}\left(1 - \frac{\pi'_{m-j}}{\pi_{m-j}}\right)\frac{T^{-p^j}}{p^j}\right). \tag{2.28}$$

There exists $\gamma \in \mathrm{Gal}(\mathbb{Q}_p(\Lambda_P)/\mathbb{Q}_p)$ such that $\pi'_j = \gamma(\pi_j)$, for all $j \geq 0$ and, by the Lubin–Tate Theorem 1.4, $\gamma(\pi_j) = [u_\gamma]_P(\pi_j)$, $u_\gamma \in \mathbb{Z}^\times$. We set [1]

$$h_\gamma(X) := 1 - [u_\gamma]_P(X)/X, \tag{2.29}$$

in order to have

$$E_m(T^{-1})/E'_m(T^{-1}) = \mathrm{e}_d([h_\gamma(\pi_m)], T^{-1}). \tag{2.30}$$

Indeed, by construction (cf. Corollary 2.1 and Definition 2.4), we have $\phi_j([h_\gamma(\pi_m)]) = 1 - \pi'_{m-j}/\pi_{m-j}$. Since $h_\gamma(0) = 1 - u_\gamma$, hence, by the Reduction Theorem 2.2 and Lemma 2.2, the series $E_m(T^{-1})/E'_m(T^{-1})$ lies in $\mathcal{R}^\times$ if and only if $|1-u_\gamma| \leq |p|^{m+1}$, i.e. $u_\gamma \in 1+p^{m+1}\mathbb{Z}_p$. Then, again by the reciprocity law Theorem 1.4, the automorphism $\gamma$ is the identity on $\mathbb{Q}_p(\Lambda_{P,m+1}) = \mathbb{Q}_p(\pi_m)$. Hence $\pi_m = \pi'_m$. □

[1] Note that the symbol $[-]_P$ was defined in Lemma 1.8 and is different from $[-]$ given in the definition 2.1.

We recall that two Lubin–Tate groups are isomorphic (as formal groups over $\mathbb{Z}_p$) if and only if they are relative to the same uniformizer w (cf. Theorem 1.5).

**Theorem 2.4** (Independence on the Lubin–Tate group) *Let $P, \widetilde{P} \in \mathfrak{F}_{\mathrm{w}}$ be two Lubin–Tate series, let $\boldsymbol{\pi} = (\pi_j)_{j\geq 0}$ and $\widetilde{\boldsymbol{\pi}} = (\pi_{\widetilde{P},j})_{j\geq 0}$ be a generator of* $\mathrm{T}(\mathfrak{G}_P)$ *and* $\mathrm{T}(\mathfrak{G}_{\widetilde{P}})$, *respectively. Let us denote by $\mathrm{M}_d^{(\widetilde{P})}(-)$, $E_m^{(\widetilde{P})}(T)$, $\mathrm{e}_d^{(\widetilde{P})}(-,T)$ the constructions attached to $\widetilde{\boldsymbol{\pi}}$, and denote in the usual way the constructions attached to $\boldsymbol{\pi}$. If $\pi_{\widetilde{P},m} = [1]_{P,\widetilde{P}}(\pi_{P,m})$, then $\mathrm{M}_d(\boldsymbol{\lambda}) \stackrel{\sim}{\to} \mathrm{M}_d^{(\widetilde{P})}(\boldsymbol{\lambda})$ over $\mathcal{R}_{H_m}$, for all $\boldsymbol{\lambda} \in \mathbf{W}_m(k_H)$.*

*Proof* Let $\boldsymbol{\lambda} \in \mathbf{W}_m(k_H)$, and let $\tilde{\boldsymbol{\lambda}} \in \mathbf{W}_m(\mathcal{O}_H)$ be a lifting of $\boldsymbol{\lambda}$. We shall show that $\mathrm{e}_d(\tilde{\boldsymbol{\lambda}}, T)/\mathrm{e}_d^{(\widetilde{P})}(\tilde{\boldsymbol{\lambda}}, T)$ belongs to $\mathcal{R}_{H_m}$. By Eq. (2.18), we reduce to showing that $E_{m-j}(T^{-1})/E_{m-j}^{(\widetilde{P})}(T^{-1}) \in \mathcal{R}_{H_{m-j}}$, for all $0 \leq j \leq m$. Since $\pi_{\widetilde{P},m} = [1]_{P,\widetilde{P}}(\pi_{P,m})$, then $\pi_{\widetilde{P},j} = [1]_{P,\widetilde{P}}(\pi_{P,j})$, for all $0 \leq j \leq m$. We have

$$E_m(T^{-1})/E_m^{(\widetilde{P})}(T^{-1}) = \exp\left(\sum_{j=0}^{m} \pi_{m-j}\left(1 - \frac{\pi_{\widetilde{P},m-j}}{\pi_{P,m-j}}\right)\frac{T^{-p^j}}{p^j}\right). \tag{2.31}$$

Let us set, as usual, $h_{P,\widetilde{P}}(X) := 1 - [1]_{P,\widetilde{P}}(X)/X$, in order to have (cf. Corollary 2.1 and Definition 2.4) $E_m(T^{-1})/E_m^{(\widetilde{P})}(T^{-1}) = \mathrm{e}_d([h_{P,\widetilde{P}}(\pi_m)], T^{-1})$. Since $[1]_{P,\widetilde{P}}(X) \equiv X \mod X^2\mathbb{Z}_p[[X]]$, we have $h_{P,\widetilde{P}}(X) \in X \cdot \mathbb{Z}_p[[X]]$, and, by the reduction Theorem 2.2 and Lemma 2.2, this exponential lies in $\mathcal{R}_{H_m}$. By the way, its inverse lies also in $\mathcal{R}_{H_m}$, so $\mathrm{M}_d(\boldsymbol{\lambda}) \stackrel{\sim}{\to} \mathrm{M}_d^{(\tilde{P})}(\boldsymbol{\lambda})$, over $\mathcal{R}_{H_m}$. □

*Remark* 2.8 If $\mathrm{w} = p$, and if $P$ is given, then, by Definition 1.15 and Remark 1.13, the isomorphism class of $\mathrm{M}_d(\boldsymbol{\lambda})$ is determined by the choice of a sequence $\{\xi_j\}_{j\geq 0}$ of $p^{j+1}$th roots of 1 such that $\xi_m^{p^j} = \xi_{m-j}$.

**Corollary 2.3** *Let $\gamma : H(\Lambda_P) \to H(\Lambda_P)$ be a continuous endomorphism of fields. Then $\gamma(E_m(T^{-1}))/E_m(T^{-1}) \in \mathcal{R}_{H_m}$ if and only if $\gamma$ is the identity on $\mathbb{Q}_p(\pi_m)$, and in this case, for all $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_{H(\Lambda_P)})$, we have*

$$\mathrm{e}_d^{\gamma}(\boldsymbol{\lambda}, T) = \mathrm{e}_d(\gamma(\boldsymbol{\lambda}), T), \tag{2.32}$$

*where, for all $f(T) = \sum a_i T^i$, we set $f^{\gamma}(T) := \sum \gamma(a_i) T^i$.*

*Proof* The proof follows the same lines as the proof of Theorem 2.3. □

*2.2.3 Frobenius structure for $\boldsymbol{\pi}$-exponentials*

**Theorem 2.5** *Let $r \geq 0$ and let $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_H)$. Let $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_H)$ be a lifting of $\bar{\boldsymbol{\lambda}}$, and let $\boldsymbol{\lambda}^{(\bar{\mathrm{F}})} \in \mathbf{W}_m(\mathcal{O}_H)$ be an arbitrary lifting of $\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}) \in \mathbf{W}_m(k_H)$. The following statements are equivalent:*

1. *The power series $\mathrm{e}_d(\boldsymbol{\lambda}^{(\bar{\mathrm{F}})}, T^p)/\mathrm{e}_d(\boldsymbol{\lambda}, T)$ is over-convergent, for all choices of $\boldsymbol{\lambda}$, $\bar{\boldsymbol{\lambda}}$ and $\boldsymbol{\lambda}^{(\bar{\mathrm{F}})}$.*
2. *The modules $\mathrm{M}_d(\bar{\boldsymbol{\lambda}})$ and $\mathrm{M}_{pd}(\mathrm{V}\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}))$ are isomorphic over $\mathcal{R}_{H_m}$, for all $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_H)$.*
3. *The power series $E_m(T^p)/E_m(T)$ is over-convergent.*
4. *We have the inequality $|\mathrm{w} - p| \leq |p|^{m+2}$.*

*Proof* (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) are evident. Let us show (3) $\Leftrightarrow$ (4). Write

$$\begin{aligned} E_m(T^p)/E_m(T) &= \exp\left(\left(\sum_{j=0}^{m} \pi_{m-j}\frac{T^{p^{j+1}}}{p^j}\right) - \left(\sum_{j=0}^{m} \pi_{m-j}\frac{T^{p^j}}{p^j}\right)\right) \\ &= \exp(-p\pi_{m+1}T)\cdot \exp\left(\sum_{j=0}^{m+1} \pi_{m+1-j}\left(p - \frac{\pi_{m-j}}{\pi_{m-j+1}}\right)\frac{T^{p^j}}{p^j}\right), \end{aligned}$$

where $\pi_{-1} := P(\pi_0) = 0$. Let $h_{\mathrm{Frob}}(X) := p - P(X)/X$, in order to have (cf. Corollary 2.1 and Definition 2.4)

$$E_m(T^p)/E_m(T) = \exp(-p\pi_{m+1}T)\cdot \mathrm{e}_{p^{m+1}}([h_{\mathrm{Frob}}(\pi_{m+1})], T). \tag{2.33}$$

Since the function $\exp(-p\pi_{m+1}T)$, and its inverse, are over-convergent, then the quotient $E_m(T^p)/E_m(T)$ is over-convergent if and only if $\mathrm{e}_{p^{m+1}}([h_{\mathrm{Frob}}(\pi_{m+1})], T)$ is over-convergent. The constant term of $h_{\mathrm{Frob}}(X)$ is $p - \mathrm{w}$. Hence, as usual, by the Reduction Theorem 2.2 and Corollary 2.1, $E_m(T^p)/E_m(T)$ is over-convergent if and only if $|p - \mathrm{w}| \leq |p|^{m+2}$. Let us now show (3) $\Rightarrow$ (1). Since (3) and (4) are equivalent, we see that $E_j(T^p)/E_j(T)$ is over-convergent, for all $j = 0, \ldots, m$. Let $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_m)$ and $\boldsymbol{\lambda}^{(\bar{\mathrm{F}})} := (\lambda_0^{(\bar{\mathrm{F}})}, \ldots, \lambda_m^{(\bar{\mathrm{F}})})$. We can suppose $\lambda_j^{(\bar{\mathrm{F}})} = \lambda_j^p$, $\forall\, j = 0, \ldots, m$. Indeed, the Witt vector $\boldsymbol{\eta} := \boldsymbol{\lambda}^{(\bar{\mathrm{F}})} - (\lambda_0^p, \ldots, \lambda_m^p) = (\eta_0, \ldots, \eta_m) \in \mathbf{W}_m(\mathcal{O}_H)$ satisfies $|\eta_j| < 1$, $\forall\, j = 0, \ldots, m$. Hence $\mathrm{e}_d(\boldsymbol{\lambda}^{(\bar{\mathrm{F}})}, T^p) = \mathrm{e}_d(\boldsymbol{\eta}, T^p)\cdot \mathrm{e}_d((\lambda_0^p, \ldots, \lambda_m^p), T^p)$, and the function $\mathrm{e}_d(\boldsymbol{\eta}, T^p)$ is over-convergent by the reduction Theorem 2.2. Now, by Eq. (2.18), we have

$$\frac{\mathrm{e}_d((\lambda_0^p, \ldots, \lambda_m^p), T^p)}{\mathrm{e}_d(\boldsymbol{\lambda}, T)} = \prod_{j=0}^{m} \frac{E_{m-j}\left(\lambda_j^p T^{np^{j+1}}\right)}{E_{m-j}\left(\lambda_j T^{np^j}\right)}. \tag{2.34}$$

Since $E_{m-j}(T^p)/E_{m-j}(T)$ is over-convergent, for all $j = 0, \ldots, m$, all factors $E_{m-j}(\lambda_j^p T^{np^{j+1}})/E_{m-j}(\lambda_j T^{np^j})$ are over-convergent. □

*Remark* 2.9 In this Theorem we do not need the existence of an absolute Frobenius on $H$. This is due to the fact that the isomorphism class of $\mathrm{M}_d(\boldsymbol{\lambda})$ depends only on the reduction $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_H)$, and $k_H$ is endowed naturally with the Frobenius given by the $p$th power map.

*Remark* 2.10 We will generalize this Theorem for all rank one differential equations (cf. Theorem 3.1). Let us show how to recover, from Theorem 2.5, the Frobenius structure Theorem in the usual sense. Let $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_H)$ be a lift of $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_H)$. Suppose that $\mathrm{w} = p$, in order to apply Theorem 2.5. Suppose that $\sigma : H_\infty \to H_\infty$ is an absolute Frobenius (cf. Definition 1.10) such that $\pi_j^\sigma = \pi_j$, for all $j \geq 0$, and such that $\sigma(H) \subseteq H$. By Corollary 2.3, we have $\varphi_\sigma(\mathrm{e}_d(\boldsymbol{\lambda}, T)) = \mathrm{e}_d(\boldsymbol{\lambda}^\sigma, T^p)$, and hence $\varphi_\sigma^*(\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})) = \widetilde{\mathrm{M}}_{pd}(\mathrm{V}(\boldsymbol{\lambda}^\sigma))$. By Theorem 2.2, the isomorphism class of $\widetilde{\mathrm{M}}_{pd}(\mathrm{V}(\boldsymbol{\lambda}^\sigma))$ depends only on the reduction $\mathrm{V}(\overline{\boldsymbol{\lambda}^\sigma}) = \mathrm{V}\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}) \in \mathbf{W}_{m+1}(k_{H_\infty})$, so $\widetilde{\mathrm{M}}_{pd}(\mathrm{V}(\boldsymbol{\lambda}^\sigma))$ is isomorphic to $\mathrm{M}_{pd}(\mathrm{V}\bar{\mathrm{F}}(\boldsymbol{\lambda}))$ over $\mathcal{R}_{H_\infty}$. Then Theorem 2.5 gives us the usual Frobenius structure. Indeed,

$$\varphi_\sigma^*(\widetilde{\mathrm{M}}_d(\boldsymbol{\lambda})) \xrightarrow[\text{Cor.2.3}]{\sim} \widetilde{\mathrm{M}}_{p\cdot d}(\mathrm{V}(\boldsymbol{\lambda}^\sigma)) \xrightarrow[\text{Th.2.2}]{\sim} \mathrm{M}_{p\cdot d}(\mathrm{V}\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}})) \xrightarrow[\text{Th.2.5}]{\sim} \widetilde{\mathrm{M}}_d(\boldsymbol{\lambda}).$$

*Remark* 2.11 Let $\varphi_p^*$ be the $p$th ramification map (cf. Sect. 1.2.4), and let $\lambda \in \mathcal{O}_H$. We observe that we cannot have an isomorphism $\mathrm{M}_1(\lambda) \xrightarrow{\sim} (\varphi_p^*)^h(\mathrm{M}_1(\lambda))$, for all $\lambda$ and all $h \geq 1$. In other words there exists module which do not admits a "ramification structure". This error is present in the papers of Christol and Mebkhout. For example, suppose that $\lambda$ is such that $\bar{\lambda}^{p^r} \neq \bar{\lambda}$ in $k_H$, for all $r \geq 0$ (i.e. $\bar{\lambda} \notin \mathbb{F}_p^{\mathrm{alg}}$). Then $\exp(\pi_0\lambda T^p)/\exp(\pi_0\lambda T)$ is not over-convergent. Indeed, for all liftings $\lambda^{(\bar{\mathrm{F}}^r)} \in \mathcal{O}_H$ of $\bar{\lambda}^{p^r}$ we have $|\lambda^{(\bar{\mathrm{F}}^r)} - \lambda| = 1$, then

$$\frac{\exp(\pi_0\lambda T^p)}{\exp(\pi_0\lambda T)} = \frac{\mathrm{e}_1(\lambda, T^p)}{\mathrm{e}_1(\lambda, T)} = \frac{\mathrm{e}_1(\lambda^{(\bar{\mathrm{F}}^r)}, T^p)}{\mathrm{e}_1(\lambda, T)} \cdot \mathrm{e}_1(\lambda - \lambda^{(\bar{\mathrm{F}}^r)}, T^p),$$

and while $\mathrm{e}_1(\lambda^{(\bar{\mathrm{F}}^r)}, T^p)/\mathrm{e}_1(\lambda, T)$ is over-convergent, the function $\mathrm{e}_1(\lambda - \lambda^{(\bar{\mathrm{F}}^r)}, T^p)$ is not over-convergent, since the reduction of $\lambda^{(\bar{\mathrm{F}}^r)} - \lambda$ is not 0 in $k_H$ (cf. Theorem 2.2).

## 2.3 A natural transformation of the Artin–Schreier complex into the Kummer complex, via Dwork's splitting functions

**Hypothesis 2.6** From now on, we will suppose $\mathrm{w} = p$ in order to have the Theorem 2.5. Then $\mathfrak{G}_P \xrightarrow{\sim} \widehat{\mathbb{G}}_m$, the formal multiplicative group (cf. Theorem 1.5). We fix moreover a generator $\boldsymbol{\pi} = (\pi_j)_{j\geq 0}$ of $\mathrm{T}(\mathfrak{G}_P)$.

In this section $L$ will be a complete valued field, containing $(\mathbb{Q}_p, |.|)$, and endowed with an absolute Frobenius $\varphi : \mathcal{O}_L \to \mathcal{O}_L$ (i.e. a lifting of the map $x \mapsto x^p$ of $k_L$).

We set as usual $L_m := L(\pi_m)$ and $L_\infty := \cup_m L_m$. We denote by $k_m$ (respectively $k_\infty$) the residue field of $L_m$ (respectively $L_\infty$). We fix an algebraic closure $L^{\mathrm{alg}}$ of $L$, then $k_L^{\mathrm{alg}} := k_{L^{\mathrm{alg}}}$ is an algebraic closure of $k_L$. Let $k_L^{\mathrm{sep}}$ (resp. $k_m^{\mathrm{sep}}$, $k_\infty^{\mathrm{sep}}$) be the separable closure of $k_L$ (respectively $k_m$, $k_\infty$) in $k_L^{\mathrm{alg}}$ (we recall that $k_L$ *is not supposed to be perfect*). We denote by $\widehat{L}^{\mathrm{unr}}$ (resp. $\widehat{L}_m^{\mathrm{unr}}$) the completion of the maximal unramified extension of $L$ (resp. $L_m$) in $L^{\mathrm{alg}}$. We set $\mathrm{G}_{k_L} := \mathrm{Gal}(k_L^{\mathrm{sep}}/k_L)$, $\mathrm{G}_{k_m} := \mathrm{Gal}(k_m^{\mathrm{sep}}/k_m)$, $\mathrm{G}_L := \mathrm{Gal}(L^{\mathrm{alg}}/L)$, and $\mathrm{G}_{L_m} := \mathrm{Gal}(L_m^{\mathrm{alg}}/L_m)$.

*Remark* 2.12 Let $k_L^0 = k_L^{\mathrm{sep}} \cap k_m$ be the separable closure of $k_L$ in $k_m$ and let $L^0 := \mathbf{W}(k_L^0) \otimes_{\mathbf{W}(k_L)} L = \widehat{L}^{\mathrm{unr}} \cap L_m$. The extension $k_m/k_L^0$ is purely inseparable (i.e. for all $x \in k_m$ there exists $r \geq 0$ such that $x^{p^r} \in k_L^0$), so $\mathrm{Gal}(k_m/k_L^0) = 1$, and we have a canonical identification $\mathrm{G}_{k_m} := \mathrm{Gal}(k_m^{\mathrm{sep}}/k_m) \xrightarrow{\sim} \mathrm{Gal}(k_L^{\mathrm{sep}}/k_L^0)$. Hence $\mathrm{G}_{k_m}$ is naturally contained in $\mathrm{G}_{k_L}$:

$$\begin{array}{ccccc} & & L_m & \subseteq & \widehat{L}_m^{\mathrm{unr}} \\ & & \cup\!| & & \cup\!| \\ L & \subseteq & L^0 & \subseteq & \widehat{L}^{\mathrm{unr}} \end{array} \qquad\qquad \begin{array}{ccccc} & & k_m & \subseteq & k_m^{\mathrm{sep}} \\ & & \cup\!| & & \cup\!| \\ k_L & \subseteq & k_L^0 & \subseteq & k_L^{\mathrm{sep}} \end{array} . \tag{2.35}$$

All these extensions are normal. We will identify $\mathrm{G}_{k_m}$ with $\mathrm{Gal}(\widehat{L}_m^{\mathrm{unr}}/L_m)$, and $\mathrm{G}_{k_L}$ with $\mathrm{Gal}(\widehat{L}^{\mathrm{unr}}/L)$. In this way $\mathrm{G}_{k_m}$ acts naturally on $\widehat{L}^{\mathrm{unr}}$.

*Remark* 2.13 The absolute Frobenius $\varphi$ extends uniquely to all unramified extensions of $L$, and hence it commutes with the action of $\mathrm{G}_{k_L}$. It extends also (not uniquely) to an absolute Frobenius $\widetilde{\varphi}$ of $L_m$. Indeed, since $\varphi$ extends uniquely to $L^0$, then to prove the existence of $\widetilde{\varphi}$ one can assume that $L = L^0$, and hence $k_m$ is a purely inseparable extension of $k_L = k_L^0$. Since the map $x \mapsto x^p$ of $k_L$ extends uniquely to $k_m$, then every field morphism $\widetilde{\varphi} : L_m \to L_m$ extending $\varphi$ is an absolute Frobenius of $L_m$. Such a $\widetilde{\varphi}$ exists since, by [2, Sect. 6, no. 1, Proposition 1], $\varphi$ extends to a $\mathbb{Q}_p$-linear morphism $\widetilde{\varphi} : L^{\mathrm{alg}} \to L^{\mathrm{alg}}$, inducing an automorphism of $\mathbb{Q}_p(\pi_m)$.

In general there is no absolute Frobenius on $L_m$ satisfying $\varphi(\pi_m) = \pi_m$. Indeed if $L$ is totally ramified over $\mathbb{Q}_p$, and if $\varphi = \mathrm{Id}_L$, then the unique extension of $\varphi$ to $L_m$, fixing $\pi_m$, is the identity. On the other hand $L_m/\mathbb{Q}_p$ is not always totally ramified, hence the identity of $L_m$ is not always an absolute Frobenius.[2]

In the sequel of the paper *we will never use such a $\widetilde{\varphi}$, hence we do not fix it.*

On the other hand, we need the existence of $\varphi$ because the functor of Witt vectors of finite length $\mathbf{W}_m(-)$ is not canonically endowed with an additive functorial Frobenius morphism (see Remark 2.16 to improve this situation).

[2] Indeed let $p = 3$, $m = 0$, and $L := \mathbb{Q}_p(\pi_{P,0})$, where $P(X)$ is the Lubin–Tate series $P(X) = -3X + X^3$. If $\xi_1^3 = 1$ is a primitive root of unity, then $L_0 = \mathbb{Q}_p(\pi_{P,0}, \xi_1)$ is not totally ramified since the element $x := \widetilde{\pi}/\pi_{P,0}$, where $\widetilde{\pi} = (\xi_1 - 1)$, verifies $|x| = 1$ and $x^6 = -1$, indeed $x^9 = (\frac{\widetilde{\pi}}{\pi_{P,0}})^9 = (\frac{-3\widetilde{\pi} - 3\widetilde{\pi}^2}{3\pi_{P,0}})^3 = -\xi_1^3 x^3 = -x^3$. But there is no element $\overline{x}$ in $\mathbb{F}_3$ verifying $\overline{x}^6 = -\overline{1}$.

**Definition 2.6** *For all* $\boldsymbol{\lambda} := (\lambda_0, \dots, \lambda_m) \in \mathbf{W}_m(\mathcal{O}_L)$, *we set*

$$\theta_d^{(\varphi)}(\boldsymbol{\lambda}, T) := \frac{\mathrm{e}_d(\varphi(\boldsymbol{\lambda}), T^p)}{\mathrm{e}_d(\boldsymbol{\lambda}, T)}. \tag{2.36}$$

*To simplify the notations, we will write* $\theta_d(\boldsymbol{\lambda}, T)$ *if no confusion is possible.*

*Example* 2.1 Let $d = 1$ and $P(X) = pX + X^p$ (cf. Remark 1.12). Then $\pi_0$ is the "$\pi$ of Dwork", and $\theta_1(1, T) = \exp(\pi_0(T^p - T))$ is the usual Dwork splitting function. While in general, if $\lambda \in \mathcal{O}_L$, we have $\theta_1(\lambda, T) = \exp(\pi_0(\varphi(\lambda)T^p - \lambda T))$.

The following Theorem shows that the over-convergent function $\boldsymbol{\lambda} \mapsto \theta_d(\boldsymbol{\lambda}, 1)$ is a splitting function in a generalized sense with respect to Dwork (cf. [13, Sect. 4, a), p. 55]). In a paper in preparation we shall analyse such functions in detail.

**Definition 2.7** *Set* $\mathcal{O}_L^{\varphi=1} := \{\lambda \in \mathcal{O}_L \mid \varphi(\lambda) = \lambda\}$ *and* $\overline{\mathcal{O}_L^{\varphi=1}} := \mathcal{O}_L^{\varphi=1}/(\mathcal{O}_L^{\varphi=1} \cap \mathfrak{p}_L)$. *We see that* $\overline{\mathcal{O}_L^{\varphi=1}} = \mathbb{F}_p$.

**Theorem 2.7** *Let* $a^p = a \in \mathcal{O}_L$, *and let* $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_L^{\varphi=1})$. *Then* $\theta_d^{(\varphi)}(\boldsymbol{\lambda}, a)$ *is a* $p^{m+1}$*-th root of* 1. *Moreover the group morphism*

$$\theta_d^{(\varphi)}(-, a) : \mathbf{W}_m(\mathcal{O}_L^{\varphi=1}) \longrightarrow \boldsymbol{\mu}_{p^{m+1}} \subset \mathbb{Z}_p[\pi_m]$$

*factorizes on* $\mathbf{W}_m(\overline{\mathcal{O}_L^{\varphi=1}}) = \mathbf{W}_m(\mathbb{F}_p) = \mathbb{Z}/p^{m+1}\mathbb{Z}$ *and defines an isomorphism*

$$\overline{\theta}_d^{(\varphi)}(-, a) : \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_{p^{m+1}}. \tag{2.37}$$

*More precisely the image of* $1 \in \mathbb{Z}/p^{m+1}\mathbb{Z}$ *is* the inverse *of the unique primitive* $p^{m+1}$*-th root of* 1, *say* $\xi_m$, *satisfying*

$$|a^n \pi_m - (\xi_m - 1)| < |a^n \pi_m|. \tag{2.38}$$

*In particular, if* $a = 1$, *then* $\xi_m$ *is the* $p^{m+1}$*th root of* 1 *defined in Remark* 1.13.

*Proof* Let $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_m) \in \mathbf{W}_m(\mathcal{O}_L^{\varphi=1})$. Let us show that $\theta_d(\boldsymbol{\lambda}, a)^{p^{m+1}} = 1$. Indeed $T \mapsto \mathrm{e}_d(\boldsymbol{\lambda}, T)^{p^{m+1}}$ is over-convergent (cf. Remark 2.14), so $\theta_d(\boldsymbol{\lambda}, a)^{p^{m+1}} = \mathrm{e}_d(\varphi(\boldsymbol{\lambda}), a^p)^{p^{m+1}}/\mathrm{e}_d(\boldsymbol{\lambda}, a)^{p^{m+1}} = 1$, since both numerator and denominator do make sense and are equal. If $|\lambda_j| < 1$, for all $j = 0, \dots, m$, then $T \mapsto \mathrm{e}_d(\boldsymbol{\lambda}, T)$ is over-convergent (cf. reduction Theorem 2.2), hence both numerator and denominator of the expression $\mathrm{e}_d(\boldsymbol{\lambda}, a^p)/\mathrm{e}_d(\boldsymbol{\lambda}, a)$ do make sense and are equal. Let us show the last assertion. By Eq. (2.33), we have

$$\begin{aligned} \theta_d((1, 0, \dots, 0), T) &= E_m(T^{np})/E_m(T^n) \\ &= \exp(-p\pi_{m+1}T^n) \cdot \mathrm{e}_{pd}([h_{\mathrm{Frob}}(\pi_{m+1})], T). \end{aligned} \tag{2.39}$$

By Eq. (2.17) this series lies in $1+\pi_{m+1}T\mathbb{Z}_p[\pi_{m+1}][[T]]$. To show that this root is $\xi_m^{-1}$ it is sufficient to show that $|\theta_d((1,0,\ldots,0),a)^{-1}-\xi_m| < |\pi_m| = |\xi_m - 1|$. We work therefore modulo the following sub group

$$C := \left\{1+\sum c_i T^i \mid c_i \in \mathbb{Z}_p[\pi_{m+1}],\ |c_i| < |\pi_m|, \quad \text{for all } i \geq 1\right\}.$$

We have $\exp(-p\pi_{m+1}T^n) \equiv 1 \mod C$. Let us consider (cf. Eq. (2.33))

$$[h_{\mathrm{Frob}}(\pi_{m+1})] = [p - P(\pi_{m+1})/\pi_{m+1}] = (\nu_0,\ldots,\nu_{m+1}). \tag{2.40}$$

Then $\nu_0 = p - (\pi_m/\pi_{m+1})$ and, since $p = \mathrm{w}$, by Corollary 2.1, we have $|\nu_j| \leq |\pi_{m+1}|$, for all $j = 0,\ldots,m+1$. By Eq. (2.18) we have $\mathrm{e}_{pd}([h_{\mathrm{Frob}}(\pi_{m+1})],T) = \prod_{j=0}^{m+1} E_{m+1-j}(\nu_j T^{np^j})$. Moreover, we know that (cf. Eq. (2.17))

$$E_{m+1-j}(\nu_j T^{np^j}) = 1 + (\text{things of valuation } \leq |\pi_{m+1-j}\cdot\nu_j|), \tag{2.41}$$

for all $j = 0,\ldots,m+1$. Then

$$\theta_d((1,0,\ldots,0),T)^{-1} \equiv E_{m+1}(\nu_0 T^n)^{-1} \mod C. \tag{2.42}$$

Since $|\nu_0^p| = |\pi_m|^{p-1}$, it follows from Eq. (2.17) that only the first $p-1$ terms of $E_{m+1}(\nu_0 T^n)^{-1}$ are greater than or equal to $|\pi_m|$, that is

$$E_{m+1}(\nu_0 T^n)^{-1} \equiv 1 + \pi_{m+1}\nu_0 T^n + \cdots + \frac{(\pi_{m+1}\nu_0 T^n)^{p-1}}{(p-1)!} \mod C. \tag{2.43}$$

Since $\pi_{m+1}\nu_0 = p\cdot\pi_{m+1} - \pi_m$, hence $\theta_d((1,0,\ldots,0),T)^{-1} \equiv 1+\pi_m T^n \mod C$. □

*Remark* 2.14 Observe that $T \mapsto \mathrm{e}_d(\boldsymbol{\lambda},T)^{p^{m+1}} = \mathrm{e}_d(p^{m+1}\boldsymbol{\lambda},T)$ is over-convergent for all $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_L)$, because the reduction of $p^{m+1}\boldsymbol{\lambda}$ in $\mathbf{W}_m(k_L)$ is 0 (cf. Theorem 2.2).

*Remark* 2.15 We recall that we do not fix an absolute Frobenius on $L_m$ (cf. Remark 2.13).

**Theorem 2.8** *The following diagram is well-defined, commutative and functorial, on the complete* (*or algebraic*) *unramified extensions of L*

$$
\begin{array}{ccccccccccc}
1 & \longrightarrow & \boldsymbol{\mu}_{p^{m+1}} & \longrightarrow & (L_m)^{\times} & \xrightarrow{f\mapsto f^{p^{m+1}}} & (L_m)^{\times} & \xrightarrow{\delta_{\mathrm{Kum}}} & \mathrm{H}^1(\mathrm{G}_{L_m},\boldsymbol{\mu}_{p^{m+1}}) & \longrightarrow & 1\\
 & & \uparrow & & \uparrow{\scriptstyle \theta_{p^m}(-,1)} & & \uparrow{\scriptstyle \mathrm{e}_{p^m}(-,1)^{p^{m+1}}} & & \vdots\uparrow{\scriptstyle \bar{\mathrm{e}}:=\overline{\mathrm{e}_{p^m}(-,1)^{p^{m+1}}}} & & \\
\wr & & \mathbf{W}_m(\mathcal{O}_L^{\varphi=1}) & \hookrightarrow & \mathbf{W}_m(\mathcal{O}_L) & \xrightarrow[\varphi-1]{} & \mathbf{W}_m(\mathcal{O}_L) & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z}/p^{m+1}\mathbb{Z} & \longrightarrow & \mathbf{W}_m(k_L) & \xrightarrow[\bar{\mathrm{F}}-1]{} & \mathbf{W}_m(k_L) & \xrightarrow{\delta} & \mathrm{H}^1(\mathrm{G}_{k_L},\mathbb{Z}/p^{m+1}\mathbb{Z}) & \longrightarrow & 0
\end{array}
\tag{2.44}
$$

*where* $\mathrm{G}_{L_m} := \mathrm{Gal}(L_m^{\mathrm{alg}}/L_m)$. *More explicitly* $\theta_{p^m}(-,1)$ *induces the identification* (*cf. Theorem* 2.7)

$$1 \mapsto \xi_m^{-1} \ : \ \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\sim} \mu_{p^{m+1}}, \tag{2.45}$$

*where* $\xi_m$ *is the unique* $p^{m+1}$*th root of* 1 *satisfying* $|(\xi_m - 1) - \pi_m| < |\pi_m|$ (*cf. Remark* 1.13). *Moreover* $\bar{\mathrm{e}}$ *sends* $\mathrm{H}^1(\mathrm{G}_{k_L}, \mathbb{Z}/p^{m+1}\mathbb{Z})$ *in* $\mathrm{H}^1(\mathrm{G}_{k_m}, \boldsymbol{\mu}_{p^{m+1}}) \subseteq \mathrm{H}^1(\mathrm{G}_{L_m}, \boldsymbol{\mu}_{p^{m+1}})$ *via the canonical diagram*

$$
\begin{array}{ccc}
\mathrm{G}_{k_L} & \hookleftarrow & \mathrm{G}_{k_m} \\
{\scriptstyle \alpha}\downarrow & & \vdots\downarrow{\scriptstyle \bar{e}(\alpha)} \\
\mathbb{Z}/p^{m+1}\mathbb{Z} & \xrightarrow[1\mapsto\xi_m^{-1}]{\sim} & \boldsymbol{\mu}_{p^{m+1}}
\end{array}
\quad . \tag{2.46}
$$

*In other words the Artin–Schreier character* $\gamma \mapsto \alpha(\gamma) : \mathrm{G}_{k_L} \to \mathbb{Z}/p^{m+1}\mathbb{Z}$ *is sent by* $\bar{\mathrm{e}}$ *into the Kummer character* $\gamma \mapsto \bar{e}(\alpha)(\gamma) = \xi_m^{-\alpha(\gamma)} : \mathrm{G}_{k_m} \to \boldsymbol{\mu}_{p^{m+1}}$. *In particular* $\bar{\mathrm{e}}(\alpha) = 1$ *if and only if* $\mathrm{G}_{k_m} \subseteq \mathrm{Ker}(\alpha)$.

*Proof* Let $L'/L$ be an unramified extension, and let $\boldsymbol{\lambda}' = (\lambda'_0, \ldots, \lambda'_m) \in \mathbf{W}_m(\mathcal{O}_{L'})$. If $L'/L$ is not complete, but algebraic, then the series $\theta_{p^m}(\boldsymbol{\lambda}', T)$, and $\mathrm{e}_{p^m}(\boldsymbol{\lambda}', T)^{p^{m+1}}$, are convergent at $T = 1$, since the finite extension $L(\{\lambda'_i\}_i)/L$ is complete. By Theorem 2.7, to show the commutativity it is enough to prove that $\bar{\mathrm{e}}$ is well-defined. Let $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_L)$ be such that $\delta(\bar{\boldsymbol{\lambda}}) = 0$ (cf. diagram (1.21)). By definition, there exist $\boldsymbol{z}, \boldsymbol{\eta} \in \mathbf{W}_m(\mathcal{O}_L)$, such that $\boldsymbol{\eta} = (\eta_0, \ldots, \eta_m)$, with $|\eta_j| < 1$, for all $j = 0, \ldots, m$, and $\boldsymbol{\lambda} = \varphi(\boldsymbol{z}) - \boldsymbol{z} + \boldsymbol{\eta}$. Hence

$$\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}} = \theta_{p^m}(\boldsymbol{z}, 1)^{p^{m+1}} \cdot \mathrm{e}_{p^m}(\boldsymbol{\eta}, 1)^{p^{m+1}} \ . \tag{2.47}$$

Then $\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}} \in (\mathcal{O}_{L_m})^{p^{m+1}}$. In other words, even if the symbol $\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)$ has no meaning, the number $\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}}$ is the $p^{m+1}$th power of the number $\theta_{p^m}(\boldsymbol{z}, 1) \cdot \mathrm{e}_{p^m}(\boldsymbol{\eta}, 1)$ of $L_m$. Hence $\delta_{\mathrm{Kum}}(\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}}) = 1$.

Let us show that the map $\bar{\mathrm{e}}$ works as indicated in the diagram (2.46). Let $\alpha = \delta(\bar{\boldsymbol{\lambda}})$, and let $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_L)$ be an arbitrary lifting of $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_L)$. By Lemma

2.4 below, an easy induction on $m$ shows that there exists $\boldsymbol{\nu} \in \mathbf{W}_m(\mathcal{O}_{\widehat{L}^{\mathrm{unr}}})$ such that

$$\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{\lambda} . \tag{2.48}$$

By definition (cf. Diagram (1.21)), for all $\gamma_1 \in \mathrm{G}_{k_L}$, we have $\alpha(\gamma_1) = \gamma_1(\overline{\boldsymbol{\nu}}) - \overline{\boldsymbol{\nu}} \in \mathbb{Z}/p^{m+1}\mathbb{Z}$. On the other hand, by definition, $\bar{\mathrm{e}}(\alpha)$ is the Kummer character of $\mathrm{G}_{L_m}$ defined by $\mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}}$, and is given by $\bar{\mathrm{e}}(\alpha)(\gamma) = \gamma(y)/y$, for all $\gamma \in \mathrm{G}_{L_m}$, where $y$ is an arbitrary root of the equation $Y^{p^{m+1}} = \mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)^{p^{m+1}}$. We let $y := \theta_{p^m}(\boldsymbol{\nu}, 1)$. Then

$$\overline{\mathrm{e}}(\alpha)(\gamma) = \gamma(y)/y = \gamma(\theta_{p^m}(\boldsymbol{\nu}, 1))/\theta_{p^m}(\boldsymbol{\nu}, 1) = \theta_{p^m}(\gamma(\boldsymbol{\nu}) - \boldsymbol{\nu}, 1) \in \boldsymbol{\mu}_{p^{m+1}}, \tag{2.49}$$

because $\gamma(\pi_m) = \pi_m$, since $\gamma \in \mathrm{G}_{L_m}$. Now $\gamma(\boldsymbol{\nu}) - \boldsymbol{\nu} \in \mathcal{O}_L^{\varphi=1}$, because $\gamma(\boldsymbol{\nu})$ is again a solution of the Eq. (2.48). By Theorem 2.7, the root $\theta_{p^m}(\gamma(\boldsymbol{\nu}) - \boldsymbol{\nu}, 1)$ depends only on the reduction of $\gamma(\boldsymbol{\nu}) - \boldsymbol{\nu}$ in $k_L^{\mathrm{sep}}$, and is equal to $\xi_m^{-\alpha(\gamma)}$. □

**Lemma 2.4** *Let $L$ have discrete valuation, and let $\widehat{L}^{\mathrm{unr}}$ be the completion of the unramified extension of $L$. Then for all $\lambda \in \mathcal{O}_{\widehat{L}^{\mathrm{unr}}}$, the equation $\varphi(\nu) - \nu = \lambda$ has a solution in $\widehat{L}^{\mathrm{unr}}$.*

*Proof* The equation $\bar{\nu}^p - \bar{\nu} = \bar{\lambda}$ has a solution in $k_L^{\mathrm{sep}}$, hence $|(\varphi(\nu) - \nu) - \lambda| < 1$, for all lifts $\nu$ of $\bar{\nu}$. Since $L$ has discrete valuation, the lemma follows from an induction on the value of the "error" $\eta$, in the equation $\varphi(\nu) - \nu = \lambda + \eta$. □

**Theorem 2.9** *Let $L$ have discrete valuation. Let $\alpha = \delta(\bar{\boldsymbol{\lambda}})$ be the Artin–Schreier character defined by $\bar{\boldsymbol{\lambda}} \in \mathbf{W}_m(k_L)$ (cf. Diagram (1.21)). Let $k_\alpha/k_L$ be the separable extension of $k_L$, defined by the kernel of $\alpha$, and let $L_\alpha/L$ be the corresponding unramified extension. Then*

$$L_\alpha(\pi_m) = L_m(\theta_{p^m}(\boldsymbol{\nu}, 1)), \tag{2.50}$$

*where $\boldsymbol{\lambda}$ is an arbitrary lifting of $\bar{\boldsymbol{\lambda}}$ in $\mathbf{W}_m(\mathcal{O}_L)$, and $\boldsymbol{\nu} \in \mathbf{W}_m(\mathcal{O}_{\widehat{L}^{\mathrm{unr}}})$ is a solution of the equation $\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{\lambda}$. In other words, up to replacing $L$ by $L_m$, the extension $L_\alpha$ is generated by $\theta_{p^m}(\boldsymbol{\nu}, 1)$.*

*Proof* Since both $L_\alpha(\pi_m)$ and $L_m(\theta_{p^s}(\boldsymbol{\nu}, 1))$ contain $L^0$ (cf. Remark 2.12), and since $\varphi$ extends uniquely to $L^0$, we can suppose $L = L^0$. In this case $\bar{\mathrm{e}}$ is injective, $L_m/L$ is totally ramified, and $\mathrm{G}_{k_L}$ can be identified with $\mathrm{G}_{k_m}$. Let us show the inclusion $L_m(\theta_{p^m}(\boldsymbol{\nu}, 1)) \subseteq L_\alpha(\pi_m)$. If $\mathrm{G}_{k_\alpha} := \mathrm{Gal}(k_L^{\mathrm{sep}}/k_\alpha) = \mathrm{Ker}(\alpha)$, then the inclusion follows from the fact that $\theta_{p^m}(\boldsymbol{\nu}, 1)$ is fixed by $\mathrm{G}_{k_\alpha}$ ($\subseteq \mathrm{G}_{k_m} \xrightarrow{\sim} \mathrm{Gal}(\widehat{L}_m^{\mathrm{unr}}/L_m)$). Indeed, for all $\gamma \in \mathrm{Gal}(\widehat{L}_m^{\mathrm{unr}}/L_m)$, we have, as in the proof of Theorem 2.8,

$$\gamma\left(\theta_{p^m}(\boldsymbol{\nu}, 1)\right) = \theta_{p^m}(\gamma(\boldsymbol{\nu}) - \boldsymbol{\nu}, 1) \cdot \theta_{p^m}(\boldsymbol{\nu}, 1) = \xi_m^{-\alpha(\gamma)} \cdot \theta_{p^m}(\boldsymbol{\nu}, 1), \tag{2.51}$$

and if $\gamma \in \mathrm{G}_{k_\alpha}$, we have $\alpha(\gamma) = 0$. Then $L_m(\theta_{p^m}(\boldsymbol{\nu}, 1)) \subseteq L_\alpha(\pi_m)$. In particular,

$$[L_m(\theta_{p^m}(\boldsymbol{\nu}, 1)) : L_m] \leq [L_\alpha(\pi_m) : L_m] = [k_{\alpha,m} : k_m], \tag{2.52}$$

where $k_{\alpha,m}$ is the smallest field in $k_m^{\mathrm{sep}}$ containing $k_m$ and $k_\alpha$ (i.e. the sub-field of $k_m^{\mathrm{sep}}$ fixed by $\mathrm{G}_{k_\alpha}$ acting on $k_m^{\mathrm{sep}}$). The inclusion $L_\alpha(\pi_m) \subseteq L_m(\theta_{p^m}(\boldsymbol{\nu}, 1))$ follows from the equality $[L_m(\theta_{p^m}(\boldsymbol{\nu}, 1)) : L_m] = [k_{\alpha,m} : k_m]$. Indeed, since $L_0 = L$, the map $\overline{\mathrm{e}}$ is injective. Hence $[L_m(\theta_{p^m}(\boldsymbol{\nu}, 1)) : L_m] = [k_\alpha : k_L]$, because these two degrees are equal to the *cardinality* of the cyclic Galois groups generated by $\overline{\mathrm{e}}(\alpha)$ and $\alpha$ respectively. On the other hand, since $k_L = k_L^0$, we have $[k_\alpha : k_L] = [k_{\alpha,m} : k_m]$. □

*Remark* 2.16 The hypothesis of discreteness of $L$, in Theorem 2.9, and the hypothesis of existence of $\varphi$ can be removed as follows. Let $\mathrm{F}_p : \mathbf{W}_m \to \mathbf{W}_m$ be the map $(\lambda_0, \ldots, \lambda_m) \mapsto (\lambda_0^p, \ldots, \lambda_m^p)$. Replace $\varphi$ by $\mathrm{F}_p$, and define $\theta_d^{(\mathrm{F}_p)}(\boldsymbol{\lambda}, T) := \mathrm{e}_d(\mathrm{F}_p(\boldsymbol{\lambda}), T^p)/\mathrm{e}_d(\boldsymbol{\lambda}, T)$. Then $\mathrm{F}_p$ is defined for *all* extensions of $L$, and commutes with the Galois action. It is easy to recover Theorems analogous to Theorems 2.7, 2.8, and 2.9. In particular the analogues of diagram (2.44) is defined and functorial, on *all* complete (or algebraic) extensions of $L$. Observe that the map $\boldsymbol{\lambda} \mapsto \theta_d^{(\mathrm{F}_p)}(\boldsymbol{\lambda}, T)$ is not a group morphism, but induces again the group morphism $1 \mapsto \xi_m^{-1} : \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_{p^{m+1}}$ (cf. Eq. (2.37)), which is the reduction of the *set* $\mathbf{W}_m(\mathcal{O}_L^{\mathrm{F}_p=1}) := \{\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_L) \mid \mathrm{F}_p(\boldsymbol{\lambda}) = \boldsymbol{\lambda}\}$, formed by Witt vectors whose entries are 0 or $p-1$ roots of 1.

### *2.3.1 Application to the field $\mathcal{E}_K^\dagger$*

*Remark* 2.17 These methods apply to obtain a description of the Kummer extensions of $\mathcal{E}_K$ (resp. $\mathcal{E}_K^\dagger$) coming by henselianity from an Artin–Schreier extension of $k((t))$ (see below). This description is really entirely explicit, since the Kummer generator $\theta_{p^m}(\boldsymbol{\nu}, 1)$ is explicitly and directly given by the vector $\boldsymbol{\lambda}$. Indeed, we will give meaning to the expression $\theta_{p^m}(\boldsymbol{\nu}, 1) = \mathrm{e}_{p^m}(\boldsymbol{\lambda}, 1)$, and we do not need to find a solution of the equation $\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{\lambda}$ (cf. Definition 3.1, and Theorem 4.6-(3)).

The precedent theory can be applied to the field $L = \mathcal{E}_K$, under the following assumptions on $K$:

$$\begin{cases} \text{(1) } K \text{ has a discrete valuation (used in Lemma 2.4).} \\ \text{(2) There exists an absolute Frobenius } \sigma : K \to K \\ \quad \text{(i.e. a lifting of the } p\text{th power map of } k). \end{cases} \tag{2.53}$$

Fixing an absolute Frobenius of $\mathcal{E}_K$, the theory applies without problems. Recall that we can suppress these two hypothesis if necessary (cf. Remark 2.16).

The situation is slightly different for the field $\mathcal{E}_K^\dagger$, because it is not complete. Nevertheless the preceding results are still true for $\mathcal{E}_K^\dagger$. Let $K$ satisfy Eq. (2.53), and fix an absolute Frobenius $\varphi : \mathcal{O}_K^\dagger \to \mathcal{O}_K^\dagger$, extending $\sigma$, by choosing $\varphi(T)$ in $\mathcal{O}_K^\dagger$, lifting $t^p \in \mathrm{E} = k((t))$ (cf. Definition 1.10).

*Remark* 2.18 Since $\varphi(T) \in \mathcal{O}_K^\dagger$ is a lifting of $t^p \in \mathrm{E}$, hence there exists $0 < \varepsilon_\varphi < 1$ such that $\varphi(\mathcal{A}_{K_m}(I^p)) \subseteq \mathcal{A}_{K_m}(I)$, where $I = ]1-\varepsilon_\varphi, 1[$ .

**Theorem 2.10** ([11, 4.2], [18, 2.2]) *If K has discrete valuation, then $\mathcal{O}_K^\dagger$ is Henselian, hence we have a bijection*

$$\left\{\text{Finite unramified ext. of } \mathcal{E}_K^\dagger\right\} \xrightarrow{\sim} \left\{\text{Finite separable ext. of } \mathrm{E} = k((t))\right\}.$$

**Proposition 2.5** *Let $\boldsymbol{f}(T) \in \mathbf{W}_m(\mathcal{O}_K^\dagger)$, then both series $\theta_{p^m}(\boldsymbol{f}(T), 1)$ and $\mathrm{e}_{p^m}(\boldsymbol{f}(T), 1)^{p^{m+1}}$ lie in $\mathcal{O}_{K_m}^\dagger$. Moreover if $\boldsymbol{u}(T) = (u_0(T), \dots, u_m(T)) \in \mathbf{W}_m(\mathcal{O}_{K_m}^\dagger)$ is such that $|u_i(T)|_1 < 1$, for all i, then $\mathrm{e}_{p^m}(\boldsymbol{u}(T), 1)$ makes sense, and lies in $\mathcal{O}_{K_m}^\dagger$.*

*Proof* Let $\varepsilon > 0$ be such that $\boldsymbol{f}(T) \in \mathbf{W}_m(\mathcal{A}_K(]1-\varepsilon, 1[))$. For all compact $J \subset ]1-\varepsilon, 1[$, the algebra $\mathcal{A}_K(J)$ is complete with respect to the absolute value $\|f(T)\|_J := \sup_{\rho \in J} |f(T)|_\rho$. Hence $\mathrm{e}_{p^m}(\boldsymbol{f}(T), 1)^{p^{m+1}} \in \mathbf{W}_m(\mathcal{A}_{K_m}(J))$, for all compact $J \subset ]1-\varepsilon, 1[$, and then $\mathrm{e}_{p^m}(\boldsymbol{f}(T), 1)^{p^{m+1}} \in \mathbf{W}_m(\mathcal{A}_{K_m}(]1-\varepsilon, 1[))$. On the other hand, $\theta_{p^m}(\boldsymbol{f}(T), Z) \in 1 + \pi_m Z \mathcal{O}_{\mathcal{E}_{K_m}}[[Z]]$ is a series in $Z$ depending only on $\boldsymbol{f}(T)$ and $\varphi(\boldsymbol{f}(T))$. By Remark 2.18, there exists $\varepsilon'$ such that both $\boldsymbol{f}(T)$ and $\varphi(\boldsymbol{f}(T))$ lie in $\mathbf{W}_m(\mathcal{A}_K(]1-\varepsilon', 1[))$. Hence as before $\theta_{p^m}(\boldsymbol{f}(T), Z) \in 1 + \pi_m \mathcal{A}_{K_m}(J)[[Z]]$, for all compact $J \subset ]1-\varepsilon', 1[$, and hence $\theta_{p^m}(\boldsymbol{f}(T), Z) \in 1 + \pi_m Z \mathcal{A}_{K_m}(]1-\varepsilon', 1[)[[Z]]$. The assertion on $\boldsymbol{u}(T)$ follows from the Reduction Theorem 2.2, and the same considerations. □

**Corollary 2.4** *The diagram* (2.44) *can be computed for $\mathcal{E}_K^\dagger$ instead of L. The other assertions of Theorems* 2.8 *and* 2.9 *remain true* (*see diagram* (0.7)). *In particular, if $\alpha = \delta(\overline{\boldsymbol{f}}(t))$ is the Artin–Schreier character defined by $\overline{\boldsymbol{f}}(t) \in \mathbf{W}_m(\mathrm{E})$, and if $\mathrm{F}_\alpha/\mathrm{E}$ is the separable extension defined by the kernel of $\alpha$, then the* (*Kummer*) *unramified extension of $\mathcal{E}_{K_m}^\dagger$, corresponding to $\mathrm{F}_\alpha$, is $\mathcal{E}_{K_m}^\dagger(\theta_{p^m}(\boldsymbol{v}, 1))$, where $\boldsymbol{v}$ is a solution of $\varphi(\boldsymbol{v}) - \boldsymbol{v} = \boldsymbol{f}(T)$, for an arbitrary lifting $\boldsymbol{f}(T)$ of $\overline{\boldsymbol{f}}(t)$.*

*Proof* Let $\mathrm{F}_\alpha/\mathrm{E}$ be the separable Artin–Schreier extension defined by $\overline{\boldsymbol{f}}(t) \in \mathbf{W}_m(\mathrm{E})$, and let $\mathcal{F}_\alpha^\dagger$ be the corresponding unramified extension of $\mathcal{E}_K^\dagger$. Let $\boldsymbol{v} \in \mathbf{W}_m(\widehat{\mathcal{E}}_K^{\mathrm{unr}})$ be a solution of $\varphi(\boldsymbol{v}) - \boldsymbol{v} = \boldsymbol{f}(T)$. The non trivial fact is that $\theta_{p^m}(\boldsymbol{v}, 1)$ lies in $\mathcal{F}_\alpha^\dagger(\pi_m)$ and not only in its completion, say $\mathcal{F}_\alpha(\pi_m)$. In other words, we shall show that $\mathcal{F}_\alpha^\dagger(\pi_m) = \mathcal{E}_{K_m}^\dagger(\theta_{p^m}(\boldsymbol{v}, 1))$. Both $\mathcal{E}_{K_m}^\dagger(\theta_{p^m}(\boldsymbol{v}, 1))$ and $\mathcal{F}_\alpha^\dagger(\pi_m)$ are unramified over $\mathcal{E}_K^{\dagger,0} = \mathcal{E}_{K_m}^\dagger \cap \mathcal{E}_K^{\dagger,\mathrm{unr}}$, since their completions are unramified. Moreover, by Theorem 2.9, they have the same residue field, since this last coincides with that of their completions. By uniqueness (cf. Theorem 2.10), they are equal. □

*Remark* 2.19 The study of a generic Artin–Schreier character, given by $\boldsymbol{f}(T) \in \mathbf{W}(\mathcal{O}_K^\dagger)$, will be reduced to the case $\boldsymbol{f}(T) \in \mathbf{W}_s(\mathcal{O}_K[T^{-1}])$ (cf. Lemma 2.5, Remark 3.1, and Proposition 3.1).

**Lemma 2.5** *Let $\boldsymbol{f}(T) \in \mathbf{W}_m(\mathcal{O}^{\dagger}_{K_m})$, then there exist $\widetilde{\boldsymbol{f}}(T) \in \mathbf{W}_m(\mathcal{O}_{K_m}[[T]][T^{-1}])$ and $\boldsymbol{u}(T) = (u_0(T), \dots, u_m(T)) \in \mathbf{W}_m(\mathcal{O}^{\dagger}_{K_m})$ such that $|u_j(T)|_1 < 1$ for all $j = 0, \dots, m$ and $\boldsymbol{f}(T) = \boldsymbol{u}(T) + \widetilde{\boldsymbol{f}}(T)$. In particular $\theta_{p^m}(\boldsymbol{\nu}, 1) = \mathrm{e}_{p^m}(\boldsymbol{u}(T), 1) \cdot \theta_{p^m}(\tilde{\boldsymbol{\nu}}, 1)$, where $\tilde{\boldsymbol{\nu}}$, and $\boldsymbol{\nu}$, are solutions of $\varphi(\tilde{\boldsymbol{\nu}}) - \tilde{\boldsymbol{\nu}} = \widetilde{\boldsymbol{f}}(T)$ and $\varphi(\boldsymbol{\nu}) - \boldsymbol{\nu} = \boldsymbol{f}(T)$ respectively.*

*Proof* This is evident for $m = 0$. By induction the lemma follows from the following relation valid for Witt vectors in general ([3, Chap. 10, Sect. 1, Lemme 4]):

$$(f_0(T), \dots, f_m(T)) = (f_0(T), 0, \dots, 0) + (0, f_1(T), \dots, f_m(T)). \qquad (2.54) \quad \square$$

(2.54)

## 3 Classification of rank one differential equations over $\mathcal{R}_{K_\infty}$

Throughout this third part, we will not need the results of Sect. 2.3. Namely, $(K, |.|)$ is only a complete ultrametric field containing $(\mathbb{Q}_p, |.|)$, and we will not suppose that $K$ satisfies Eq. (2.53), nor that its residue field is perfect. We fix a Lubin–Tate group $\mathfrak{G}_P$, isomorphic to $\widehat{\mathbb{G}}_m$, and fix a generator $\boldsymbol{\pi} = (\pi_j)_{j \geq 0}$ of the Tate module $T(\mathfrak{G}_P)$.

We recall that $K_s = K(\pi_s)$, and that $k_s$ is its residue field (cf. Definition 1.14). For all algebraic extension $H/K$, we set $H_s := H(\pi_s)$. The residue fields of $H$ and $H_s$ are denoted by $k_H$ and $k_{H_s}$, respectively. We set $\mathrm{E}_s := k_s((t))$.

*3.0.2*

The starting point of the classification is the equation

$$\theta_{p^s}(\boldsymbol{\nu}, 1)^{p^{s+1}} = \mathrm{e}_{p^s}(\boldsymbol{f}(T), 1)^{p^{s+1}}, \qquad (3.1)$$

with the notations of Corollary 2.4 and diagram (0.7). In some cases the symbol $\mathrm{e}_{p^s}(\boldsymbol{f}(T), 1)$ does make sense, and the interesting "Kummer generator" $\theta_{p^s}(\boldsymbol{\nu}, 1)$ is equal to $\mathrm{e}_{p^s}(\boldsymbol{f}(T), 1)$. We will show that all rank one solvable differential equations over $\mathcal{R}_{K_m}$ admit, in some basis, such an exponential as solution.

**Definition 3.1** *Let $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_K[T^{-1}])$, then we set*

$$\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1) := \exp\left(\pi_s \phi_0^-(T) + \pi_{s-1}\frac{\phi_1^-(T)}{p} + \cdots + \pi_0 \frac{\phi_s^-(T)}{p^s}\right), \qquad (3.2)$$

*where $\phi_j^-(T)$ is the $j$-th phantom component of $\boldsymbol{f}^-(T) = (f_0^-(T), \dots, f_s^-(T))$.*

*Remark* 3.1 Clearly $\phi_j^-(T)$ lies in $T^{-1}\mathcal{O}_K[T^{-1}]$, for all $j = 0, \dots, s$, and hence the expression (3.2) converges $T^{-1}$-adically. Moreover,

$$\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1) = \prod_{j=0}^{s} E_{s-j}(f_j^-(T)) \in 1 + \pi_s T^{-1}\mathcal{O}_{K_s}[[T^{-1}]]. \qquad (3.3)$$

In particular, $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ is convergent for $|T|>1$. As mentioned in Remark 2.19, Lemmas 2.5, 3.4, and 4.5, will be useful to reduce the study of $\theta_{p^s}(\boldsymbol{\nu},1)$, with $\varphi(\boldsymbol{\nu})-\boldsymbol{\nu}=\boldsymbol{f}(T)$, to the case in which $\boldsymbol{f}(T)\in\mathbf{W}_s(\mathcal{O}_K[T^{-1}])$.

## 3.1 Survey of the results

*Remark* 3.2 For all algebraic extensions $H/K$, the function (cf. 3.1)

$$\boldsymbol{f}^-(T)\mapsto \mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1) \tag{3.4}$$

defines a group morphism (as we can see by considering the phantom components)

$$\mathrm{e}_{p^s}(-,1):\mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])\longrightarrow 1+\pi_sT^{-1}\mathcal{O}_{H_s}[[T^{-1}]]\,. \tag{3.5}$$

Indeed $\boldsymbol{f}^-(T)$ involve only a finite numbers of coefficients of $H$, then the series $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ lies in a finite (and hence complete) extension of $K$.

Let $\boldsymbol{f}^-(T)\in\mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$, we set

$$\mathrm{L}(0,\boldsymbol{f}^-(T))=\partial_T-\partial_{T,\log}(\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)). \tag{3.6}$$

Observe that $1+\pi_sT^{-1}\mathcal{O}_{H_s}[[T^{-1}]]$ is not contained in $\mathcal{E}_{H_s}=\mathcal{E}_K\otimes_K H_s$. However, every series in this multiplicative group is convergent for $|T|>1$ (cf. Remark 3.1). Then, by the transfer Theorem 1.1 and by continuity of the radius, $\mathrm{L}(0,\boldsymbol{f}^-(T))$ is solvable over $\mathcal{R}_{H_s}$.

**Theorem 3.1** (Main theorem) *Let $M$ be a rank one solvable differential module over $\mathcal{R}_{K_\infty}$ (i.e. over $\mathcal{R}_{K_m}$, for some $m\geq 0$, or over $\mathcal{R}_K$ (cf. Definition* 1.9)*). Then there exists a basis of $M$ such that*

1. *The $1\times 1$ matrix of the derivation of $M$ lies in $\mathcal{O}_K[T^{-1}]$;*
2. *There exist an $s\geq 0$, and a Witt vector $\boldsymbol{f}^-(T)\in\mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$ such that the Taylor solution (cf. Eq.* (1.6)*) of $M$, at $\infty$, is*

$$T^{a_0}\cdot \mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1), \tag{3.7}$$

*with $a_0\in\mathbb{Z}_p$. In particular $M$ is defined (in this basis) by the operator*

$$\begin{aligned}\mathrm{L}(a_0,\boldsymbol{f}^-(T)) &:= \partial_T-\partial_{T,\log}\left(T^{a_0}\cdot \mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)\right)\\ &= \partial_T - a_0+\sum_{j=0}^{s}\pi_{s-j}\sum_{i=0}^{j} f_i^-(T)^{p^{j-i}}\partial_{T,\log}(f_i^-(T)).\end{aligned} \tag{3.8}$$

*Moreover the isomorphism class of $M$ depends bijectively on*

- *The class of $a_0$ in $\mathbb{Z}_p/\mathbb{Z}$;*
- *The Artin–Schreier character $\alpha:=\delta(\overline{\boldsymbol{f}^-}(t))$ defined by the reduction $\overline{\boldsymbol{f}^-}(t)\in\mathbf{W}_s(\mathrm{E}_s)$ of $\boldsymbol{f}^-(T)$.*

**Definition 3.2** *We will denote indifferently by* $\mathrm{M}(a_0,\alpha)$, $\mathrm{M}(a_0,\overline{\boldsymbol{f}^-}(t))$ *or* $\mathrm{M}(a_0, \boldsymbol{f}^-(T))$*, the differential module defined by* $\mathrm{L}(a_0,\boldsymbol{f}^-(T))$.

Assume the point (1) and (2) of the Theorem 3.1. Then the last assertion can be translated in terms of $\boldsymbol{\pi}$-exponentials as follow. Recall that $p = w$ (cf. Theorem 2.5).

**Theorem 3.2** *Let* $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$*, and let* $\overline{\boldsymbol{f}^-}(t) \in \mathbf{W}_s(t^{-1}k_s[t^{-1}])$ *be its reduction. Then*

3. *If* $\widetilde{\boldsymbol{f}}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$ *is another lifting of* $\overline{\boldsymbol{f}^-}(t)$*, then*

$$\frac{\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)}{\mathrm{e}_{p^s}(\widetilde{\boldsymbol{f}}^-(T),1)} = \mathrm{e}_{p^s}(\boldsymbol{f}^-(T) - \widetilde{\boldsymbol{f}}^-(T),1) \tag{3.9}$$

*is convergent for* $|T| > 1-\varepsilon$*, for some* $\varepsilon > 0$ *(i.e. lies in* $\mathcal{R}_{K_s}$*).*

4. *If* $\boldsymbol{f}^-_{(\bar{\mathrm{F}})}(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$ *is an arbitrary lifting of* $\bar{\mathrm{F}}(\overline{\boldsymbol{f}^-}(t))$*, then*

$$\frac{\mathrm{e}_{p^s}(\boldsymbol{f}^-_{(\bar{\mathrm{F}})}(T),1)}{\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)} = \mathrm{e}_{p^s}(\boldsymbol{f}^-_{(\bar{\mathrm{F}})}(T) - \boldsymbol{f}^-(T),1) \tag{3.10}$$

*is convergent for* $|T| > 1-\varepsilon'$*, for some* $\varepsilon' > 0$ *(i.e. lies in* $\mathcal{R}_{K_s}$*).*

5. *Conversely the function* $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ *lies in* $\mathcal{R}_{K_s}$ *if and only if the equation* $\bar{\mathrm{F}}\left(\bar{\boldsymbol{\nu}}^-\right) - \bar{\boldsymbol{\nu}}^- = \overline{\boldsymbol{f}^-}(t)$ *has a solution* $\bar{\boldsymbol{\nu}}^- \in \mathbf{W}_s(t^{-1}k_s[t^{-1}])$*.*

**Notation 3.3** The point (5) will be called the Frobenius structure theorem.

*3.1.1*

By the Main Theorem 3.1, Definition 3.1 and by the rules introduced in Sect. 1.2, it follows that, for all $s \geq 0$, and for all algebraic extensions $H/K$, we have an exact sequence of abelian groups (functorial on the algebraic extensions $H$ of $K$)

$$\mathbf{W}_s(t^{-1}k_H[t^{-1}]) \xrightarrow{\bar{\mathrm{F}}-1} \mathbf{W}_s(t^{-1}k_H[t^{-1}]) \xrightarrow{\mathrm{M}(0,-)} \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_s}). \tag{3.11}$$

On the other hand, it follows from the Definition 3.1, that we have

$$\mathrm{e}_{p^{s+1}}(\mathrm{V}(\boldsymbol{f}^-(T)),1) = \mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1). \tag{3.12}$$

Hence, for all $s \geq 0$, we have the following functorial commutative diagram

$$\begin{array}{ccccc} \mathbf{W}_s(t^{-1}k_H[t^{-1}]) & \xrightarrow{\bar{\mathrm{F}}-1} & \mathbf{W}_s(t^{-1}k_H[t^{-1}]) & \xrightarrow{\mathrm{M}(0,-)} & \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_{s+1}}) \\ \mathrm{V}\downarrow & \odot & \mathrm{V}\downarrow & \nearrow{\scriptstyle \mathrm{M}(0,-)} & \\ \mathbf{W}_{s+1}(t^{-1}k_H[t^{-1}]) & \xrightarrow{\bar{\mathrm{F}}-1} & \mathbf{W}_{s+1}(t^{-1}k_H[t^{-1}]) & & \end{array} \tag{3.13}$$

This shows, by passing to the inductive limit, that we again have an exact sequence

$$\mathbf{CW}(t^{-1}k_H[t^{-1}]) \xrightarrow{\bar{\mathrm{F}}-1} \mathbf{CW}(t^{-1}k_H[t^{-1}]) \xrightarrow{\mathrm{M}(0,-)} \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_\infty}). \tag{3.14}$$

The group $\mathbb{Z}_p/\mathbb{Z}$ has no $p$-torsion element. On the other hand, every element of $\mathbf{CW}(t^{-1}k_H[t^{-1}])$ is killed by a power of $p$. Since we are assuming that all solutions are of the form Eq. (3.7), this proves the following:

**Lemma 3.1** *Let $H$ be an algebraic extension of $K_\infty$. The image of* $\mathrm{M}(0,-)$ *is the sub-group of the $p$-torsion elements of* $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_H)$*, and if $H/K_\infty$ is Galois, then* $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_H)$ *is isomorphic, as* $\mathrm{Gal}(H/K_\infty)$*-module, to the direct sum of $\mathbb{Z}_p/\mathbb{Z}$ with the image of* $\mathrm{M}(0,-)$.

**Corollary 3.1** *The map* $(a_0,\alpha) \mapsto \mathrm{M}(a_0,\alpha)$ *induces an isomorphism*

$$\mathbb{Z}_p/\mathbb{Z} \oplus \frac{\mathbf{CW}(t^{-1}k_\infty[t^{-1}])}{(\bar{\mathrm{F}}-1)\mathbf{CW}(t^{-1}k_\infty[t^{-1}])} \quad \xrightarrow[\sim]{\mathrm{M}(-,-)} \quad \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty}). \tag{3.15}$$

*Proof* By Galois descent $\mathrm{M}(-,-)$ induces an isomorphism, with $k_\infty^{\mathrm{perf}} := (k^{\mathrm{alg}})^{\mathrm{Gal}(k^{\mathrm{alg}}/k_\infty)}$ instead of $k_\infty$. But actually, the co-vector quotient is invariant under inseparable extension of $k_\infty$ as explained in Subsect. 3.1.3 below. □

*3.1.2*

On the other hand, it is well known that (cf. Lemma 3.4 and Proposition 3.1)

$$\mathrm{H}^1\left(\mathrm{Gal}\big(k_\infty((t))^{\mathrm{sep}}/k_\infty((t))\big), \mathbb{Q}_p/\mathbb{Z}_p\right) = \mathbf{P}(k_\infty) \oplus \mathrm{H}^1\left(\mathrm{Gal}\big(k_\infty^{\mathrm{sep}}/k_\infty\big), \mathbb{Q}_p/\mathbb{Z}_p\right),$$

where $\mathbf{P}(k_\infty)$ is the character group of $\mathcal{P}_{\mathrm{E}_\infty}$, with $\mathrm{E}_\infty = k_\infty((t))$ (cf. Remark 1.11). More precisely we have the following (for a more convenient description of $\mathbf{P}(\kappa)$ see Lemma 4.1)

**Lemma 3.2** *For all fields $\kappa$ of characteristic $p > 0$, one has*

$$\mathbf{P}(\kappa) = \frac{\mathbf{CW}(t^{-1}\kappa[t^{-1}])}{(\bar{\mathrm{F}}-1)\mathbf{CW}(t^{-1}\kappa[t^{-1}])}. \tag{3.16}$$

*Proof* This will follow from Lemma 3.4 and Proposition 3.1. □

*3.1.3*

Furthermore we have $\mathbf{P}(k_\infty^{\mathrm{perf}}) = \mathbf{P}(k_\infty)$, because, by Remark 1.8 (or Remark 1.9), the Artin–Schreier complex is stable under purely inseparable extensions, that is $\mathrm{Gal}(k_\infty^{\mathrm{perf,sep}}/k_\infty^{\mathrm{perf}}) = \mathrm{Gal}(k_\infty^{\mathrm{sep}}/k_\infty)$. In other words, for all $r \geq 0$, the co-vectors $\overline{\boldsymbol{f}^-}(t) = (\ldots,0,\overline{f_0^-}(t),\ldots,\overline{f_s^-}(t))$ and $\bar{\mathrm{F}}^r(\overline{\boldsymbol{f}^-}(t)) = (\ldots,0,\overline{f_0^-}(t)^{p^r},\ldots,\overline{f_s^-}(t)^{p^r})$ have the same image in the right hand quotient of Eq. (3.16).

## 3.2 Proofs of the statements

We first prove the statements (3), (4), and (5) of Theorem 3.2. The idea is to express $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ as a product of $\boldsymbol{\pi}$-exponentials of the type $\mathrm{e}_d(\boldsymbol{\lambda},T^{-1})$. The main tool will be the notion of *s-co-monomial* which reduce the study to $\boldsymbol{\pi}$-exponentials (see Eq. (3.21)). The principal lemma will be Lemma 3.3.

**Definition 3.3** *Let $H/K$ be an algebraic extension. Let $d = np^m > 0$, $(n,p) = 1$. Let $s \geq 0$. We will call $s$-*co-monomial *of degree $-d$ relative to $\boldsymbol{\lambda} := (\lambda_0,\ldots,\lambda_m) \in \mathbf{W}_m(\mathcal{O}_H)$ the Witt vector in $\mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$*

$$\begin{aligned} \boldsymbol{\lambda}T^{-d} &:= \Big(\overbrace{0,\ldots,0}^{s-m},\lambda_0 T^{-n},\lambda_1 T^{-np},\ldots,\lambda_m T^{-d}\Big) && \text{if } m \leq s, \\ \boldsymbol{\lambda}T^{-d} &:= \Big(\lambda_{m-s}T^{-np^{m-s}},\lambda_{m-s+1}T^{-np^{m-s+1}},\ldots,\lambda_m T^{-d}\Big) && \text{if } m \geq s. \end{aligned} \tag{3.17}$$

*We denote by $\mathbf{W}_s^{(-d)}(\mathcal{O}_H)$ the sub-group of $\mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$ formed by s-co-monomials of degree $-d$, and by $\mathbf{W}_s^{(-d)}(k_H)$ its image in $\mathbf{W}_s(t^{-1}k_H[t^{-1}])$.*

*Remark* 3.3 By looking at the phantom components we find an isomorphism of groups $\mathbf{W}_s^{(-d)}(\mathcal{O}_H) \xrightarrow{\sim} \mathbf{W}_{\min(s,m)}(\mathcal{O}_H)$, and hence $\mathbf{W}_s^{(-d)}(k_H) \xrightarrow{\sim} \mathbf{W}_{\min(s,m)}(k_H)$.

**Lemma 3.3** *Let now $H/K$ be an algebraic extension. Let $d = np^m > 0$, $(n,p) = 1$, let $s \geq 0$, and let $\boldsymbol{\lambda} := (\lambda_0,\ldots,\lambda_m) \in \mathbf{W}_m(\mathcal{O}_H)$. If $m \leq s$, we have*

$$\mathrm{e}_{p^s}(\boldsymbol{\lambda}T^{-d},1) = \mathrm{e}_d(\boldsymbol{\lambda},T^{-1}). \tag{3.18}$$

*Proof* The phantom vector of $(\underbrace{0,\ldots,0}_{s-m},\lambda_0 T^{-n},\lambda_1 T^{-np},\ldots,\lambda_m T^{-d})$, is

$$\Big\langle 0,\ldots,0,p^{s-m}\phi_0 T^{-n},p^{s-m}\phi_1 T^{-np},\ldots,p^{s-m}\phi_m T^{-d}\Big\rangle, \tag{3.19}$$

where $\langle\phi_0,\ldots,\phi_m\rangle$ is the phantom vector of $(\lambda_0,\ldots,\lambda_m)$. The proof follows immediately from the Definitions 3.1 and 2.4. □

**Definition 3.4** *For all algebraic extensions $H/K$ we set* $\mathrm{E}_H := k_H(\!(t)\!)$.

**Lemma 3.4** *For all $s \geq 0$, there is a (functorial) decomposition*

1. $\mathbf{W}_s(\mathrm{E}_H) = \oplus_{d>0}\mathbf{W}_s^{(-d)}(k_H) \oplus \mathbf{W}_s(k_H) \oplus \mathbf{W}_s(tk_H[[t]])$ ;
2. $\mathbf{W}_s(\mathcal{O}_H[[T]][T^{-1}]) = \oplus_{d>0}\mathbf{W}_s^{(-d)}(\mathcal{O}_H) \oplus \mathbf{W}_s(\mathcal{O}_H) \oplus \mathbf{W}_s(T\mathcal{O}_H[[T]])$.

*Proof* Let $s = 0$, then $k_H((t)) = \oplus_{d>0}k_H t^{-d} \oplus k_H \oplus tk_H[[t]]$. The proof follows easily by induction from Eq. (2.54). □

*Remark* 3.4 Witt vectors in $\oplus_{d>0}\mathbf{W}_s^{(-d)}(\mathcal{O}_H)$ (respectively $\mathbf{W}_s(\mathcal{O}_H)$, $\mathbf{W}_s(T\mathcal{O}_K[[T]])$) have their phantom components in $T^{-1}\mathcal{O}_H[T^{-1}]$ (respectively $\mathcal{O}_H$, $T\mathcal{O}_K[[T]]$).

**Corollary 3.2** *We have a* (*functorial*) *decomposition*

$$\mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}]) = \oplus_{d>0}\mathbf{W}_s^{(-d)}(\mathcal{O}_H)\ , \quad \mathbf{W}_s(t^{-1}k_H[t^{-1}]) = \oplus_{d>0}\mathbf{W}_s^{(-d)}(k_H).$$

*Proof* The inclusion $\subseteq$ follows by Remark 3.4. Since all monomials belong to $\mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$ we have the inclusion $\supseteq$. The right hand equality follows from the first one by reduction. □

**Definition 3.5** *For all* $\boldsymbol{f}(T) \in \mathbf{W}_s(\mathcal{O}_H[[T]][T^{-1}])$, *we will denote by*

$$\boldsymbol{f}(T) = \boldsymbol{f}^-(T) + \boldsymbol{f}_0 + \boldsymbol{f}^+(T) \tag{3.20}$$

*the unique decomposition of* $\boldsymbol{f}(T)$ *satisfying* $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$, $\boldsymbol{f}_0 \in \mathbf{W}_s(\mathcal{O}_H)$, $\boldsymbol{f}^+(T) \in \mathbf{W}_s(T\mathcal{O}_H[[T]])$ (*cf. Lemma* 3.4). *The same notation will be used for a Witt vector* $\overline{\boldsymbol{f}}(t) \in \mathbf{W}_s(\mathrm{E}_H)$.

*Remark* 3.5 By Eq. (1.21), we then have a corresponding decomposition of $\alpha := \delta(\overline{\boldsymbol{f}}(t))$, i.e. $\alpha = \alpha^- + \alpha_0$, ($\alpha^+ = 0$ by Proposition 3.1), with $\alpha^- = \delta(\overline{\boldsymbol{f}^-}(t))$, and $\alpha_0 = \delta(\overline{\boldsymbol{f}}_0)$. This shows that $\mathrm{Gal}(\mathrm{E}_H^{\mathrm{sep}}/\mathrm{E}_H)^{\mathrm{ab}} = \mathrm{Gal}(k_H^{\mathrm{sep}}/k_H)^{\mathrm{ab}} \oplus \mathcal{I}_{\mathrm{E}_H}^{\mathrm{ab}}$, where $\mathrm{E}_H = k_H(\!(t)\!)$ .

**Proposition 3.1** $\mathbf{W}_s(tk_H[[t]]) \subseteq (\bar{\mathrm{F}} - 1)\mathbf{W}_s(tk_H[[t]])$, *for all* $s \geq 0$.

*Proof* Since $\mathrm{E}_H$ is complete, by Corollary 1.4, $\mathbf{W}_s(\mathrm{E}_H)$ is complete. Let $\overline{\boldsymbol{f}}^+(t) \in \mathbf{W}_s(tk_H[[t]])$. Then the series $\overline{\boldsymbol{g}}^+(t) := -\sum_{i\geq 0}\bar{\mathrm{F}}^i(\overline{\boldsymbol{f}}^+(t))$ is Cauchy for this topology, and hence converges in $\mathbf{W}_s(\mathrm{E}_H)$. Moreover $\overline{\boldsymbol{f}}^+(t) = \bar{\mathrm{F}}(\overline{\boldsymbol{g}}^+(t)) - \overline{\boldsymbol{g}}^+(t)$. □

*Remark* 3.6 Let $H/K$ be an algebraic extension and let $\boldsymbol{f}^-(T) \in \mathbf{W}_s(t^{-1}\mathcal{O}_H[t^{-1}])$. Let $v_p(-)$ be the $p$-adic valuation (namely $v_p(d) = m$ if $d = np^m$, $(n,p) = 1$). Let $\boldsymbol{f}^-(T) = \sum_{d>0}\boldsymbol{\lambda}_d T^{-d}$, with $\boldsymbol{\lambda}_d \in \mathbf{W}_{v_p(d)}(k_H)$ be its decomposition in $s$-co-monomials of degree $-d$. We can suppose $s \gg 0$ (cf. Eq. (3.12)), then

$$\mathrm{e}_{p^s}\left(\boldsymbol{f}^-(T),1\right) = \mathrm{e}_{p^s}\left(\sum_{d>0}\boldsymbol{\lambda}_d T^{-d},1\right) = \prod_{d>0}\mathrm{e}_{p^s}(\boldsymbol{\lambda}_d T^{-d},1) \overset{\text{Lemma 3.3}}{=} \prod_{d>0}\mathrm{e}_d(\boldsymbol{\lambda}_d, T^{-1}). \tag{3.21}$$

Then $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ is a (finite) product of elementary $\boldsymbol{\pi}$-exponentials. In terms of differential modules, we have $\mathrm{M}(0,\boldsymbol{f}^-(T)) = \otimes_{d>0}\mathrm{M}(0,\boldsymbol{\lambda}_d T^{-d})$. Hence, by the basic rules introduced in Sect. 1.2, the study can be reduced to $\boldsymbol{\pi}$-exponentials.

*3.2.1 Proof of the statements* (3), (4), (5) *of Theorem 3.2*

**Notation 3.4** For all $d > 0$, we set $d = np^m$, with $(n,p) = 1$ and $v_p(d) := m$. In the sequel the letters $n$ and $m$ will indicate always this decomposition.

By Lemma 3.3, for all $d$ appearing in the product (3.21), we have (cf. Definition 2.5)

$$\mathrm{L}_d(\boldsymbol{\lambda}_d) = \mathrm{L}(0,\boldsymbol{\lambda}_d T^{-d}) , \quad \mathrm{M}_d(\boldsymbol{\lambda}_d) = \mathrm{M}(0,\boldsymbol{\lambda}_d T^{-d}), \tag{3.22}$$

where $\boldsymbol{\lambda}_d T^{-d}$ is the $s$-co-monomial of degree $-d$ attached to $\boldsymbol{\lambda}_d \in \mathbf{W}_{v_p(d)}(\mathcal{O}_H)$ (cf. Definition 3.3). Actually, by the rule (3.12), we can suppose $s \gg v_p(d) = m$, for all $d > 0$ appearing in the (finite) product (3.21).

The assertions (3) and (4) are consequences of the Reduction Theorem 2.2, and the Frobenius Structure Theorem 2.5 for $\boldsymbol{\pi}$-exponentials, respectively. Let us prove the assertion (3). We decompose $\boldsymbol{f}^-(T) - \widetilde{\boldsymbol{f}}^-(T)$ in $s$-co-monomials of degree $-d$, $\boldsymbol{f}^-(T) - \widetilde{\boldsymbol{f}}^-(T) = \sum_d \boldsymbol{\lambda}_d T^{-d}$, with $\boldsymbol{\lambda}_d \in \mathbf{W}_{v_p(d)}(\mathcal{O}_H)$ (cf. Lemma 3.4). Then

$$\mathrm{e}_{p^s}(\boldsymbol{f}^-(T) - \widetilde{\boldsymbol{f}}^-(T),1) = \prod_{d>0} \mathrm{e}_{p^s}(\boldsymbol{\lambda}_d T^{-d},1) \overset{\text{Lemma 3.3}}{=} \prod_{d>0} \mathrm{e}_d(\boldsymbol{\lambda}_d, T^{-1}). \tag{3.23}$$

The over-convergence of $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T) - \widetilde{\boldsymbol{f}}^-(T),1)$ will result from the over-convergence of every $\mathrm{e}_d(\boldsymbol{\lambda}_d, T^{-1})$. In order to apply the reduction Theorem 2.2, we shall show that the reduction $\overline{\boldsymbol{\lambda}_d}$ of $\boldsymbol{\lambda}_d$ is 0, for all $d > 0$. Since the reduction of $\boldsymbol{f}^-(T) - \widetilde{\boldsymbol{f}}^-(T)$ is 0, it follows from Lemma 3.4 that the reduction of $\boldsymbol{\lambda}_d T^{-d}$ in $\mathbf{W}_s^{(-d)}(k_H)$ is 0, for all $d > 0$. By Remark 3.3, for all $d > 0$, we have an isomorphism $\boldsymbol{\lambda}_d T^{-d} \mapsto \boldsymbol{\lambda}_d : \mathbf{W}_s^{(-d)}(\mathcal{O}_H) \xrightarrow{\sim} \mathbf{W}_{v_p(d)}(\mathcal{O}_H)$. Hence $\overline{\boldsymbol{\lambda}_d} = 0$ in $\mathbf{W}_{v_p(d)}(k_H)$, for all $d > 0$.

The proof of (4) follows the same lines. Namely, by the assertion (3), the isomorphism class of $\mathrm{M}(0,\boldsymbol{f}^-(T))$ depends only on the reduction $\overline{\boldsymbol{f}^-}(t) \in \mathbf{W}_s(t^{-1}k_H[t^{-1}])$ of $\boldsymbol{f}^-(T)$. As usual, we decompose $\overline{\boldsymbol{f}^-}(t) = \sum_{d>0} \bar{\boldsymbol{\lambda}}_d t^{-d}$, with $\bar{\boldsymbol{\lambda}}_d t^{-d} \in \mathbf{W}_s^{(-d)}(k_H)$. The morphism $\bar{\mathrm{F}} : \mathbf{W}_s(\mathrm{E}_H) \to \mathbf{W}_s(\mathrm{E}_H)$ sends the monomial $\bar{\boldsymbol{\lambda}}_d t^{-d}$ into $\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}_d)t^{-pd}$, Hence $\bar{\mathrm{F}}(\overline{\boldsymbol{f}^-}(t)) = \sum_{d>0} \bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}_d)t^{-pd}$. Then

$$\mathrm{M}(0,\overline{\boldsymbol{f}^-}(t)) \xrightarrow{\sim} \otimes_{d>0}\mathrm{M}_d(\bar{\boldsymbol{\lambda}}_d) \xrightarrow[\text{Th. 2.5}]{\sim} \otimes_{d>0}\mathrm{M}_{pd}(\mathrm{V}\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}_d)) \xrightarrow{\sim} \mathrm{M}(0,\bar{\mathrm{F}}(\overline{\boldsymbol{f}^-}(t))),$$

where the last isomorphism follows from the fact that $\mathrm{V}(\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}_d)t^{-pd})$ and $\bar{\mathrm{F}}(\bar{\boldsymbol{\lambda}}_d)t^{-pd}$ define the same differential module (cf. Eq. (3.12)).

The proof of the assertion (5) of Theorem 3.2 follows from assertions (3) and (4) of Theorem 3.2 in the following way. Suppose that $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ is over-convergent. We want to show that the equation $\bar{\mathrm{F}}(\bar{\boldsymbol{\nu}}) - \bar{\boldsymbol{\nu}} = \overline{\boldsymbol{f}^-}(t)$ has a solution $\bar{\boldsymbol{\nu}} \in \mathbf{W}_s(t^{-1}k_H[t^{-1}])$. In other words, we shall show that $\overline{\boldsymbol{f}^-}(t)$ belongs to $(\bar{\mathrm{F}}-1)\mathbf{W}_s(t^{-1}k_H[t^{-1}])$. Let us write $\boldsymbol{f}^-(T) = \sum_{d>0} \boldsymbol{\lambda}_d T^{-d}$ as a (finite) sum of $s$-co-monomials. We need to replace $\boldsymbol{f}^-(T)$ by a more convenient Witt vector.

**Definition 3.6** *A Witt vector* $\boldsymbol{f}_p^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$ *is called* pure *if its decomposition in s-co-monomials is a (finite) sum of the type*

$$\boldsymbol{f}_p^-(T) = \sum_{n \in \mathrm{J}_p} \boldsymbol{\lambda}_{np^{m(n)}} T^{-np^{m(n)}}, \tag{3.24}$$

*where* $J_p = \{n \in \mathbb{Z} \mid (n,p) = 1 \ , \ n > 0\}$*, and* $\boldsymbol{\lambda}_{np^{m(n)}} \in \mathbf{W}_{m(n)}(\mathcal{O}_H)$.

*Remark* 3.7 $\partial_{T,\log}(\mathrm{e}_{p^s}(\boldsymbol{f}_p^-(T),1)) = \sum_{n\in \mathrm{J}_p} -n \sum_{j=0}^{m(n)} \pi_{m(n)-j}\phi_{np^{m(n)},j} T^{-np^j}$, where $\langle \phi_{np^{m(n)},0}, \ldots, \phi_{np^{m(n)},m(n)} \rangle$ is the phantom vector of $\boldsymbol{\lambda}_{np^{m(n)}}$. In this case the coefficients of the differential equation are simpler and directly related to $\boldsymbol{\lambda}_{np^{m(n)}}$ instead of $\boldsymbol{\lambda}_{np^{m(n)}} T^{-np^{m(n)}}$. This will be useful for explicit computations (cf. Corollary 4.7).

**Lemma 3.5** *Let* $\boldsymbol{f}_p^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$ *be a pure Witt vector. The exponential* $\mathrm{e}_{p^s}(\boldsymbol{f}_p^-(T),1)$ *is over-convergent if and only if* $\overline{\boldsymbol{f}_p^-}(t) = 0$. *Moreover,*

$$\mathrm{Irr}\left(\mathrm{M}(0,\boldsymbol{f}_p^-(T))\right) = \max_{n \in J_p} \ \mathrm{Irr}\left(\mathrm{M}_{np^{m(n)}}(\boldsymbol{\lambda}_{np^{m(n)}})\right). \tag{3.25}$$

*Proof* Write $\mathrm{M}(0,\boldsymbol{f}_p^-(T)) = \otimes_{n\in \mathrm{J}_p} \mathrm{M}(0,\boldsymbol{\lambda}_{np^{m(n)}} T^{-np^{m(n)}})$. The irregularity of $\mathrm{M}(0,\boldsymbol{\lambda}_{np^{m(n)}} T^{-np^{m(n)}}) \xrightarrow[\text{Eq. (3.22)}]{\sim} \widetilde{\mathrm{M}}_{np^{m(n)}}(\boldsymbol{\lambda}_{np^{m(n)}})$ is, by Theorem 2.2, a number belonging to the set $\{0\} \cup \{n \cdot p^m \mid m \geq 0\}$. Hence, for different values of $n$, we have different values of the $p$-adic slope of $\mathrm{M}_{np^{m(n)}}(\boldsymbol{\lambda}_{np^{m(n)}})$. Corollary 1.1 then implies the Eq. (3.25). Suppose now that $\mathrm{e}_{p^s}(\boldsymbol{f}_p^-(T),1)$ is over-convergent, then this irregularity is equal to 0. Hence all $\mathrm{M}_{np^{m(n)}}(\boldsymbol{\lambda}_{np^{m(n)}})$ are trivial, and $\mathrm{e}_{p^s}(\boldsymbol{\lambda}_{np^{m(n)}} T^{-np^{m(n)}},1)$ is over-convergent (i.e. lies in $\mathcal{R}_H$), for all $n \in \mathrm{J}_p$. By Theorem 2.2 this implies $\overline{\boldsymbol{\lambda}_{np^{m(n)}}} = 0$, for all $n \in \mathrm{J}_p$. □

Assertion (5) of Theorem 3.2 follows then by point (1) of the following:

**Lemma 3.6** *Let* $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$. *Then*

1. *There exists a pure Witt vector* $\boldsymbol{f}_p^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}])$ *such that*

$$\overline{\boldsymbol{f}^-(T) - \boldsymbol{f}_p^-(T)} \in (\bar{\mathrm{F}}-1)\mathbf{W}_s(t^{-1}k_H[t^{-1}]). \tag{3.26}$$

*In particular, by assertion* (4) *of Theorem 3.2,* $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T)-\boldsymbol{f}^-_p(T),1)$ *is overconvergent, and* $\mathrm{M}(0,\boldsymbol{f}^-(T)) \xrightarrow{\sim} \mathrm{M}(0,\boldsymbol{f}^-_p(T))$*, over* $\mathcal{R}_{H_s}$*;*

2. *There exists a pure Witt vector* $\boldsymbol{h}^-_p(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{H_\infty}[T^{-1}])$ *such that*

$$\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1) = \mathrm{e}_{p^s}(\boldsymbol{h}^-_p(T),1). \tag{3.27}$$

*Proof* Let us write $\boldsymbol{f}^-(T) = \sum_{d>0} \boldsymbol{\lambda}_d T^{-d}$ as a (finite) sum of $s$-co-monomials. Write

$$\boldsymbol{\lambda}_d T^{-d} = (0,\dots,0,\lambda_{d,0}T^{-n},\dots,\lambda_{d,m}T^{-np^m}) \in \mathbf{W}_s(T^{-1}\mathcal{O}_H[T^{-1}]),$$

where, for all $d > 0$, we set $d = np^m$, $m = v_p(d)$. Now set

$$\boldsymbol{\lambda}^{(\bar{\mathrm{F}})}_{pd} T^{-pd} := (0,\dots,0,\lambda^p_{d,0}T^{-np},\dots,\lambda^p_{d,m}T^{-np^{m+1}}),$$

then the reduction $\overline{\boldsymbol{\lambda}^{(\bar{\mathrm{F}})}_{pd} T^{-pd} - \boldsymbol{\lambda}_d T^{-d}}$ lies in $(\bar{\mathrm{F}}-1)\mathbf{W}_s(k((t)))$. Hence we can replace $\boldsymbol{\lambda}_d T^{-d}$ with $\boldsymbol{\lambda}^{(\bar{\mathrm{F}})}_{pd} T^{-pd}$. Replacing in this way $\boldsymbol{\lambda}_{np^m} T^{-np^m}$ with $\boldsymbol{\lambda}^{(\bar{\mathrm{F}})}_{np^m} T^{-np^{m+1}}$, step by step, we obtain a pure Witt vector. In other words, we can suppose that for all $n \in \mathrm{J}_p$ there exists a unique $m(n) \geq 0$ such that $\boldsymbol{\lambda}_{np^{m(n)}} T^{-np^{m(n)}} \neq 0$. Now let us construct $\boldsymbol{h}^-_p(T)$. First we arrange the sum $\boldsymbol{f}^-(T) = \sum_{n\in\mathrm{J}_p}\sum_{m\geq 0} \boldsymbol{\lambda}_{np^m} T^{-np^m}$. Then we construct, for all $n \in \mathrm{J}_p$, a natural number $m(n) \geq 0$, and a Witt vector $\boldsymbol{\nu}_{np^{m(n)}} \in \mathbf{W}_s(\mathcal{O}_H)$, satisfying $\mathrm{e}_{p^s}(\boldsymbol{\nu}_{np^{m(n)}} T^{-np^{m(n)}},1) = \mathrm{e}_{p^s}(\sum_{m\geq 0} \boldsymbol{\lambda}_{np^m} T^{-np^m},1)$. Let $m(n) = \sup\{m \mid \boldsymbol{\lambda}_{np^m} \neq 0\}$. By Eq. (3.12), we can suppose $s \geq m(n)$. Let $\boldsymbol{\lambda}_{np^m} = (\lambda_{np^m,0},\dots,\lambda_{np^m,m})$, and let $\langle \phi_{np^m,0},\dots,\phi_{np^m,m}\rangle$ be its phantom vector. Then $\mathrm{e}_{p^s}(\sum_{m=0}^{m(n)} \boldsymbol{\lambda}_{np^m} T^{-np^m},1) = \exp\left(\pi_{m(n)}a_0 T^{-n} + \cdots + \pi_0 a_{m(n)} \frac{T^{-np^{m(n)}}}{p^{m(n)}}\right)$, where, for all $j = 0,\dots,m(n)$, we have

$$a_j = \frac{\pi_0}{\pi_{m(n)-j}} \cdot \phi_{np^j,j} + \frac{\pi_1}{\pi_{m(n)-j}} \cdot \phi_{np^{j+1},j} + \cdots + \frac{\pi_{m(n)-j}}{\pi_{m(n)-j}} \cdot \phi_{np^{m(n)},j}. \tag{3.28}$$

Let $P(X)$ be the chosen Lubin–Tate series. Denote by $P^{(1)}(X) := P(X)$, $P^{(r)}(X) := P(P(\cdots P(X)\cdots))$, $r$-times. We set $h_0(X) := 1$, and $h_r(X) := P^{(r)}(X)/X$, for $r = 1,\dots,m(n)$. The phantom vector of $[h_r(\pi_{m(n)})] \in \mathbf{W}_{m(n)}(\mathcal{O}_H)$ is as usual $\langle h_r(\pi_{m(n)}), h_r(\pi_{m(n)-1}),\dots,h_r(\pi_0)\rangle$ and is then equal to

$$\left\langle \frac{\pi_{m(n)-r}}{\pi_{m(n)}}, \frac{\pi_{m(n)-r-1}}{\pi_{m(n)-1}},\dots,\frac{\pi_0}{\pi_r},0,\dots,0\right\rangle \in \mathcal{O}_H^{m(n)+1}, \quad \text{if } r > 0, \tag{3.29}$$

while $[h_0(\pi_{m(n)})] = 1$, and its phantom vector is $\langle 1, \dots, 1\rangle$. Hence we have

$$a_j = h_{m(n)}(\pi_{m(n)-j})\phi^*_{n,j} + h_{m(n)-1}(\pi_{m(n)-j})\phi^*_{np,j} + \cdots + h_0(\pi_{m(n)-j})\phi^*_{np^{m(n)},j},$$

where, for all $k = 0, \dots, m(n)$, $\langle \phi^*_{np^k,0}, \dots, \phi^*_{np^k,m(n)}\rangle$ is the phantom vector of $\boldsymbol{\lambda}^*_{np^k} := (\lambda_{np^k,0}, \dots, \lambda_{np^k,k}, *, \dots, *) \in \mathbf{W}_{m(n)}(\mathcal{O}_H)$, where the last $m(n) - k$ components are arbitrarily chosen. Observe that $\phi^*_{np^k,j} = \phi_{np^k,j}$, for all $j = 0, \dots, k$, while, if $j > k$ we have $h_{m(n)-k}(\pi_{m(n)-j}) = 0$. This shows that

$$\boldsymbol{\nu}_{np^{m(n)}} := [h_{m(n)}(\pi_{m(n)})]\boldsymbol{\lambda}^*_n + [h_{m(n)-1}(\pi_{m(n)})]\boldsymbol{\lambda}^*_{np} + \cdots + [h_0(\pi_{m(n)})]\boldsymbol{\lambda}^*_{np^{m(n)}}. \quad \square$$

*3.2.2 Proof of* (1) *and* (2)

The assertions (1) and (2) of Theorem 3.1 will be a direct consequence of the following Theorem, and standard considerations (cf. Sec. 3.1.1 and Cor. 1.2). The algorithm employed is due to Robba [22, 10.10] (see also [10, 13.3]). We translate his techniques in terms of Witt vectors. Recall that, by Corollary 1.2, every rank one solvable equation has a basis in which the matrix is a polynomial in $T^{-1}$ with coefficients in $\mathcal{O}_K$.

**Theorem 3.5** *Let $H/K$ be a finite extension. Let $M$ be a solvable rank one differential module over $\mathcal{R}_H$, defined by an operator $\partial_T - g(T)$, $g(T) = \sum_{-d\le i\le -1} a_i T^i \in \mathcal{O}_H[T^{-1}]$. Then there exists a Witt vector $\boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{H'}[T^{-1}])$, whose coefficients lies in a finite extension $H'/H$, such that $\partial_T - g(T) = \mathrm{L}(0, \boldsymbol{f}^-(T))$. More explicitly we have* (*cf. Eq.* (3.8))

$$\sum_{-d\le i\le -1} a_i T^i = -\sum_{j=0}^{s} \pi_{s-j} \sum_{i=0}^{j} f_i^-(T)^{p^{j-i}} \partial_{T,\log}(f_i^-(T)),$$

*and in particular* $\exp(\sum_{-d\le i\le -1} a_i T^i/i) = \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$.

*Proof* We shall express $\exp\left(\sum_{-d\le i\le -1} a_i T^i/i\right)$ as a product of elementary $\boldsymbol{\pi}$-exponentials, with coefficients in $H^{\mathrm{alg}}$. Observe that solvability does not change by scalar extension of $H$. Let $d = np^m$, $(n,p) = 1$, and let $b_d \in H^{\mathrm{alg}}$ be such that $b_d^{p^m} = a_{-d}/(n\pi_0)$. By Lemma 1.3, $|a_{-d}| \le \omega < 1$, hence $|b_d| \le 1$. We consider the Witt vector $\boldsymbol{\lambda}_d := (b_d, 0, \dots, 0) \in \mathbf{W}_m(\mathcal{O}_{H^{\mathrm{alg}}})$, whose phantom vector is $\langle b_d, b_d^p, \dots, b_d^{p^m}\rangle$. By construction, we have

$$\mathrm{L}_d(\boldsymbol{\lambda}_d) = \partial_T + n \cdot \left(\pi_0 b_d^{p^m} T^{-d} + \pi_1 b_d^{p^{m-1}} T^{-d/p} + \cdots + \pi_m b_d T^{-n}\right). \quad (3.30)$$

Then $M \otimes \mathrm{M}_d(b_d, 0, \dots, 0)$ is defined by an operator of the form $\partial_T - \sum_{-d+1\le i\le -1} \tilde{a}_i T^i$, $\exists\ \tilde{a}_i \in H^{\mathrm{alg}}$ (cf. Sect. 1.2). Moreover $M \otimes \mathrm{M}_d(b_d, 0, \dots, 0)$ is again solvable, so, by Lemma 1.3, we have again $|\tilde{a}_{-d+1}| \le \omega$. This shows that we can iterate this process. More precisely there exist $\boldsymbol{\lambda}_i = (b_i, 0, \ldots, 0) \in \mathbf{W}_{v_p(i)}(\mathcal{O}_{H^{\mathrm{alg}}})$, $i = 1, \ldots, d$, such that

$$\epsilon(T) := \prod_{i=1,\ldots,d} \mathrm{e}_i(\boldsymbol{\lambda}_i, T^{-1}) = \mathrm{e}_{p^s}\left(\sum_{i=1}^{d} \boldsymbol{\lambda}_i T^{-i}, 1\right), \quad s \gg 0, \tag{3.31}$$

satisfies $\partial_{T,\log}(\epsilon(T)) = \sum_{-d \le i \le -1} a_i T^i$. Then $\boldsymbol{f}^-(T) := \sum_{1 \le i \le d} \boldsymbol{\lambda}_i T^{-i}$ (cf. Eq. (3.21)). □

# 4 Applications

## 4.1 Description of character group

**Lemma 4.1** *Let* $\mathrm{J}_p := \{n \in \mathbb{Z} \mid (n,p) = 1, n > 0\}$. *For all fields* $\kappa$ *of characteristic* $p$*, one has the following isomorphisms of additive groups* (*cf. Sect.* 1.3.4):

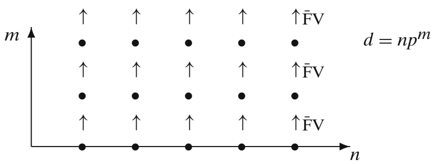
$$\mathbf{CW}(t^{-1}\kappa[t^{-1}]) \cong \oplus_{d>0} \mathbf{W}_{v_p(d)}(\kappa)\,; \quad \mathbf{P}(\kappa) \cong \widetilde{\mathbf{CW}}(\kappa)^{(\mathrm{J}_p)}, \tag{4.1}$$

*where* $\widetilde{\mathbf{CW}}(\kappa)^{(\mathrm{J}_p)}$ *means* $\oplus_{n \in \mathrm{J}_p} \widetilde{\mathbf{CW}}(\kappa)$ (*direct sum of copies of* $\widetilde{\mathbf{CW}}(\kappa)$).

*Proof* We have $\mathbf{CW}(t^{-1}\kappa[t^{-1}]) := \lim_{\to s} \mathbf{W}_s(t^{-1}\kappa[t^{-1}]) \overset{\text{Lemma 3.4}}{=} \lim_{\to s} \oplus_{d>0} \mathbf{W}_s^{(-d)}(\kappa)$. Observe that $\mathbf{W}_s^{(-d)}(\kappa) = \mathbf{W}_{\min(s, v_p(d))}(\kappa)$ (cf. Remark 3.3), hence

$$\mathbf{CW}(t^{-1}\kappa[t^{-1}]) = \oplus_{d>0} \varinjlim_{s} \mathbf{W}_{\min(s, v_p(d))}(\kappa) = \oplus_{d>0} \mathbf{W}_{v_p(d)}(\kappa)\,. \tag{4.2}$$

Now we write $d = np^m$, $n \in \mathrm{J}_p = \{n \in \mathbb{Z} \mid (n,p) = 1, n > 0\}$ and $m \ge 0$, then on the right hand side we have $\oplus_{d>0} \mathbf{W}_{v_p(d)}(\kappa) = \oplus_{n \in \mathrm{J}_p} (\oplus_m \mathbf{W}_{v_p(np^m)}(\kappa))$. The Frobenius morphism $\bar{\mathrm{F}}$ sends $\mathbf{W}_s^{(-d)}(\kappa)$ into $\mathbf{W}_s^{(-pd)}(\kappa)$, and, under the isomorphism $\mathbf{W}_s^{(-d)}(\kappa) \xrightarrow{\sim} \mathbf{W}_{\min(s, v_p(d))}(\kappa)$ (cf. Remark 3.3), it becomes the morphism $\bar{\mathrm{F}}\mathrm{V} : \mathbf{W}_{v_p(np^m)}(\kappa) \to \mathbf{W}_{v_p(np^{m+1})}(\kappa)$ as illustrated in the picture




Then

$$\begin{aligned}\mathbf{P}(\kappa) &\cong \oplus_{n \in \mathrm{J}_p} \left(\oplus_{m \ge 0} \mathbf{W}_{v_p(np^m)}(\kappa) / (\bar{\mathrm{F}}\mathrm{V} - 1)(\oplus_{m \ge 0} \mathbf{W}_{v_p(np^m)}(\kappa))\right) \\ &\cong \left(\oplus_{m \ge 0} \mathbf{W}_m(\kappa) / (\bar{\mathrm{F}}\mathrm{V} - 1)(\oplus_{m \ge 0} \mathbf{W}_m(\kappa))\right)^{(\mathrm{J}_p)}.\end{aligned} \tag{4.3}$$

One sees that $\oplus_{m\geq 0}\mathbf{W}_m(\kappa)/(\bar{\mathrm{F}}\mathrm{V}-1)(\oplus_{m\geq 0}\mathbf{W}_m(\kappa))$ is isomorphic to $\widetilde{\mathbf{CW}}(\kappa) = \varinjlim(\mathbf{W}_m(\kappa)\xrightarrow{\bar{\mathrm{F}}\mathrm{V}}\mathbf{W}_{m+1}(\kappa)\xrightarrow{\bar{\mathrm{F}}\mathrm{V}}\cdots)$. □

## 4.2 Equations killed by an abelian extension

### *4.2.1 Extension of the field of constants*

**Corollary 4.1** *The natural morphism*

$$M\mapsto M\otimes K^{\mathrm{alg}}:\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K)\to\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K^{\mathrm{alg}}})$$

*is a monomorphism. In other words, two $\mathcal{R}_K$-differential modules are isomorphic if and only if they are isomorphic over $\mathcal{R}_{K^{\mathrm{alg}}}$ after scalar extension.*

*Proof* We show that the kernel of $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K)\to\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K^{\mathrm{alg}}})$ is equal to 0. Let $M$ be defined by the operator $L=\partial_T-g(T)$, $g(T):=\sum_i a_iT^i\in\mathcal{R}_K$, and suppose that $M\otimes K^{\mathrm{alg}}$ is trivial over $\mathcal{R}_{K^{\mathrm{alg}}}$. By Corollary 1.2, we can suppose $a_i=0$, for all $i\neq -d,\ldots,0$. We know that $M\otimes K^{\mathrm{alg}}\xrightarrow{\sim}\mathrm{M}(a_0,\boldsymbol{f}^-(T))=\mathrm{M}(a_0,0)\otimes\mathrm{M}(0,\boldsymbol{f}^-(T))$, for a suitable $\boldsymbol{f}^-(T)\in\mathbf{W}_s(T^{-1}\mathcal{O}_{K_s}[T^{-1}])$. Then $M\otimes K^{\mathrm{alg}}$ is trivial only if both $\mathrm{M}(a_0,0)$ and $\mathrm{M}(0,\boldsymbol{f}^-(T))$ are trivial over $K^{\mathrm{alg}}$. This implies that $a_0\in\mathbb{Z}$, and hence $\mathrm{M}(a_0,0)$ is trivial also over $\mathcal{R}_K$. On the other hand, $\mathrm{M}(0,\boldsymbol{f}^-(T))$ is trivial if and only if $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ lies in $\mathcal{R}_{K^{\mathrm{alg}}}$. By Theorem 3.5, the series $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ has its coefficients in $K$, and $\mathrm{M}(0,\boldsymbol{f}^-(T))\in\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K)$. Since the convergence does not change by scalar extension of $K$, it follows that $\mathrm{M}(0,\boldsymbol{f}^-(T))$ is trivial over $\mathcal{R}_K$. □

**Corollary 4.2** *We have* $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K)=\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty})^{\mathrm{Gal}(K_\infty/K)}$.

### *4.2.2 Frobenius structure*

Assume now that $K$ has an absolute Frobenius $\sigma:K\to K$ (cf. Definition 1.10), and fix an absolute Frobenius $\varphi:\mathcal{R}_K\to\mathcal{R}_K$. By Theorem 3.2-(5), for any Artin–Schreier characters $\alpha$, the module $\mathrm{M}(0,\alpha)$ has a Frobenius structure of order 1 over $K_\infty$ (with respect to one, and hence any absolute Frobenius, cf. Sect. 1.2.3). By Corollary 4.1, this isomorphism descends to $K$.

**Lemma 4.2** $\mathrm{M}(a_0,0)$ *has a Frobenius structure of order $h$* (*cf. Definition* 1.11) *if and only if $a_0\in\mathbb{Z}_{(p)}$. Moreover let $a_0=a/b$, $a,b\in\mathbb{Z}$, and let $b=\prod_i q_i^{r_i}$ be the factorization of $b$ in positive prime numbers. For all $q,r\in\mathbb{Z}$, $r>0$, we define $[q]_r:=q^{q^{\cdot^{\cdot^{\cdot q}}}}$, r-times,(i.e. $[q]_1=q$ and $[q]_{r+1}=q^{[q]_r}$). Then $(\varphi^*)^h(\mathrm{M}(a_0,0))\xrightarrow{\sim}\mathrm{M}(a_0,0)$, with $h=\prod_i([q_i]_{r_i}-1)$.*

*Proof* By Sect. 1.2.3 we can suppose $\varphi=\varphi_\sigma$. Suppose that $\mathrm{M}(a_0,0)$ has a Frobenius structure of order $h$. Since $\mathrm{M}(a_0,0)$ is solvable (cf. Remark 1.5), hence $a_0\in\mathbb{Z}_p$. By Definition 1.11, $p^h{\cdot}a_0-a_0\in\mathbb{Z}$, hence $a_0\in\mathbb{Q}$. Conversely, let

$a_0 = a/b \in \mathbb{Z}_{(p)}, b > 0$. We have $p^{[q]_r - 1} \equiv 1 \pmod{q^r}$. Then if $h = \prod_i([q_i]_{r_i} - 1)$ we have $(p^h - 1)a_0 \in \mathbb{Z}$. □

*Remark* 4.1 Let $L = \partial_T + \sum_{i \in \mathbb{Z}} a_i T^i$, be an operator over $\mathcal{R}_K$ with Frobenius structure. The order $h$ of the Frobenius structure depends only on the exponent $a_0 \in \mathbb{Z}_{(p)}$. Explicitly, if $a_0 = a/b$, $a, b \in \mathbb{Z}$, $(b, p) = 1$, and if $b = \prod_i q_i^{r_i} > 0$, $q_i > 0$, is a factorization of $b$ in prime numbers, then, by Lemma 4.2, we have $h \leq \prod_i([q_i]_{r_i} - 1)$.

**Definition 4.1** *We denote by* $\mathrm{Pic}^{\mathrm{Frob}}(\mathcal{R}_{K_\infty}) \subseteq \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty})$ *the sub-group of differential modules having a Frobenius structure of some order.*

**Corollary 4.3** $\mathrm{Pic}^{\mathrm{Frob}}(\mathcal{R}_{K_\infty}) \cong \mathbb{Z}_{(p)}/\mathbb{Z} \oplus \mathbf{P}(k_\infty)$.

*4.2.3 Artin–Schreier extensions*

In order to apply Theorem 2.10, and Proposition 4.1 below, in this subsection $K$ has a discrete valuation, and $k$ will be perfect.

**Proposition 4.1** ([18, 3.4], [25, 2.2.2]) *Let* $\mathrm{F}/k((t))$ *be a finite separable extension. Let* $\mathcal{F}^\dagger$ *be the corresponding unramified extension of* $\mathcal{E}^\dagger_{K,T}$*. Then*

1. *There exist a finite unramified extension* $\widetilde{K}/K$*, a new variable* $\widetilde{T}$ *and an isometric isomorphism* $\tau : (\mathcal{F}^\dagger, |.|) \xrightarrow{\sim} (\mathcal{E}^\dagger_{\widetilde{K},\widetilde{T}}, |.|_{\widetilde{T},1})$*, where* $|.|_{\widetilde{T},1}$ *is the Gauss norm with respect to* $\widetilde{T}$*. In particular, for all* $f(T) \in \mathcal{E}^\dagger_{K,T}$*, one has* $|f(T)|_{T,1} = |f(T)|_{\widetilde{T},1}$*.*
2. *Let* $\tilde{t}$ *and* $t$ *be the reductions of* $\widetilde{T}$ *and* $T$ *respectively. Let* $\mathrm{F} = \tilde{k}((\tilde{t}))$*. Let* $r$ *be the ramification index of* $\mathrm{F}/k((t))$*. Write* $t = \bar{a}_r\tilde{t}^r + \bar{a}_{r+1}\tilde{t}^{r+1} + \cdots$*, with* $\bar{a}_i \in \tilde{k}$*. Then* $\widetilde{T}$ *can be chosen such that* $\tau(T) = a_r\widetilde{T}^r + a_{r+1}\widetilde{T}^{r+1} + \cdots$*,* $a_i \in \mathcal{O}_{\widetilde{K}}$*, where the* $a_i$*'s are liftings in* $\mathcal{O}_{\widetilde{K}}$ *of the* $\bar{a}_i$*'s.*

*Proof* Let $Q(\widetilde{T}) := a_r\widetilde{T}^r + a_{r+1}\widetilde{T}^{r+1} + \cdots$. The proof consists in showing that $f(T) \mapsto \tau(f(T)) := f(Q(\widetilde{T})) : \mathcal{E}^\dagger_{K,T} \to \mathcal{E}^\dagger_{\widetilde{K},\widetilde{T}}$ is étale (cf. [18, 3.4]). □

**Notation 4.1** We denote by $\mathcal{R}_{\widetilde{K},\widetilde{T}}$ the corresponding Robba ring.

*Remark* 4.2 We have $(\partial_{\widetilde{T}} \circ \tau)(f(T)) = \partial_{\widetilde{T},\log}(Q(\widetilde{T})) \cdot (\tau \circ \partial_T)(f(T))$, where as usual $\partial_{\widetilde{T},\log}(Q(\widetilde{T})) = \frac{\partial_{\widetilde{T}}(Q(\widetilde{T}))}{Q(\widetilde{T})}$. Then, after scalar extension, a generic differential operator $\partial_T - g(T)$ becomes $\partial_{\widetilde{T}} - \partial_{\widetilde{T},\log}(Q(\widetilde{T})) \cdot g(Q(\widetilde{T}))$. Indeed the unique $K_\infty$ derivation of the étale extension $\mathcal{R}_{K_\infty,\widetilde{T}}$ extending $\partial_T$ is $\partial_{\widetilde{T},\log}(Q(\widetilde{T}))^{-1} \cdot \partial_{\widetilde{T}}$. The solutions of this operator are the same as those of $\partial_T - g(T)$.

**Corollary 4.4** *Let* $\mathrm{E} = k((t))$*. Let* $\mathrm{F}/\mathrm{E}$ *be the Artin–Schreier extension defined by the kernel of* $\alpha = \delta(\overline{\boldsymbol{f}}(t))$*, with* $\overline{\boldsymbol{f}}(t) \in \mathbf{W}_s(\mathrm{E})$*. Let* $\mathcal{R}_{K,T} \to \mathcal{R}_{\widetilde{K},\widetilde{T}}$ *be the corresponding étale extension. Then the kernel of the scalar extension map*

$$\mathrm{Res} : \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty,T}) \longrightarrow \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{\widetilde{K}_\infty,\widetilde{T}}) \tag{4.4}$$

*is the (finite and cyclic) sub-group of* $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty,T})$*, formed by (isomorphism classes of) modules of the type* $\mathrm{M}(0,\boldsymbol{f}^-(T))^{\otimes k}$*,* $k \geq 0$*, (cf. Definition 3.5), where* $\boldsymbol{f}(T) \in \mathbf{W}_s(\mathcal{O}_K[[T]][T^{-1}])$ *is an arbitrary lifting of* $\overline{\boldsymbol{f}}(t)$*. This kernel has order* $[\mathrm{F}:\mathrm{E}]$.

*Proof* By Corollary 4.1, we can suppose $K = K^{\mathrm{alg}}$. We decompose $\overline{\boldsymbol{f}}(t) = \overline{\boldsymbol{f}^-}(t) + \overline{\boldsymbol{f}}_0 + \overline{\boldsymbol{f}^+}(t)$ (cf. Definition 3.5). Since $k = \tilde{k}$, we have $\delta(\overline{\boldsymbol{f}}_0) = 0$ (cf. Eq. (1.21)). On the other hand, by Proposition 3.1, we always have $\delta(\overline{\boldsymbol{f}^+}(t)) = 0$. Hence we can suppose $\overline{\boldsymbol{f}}(t) = \overline{\boldsymbol{f}^-}(t) = (\overline{f_0^-}(t), \ldots, \overline{f_s^-}(t))$. Since the Artin–Schreier complex is invariant by V (cf. Eq. (1.21)), we can suppose $\overline{f_0^-}(t) \neq 0$ (i.e. the degree $[\mathrm{F}:\mathrm{E}]$ is $p^{s+1}$). By Corollary 3.1, the morphism in the Eq. (4.4) can be viewed as a map

$$\mathbb{Z}_p/\mathbb{Z} \oplus \frac{\mathbf{CW}(t^{-1}k[t^{-1}])}{(\bar{\mathrm{F}}-1)\mathbf{CW}(t^{-1}k[t^{-1}])} \xrightarrow{\mathrm{Res}} \mathbb{Z}_p/\mathbb{Z} \oplus \frac{\mathbf{CW}(\tilde{t}^{-1}k[\tilde{t}^{-1}])}{(\bar{\mathrm{F}}-1)\mathbf{CW}(\tilde{t}^{-1}k[\tilde{t}^{-1}])}\,, \tag{4.5}$$

where $\tilde{t}$ is the reduction of $\widetilde{T}$. We start by studying the term $\mathbb{Z}_p/\mathbb{Z}$. By Proposition 4.1, $T = Q(\widetilde{T})$, with $Q(\widetilde{T}) = a_{p^{s+1}}\widetilde{T}^{p^{s+1}} + \cdots$, with $a_i \in \mathcal{O}_{K_\infty}$. The differential operator $\partial_T - a_0$, $a_0 \in \mathbb{Z}_p$ is sent to $\partial_{\widetilde{T}} - \partial_{\widetilde{T},\log}(Q(\widetilde{T})) \cdot a_0$. Observe that

$$\partial_{\widetilde{T},\log}(Q(T)) = p^{s+1} + Q_1(\widetilde{T}), \quad Q_1(\widetilde{T}) \in \widetilde{T} \cdot \mathcal{O}_K[[\widetilde{T}]]\,. \tag{4.6}$$

Hence the new operator is $\partial_{\widetilde{T}} - p^{s+1} \cdot a_0 - Q_1(\widetilde{T}) \cdot a_0$. By Proposition 1.2, this operator is isomorphic to $\partial_{\widetilde{T}} - p^{s+1}a_0$. Then the morphism in the Eq. (4.5) sends $\mathbb{Z}_p/\mathbb{Z}$ into itself by multiplication by $p^{s+1} = [\mathrm{F}:\mathrm{E}]$, and so is bijective on $\mathbb{Z}_p/\mathbb{Z}$.

On the co-vectors quotient, the morphism in the Eq. (4.5) is the usual functorial map corresponding to the inclusion $t^{-1}k[t^{-1}] \longrightarrow \tilde{t}^{-1}k[\tilde{t}^{-1}]$. The module $\mathrm{M}(0,\boldsymbol{f}(T)) \xrightarrow{\sim} \mathrm{M}(0,\boldsymbol{f}^-(T))$ then lies in the kernel. Indeed, by definition of F/E, there exists $\boldsymbol{\nu}(\tilde{t}) \in \mathbf{W}_s(\tilde{t}^{-1}k[\tilde{t}^{-1}])$ such that (cf. Remark 1.10)

$$\bar{\mathrm{F}}(\overline{\boldsymbol{\nu}}(\tilde{t})) - \overline{\boldsymbol{\nu}}(\tilde{t}) = \overline{\boldsymbol{f}^-}(t), \tag{4.7}$$

hence, by Theorem 3.2, $\mathrm{e}_{p^s}(\boldsymbol{f}(T),1)$ lies in $\mathcal{R}_{K,\widetilde{T}}$. In other words, this exponential is over-convergent in the new variable $\widetilde{T}$. Conversely, a module $\mathrm{M}(0,\boldsymbol{g}^-(T))$ lies in the kernel, if and only if the exponential $\mathrm{e}_{p^s}(\boldsymbol{g}^-(T),1)$ belongs to $\mathcal{R}_{K,\widetilde{T}}$. By Theorem 3.2, this happens if and only if the equation $\bar{\mathrm{F}}(\boldsymbol{\nu}) - \boldsymbol{\nu} = \overline{\boldsymbol{g}^-}(t)$ has a solution $\boldsymbol{\nu} \in \mathbf{W}_s(k((\tilde{t})))$. This happens if and only if the kernel of $\delta(\overline{\boldsymbol{g}^-}(t))$ contains the kernel of $\alpha = \delta(\overline{\boldsymbol{f}^-}(t))$. Since the quotient $\mathrm{G}_{\mathrm{E}}/\mathrm{Ker}(\delta(\overline{\boldsymbol{f}^-}(t)))$ is cyclic, this implies that $\delta(\overline{\boldsymbol{g}^-}(t)) = m \cdot \delta(\overline{\boldsymbol{f}^-}(t))$, for some $m \geq 0$. Hence $\mathrm{M}(0,\boldsymbol{g}^-(T)) \xrightarrow{\sim} \mathrm{M}(0,\boldsymbol{f}^-(T))^{\otimes m}$. □

*Remark* 4.3 As suggested by the referee, this corollary is in relation with the Proposition 4.11 of [11].

*4.2.4 Kummer extensions*

**Corollary 4.5** *Let* $\mathrm{F}/\mathrm{E}$ *be an abelian totally ramified extension of degree* $[\mathrm{F} : \mathrm{E}] = n$*, with* $(n, p) = 1$*. Let* $\mathcal{R}_{K,T} \mapsto \mathcal{R}_{K,\widetilde{T}}$ *be the corresponding étale extension. Then the scalar extension morphism* $\mathrm{Res} : \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty,T}) \longrightarrow \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty,\widetilde{T}})$ *is multiplication by n, and so its kernel is* $(\frac{1}{n}\mathbb{Z})/\mathbb{Z}$.

*Proof* Indeed, in this case we can choose $\widetilde{t}$ satisfying $t = Q(\tilde{t}) = \tilde{t}^n$. □

## 4.3 A criterion of solvability

This sub-section is devoted to proving the Corollary 4.6. The aim of this result is to characterize the solvability of the differential equation $\partial_T - g(T)$, with $g(T) = \sum a_i T^i$ giving an explicit condition on the coefficients "$a_i$". Roughly this Theorem shows that every solvable differential equation over $\mathcal{E}_K$ has, *without change of basis*, a solution which can be represented by the symbol

$$E(\boldsymbol{f}^-(T), 1) \cdot T^{a_0} \cdot E(\boldsymbol{f}^+(T), 1), \tag{4.8}$$

where $\boldsymbol{f}^-(T) \in \mathbf{W}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ and $\boldsymbol{f}^-(T) \in \mathbf{W}(T\mathcal{O}_K[[T]])$ are certain (*infinite*) Witt vectors, satisfying some convergence properties which ensure that the series $E(\boldsymbol{f}^-(T), 1)$ makes sense (cf. Lemma 4.3). Similarly to the previous situation, this Witt vector will be a sum of *monomials* (dual notion of $s$-co-monomial, cf. Definition 4.3). If a Lubin–Tate group $\mathfrak{G}_P$ is chosen, then this classification is a generalization of Theorem 3.1, because $\mathbf{W}(T^{-1}\mathcal{O}_{K_\infty}[[T^{-1}]])$ contains $\mathbf{CW}(T^{-1}\mathcal{O}_{K_\infty}[[T^{-1}]])$, via the choice of a generator $\boldsymbol{\pi} \in \mathrm{T}(\mathfrak{G}_P)$ (cf. diagram (4.15)), and the exponential $E(\boldsymbol{f}^-(T), 1)$ becomes $\mathrm{e}_{p^s}(-, 1)$ if applied to the image of a co-vector (cf. Eq. (4.14)).

We maintain the notations of Sect. 2.1. In the sequel we will work both with $T\mathcal{O}_K[[T]]$ and $T^{-1}\mathcal{O}_K[[T^{-1}]]$. Almost all assertions have a dual meaning.

**Lemma 4.3** *Let* $E(-, Y) : \mathbf{W}(\mathcal{O}_K[[T]]) \to 1 + Y\mathcal{O}_K[[T]][[Y]]$ *be the Artin Hasse exponential (cf. Definition* 2.3*). Let* $v_T$ *be the T-adic valuation. Let* $\boldsymbol{f}(T) = (f_0(T), f_1(T), \ldots) \in \mathbf{W}(T\mathcal{O}_K[[T]])$*, and let* $\phi_j(T)$ *be its j-th phantom component. If* $\lim_{j\to\infty} v_T(f_j(T)) = +\infty$*, then* $\lim_{j\to\infty} v_T(\phi_j(T)) = +\infty$*, and* $E(\boldsymbol{f}(T), Y)$ *converges T-adically at* $Y = 1$. □

**Definition 4.2** *We denote by* $\mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$ *the ideal of* $\mathbf{W}(\mathcal{O}_K[[T]])$ *formed by series satisfying the condition of Lemma* 4.3.

*Remark* 4.4 For all $\boldsymbol{f}^+(T) = (f_0^+(T), f_1^+(T), \ldots) \in \mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$, we have

$$E(\boldsymbol{f}^+(T), 1) := \prod_{j \geq 0} E(f_j^+(T)) = \exp\left(\phi_0^+(T) + \frac{\phi_1^+(T)}{p} + \frac{\phi_2^+(T)}{p^2} + \cdots\right), \tag{4.9}$$

where $\phi_j^+(T)$ is the $j$th phantom component of $\boldsymbol{f}^+(T)$. The $T$-adic convergence of this product is guaranteed by Lemma 4.3.

**Definition 4.3** (Monomials) *Let* $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots) \in \mathbf{W}(\mathcal{O}_K)$ *and* $d$ *a positive integer. We will call* $\boldsymbol{\lambda} T^d := (\lambda_0 T^d, \lambda_1 T^{dp}, \lambda_2 T^{dp^2}, \ldots) \in \mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$ *the* monomial *of degree* $d$ *relative to the Witt vector* $\boldsymbol{\lambda}$.[3] *In analogy with Definition* 3.3, *we call* $\mathbf{W}^{(d)}(\mathcal{O}_K)$ *the sub-group of* $\mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$, *formed by monomials of degree* $d$.

**Lemma 4.4** *Let* $\mathrm{J}_p := \{n \in \mathbb{Z} \mid (n,p) = 1, n > 0\}$. *There is an injection*

$$\prod_{n \in \mathrm{J}_p} \mathbf{W}^{(n)}(\mathcal{O}_K) \subset \mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]]) \,, \tag{4.10}$$

*given by* $(\boldsymbol{\lambda}_n T^n)_{n \in \mathrm{J}_p} \mapsto \sum_{n \in \mathrm{J}_p} \boldsymbol{\lambda}_n T^n$.

*Proof* If $\phi_n = (\phi_{n,0}, \phi_{n,1}, \ldots)$ is the phantom vector of $\boldsymbol{\lambda}_n$, then the phantom vector of $\boldsymbol{\lambda}_n T^n$ is $(\phi_{n,0} T^n, \phi_{n,1} T^{np}, \phi_{n,2} T^{np^2}, \ldots)$. Hence all terms have different degree and they do not "blend" when we sum the phantom components. □

*Remark* 4.5

1. Let $\boldsymbol{f}^+(T) \in \mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$, let $\boldsymbol{\lambda}, \boldsymbol{\lambda}_d \in \mathbf{W}(\mathcal{O}_K)$, $d > 0$. Then we have

$$E(\mathrm{V}(\boldsymbol{f}^+(T)), 1) = E(\boldsymbol{f}^+(T), 1), \tag{4.11}$$

$$E(\boldsymbol{\lambda}, T^d) = E(\boldsymbol{\lambda} T^d, 1), \tag{4.12}$$

$$\prod_{d \geq 1} E(\boldsymbol{\lambda}_d, T^d) = E\Big(\sum_{d \geq 1} \boldsymbol{\lambda}_d T^d, 1\Big). \tag{4.13}$$

2. If $\phi_{-n} = (\phi_{-n,0}, \phi_{-n,1}, \ldots)$ is the phantom vector of $\boldsymbol{\lambda}_{-n}$, then we have

$$E\left(\sum_{n \in \mathrm{J}_p} \boldsymbol{\lambda}_{-n} T^{-n}, 1\right) = \exp\left(\sum_{n \in \mathrm{J}_p} \sum_{m \geq 0} \phi_{-n,m} \frac{T^{-np^m}}{p^m}\right)$$

3. If $\boldsymbol{f}^-(T) = (f_0^-(T), f_1^-(T), \ldots) \in \mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ and if $\mathrm{pr}_m(\boldsymbol{f}^-(T))$ is the image of $\boldsymbol{f}^-(T)$ in $\mathbf{W}_m(T^{-1}\mathcal{O}_K[[T^{-1}]])$, then (cf. Definition 2.4)

$$E([\pi_m] \cdot \boldsymbol{f}^-(T), 1) = \mathrm{e}_{p^m}\big(\mathrm{pr}_m(\boldsymbol{f}^-(T)), 1\big) = \prod_{j=0}^{m} E_{m-j}(f_j^-(T)). \tag{4.14}$$

The exponentials used in the preceding section are then a particular case of $E(-, 1)$.

[3] Observe that if $\boldsymbol{\lambda} T^{-d}$ is a monomial in $\mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[[T^{-1}]])$, its reduction in $\mathbf{W}_m(T^{-1}\mathcal{O}_K[[T^{-1}]])$ is NOT a co-monomial of degree $-d$, but it is a co-monomial of degree $-dp^m$.

*Remark* 4.6 Recall that $\mathbf{W}_m(T^{-1}\mathcal{O}_{K_m}[T^{-1}]) \xrightarrow{\sim} [\pi_m]\mathbf{W}(T^{-1}\mathcal{O}_{K_m}[T^{-1}]) \subset \mathbf{W}(T^{-1}\mathcal{O}_{K_m}[[T^{-1}]])$ (see Eq. (2.9)). We have the following commutative diagram

$$\begin{array}{ccccc} \mathbf{W}_{m+1}(T^{-1}\mathcal{O}_{K_{m+1}}[T^{-1}]) & \xrightarrow[{[\pi_{m+1}]\cdot}]{} & \mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_{K_{m+1}}[[T^{-1}]]) & \xrightarrow{E(-,1)} & 1+T^{-1}\mathcal{O}_{K_\infty}[[T^{-1}]] \\ \uparrow \mathrm{V} & & \uparrow \mathrm{V} & \nearrow E(-,1) & \\ \mathbf{W}_m(T^{-1}\mathcal{O}_{K_m}[T^{-1}]) & \xrightarrow[{[\pi_m]\cdot}]{} & \mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_{K_m}[[T^{-1}]]) & & . \end{array} \tag{4.15}$$

Indeed, we see, looking at the phantom components, that

$$[\pi_m](f_0^-,\ldots,f_m^-,f_{m+1}^-,\ldots) = [\pi_m](f_0^-,\ldots,f_m^-,0,0,\ldots), \tag{4.16}$$

for all $\boldsymbol{f}^-(T) = (f_0^-,\ldots,f_m^-,f_{m+1}^-,\cdots) \in \mathbf{W}(T^{-1}\mathcal{O}_K[T^{-1}])$. Hence $[\pi_m]\boldsymbol{f}^-(T)$ lies in $\mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[T^{-1}])$, for all $\boldsymbol{f}^-(T) \in \mathbf{W}(T^{-1}\mathcal{O}_K[T^{-1}])$.

*Remark* 4.7 By definition one has $\mathbf{W}^{(d)}(\mathcal{O}_K) \subset \mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$, for all $d \geq 1$. The group $\mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$ is not generated by the family $\{\mathbf{W}^{(d)}(\mathcal{O}_K)\}_{d\geq 0}$ of subgroups. Indeed, for example, the $m$-th phantom component $\phi_m(T)$ of a Witt vector of the form $\sum_{d>0} \boldsymbol{\lambda}_d T^d$ is always of the type $\phi_m(T) = h(T^{p^m})$, for some $h(T) \in \mathcal{O}_K[[T]]$.

However, the basic fact is that, for all $\boldsymbol{f}^+(T) \in \mathbf{W}^{\downarrow}(T\mathcal{O}_K[[T]])$ (resp. $\boldsymbol{f}^-(T) \in \mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[[T^{-1}]])$), there exists an (infinite) family of monomials $\{\boldsymbol{\lambda}_n T^n\}_{n\in \mathrm{J}_p} \in \prod_{n\in\mathrm{J}_p} \mathbf{W}^{(n)}(\mathcal{O}_K)$ (resp. $\{\boldsymbol{\lambda}_{-n}T^{-n}\}_{n\in\mathrm{J}_p} \in \prod_{n\in\mathrm{J}_p} \mathbf{W}^{(-n)}(\mathcal{O}_K)$) satisfying

$$E(\boldsymbol{f}^+(T),1) = E\left(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_n T^n, 1\right); \quad E(\boldsymbol{f}^-(T),1) = E\left(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_{-n} T^{-n}, 1\right).$$

In other words, a general Witt vector is not an infinite sum of monomials, but the Artin–Hasse exponential of this Witt vector is always equal to the Artin–Hasse exponential of an infinite sum of monomials with support in $\mathrm{J}_p$ (cf. Lemma 4.5).

**Lemma 4.5** *The differential equation $\partial_T - g^+(T)$, $g^+(T) = \sum_{i\geq 1} a_i T^i \in \mathcal{R}_K$ is solvable if and only if there exists a family $\{\boldsymbol{\lambda}_n\}_{n\in\mathrm{J}_p}$, $\boldsymbol{\lambda}_n \in \mathbf{W}(\mathcal{O}_K)$, with phantom components $\phi_n = (\phi_{n,0},\phi_{n,1},\ldots)$ satisfying*

$$a_{np^m} = n\phi_{n,m}, \quad \text{for all } n \in \mathrm{J}_p, \quad m \geq 0\,. \tag{4.17}$$

*In other words we have* $\exp\left(\sum_{i\geq 1} a_i \frac{T^i}{i}\right) = E(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_n T^n, 1)$.

*Proof* The formal series $E\left(\sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_n T^n, 1\right) \in 1 + T\mathcal{O}_K[[T]]$ is a solution of the equation $L := \partial_T - \sum_{n\in\mathrm{J}_p}\sum_{m\geq 0} n\phi_{n,m}T^{np^m}$. Since this exponential converges in the unit disk, then $\mathrm{Ray}(L,\rho)=\rho$, for all $\rho<1$ and $L$ is solvable. Conversely if $\partial_T-g^+(T)$ is solvable, then the Witt vectors $\boldsymbol{\lambda}_n=(\lambda_{n,0},\lambda_{n,1},\dots)$ are defined by the relation (4.17) (cf. Lemma 1.5). For example for all $n\in\mathrm{J}_p$ we have

$$\lambda_{n,0}=\frac{a_n}{n},\quad \lambda_{n,1}=\frac{1}{p}\left(\frac{a_{np}}{n}-\left(\frac{a_n}{n}\right)^p\right). \tag{4.18}$$

We must show that $|\lambda_{n,m}|\leq 1$ for all $n\in\mathrm{J}_p$, $m\geq 0$.

*Step 1* By the Small Radius Lemma 1.1, we have $|a_i|\leq 1$, for all $i\geq 1$. Hence, for all $n\in\mathrm{J}_p$, we have $|\lambda_{n,0}|\leq 1$. Then the exponential

$$E\left(\sum_{n\in\mathrm{J}_p}(\lambda_{n,0},0,0,\dots)T^n,1\right)=\exp\left(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\lambda_{n,0}^{p^m}\frac{T^{p^m}}{p^m}\right)$$

converges in the unit disk and is a solution of the operator $Q^{(0)}:=\partial_T-h^{(0)}(T)$, with $h^{(0)}(T)=\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\lambda_{n,0}^{p^m}T^{p^m}$, which is therefore solvable.

*Step 2* The tensor product operator $\partial_T-(g^+(T)-h^{(0)}(T))$ is again solvable and satisfies $g^+(T)-h^{(0)}(T)=p\cdot g^{(1)}(T^p)$, for some $g^{(1)}(T)\in TK[[T]]$. In other words the "antecedent by ramification" $\varphi_p^*$ (cf. Sect. 1.2.4) of the equation $\partial_T-(g^+(T)-h^{(0)}(T))$ is given by $\partial_T-g^{(1)}(T)$, which is therefore solvable.

*Step 3* We observe that $g^{(1)}(T)=\frac{1}{p}\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}(a_{np^{m+1}}-n(\frac{a_n}{n})^{p^{m+1}})T^{np^m}$, and again by the Small Radius Lemma we have $|\frac{1}{p}||a_{np}-n(\frac{a_n}{n})^p|\leq 1$, which implies $|\lambda_{n,1}|\leq 1$. The process can be iterated indefinitely. □

*Remark* 4.8 We shall now consider the general case of an equation $\partial_T-g(T)$, with $g(T)=\sum_{i\in\mathbb{Z}}a_iT^i\in\mathcal{R}_K$, and get a criterion of solvability. Suppose that $\partial_T-g(T)$ is solvable. We know that $\partial_T-g^-(T)$, $\partial_T-a_0$ and $\partial_T-g^+(T)$ are all solvable (cf. Proposition 1.1). We can then consider $\partial_T-g^-(T)$ as an operator on $]1,\infty]$ (instead of $]1-\varepsilon,\infty]$) and Lemma 4.5 gives us the existence of a family of Witt vector $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ satisfying $a_{-np^m}=-n\phi_{-n,m}$, for all $n\in\mathrm{J}_p$, and all $m\geq 0$. Conversely suppose that we are given two families $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ and $\{\boldsymbol{\lambda}_n\}_{n\in\mathrm{J}_p}$, with $\boldsymbol{\lambda}_n\in\mathbf{W}(\mathcal{O}_K)$. Since the phantom components of $\boldsymbol{\lambda}_n$ are bounded by 1, then $g^+(T)$ is bounded and belongs to $\mathcal{R}_K$. Now we need a condition on the family $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ in order that the series $g^-(T):=\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}-n\phi_{-n,m}T^{-np^m}$ belongs to $\mathcal{R}_K$.

**Lemma 4.6** *Let $c\leq\omega=|p|^{\frac{1}{p-1}}$, $n\in\mathrm{J}_p$, $\rho\leq 1$ be fixed. Let $(\lambda_0,\lambda_1,\dots)\in\mathbf{W}(\mathcal{O}_K)$ and let $\phi=(\phi_0,\phi_1,\dots)$ be its phantom vector. Then $|\phi_i/p^i|\leq c\rho^{np^i}$ for all $i\geq 0$ if and only if $|\lambda_i|\leq c\rho^{np^i}$ for all $i\geq 0$.*

*Proof* Recall that $c^{p^i}\leq|p|^ic$, for all $i\geq 0$. Suppose that $|\phi_i/p^i|\leq c\rho^{np^i}$ for all $i\geq 0$. Then $|\lambda_0|=|\phi_0|\leq c\rho^n$. By induction suppose that $|\lambda_j|\leq c\rho^{np^j}$ for all

$j=0,\dots,i-1$, then $|\lambda_i|=|\frac{1}{p^i}(\phi_i-\lambda_0^{p^i}-p\lambda_1^{p^{i-1}}-\dots-p^{i-1}\lambda_{i-1}^{p})|$. By induction $|\phi_i|\leq|p|^i c\rho^{np^i}$ and $|p^k\lambda_k^{p^{i-k}}|\leq|p|^k(c\rho^{np^k})^{p^{i-k}}=|p|^k c^{p^{i-k}}\rho^{np^i}\leq|p^i|c\rho^{np^i}$, hence $|\lambda_i|\leq c\rho^{np^i}$. Conversely suppose that $|\lambda_i|\leq c\rho^{np^i}$ for all $i\geq 0$. Then $|\phi_i|=|\lambda_0^{p^i}+p\lambda_1^{p^{i-1}}+\dots+p^i\lambda_i|\leq\sup((c\rho^n)^{p^i},|p|(c\rho^{np})^{p^{i-1}},\dots,|p|^i(c\rho^{np^i}))\leq|p|^i c\rho^{np^i}$. □

**Definition 4.4** *Let $c\leq\omega$ and $\rho\leq 1$. We denote by*

$$\mathbf{W}^{\downarrow}_{c,\rho}(T^{-1}\mathcal{O}_K[[T^{-1}]])\subset\prod_{n\in\mathrm{J}_p}\mathbf{W}^{(-n)}(\mathcal{O}_K)\overset{\text{Lemma 4.4}}{\subset}\mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[[T^{-1}]])$$

*the sub-group formed by the sums $\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n}$ such that*

$$\boldsymbol{\lambda}_{-n}=(\lambda_{-n,0},\lambda_{-n,1},\dots)\in\mathbf{W}(\mathcal{O}_K)$$

*verifies the conditions of Lemma 4.6. In other words $|\lambda_{-n,m}|\leq c\rho^{np^m}$.*

*Remark* 4.9 Observe that, by Lemma 4.6, a Witt vector $\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n}$ belongs to the subgroup $\mathbf{W}^{\downarrow}_{c,\rho}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ if and only if the argument of the exponential $E(\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n},1)=\exp(\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{-n,m}\frac{T^{-np^m}}{p^m})$ satisfies

$$\left|\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{-n,m}\frac{T^{-np^m}}{p^m}\right|_{\rho}:=\sup_{n\in\mathrm{J}_p,m\geq 0}\left(\frac{|\phi_{-n,m}|}{|p|^m}\rho^{-np^m}\right)\leq c\leq\omega. \tag{4.19}$$

**Definition 4.5** *Let $\mathbf{W}^{\dagger}(T^{-1}\mathcal{O}_K[[T^{-1}]])\subset\mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ be the subgroup defined as the sum of the sub-group $\bigcup_{c<\omega,\rho<1}\mathbf{W}^{\downarrow}_{c,\rho}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ with the sub-group $\left(\bigcup_{j\geq 0}[\pi_j]\cdot\mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_{K^{\mathrm{alg}}}[T^{-1}])\right)\bigcap\mathbf{W}^{\downarrow}(T^{-1}\mathcal{O}_K[T^{-1}])$ .*

**Corollary 4.6** (Solvability criterion) *Let $\partial_T-g(T)$, $g(T):=\sum_{i\in\mathbb{Z}}a_iT^i\in\mathcal{R}_K$ be a solvable equation. Then $a_0\in\mathbb{Z}_p$ and there exist two families $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ and $\{\boldsymbol{\lambda}_n\}_{n\in\mathrm{J}_p}$, with $\boldsymbol{\lambda}_{-n},\boldsymbol{\lambda}_n\in\mathbf{W}(\mathcal{O}_K)$ such that for all $n\in\mathrm{J}_p$, and all $m\geq 0$ we have*

$$a_{-np^m}=-n\phi_{-n,m}\,;\quad a_{np^m}=n\phi_{n,m}\ ,$$

*where $(\phi_{-n,0},\phi_{-n,1},\dots)$ (resp. $(\phi_{n,0},\phi_{n,1},\dots)$) is the phantom vector of $\boldsymbol{\lambda}_{-n}$ (resp. $\boldsymbol{\lambda}_n$). Moreover $\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n}$ belongs to $\mathbf{W}^{\dagger}(T^{-1}\mathcal{O}_K[[T^{-1}]])$.*

*Conversely given a triplet $(\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n},a_0,\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_nT^n)$, with*

$$\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n}\in\mathbf{W}^{\dagger}(T^{-1}\mathcal{O}_K[[T^{-1}]]),\quad a_0\in\mathbb{Z}_p,\quad\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_nT^n\in\mathbf{W}(T\mathcal{O}_K[[T]]),$$

*then*

$$g(T) := \sum_{n\in \mathrm{J}_p}\sum_{m\geq 0} -n\phi_{-n,m}T^{-np^m} + a_0 + \sum_{n\in \mathrm{J}_p}\sum_{m\geq 0} n\phi_{n,m}T^{np^m}$$

*belongs to* $\mathcal{R}_K$*, and the equation* $\partial_T - g(T)$ *is solvable.*

*Remark* 4.10 This corollary asserts that $\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n}$ is a sum of a "small" vector, i.e. verifying the relation (4.19), and a vector of "type Robba", i.e. of the type $[\pi_j]\boldsymbol{f}^-(T), \boldsymbol{f}^-(T) \in \mathbf{W}(T^{-1}\mathcal{O}_{K^{\mathrm{alg}}}[[T^{-1}]])$, for some $j \geq 0$ and such that the product $[\pi_j]\boldsymbol{f}^-(T)$ lies in $\mathbf{W}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ i.e. has its coefficients in $K$. Actually the proof will show that $\boldsymbol{f}^-(T)$ can be chosen pure (see below).

*Proof of Corollary 4.6* Let $\partial_T - g(T)$ be solvable. By Lemma 4.5, we know the existence of $\{\boldsymbol{\lambda}_{-n}\}_{n\in\mathrm{J}_p}$ and $\{\boldsymbol{\lambda}_n\}_{n\in\mathrm{J}_p}$ (cf. Remark 4.8). We must show that $\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}_{-n}T^{-n}$ lies in $\mathbf{W}^\dagger(T^{-1}\mathcal{O}_K[[T^{-1}]])$. Let $d > 0$ be such that $|\sum_{i<-d} a_iT^i/i|_\rho < \omega$ for some $\rho < 1$ (cf. Lemma 1.2). Write $g^-(T) = \sum_{i<-d} a_iT^i + \sum_{-d\leq i\leq -1} a_iT^i$. By Lemma 1.2 we know that $\exp(\sum_{i<-d} a_iT^i/i) \in \mathcal{R}_K$, hence the equation $\partial_T - \sum_{i<-d} a_iT^i$ is solvable (and actually trivial). In particular $\partial_T - \sum_{-d\leq i\leq -1} a_iT^i$ is solvable and hence, again by Remark 4.8, there exists a family $\{\boldsymbol{\lambda}'_{-n}\}_{n\in\mathrm{J}_p}$, such that

$$-n\phi'_{-n,m} = \begin{cases} a_{-np^m} \text{ if } & -np^m < -d, \\ 0 \text{ if } & -d \leq -np^m \leq -1, \end{cases}$$

where $(\phi_{-n,0}, \phi_{-n,1}, \ldots)$ is the phantom vector of $\boldsymbol{\lambda}_{-n}$. Since, by construction $|\sum_{i<-d} a_iT^i/i|_\rho < \omega$, this implies $|\sum_{n\in\mathrm{J}_p}\sum_{m\geq 0}\phi_{-n,m}T^{-np^m}/p^m|_\rho < \omega$, hence $\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}'_{-n}T^{-n}$ lies in $\mathbf{W}^{\downarrow}_{c,\rho}(T^{-1}\mathcal{O}_K[[T^{-1}]])$ for some $c < \omega$. Now we consider $\boldsymbol{\lambda}''_{-n} := \boldsymbol{\lambda}_n - \boldsymbol{\lambda}'_n$, the family $\{\boldsymbol{\lambda}''_{-n}\}_{n\in\mathrm{J}_p}$ then satisfies

$$-n\phi''_{-n,m} = \begin{cases} 0 \text{ if } & -np^m < -d, \\ a_{-np^m} \text{ if } & -d \leq -np^m \leq -1. \end{cases}$$

By Theorem 3.5, and by Lemma 3.6 there exists a *pure* Witt vector $\boldsymbol{f}^-(T) = (f_0^-(T), \ldots, f_s^-(T)) \in \mathbf{W}_s(T^{-1}\mathcal{O}_{K^{\mathrm{alg}}}[T^{-1}])$ such that $\mathrm{L}(0, \boldsymbol{f}^-(T)) = \partial_T - \sum_{-d\leq i\leq -1} a_iT^i$. Hence $[\pi_s]\boldsymbol{f}^-(T)$ and $\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}''_{-n}T^{-n}$ have the "same" phantom vector because they are both pure. Then $\sum_{n\in\mathrm{J}_p}\boldsymbol{\lambda}''_nT^{-n}$ lies in the image of the morphism $\mathbf{W}_s(T^{-1}\mathcal{O}_{K^{\mathrm{alg}}}[T^{-1}]) \xrightarrow{\sim} [\pi_s]\cdot\mathbf{W}(T^{-1}\mathcal{O}_{K^{\mathrm{alg}}}[T^{-1}])$. □

*Remark* 4.11 Let $L := \partial_T - \sum_{i\in\mathbb{Z}} a_iT^i, g(T) \in \mathcal{R}_K$ be a given equation. Then $L$ is solvable if and only if $a_0 \in \mathbb{Z}_p$, and, for all $n \in \mathrm{J}_p$, both the operators $\partial_T - \sum_{m\geq 0} a_{np^m}T^{np^m}$ and $\partial_T - \sum_{m\geq 0} a_{-np^m}T^{-np^m}$ are solvable.

**Corollary 4.7** *If $K$ is unramified over $\mathbb{Q}_p$, then every solvable differential module over $\mathcal{R}_K$ of rank one is isomorphic to a moderate module* (*cf. Sect.* 1.2.5). *In other words,*

$$\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_K) = \begin{cases} \mathbb{Z}_p/\mathbb{Z} & \text{if } p \neq 2, \\ \mathbb{Z}_p/\mathbb{Z} \oplus k((t))/(\bar{F}-1)(k((t))) & \text{if } p = 2. \end{cases} \tag{4.20}$$

*Proof* We must show that all $\boldsymbol{\pi}$-exponential $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)$ whose logarithmic derivative has its coefficients in $K$ is trivial. Actually we can suppose that the co-monomial $\boldsymbol{f}^-(T)$ is *pure* (cf. Lemma 3.6). Write $\partial_{T,\log}(\mathrm{e}_{p^s}(\boldsymbol{f}^-(T),1)) = \sum_{-d\leq i\leq -1} a_i T^i$ with $a_i \in \mathcal{O}_K$ for all $i = -d,\ldots,-1$. Write $\boldsymbol{f}^-(T) = \sum_{n\in \mathrm{J}_p} \boldsymbol{\lambda}_{-np^{m(n)}} T^{-np^{m(n)}}, \boldsymbol{\lambda}_{-np^{m(n)}} = (\lambda_{-np^{m(n)},0},\ldots,\lambda_{-np^{m(n)},m(n)}) \in \mathbf{W}_{m(n)}(\mathcal{O}_{K^{\mathrm{alg}}})$. Since $\boldsymbol{f}^-(T)$ is pure, one has (cf. Remark 3.7)

$$a_{-np^j} = -n\pi_{m(n)-j}\phi_{-np^{m(n)},j}, \tag{4.21}$$

for all $j = 0,\ldots,m(n)$, where $\langle \phi_{-np^{m(n)},0},\ldots,\phi_{-np^{m(n)},m(n)}\rangle$ is the phantom vector of $\boldsymbol{\lambda}_{-np^{m(n)}}$. On the other hand the criterion of solvability (Corollary 4.6) asserts the existence of a family $\{\boldsymbol{\lambda}'_{-n}\}_{n\in\mathrm{J}_p}$ with phantom vectors $\{\boldsymbol{\phi}'_{-n}\}_{n\in\mathrm{J}_p}$, with $\boldsymbol{\phi}'_{-n} := \langle \phi'_{1-n,0}, \phi'_{1-n,1},\ldots\rangle$, such that $a_{-np^m} = -n\phi'_{-n,m}$ for all $n \in \mathrm{J}_p$, $m \geq 0$. Observe that $\phi'_{-n,m} \in \mathcal{O}_K$. Since $K$ is unramified over $\mathbb{Q}_p$, then we can employ Lemma 1.6. Then $\phi'_{-n,m} \equiv \sigma(\phi'_{-n,m-1}) \mod p^m\mathcal{O}_K$ for all $n \in \mathrm{J}_p$, $m \geq 0$, that is

$$a_{-np^j} \equiv \sigma(a_{-np^{j-1}}) \qquad \mathrm{mod}\ p^j\mathcal{O}_K \quad \text{for all } j \geq 0. \tag{4.22}$$

Since $a_{-np^{m(n)+1}} = 0$ we obtain, by equation (4.21), the estimate $|\pi_{m(n)-j}\phi_{-np^{m(n)},j}| \leq |p|^{j+1}$, for all $j = 0,\ldots,m(n)$. Then we have a system of conditions

$$\left|\lambda^{p^j}_{-np^{m(n)},0} + p\lambda^{p^{j-1}}_{-np^{m(n)},1} + \cdots + p^j\lambda_{-np^{m(n)},j}\right| \leq |p|^{j+1}|\pi_{m(n)-j}|^{-1},$$

which easily gives $|\lambda_{-np^{m(n)},j}| \leq |p||\pi_{m(n)-j}|^{-1}$, this last is $\leq 1$, and is $= 1$ if and only if $p = 2$, and $m(n) = j = 0$. If $p \neq 2$, then by Theorem 2.2, and Corollary 4.1, $\mathrm{e}_d(\boldsymbol{\lambda}_{-np^{m(n)}}, T^{-1})$ lies in $\mathcal{R}_K$, for all $n \in \mathrm{J}_p$, and $\mathrm{L}_d(0,\boldsymbol{f}^-(T))$ is trivial. □

## 4.4 Explicit computation of the irregularity in some cases

Let $v_t$ be the $t$-adic valuation of $k((t))$.

**Lemma 4.7** *Let $f(T) \in T^{-1}\mathcal{O}_K[T^{-1}]$ be a polynomial in $T^{-1}$, and let $\bar{f}(t) \in t^{-1}k[t^{-1}]$ be the reduction of $f(T)$. Let $n := -v_t(\bar{f}(t)) > 0$. If $(n,p) = 1$, then*

$$\mathrm{Irr}\Big(\mathrm{M}\Big(0,\Big(\underbrace{0,\ldots,0,f(T),0,\ldots,0}_{\ell+1}\Big)\Big)\Big)=n\cdot p^{\ell},\tag{4.23}$$

*where* $\ell=\ell(0,\ldots,0,f(T),0,\ldots,0)$ (*cf. Sect.* 1.3.3).

*Proof* We have $\mathrm{M}\big(0,(0,\ldots,0,f(T),0,\ldots,0)\big)=\mathrm{M}(0,(f(T),0,\ldots,0))$ . (cf. Eq. (3.12)). Moreover, the isomorphism class of this module depends only on $\overline{f}(t)$, hence we can suppose that $f(T)=a_{-n}T^{-n}+\cdots+a_{-1}T^{-1}$, with $|a_{-n}|=1$. Then

$$L(0,(f(T),0,\ldots,0))=\partial_T+\partial_{T,\log}(f(T))\\ \cdot\Big[\pi_s f(T)+\pi_{s-1}f(T)^p+\cdots+\pi_0 f(T)^{p^s}\Big].$$

We have $\partial_{T,\log}(f(T))=-n+TQ(T)$, with $Q(T)\in\mathcal{O}_K[[T]]$, so that

$$g(T)=-\pi_0\cdot n\cdot a_{-n}^{p^{\ell}}\cdot T^{-n\cdot p^{\ell}}+(\text{terms of degree}>-np^{\ell})\ .\tag{4.24}$$

Since $(n,p)=1$, we can apply Lemma 1.3 and $\mathrm{Irr}(\partial_T+g(T))=\mathrm{Irr}_F(\partial_T+g(T))=np^{\ell}$. □

**Corollary 4.8** *Let* $\overline{\boldsymbol{f}^-}(t)=(\overline{f_0^-},\ldots,\overline{f_s^-})\in\mathbf{W}_s(t^{-1}k[t^{-1}])$. *Let* $n_j:=-v_t\left(\overline{f_j^-}\right)$. *If* $(n_j,p)=1$, *or* $n_j=0$, *for all* $j=0,\ldots,s$, *then*

$$\mathrm{Irr}\left(\mathrm{M}(0,\overline{\boldsymbol{f}^-}(t))\right)=\max_{0\le j\le s}\left(n_j\cdot p^{s-j}\right).\tag{4.25}$$

*Proof* Let $M_j$ be the differential module defined by $(0,\ldots,0,\overline{f_j^-}(t),0,\ldots,0)$. By Lemma 4.7, $\mathrm{Irr}(M_j)=n_jp^{s-j}$. Since $\mathrm{M}(0,\boldsymbol{f}^-(T))=\otimes_j M_j$ (cf. Eq. (2.54)), and since $n_jp^{s-j}$ are all different, then, by Corollary 1.1, we have the desired conclusion. □

### 4.5 Tannakian group

In this section we study the category of solvable differential modules over $\mathcal{R}_K$ which are extensions of rank one sub objects. We remove the hypothesis "$K$ is spherically complete", present in the literature. Let $H/K$ be an arbitrary algebraic extension. We set $\mathcal{H}_H^{\dagger}:=\cup_{\varepsilon}\mathcal{A}_H(]1-\varepsilon,\infty[)$. Let $S$ be a sub-group of $\mathbb{Z}_p$ without Liouville numbers and containing $\mathbb{Z}$.

**Definition 4.6** *Let* $\mathcal{C}$ *be an additive category. If there exists a function* rank *on* $\mathcal{C}$, *then we denote by* $\mathcal{C}_{\oplus\text{-}1}$ *(resp.* $\mathcal{C}_{\text{ext-}1}$*) the full sub-category of* $\mathcal{C}$ *whose objects are finite direct sum (resp. finite successive extension) of rank one objects.*

**Definition 4.7** *An object is said to be* simple *if it has no non trivial sub-objects. It is said* indecomposable *if it is not a direct sum of non trivial objects.*

**Definition 4.8** *Let* $\mathrm{MLS}(\mathcal{H}_H^\dagger)$ *be the category of (free) differential modules over* $\mathcal{H}_H^\dagger$ *solvable at* 1 (*i.e.* $\mathrm{Ray}(N,1) = 1$, *cf. Sect.* 1.2.2). *Recall that, by definition, such a module comes, by scalar extension, from a module over* $\mathcal{H}_L^\dagger$, *for some finite extension* $L/K$ (*cf. Eq.* (1.2)). *Let* $N \in \mathrm{MLS}_{\text{ext-1}}(\mathcal{H}_H^\dagger)$ *be extension of rank one modules, say* $\{N_i\}_{i=1,\dots,k}$. *We will say that N is regular at* $\infty$, *write* $N \in \mathrm{MLS}^{\mathrm{reg}}_{\text{ext-1}}(\mathcal{H}_H^\dagger)$, *if, for all i, the module* $N_i$ *is defined, in some basis, by an operator* $\partial_T + g_i(T)$, *satisfying*

$$g_i(T) = \sum a_{i,j} T^j \ , \quad \text{with } a_{i,j} = 0 \ , \quad \text{for all } j \geq 1 \ . \tag{4.26}$$

*We will say that* $N \in \mathrm{MLS}^{\mathrm{reg}}_{\text{ext-1}}(\mathcal{H}_H^\dagger, S)$, *if* $N \in \mathrm{MLS}^{\mathrm{reg}}_{\text{ext-1}}(\mathcal{H}_H^\dagger)$, *and if* $a_{i,0} \in S, \forall i$.

**Lemma 4.8** (Schur's lemma) *Let* $M_1, M_2$ *be two rank one objects in* $\mathrm{MLS}(\mathcal{R}_H)$ (*resp.* $\mathrm{MLS}^{\mathrm{reg}}_{\text{ext-1}}(\mathcal{H}_H^\dagger)$). *Every non zero morphism* $\varrho : M_1 \to M_2$ *is an isomorphism.*

*Proof* Let $a_{0,i} \in \mathbb{Z}_p$ be the exponent of $M_i$. In Theorem 3.5 we have seen that $M_i$ has a basis $\mathrm{e}_i \in M_i$ in which the solution is of the type $T^{a_{0,i}}\epsilon_i(T)$, where $\epsilon_i(T) \in \mathcal{A}_H(]1,\infty])$ is a series with coefficients in $H$. We have then $\varrho(\mathbf{e}_1) = h(T)\mathbf{e}_2$, with $h(T) = T^{a_{0,2}-a_{0,1}}\epsilon_2(T)\epsilon_1(T)^{-1} \in \mathcal{R}_H$. Then $a_{0,2} - a_{0,1} \in \mathbb{Z}$, and $\epsilon_2(T)\epsilon_1(T)^{-1} \in \mathcal{R}_H$. Since $\epsilon_i(T)$ is a product of $\boldsymbol{\pi}$-exponentials, both $h(T)$ and its inverse lie in $\mathcal{R}_H$. If $M_1, M_2 \in \mathrm{MLS}^{\mathrm{reg}}_{\oplus\text{-1}}(\mathcal{H}_H^\dagger)$, then, by the proof of Lemma 1.2, the base change necessary to obtain $\mathbf{e}_i$ lies in $(\mathcal{H}_H^\dagger)^\times$. □

*Remark* 4.12 By Lemma 4.8, rank one objects in $\mathrm{MLS}(\mathcal{R}_H)$ are simple in Mod-$\mathcal{R}_H[\partial_T]$. Then, by the Jordan–Hölder Theorem in Mod-$\mathcal{R}_H[\partial_T]$, the categories $\mathrm{MLS}_{\text{ext-1}}(\mathcal{R}_H)$, and $\mathrm{MLS}_{\text{ext-1}}(\mathcal{R}_H, S)$ are abelian, and, for all objects $M$, the set of rank one objects appearing in a decomposition series of $M$ does not depend, up to the order, on the chosen decomposition. Moreover the sub-categories $\mathrm{MLS}_{\oplus\text{-1}}(\mathcal{R}_H)$ and $\mathrm{MLS}_{\oplus\text{-1}}(\mathcal{R}_H, S)$ are abelian and semi-simple. The same facts are true for $\mathrm{MLS}^{\mathrm{reg}}_{\text{ext-1}}(\mathcal{H}_H^\dagger)$, $\mathrm{MLS}^{\mathrm{reg}}_{\text{ext-1}}(\mathcal{H}_H^\dagger)$ and $\mathrm{MLS}^{\mathrm{reg}}_{\oplus\text{-1}}(\mathcal{H}_H^\dagger, S)$.

**Theorem 4.2** *Let* $N \in \mathrm{MLS}_{\text{ext-1}}(\mathcal{R}_H, S)$ (*resp.* $N \in \mathrm{MLS}^{\mathrm{reg}}_{\text{ext-1}}(\mathcal{H}_H^\dagger, S)$). *There exists a basis of N in which the matrix of the derivation is in Jordan canonical form. In other words, N is a direct sum of objects of the form* $M \otimes U_m$, *where M is a rank one object and* $U_m$ *is defined by the operator* $\partial_T^m$.

*Proof* Let $M_1, M_2 \in \mathrm{MLS}_{\text{ext-1}}(\mathcal{R}_H)$. By the Robba's index Theorem for rank one operators whose matrix is a rational fraction [21], we have $\dim \mathrm{Hom}(M_1, M_2) = \dim \mathrm{Ext}^1_{\mathcal{R}_K[\partial_T]}(M_1, M_2)$. This fact does not need the "spherically complete" hypothesis on the field $K$. The Theorem results then by classical considerations. □

*Remark* 4.13 $\partial_T : \mathcal{E}_K^\dagger \to \mathcal{E}_K^\dagger$ has a big co-kernel, hence $\mathrm{Ext}^1_{\mathcal{E}_K^\dagger[\partial_T]}(\mathcal{E}_K^\dagger, \mathcal{E}_K^\dagger)$ is not one dimensional (see [9, pp. 133–134]). While the theory of rank one equation over $\mathcal{R}_K$ coincide with the theory over $\mathcal{E}_K^\dagger$, this is false for rank $\geq 2$.

**Theorem 4.3** (Canonical extension) *The canonical restriction functor* Res : $\mathrm{MLS}^{\mathrm{reg}}_{\text{ext-1}}(\mathcal{H}^{\dagger}_{H},S) \to \mathrm{MLS}_{\text{ext-1}}(\mathcal{R}_H,S)$ *is an equivalence.*

*Proof* By Theorem 3.1, Res : $\mathrm{MLS}^{\mathrm{reg}}_{\text{ext-1}}(\mathcal{H}^{\dagger}_{H},S) \to \mathrm{MLS}_{\text{ext-1}}(\mathcal{R}_H,S)$ is essentially surjective. Indeed $\mathrm{L}(a_0,\boldsymbol{f}^-(T))$ has its coefficients in $\mathcal{H}^{\dagger}_{H}$. By Lemma 4.8, two rank one modules in $\mathrm{MLS}^{\mathrm{reg}}_{\text{ext-1}}(\mathcal{H}^{\dagger}_{H},S)$ are isomorphic if and only if they are isomorphic over $\mathcal{R}_H$, because the base change is given by an over-convergent exponential in $1+T^{-1}\mathcal{O}_H[[T^{-1}]]$. Hence, by the Schur Lemma 4.8, Res is also fully-faithful. □

**Corollary 4.9** *The Tannakian category* $\mathrm{MLS}_{\text{ext-1}}(\mathcal{R}_H,S)$ *is neutral.*

*Proof* Let $\omega_S : \mathrm{MLS}^{\mathrm{reg}}_{\oplus\text{-1}}(\mathcal{H}^{\dagger}_{H},S) \to \underline{\mathrm{Vect}}^{\mathrm{fin}}(H)$ be the fiber functor sending a rank one object in the vector space of its Taylor solutions at 1 (cf. Eq. (1.6)). An $H$-linear fiber functor of $\mathrm{MLS}_{\oplus\text{-1}}(\mathcal{R}_H,S)$ is given by composing $\omega_S$ with a quasi-inverse of Res. □

**Definition 4.9** *An affine group scheme* $\mathcal{H}$ (*over* $H$) *is* linear *if there exists a closed immersion* $\mathcal{H}\to \mathrm{GL}_H(V)$*, for some finite dimensional vector space* $V$.

**Definition 4.10** *Let* $\omega_S : \mathrm{MLS}_{\oplus\text{-1}}(\mathcal{R}_H,S) \to \underline{\mathrm{Vect}}^{\mathrm{fin}}(H)$ *be a fiber functor. We denote by* $\mathcal{G}_H := \mathrm{Aut}^{\otimes}(\omega_S)$ *the Tannakian group of* $\mathrm{MLS}_{\oplus\text{-1}}(\mathcal{R}_H,S)$.

*Remark* 4.14 By Theorem 4.2, the Tannakian group of $\mathrm{MLS}_{\text{ext-1}}(\mathcal{R}_H,S)$ is $\mathcal{G}_H\times\mathbb{G}_a$.

### *4.5.1 Study of* $\mathcal{G}_H$

For all (finite dimensional) representations $\rho_V : \mathcal{G}_H \to \mathrm{GL}_H(V)$, we set $\mathcal{G}_{H,V} := \rho_V(\mathcal{G}_H)$. The group $\mathcal{G}_{H,V}$ is then linear and affine. Moreover, $\mathcal{G}_{H,V}$ is diagonalizable (i.e. closed subgroup of the group of diagonal matrices). The group $\mathcal{G}_H$ is the inverse limit of its linear (compact) quotients $\mathcal{G}_{H,V}$, and is endowed with the limit topology. Hence $\mathcal{G}_H$ is abelian, because every $V$ is a direct sum of rank one objects, and $\mathcal{G}_{H,V}$ is abelian.

*Remark* 4.15 Let $I$ be a non empty directed set. The functor $\varinjlim_{i\in I}$ is exact if applied to exact sequences of compact algebraic groups (see [4, Chap.3, Sect. 7, Corollary.1]). All exact sequences in the sequel will be studied at level $\mathcal{G}_{H,V}$.

**Definition 4.11** *We set* $\mathbf{X}(\mathcal{G}_H) := \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{gr}}(\mathcal{G}_H,\mathbb{G}_m\otimes H)$*, where "cont" means that such a morphism* $\mathcal{G}_H\to\mathbb{G}_m\otimes H$ *factors on a linear quotient* $\mathcal{G}_{H,V}$.

Let $\mathrm{Pic}^{\mathrm{sol}}_S(\mathcal{R}_{H_\infty})$ be the sup-group of $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{H_\infty})$ formed by modules whose residue lies in $S$. By Tannakian equivalence, we have an isomorphism of groups

$$\mathbf{X}(\mathcal{G}_{H_\infty}) \cong \mathrm{Pic}^{\mathrm{sol}}_S(\mathcal{R}_{H_\infty}) \xrightarrow{\sim} S/\mathbb{Z}\oplus \mathbf{P}(k_{H_\infty}). \tag{4.27}$$

This leads us to recover the group $\mathcal{G}_{H_\infty}$ itself (cf. [24, 3.2.6]). Let us write

$$S/\mathbb{Z} = S/\mathbb{Z}_{(p)}\oplus\mathbb{Z}_{(p)}/\mathbb{Z}\,. \tag{4.28}$$

**Theorem 4.4** *$\mathcal{G}_{H_\infty}$ is the product of a torus $\mathbf{T}_{H_\infty}$ (dual of $S/\mathbb{Z}_{(p)}$) with a profinite group $\mathcal{I}_{H_\infty}$ (dual of $\mathbb{Z}_{(p)}/\mathbb{Z}\oplus\mathbf{P}(k_{H_\infty})$). This last is isomorphic to the Galois group $\mathcal{I}^{\mathrm{ab}}_{E_{H_\infty}} := \mathrm{Gal}(\mathrm{E}^{\mathrm{sep}}_{H_\infty}/\mathrm{E}_{H_\infty})^{\mathrm{ab}}$, where $\mathrm{E}_{H_\infty} = k_{H_\infty}((t))$ (cf. Remark 1.11)*

$$\mathcal{I}^{\mathrm{ab}}_{\mathrm{E}_{H_\infty}} \xrightarrow{\sim} \mathcal{I}_{H_\infty} . \tag{4.29}$$

*Proof* The proof is standard. These two groups have the same character groups. Namely, by Tannakian equivalence and by Corollary 3.1, the character group of $\mathcal{I}_{H_\infty}$ is $\mathbb{Z}_{(p)}/\mathbb{Z} \oplus \mathbf{P}(k_{H_\infty})$. By Artin–Schreier theory, and Kummer theory, this last is also the character group of $\mathcal{I}_{H_\infty}$. □

*Remark* 4.16 We will see in the next section that this isomorphism is induced by the Fontaine–Katz functor $\mathbf{M}^\dagger$. Actually, this isomorphism exists even without the hypothesis required in the definition of this functor.

### 4.6 Differential equations and $\varphi$-modules over $\mathcal{E}^\dagger_K$ in the abelian case

In this section the notations, and hypotheses, will follow [25]. We recall that $\mathrm{w} = p$ (cf. Hypothesis 2.6). We suppose $k$ perfect (used in Proposition 4.1). Let $\Lambda/\mathbb{Q}_p$ be a finite extension containing $\mathbb{Q}_p(\xi_s)$. Let $\mathbb{F}_q$, $q := p^r$, be the residue field of $\Lambda$.

**Hypothesis 4.5** We assume the existence of an *absolute* Frobenius $\sigma_0 : \Lambda \to \Lambda$ (i.e. lifting of the $p$-th power map $x \mapsto x^p$ of $\mathbb{F}_q$), satisfying $\sigma_0^r = \mathrm{Id}_\Lambda$ and $\sigma_0(\pi_s) = \pi_s$. This is always possible if $\Lambda/\mathbb{Q}_p$ is Galois.

We let $K := \Lambda \otimes_{\mathbf{W}(\mathbb{F}_q)} \mathbf{W}(k)$ and $\sigma := \mathrm{Id}_\Lambda \otimes \mathrm{F}^r$. We denote again by $\sigma_0$ the morphism $(\sigma_0 \otimes \mathrm{F})$ on $K$, then $\sigma = \sigma_0^r$. We fix a continuous absolute Frobenius $\varphi_0$ on $\mathcal{O}^\dagger_K$, by setting $\varphi_0(\sum a_i T^i) := \sum \sigma_0(a_i)\varphi_0(T)^i$, where $\varphi_0(T) \in \mathcal{O}^\dagger_\Lambda$ is a lifting of $t^p \in k(\!(t)\!)$ (see Definition 1.10). Then $\varphi_0$ verifies $\varphi_0(\pi_s) = \pi_s$, and $\varphi_0(\mathcal{E}^\dagger_\Lambda) \subseteq \mathcal{E}^\dagger_\Lambda$. We set $\varphi = \varphi_0^r$. Both $\varphi$ and $\varphi_0$ extend uniquely to all unramified extensions of $\mathcal{E}^\dagger_K$, hence they commute with the action of $\mathrm{G_E} := \mathrm{Gal}(\mathrm{E}^{\mathrm{sep}}/\mathrm{E})$, with $\mathrm{E} = k(\!(t)\!)$.

**Definition 4.12** *Let $\underline{\mathrm{Rep}}^{\mathrm{fin}}_\Lambda(\mathrm{G_E})$ be the category of continuous (finite dimensional) representations $\alpha : \mathrm{G_E} \to \mathrm{GL}_\Lambda(\mathrm{V})$, such that $\alpha(\mathcal{I}_\mathrm{E})$ is finite.*

**Definition 4.13** *Let $\alpha : \mathrm{G_E} \to \Lambda^\times$ be a character such that $\alpha(\mathcal{I}_\mathrm{E})$ is finite. Then we denote by $\mathrm{V}_\alpha \in \underline{\mathrm{Rep}}^{\mathrm{fin}}_\Lambda(\mathrm{G_E})$ the rank one representation of $\mathrm{G_E}$ given by*

$$\gamma(\mathbf{e}) := \alpha(\gamma)\cdot\mathbf{e}, \quad \text{for all } \gamma \in \mathrm{G_E},$$

*where $\mathbf{e} \in \mathrm{V}_\alpha$ is a basis. We denote by $\mathbf{D}^\dagger(\mathrm{V}_\alpha)$ (respectively $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$) the $\varphi - \nabla$-module over $\mathcal{E}^\dagger_K$ (resp. $\nabla$-module over $\mathcal{R}_K$) attached to $\mathrm{V}_\alpha$. Namely*

$$\mathbf{D}^\dagger(\mathrm{V}_\alpha) = (\mathrm{V}_\alpha \otimes_\Lambda \mathcal{E}^{\dagger,\mathrm{unr}}_K)^{\mathrm{G_E}} \quad , \quad \mathbf{M}^\dagger(\mathrm{V}_\alpha) = \mathbf{D}^\dagger(\mathrm{V}_\alpha) \otimes_{\mathcal{E}^\dagger_K} \mathcal{R}_K . \tag{4.30}$$

*We recall that* $\mathbf{M}^{\dagger}(\mathrm{V}_{\alpha})$ *is endowed with the unique derivation "commuting" with* $\varphi$ (*cf.* [14, 2.2.4]). *This derivation is* $\nabla = 1 \otimes \partial_T$. *By Remark* 1.5, $\mathbf{M}^{\dagger}(\mathrm{V}_{\alpha}) \in \mathrm{MLS}(\mathcal{R}_K)$.

**Definition 4.14** *We will identify* $\mathbb{Z}/p^{s+1}\mathbb{Z}$ *with* $\boldsymbol{\mu}_{p^{s+1}}$, *by sending*

$$1 \mapsto \xi_s \ : \ \mathbb{Z}/p^{s+1}\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_{p^{s+1}},$$

*where* $\xi_s$ *is the unique* $p^{s+1}$*th root of* 1 *verifying* (*cf. Remark* 1.13)

$$|(\xi_s - 1) - \pi_s| < |\pi_s| \,. \tag{4.31}$$

*If* $\alpha \in \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G_E}, \mathbb{Z}/p^{s+1}\mathbb{Z})$, *we again denote by* $\mathrm{V}_{\alpha}$ *the representation given by*

$$\gamma(\mathbf{e}) := \xi_s^{\alpha(\gamma)} \cdot \mathbf{e}, \quad \text{for all } \gamma \in \mathrm{G_E}.$$

*Remark* 4.17 This definition is chosen "ad hoc" to be the inverse of the action of $\mathrm{G_E}$ described by the Eq. (2.45).

*Remark* 4.18 Let $\alpha : \mathrm{G_E} \to \Lambda^{\times}$ be a continuous character, then $\alpha$ factors on the abelianized $\mathrm{G_E}^{\mathrm{ab}}$. Let $\mathcal{I}_{\mathrm{E}}^{\mathrm{ab}}$ be the inertia of $\mathrm{G_E}^{\mathrm{ab}}$, and $\mathrm{G}_k^{\mathrm{ab}}$ be the abelianized of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. Since $k$ is perfect, then, by Remark 3.5, the exact sequence $1 \to \mathcal{I}_{\mathrm{E}}^{\mathrm{ab}} \to \mathrm{G}_{\mathrm{E}}^{\mathrm{ab}} \to \mathrm{G}_k^{\mathrm{ab}} \to 1$ is split, hence

$$\mathrm{G_E}^{\mathrm{ab}} = \mathcal{I}_{\mathrm{E}}^{\mathrm{ab}} \oplus \mathrm{G}_k^{\mathrm{ab}} \,, \tag{4.32}$$

and $\alpha = \alpha^{-} \cdot \alpha_0$, where $\alpha^{-} : \mathcal{I}_{\mathrm{E}}^{\mathrm{ab}} \to \Lambda^{\times}$ and $\alpha_0 : \mathrm{G}_k^{\mathrm{ab}} \to \Lambda^{\times}$. Then

$$\mathrm{V}_{\alpha} = \mathrm{V}_{\alpha^{-}} \otimes \mathrm{V}_{\alpha_0}.$$

We observe that $\mathbf{M}^{\dagger}(\mathrm{V}_{\alpha_0}) \xrightarrow{\sim} \mathcal{R}_K$, is trivial because its solution is a constant. Indeed, the extension of $\mathcal{O}_K^{\dagger}$ defined by $\alpha_0$ is $\mathcal{O}_K^{\dagger} \otimes_K H$, for some unramified extensions $H/K$. In the sequel we will treat only characters $\alpha : \mathrm{G_E} \to \Lambda^{\times}$ with *finite image*, this will be restrictive in terms of $\varphi$-modules but not in terms of differential modules. Indeed,

$$\mathbf{D}^{\dagger}(\mathrm{V}_{\alpha}) = \mathbf{D}^{\dagger}(\mathrm{V}_{\alpha^{-}}) \otimes \mathbf{D}^{\dagger}(\mathrm{V}_{\alpha_0})\,; \quad \mathbf{M}^{\dagger}(\mathrm{V}_{\alpha}) = \mathbf{M}^{\dagger}(\mathrm{V}_{\alpha^{-}}). \tag{4.33}$$

*Remark* 4.19 Points (4) and (5) of the following theorem have been already proved in [18] in the case $p \neq 2$ and rank one, and in [12], [19], [26] in the general case. Moreover we thank the referee to pointed out to us that the explicit form of the differential operator (answer to (5) of Sect. 0.1) was probably written in the proof of Lemma 5.2 of [18], in the case $p \neq 2$.

**Theorem 4.6** *Let $\overline{\boldsymbol{f}}(t) \in \mathbf{W}_s(\mathrm{E})$ and let $\alpha = \delta(\overline{\boldsymbol{f}}(t))$ be the Artin–Schreier character defined by $\overline{\boldsymbol{f}}(t)$ (cf. Eq. (1.21)). Let $(\mathcal{E}_K^\dagger)'$ be the unramified extension of $\mathcal{E}_K^\dagger$ corresponding, by henselianity, to the separable extension of $k((t))$ defined by $\alpha$. Then*

1. *A basis of $\mathbf{D}^\dagger(\mathrm{V}_\alpha)$ is given by*

$$y := \mathbf{e} \otimes \theta_{p^s}(\boldsymbol{v}, 1), \tag{4.34}$$

*where $\mathbf{e} \in \mathrm{V}_\alpha$ is the basis of Definition 4.14 and $\boldsymbol{v} \in \mathbf{W}_s(\widehat{\mathcal{E}}_K^{\mathrm{unr}})$ is a solution of*

$$\varphi_0(\boldsymbol{v}) - \boldsymbol{v} = \boldsymbol{f}(T), \tag{4.35}$$

*where $\boldsymbol{f}(T) \in \mathbf{W}_s(\mathcal{O}_K[[T]][T^{-1}])$ is an arbitrary lifting of $\overline{\boldsymbol{f}}(t)$;*

2. *The Frobenius $\varphi_0$ acts on $\mathrm{V}_\alpha$, moreover $\varphi_0(y) = \theta_{p^s}(\boldsymbol{f}(T), 1) \cdot y$. Hence, if $\mathrm{Tr}(\boldsymbol{f}(T)) := \boldsymbol{f}(T) + \varphi_0(\boldsymbol{f}(T)) + \cdots + \varphi_0^{r-1}(\boldsymbol{f}(T))$, we have*

$$\varphi(y) = \theta_{p^s}(\mathrm{Tr}(\boldsymbol{f}(T)), 1) \cdot y; \tag{4.36}$$

3. *By Corollary 2.4, since $K = K_s$, one has $(\mathcal{E}_K^\dagger)' = \mathcal{E}_K^\dagger[\theta_{p^s}(\boldsymbol{v}, 1)]$ (cf. Definition 3.5). If $\overline{\boldsymbol{f}}_0 \in (\mathrm{F}-1)\mathbf{W}_s(k((t)))$, i.e. if $\alpha_0 = 0$, then this extension can be identified with the extension*

$$\mathcal{E}_K^\dagger[\theta_{p^s}(\boldsymbol{v}, 1)] = \mathcal{E}_K^\dagger[\theta_{p^s}(\boldsymbol{v}^-, 1)] \xrightarrow{\sim} \mathcal{E}_K^\dagger[\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)] \tag{4.37}$$

*by sending $\theta_{p^s}(\boldsymbol{v}^-, 1)$ into $\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$. In particular, since $\pi_s \in K$, one has*

$$\widetilde{y} = \mathbf{e} \otimes \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1). \tag{4.38}$$

*Moreover $\varphi_0(\widetilde{y}) = \theta_{p^s}(\boldsymbol{f}^-(T), 1) \cdot \widetilde{y}$, and $\varphi(\widetilde{y}) = \theta_{p^s}(\mathrm{Tr}(\boldsymbol{f}^-(T)), 1) \cdot \widetilde{y}$.*

4. *The isomorphism class of $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$ depends only on $\alpha^-$ and*

$$\mathbf{M}^\dagger(\mathrm{V}_\alpha) \xrightarrow{\sim} \mathrm{M}(0, \alpha^-); \tag{4.39}$$

5. *The irregularity of $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$ is equal to the Swan conductor of $\mathrm{V}_\alpha$.*

*Proof* Let $\mathrm{E} = k((t))$. For all $\gamma \in \mathrm{G}_\mathrm{E} = \mathrm{Gal}(\mathrm{E}^{\mathrm{sep}}/\mathrm{E})$, we have (cf. Eq. (2.51))

$$\gamma(\mathbf{e} \otimes \theta_{p^s}(\boldsymbol{v}, 1)) = (\xi_s^{\alpha(\gamma)} \cdot \mathbf{e}) \otimes (\xi_s^{-\alpha(\gamma)} \cdot \theta_{p^s}(\boldsymbol{v}, 1)) = \mathbf{e} \otimes \theta_{p^s}(\boldsymbol{v}, 1), \tag{4.40}$$

hence $\mathbf{e} \otimes \theta_{p^s}(\boldsymbol{v}, 1) \in \mathbf{D}^\dagger(\mathrm{V}_\alpha)$. Moreover, $\varphi_0(\mathbf{e} \otimes \theta_{p^s}(\boldsymbol{v}, 1)) = \mathbf{e} \otimes \varphi_0(\theta_{p^s}(\boldsymbol{v}, 1))$ and

$$\varphi_0(\theta_{p^s}(\boldsymbol{v}, 1)) \stackrel{(*)}{=} \theta_{p^s}(\varphi_0(\boldsymbol{v}), 1) = \theta_{p^s}(\varphi_0(\boldsymbol{v}) - \boldsymbol{v}, 1) \cdot \theta_{p^s}(\boldsymbol{v}, 1)\ . \tag{4.41}$$

The equality (∗) is true because $\varphi_0(\pi_s) = \sigma_0(\pi_s) = \pi_s$, for all $s \geq 0$. The derivation on $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$ arises from the derivation on $\mathcal{E}_K^{\dagger,\mathrm{unr}}$ (cf. Eq. (4.30)), hence the operator attached to the basis $\mathbf{e} \otimes \theta_{p^s}(\boldsymbol{\nu}, 1) \in \mathbf{M}_K^\dagger(\mathrm{V}_\alpha)$ is (cf. Sect. 1.2) $\partial_T - \partial_{T,\log}(\theta_{p^s}(\boldsymbol{\nu}, 1))$. As explained in Remark 4.18, the isomorphism class of $\mathbf{M}^\dagger(\mathrm{V}_\alpha)$ depends only on $\alpha^- = \delta(\overline{\boldsymbol{f}^-(t)})$. Hence, we can suppose $\alpha = \alpha^-$ and $\boldsymbol{f}(T) = \boldsymbol{f}^-(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_K[T^{-1}])$ (cf. Definition 3.5). Let us write $\boldsymbol{\nu}^-$ instead of $\boldsymbol{\nu}$. By Eq. (3.1), we have

$$\theta_{p^s}(\boldsymbol{\nu}^-, 1)^{p^{s+1}} = \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)^{p^{s+1}}, \tag{4.42}$$

hence $\partial_{T,\log}(\theta_{p^s}(\boldsymbol{\nu}^-, 1)) = \partial_{T,\log}(\mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1))$. This establishes point (3) and (4).

Both the Swan conductor and the irregularity are stable under extension of the constant field $K$, hence we can suppose $K = K^{\mathrm{alg}}$. We can suppose that $\boldsymbol{f}^-(T)$ is pure (cf. Definition 3.6), because both the Swan conductor and the irregularity depend only on $\alpha^-$. Write $\boldsymbol{f}^-(T) = \sum_{n\in\mathrm{J}_p} \boldsymbol{\lambda}_{np^{m(n)}} T^{-np^{m(n)}}$. Since the irregularities of the $\boldsymbol{\lambda}_{np^{m(n)}} T^{-np^{m(n)}}$'s are all different we can suppose $\boldsymbol{f}^-(T) = \boldsymbol{\lambda}_{np^{m(n)}} T^{-np^{m(n)}}$. Now write explicitly (cf. Definition 3.3)

$$\begin{aligned}\boldsymbol{\lambda}_{np^{m(n)}} T^{-np^{m(n)}} &= (\lambda_r T^{-np^r}, \lambda_{r+1} T^{-np^{r+1}}, \ldots, \lambda_m T^{-np^{m(n)}}) \\ &= (\lambda_r T^{-np^r}, 0 \ldots, 0) + \cdots + (0, \ldots, 0, \lambda_m T^{-np^{m(n)}}) .\end{aligned}$$

Since, by reduction Theorem 2.2, the irregularities of the these vectors are all different, and since both the irregularity and the Swan conductor are invariant by V (cf. Eq. (4.23)), we can suppose $\boldsymbol{f}^-(T) = (\lambda T^{-n}, 0, \ldots, 0)$, with $|\lambda| = 1$. Moreover, since $K = K^{\mathrm{alg}}$, the residue field is perfect and, replacing $\lambda T^{-n}$ with $\lambda^{1/p^k} T^{-n/p^k}$, we can suppose $(n, p) = 1$ (cf. Remark 1.9). The irregularity is then $np^s$ (cf. Lemma 4.7), and it is equal to the Swan conductor (see for example [5]). This Theorem is the analogue of Corollary 4.8 for Artin–Schreier characters of $\mathrm{G_E}$). □

*Remark* 4.20 Suppose that the character is totally ramified, and choose $\boldsymbol{f}^-(T)$ in $\mathbf{W}_s(T^{-1}\mathcal{O}_K[T^{-1}])$, then $\theta_{p^s}(\boldsymbol{\nu}, 1) = \mathrm{e}_{p^s}(\boldsymbol{f}^-(T), 1)$.

*Remark* 4.21 Let $a_0 = \frac{m}{n} \in \mathbb{Z}_p \cap \mathbb{Q}$. Suppose that $\boldsymbol{\mu}_n \subset k$. Let $\beta_{a_0} : \mathrm{G} \to \boldsymbol{\mu}_n \subset \Lambda^\times$ be the Kummer character defined by $t^{a_0}$. We have $\beta_{a_0}(\gamma) = \gamma(t^{a_0})/t^{a_0}$. As before, a basis of $\mathbf{D}^\dagger(\mathrm{V}_{\beta_{a_0}})$ is given by $\mathrm{e} \otimes T^{-a_0} \in \mathrm{V}_{\beta_{a_0}} \otimes \mathcal{E}_K^{\dagger,\mathrm{unr}}$, because $\gamma(\mathrm{e}) := \beta_{a_0}(\gamma)\mathrm{e}$. Then $\varphi(\mathrm{e} \otimes T^{-a_0}) = T^{a_0}\varphi(T^{-a_0}) \cdot (\mathrm{e} \otimes T^{-a_0})$, and $\mathbf{M}^\dagger(\mathrm{V}_{\beta_0}) = \mathrm{M}(a_0, 0)$. We do not necessarily have an action of $\varphi_0$, because $\sigma_0$ does not fix the $n$th root of 1.

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