Arithmetic and differential Swan conductors of rank one representations with finite local monodromy.

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Abstract

We consider a complete discrete valuation field of characteristic p, with possibly non perfect residue field. Let V be a rank one continuous representation of its absolute Galois group with finite local monodromy. We will prove that the *arithmetic Swan conductor* of V (defined after K.Kato in [Kat89] which fits in the more general theory of [AS02] and [AS06]) coincides with the *differential Swan conductor* of the associated differential module $\mathrm{D}^\dagger(\mathrm{V})$ defined by K.Kedlaya in [Ked07]. This construction is a generalization to the non perfect residue case of the Fontaine's formalism as presented in [Tsu98a]. Our method of proof will allow us to give a new interpretation of the refined Swan conductor.

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Contents

Introduction

Let E be a complete discrete valuation field of characteristic p, and let $\mathrm{G_E}$ be its absolute Galois group. It has been known now for some time (cf. [Fon90], [Kat73]) that one has an equivalence between the category of continuous p-adic representation of $\mathrm{G_E}$ with coefficients in a finite extension K of $\mathbb{Q}_p$, and the category of unit-root φ-modules over a Cohen ring $\mathcal{E}_L$ of E. If k were perfect, then the introduction of a finiteness hypothesis (finite local monodromy) on the representation has allowed R.Crew, S.Matsuda and N.Tsuzuki to associate a (φ,∇)-module (with one derivative) not only on the "arithmetic ring" $\mathcal{E}_L$, but in some more geometric ring $\mathcal{E}_L^\dagger$, which we could indicated as "*the bounded Robba ring*" (cf. [Tsu98a], [Tsu98b], [Mat95], [Mat97], [Cre98] ...). Here "geometric" means that the elements of this ring can be understood as analytic functions converging in some annulus. S.Matsuda and N.Tsuzuki actually provide a more general framework: they allow the field of constants L to be a certain totally ramified extension of the field of fractions of $\mathbf{W}(k)$. In this context we would find ourselves with two theories of ramification between two parallel worlds. In the arithmetic setting we have the classical ramification theory as presented in [Ser62] and [Kat88]. In particular, we have the notion of *arithmetic Swan conductor* of a representation V with finite local monodromy. In the differential framework, G.Christol, Z.Mebkhout, and P.Robba obtained a theory of *p-adic slopes* of a solvable ∇-module M (cf. [Rob85], [CR94], [CM00], [CM01], for example). In particular we have the notion of *p-adic Irregularity* of the ∇-module underling the (φ,∇)-module $\mathrm{D}^{\dagger}(\mathrm{V})$ associated to V. The works of R.Crew, S.Matsuda and N.Tsuzuki have shown that the *arithmetic Swan conductor* of V coincides with the *p-adic irregularity* of $\mathrm{D}^{\dagger}(\mathrm{V})$ (cf. [Cre00], [Tsu98b], [Mat95]).

If, now, we allow k be non-perfect, then K.Kedlaya has recently shown that the category of p-adic representations of $\mathrm{G_E}$ with finite local monodromy is again equivalent to a category of unit-root (φ,∇)-modules over $\mathcal{E}^{\dagger}_L$ in a slightly generalized sense with respect to the work of S.Matsuda and N.Tsuzuki (cf. [Ked07]). In this new context we can no longer expect to have a single derivative, indeed since the residual field k is not necessarily perfect, the number of independent derivatives will depend on the cardinality of a p-basis of the residue field. In fact K.Kedlaya is able to associate to a representation V, with finite local monodromy, a (φ,∇)-module, denoted again by $\mathrm{D}^{\dagger}(\mathrm{V})$, where now $\nabla:\mathrm{D}^{\dagger}(\mathrm{V})\to\mathrm{D}^{\dagger}(\mathrm{V})\otimes\widehat{\Omega}^1_{\mathcal{E}^{\dagger}_L/K}$ is an integrable connection. If we fix a uniformizer parameter t of E and an isomorphism $\mathrm{E}\xrightarrow{\sim}k(\!(t)\!)$, then $\widehat{\Omega}^1_{\mathcal{E}^{\dagger}_L/K}$ is a finite free module over $\mathcal{E}^{\dagger}_L$ generated by dT, du_1, $\ldots$, du_r, where T is a lifting of t, and $u_1,\ldots,u_r$ are lifting of a p-basis of the residual field k. If k is perfect, then $\widehat{\Omega}^1_{\mathcal{E}^{\dagger}_L/K}=\widehat{\Omega}^1_{\mathcal{E}^{\dagger}_L/L}$ and we obtain exactly the theory as presented by S.Matsuda and N.Tsuzuki. From this K.Kedlaya obtains a generalized definition of irregularity involving all the derivations. We will call it *differential Swan conductor* of $\mathrm{D}^{\dagger}(\mathrm{V})$. This new notion of irregularity should be seen as a re-interpretation of the (now) classical definition of irregularity by means of slopes of the radius of solutions given by P.Robba, G.Christol, and Z.Mebkhout. Unfortunately K.Kedlaya does not have any interpretation of this new irregularity in term of index. We recall that on the arithmetic side we have now a good theory of ramification and relative Swan conductors for non perfect residue field obtained by A.Abbes and T.Saito (cf. [AS02]). For rank one representations it has been shown that the new definition coincides with the older one given by K.Kato (cf. [AS06], [Kat89]).

In this paper we prove that for a rank one p-adic representation V of $\mathrm{G_E}$ with finite local monodromy, the *arithmetic Swan conductor* of V, defined by A.Abbes, T.Saito and K.Kato, coincides with the *differential Swan conductor* of $\mathrm{D}^{\dagger}(\mathrm{V})$, defined by K.Kedlaya. This will be done by understanding Kato's definition via the techniques introduced in [Pul07], where one has a description of the character group $\mathrm{H}^1(\mathrm{E},\mathbb{Q}_p/\mathbb{Z}_p)$, and one computes the functor $\mathrm{D}^{\dagger}$ in rank one case. Moreover within this framework, we will interpretate the notion of refined Swan conductor (as in [Kat89], see also [Mat97] and [AS06]).

We expect a generalization of the previous results to representations of any rank by comparison between irregularity and Abbes-Saito Swan conductor. Recently Liang Xiao (cf. [Xia08]) announced the proof of such a generalization: his methods are completely different from ours.

Plan of the article:

After introducing some notation and basic definitions in section 1, in section 2 we will recall some known facts about (local) irregularity of differential operators in one variable over a non archimedean complete valued field and its various interpretations. In section 3 we introduce, Kedlaya's definition of the p-adic differential Swan conductor for (φ, ∇)-modules.

In section 4, we recall Kato's definition of Swan conductor for characters $\alpha : \mathrm{G_E} \to \mathbb{Q}/\mathbb{Z}$ *with finite image*. We then interpret an explicit description of $\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p)$ obtained in [Pul07, Section 4.1] in term of Kato's filtration of $\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}/\mathbb{Z})$ (cf. Theorem 4.9). As a corollary we prove that

$$\mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p)) \;\cong\; \begin{cases} \mathbf{W}_{v_p(d)}(k)/p\mathbf{W}_{v_p(d)}(k) & \text{if} \quad d > 0 \;, \\ \mathrm{H}^1(k, \mathbb{Q}_p/\mathbb{Z}_p) & \text{if} \quad d = 0 \;, \end{cases} \tag{0.0.1}$$

where Gr_d means the d-th graded piece with respect to the Kato's filtration, and $v_p(d)$ is the p-adic valuation of d. This simple expression of $\mathrm{Gr}_d(\ \mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p)\ )$ is obtained by considering any such a character as the image of a Witt co-vector in the quotient $\mathbf{CW}(\mathrm{E})/(\bar{\mathrm{F}} - 1)\mathbf{CW}(\mathrm{E})$, which is isomorphic to $\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p)$ via the Artin-Schreier-Witt theory (cf. Sequence (1.3.3)).

In the same section we recall (and precise) a decomposition of $\mathrm{G_E^{ab}}$ obtained in [Pul07, Equation (4.32)]. This allows us to generalize the Kato's definition of the Swan conductor to rank one representations *with finite local monodromy* (not necessarily with finite image) (cf. Definition 4.17). In section 5, by means of the isomorphism (0.0.1), we will give a new construction of Kato's refined Swan conductor according to [Kat89] and [AS06].

Finally, in the last sections 6 and 7, we prove our theorem: this will be done by computing $\mathrm{D}^\dagger$ in the rank one case along the lines of [Pul07]. Firstly we reduce the problem to the case of a representation defined by a "*pure co-monomial*" (cf. Def. 4.2.3), then we prove that in this case the differential Swan conductor (defined as the slope of $T(\mathrm{M}, \rho)$ at 1^- (cf. Section 3.13)) coincides with the slope of $T(\mathrm{M}, \rho)$ at 0^+ which can be computed explicitly in term of the coefficients of the equation (Lemma 7.4).

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1 Generalities and Notation

1.1 Rings

Let $(L, |\cdot|)$ be an ultrametric complete valued field, and let $I \subseteq \mathbb{R}_{\geq 0}$ be an interval. We set

$$\mathcal{A}_L(I) \;:=\; \{\sum_{i \in \mathbb{Z}} a_i T^i \mid a_i \in L \;,\; |a_i|\rho^i \to 0 \;,\; \text{ for } i \to \pm\infty \;,\; \forall \rho \in I \;\} \;, \tag{1.1.1}$$

$$\begin{aligned}
\mathcal{E}_L &:= \{ f = \sum_{i\in\mathbb{Z}} a_i T^i \mid a_i \in L \ , \ \sup_{i\in\mathbb{Z}} |a_i| < +\infty \ , \ \lim_{i\to-\infty} a_i = 0 \} \ , && (1.1.2)\\
\mathcal{R}_L &:= \bigcup_{\varepsilon>0} \mathcal{A}_L(]\varepsilon, 1[) \ , && (1.1.3)\\
\mathcal{E}_L^\dagger &:= \mathcal{R}_L \cap \mathcal{E}_L \ . && (1.1.4)
\end{aligned}$$

The ring $\mathcal{A}_L(I)$ is complete with respect to the topology given by the family of norms $\{|\cdot|_\rho\}_{\rho\in I}$, where

$$|\sum_i a_i T^i|_\rho := \sup_i |a_i|\rho^i \ . \qquad (1.1.5)$$

The Robba ring $\mathcal{R}_L$ is complete with respect to the limit-Frechet topology induced by the family of topologies of $\mathcal{A}_L(]\varepsilon, 1[)$. The ring $\mathcal{E}_L$ is complete for the topology given by the Gauss norm $|\cdot|_1 = |\cdot|_{\mathcal{E}_L} = |\cdot|_{\mathrm{Gauss}}$: $|\sum a_i T^i|_1 := \sup_i |a_i|$. The ring $\mathcal{E}_L^\dagger$ has two topologies arising from $\mathcal{R}_L$ and $\mathcal{E}_L$ respectively, moreover $\mathcal{E}_L^\dagger$ is dense in $\mathcal{E}_L$ and in $\mathcal{R}_L$ for the respective topologies. If the valuation on L is discrete, then $\mathcal{E}_L^\dagger$ is a henselian field.

Let $[\rho] := \{\rho\}$, $\rho > 0$, be the (closed) interval which consists of only the point ρ. The completion of the fraction field of $\mathcal{A}_L([\rho])$ will be denoted by

$$(\mathcal{F}_{L,\rho}, |\cdot|_\rho) \ := \ (\ \mathrm{Frac}(\mathcal{A}_L([\rho]))\ , \ |\cdot|_\rho\ )^{\widehat{\ }} \ . \qquad (1.1.6)$$

Since rational fractions, without poles of norm ρ, are dense in $\mathcal{A}_L([\rho])$ with respect to the norm $|\cdot|_\rho$, one has that $\mathcal{F}_{L,\rho}$ is the completion of the field of rational fraction $L(T)$ with respect to the norm $|\cdot|_\rho$. If $\rho \in I$, one has the inclusions

$$\mathcal{A}_L(I) \ \subset \ \mathcal{A}_L([\rho]) \ \subset \ \mathcal{F}_{L,\rho} \ . \qquad (1.1.7)$$

1.2 Witt vectors and Witt covectors

For all rings R (not necessarily with unit element), we denote by $\mathbf{W}_m(R)$ the ring of Witt vectors with coefficients in R. The notation follows from [Bou83], with the exception that in our setting $\mathbf{W}_m(R) := \mathbf{W}(R)/\mathrm{V}^{m+1}(\mathbf{W}(R))$, where $\mathrm{V}(\lambda_0, \lambda_1, \ldots) = (0, \lambda_0, \lambda_1, \ldots)$ is the Verschiebung. We denote by $\phi_j := \phi_j(\lambda_0, \lambda_1, \ldots) := \lambda_0^{p^j} + p\lambda_1^{p^{j-1}} + \cdots + p^j\lambda_j$ the j-th phantom component of $(\lambda_0, \lambda_1, \ldots)$. We distinguish phantom vectors from Witt vectors by using the notation $\langle\phi_0, \phi_1, \ldots\rangle \in R^{\mathbb{N}}$ instead of $(\phi_0, \phi_1, \ldots)$. We denote by F the Frobenius of $\mathbf{W}(R)$ (i.e. the one that induce the map $\langle\phi_0, \phi_1, \ldots\rangle \mapsto \langle\phi_1, \phi_2, \ldots\rangle$ on the phantom components). If R is an $\mathbb{F}_p$-algebra, we define a Frobenius $\bar{\mathrm{F}} : \mathbf{W}_m(R) \to \mathbf{W}_m(R)$ by $\bar{\mathrm{F}}(\lambda_0, \lambda_1, \ldots) := (\lambda_0^p, \lambda_1^p, \ldots)$.

1.2.1

We recall that if $I \subseteq R$ is a subgroup closed by multiplication, then $\mathbf{W}(I) := \{(\lambda_0, \lambda_1, \ldots) \in \mathbf{W}(R) \mid \lambda_i \in I, \text{ for all } i \geq 0\}$ is a subgroup closed by multiplication of $\mathbf{W}(R)$. In the sequel we will apply this to the ring $R[[t]][t^{-1}]$, and $I = tR[[t]]$, or $I = t^{-1}R[t^{-1}]$.

1.2.2 The R-modules $\mathbf{CW}(R)$ and $\widetilde{\mathbf{CW}}(R)$.

We set $\mathbf{CW}(R) := \varinjlim(\mathbf{W}_m(R) \xrightarrow{\mathrm{V}} \mathbf{W}_{m+1}(R) \xrightarrow{\mathrm{V}} \cdots)$ and, if R is an $\mathbb{F}_p$-algebra, we set $\widetilde{\mathbf{CW}}(R) := \varinjlim(\mathbf{W}_m(R) \xrightarrow{p} \mathbf{W}_{m+1}(R) \xrightarrow{p} \cdots)$, where $p : \mathbf{W}_m(R) \to \mathbf{W}_{m+1}(R)$ denotes the morphism $\mathrm{V}\bar{\mathrm{F}} : (\lambda_0, \ldots, \lambda_m) \mapsto (0, \lambda_0^p, \ldots, \lambda_m^p)$.

Remark 1.1. *If R is an $\mathbb{F}_p$-algebra, and if the Frobenius $x \mapsto x^p : R \to R$ is a bijection, then $\widetilde{\mathbf{CW}}(R)$ and $\mathbf{CW}(R)$ are isomorphic (cf. [Pul07, Section 1.3.4]). In general they are not isomorphic.*

1.2.3 Canonical Filtration of $\mathbf{CW}(R)$ and $\widetilde{\mathbf{CW}}(R)$.

Both $\mathbf{CW}(R)$ and $\widetilde{\mathbf{CW}}(R)$ have a natural *filtration* given by $\mathrm{Fil}_m(\mathbf{CW}(R)) := \mathbf{W}_m(R) \subset \mathbf{CW}(R)$ and $\mathrm{Fil}_m(\widetilde{\mathbf{CW}}(R)) := \mathbf{W}_m(R) \subset \widetilde{\mathbf{CW}}(R)$, respectively. Defining Gr_m as $\mathrm{Fil}_m/\mathrm{Fil}_{m-1}$, one has :

$$\mathrm{Gr}_m(\mathbf{CW}(R)) = \mathbf{W}_m(R)/\mathrm{V}(\mathbf{W}_{m-1}(R)) \cong R\,, \tag{1.2.1}$$
$$\mathrm{Gr}_m(\widetilde{\mathbf{CW}}(R)) = \mathbf{W}_m(R)/p\cdot\mathbf{W}_m(R)\,. \tag{1.2.2}$$

Indeed $p\cdot\mathbf{W}_m(R) = \mathrm{V}\bar{\mathrm{F}}(\mathbf{W}_{m-1}(R))$. We recall that if R is an $\mathbb{F}_p$-algebra then

$$p\cdot\mathbf{W}_m(R) = \{\ \boldsymbol{\lambda}\in\mathbf{W}_m(R)\ |\ \boldsymbol{\lambda} = (0,\lambda_1^p,\ldots,\lambda_m^p)\ ,\ \lambda_1,\ldots,\lambda_m\in R\ \}\,. \tag{1.2.3}$$

1.3 Notation on Artin-Schreier-Witt theory

Let κ be a field of characteristic $p>0$ and let $\kappa^{\mathrm{sep}}/\kappa$ be a fixed separable closure of κ. We set

$$\mathrm{G}_\kappa := \mathrm{Gal}(\kappa^{\mathrm{sep}}/\kappa)\,. \tag{1.3.1}$$

If κ is a complete discrete valuation field, we denote by $\mathcal{I}_{\mathrm{G}_\kappa}$ the inertia group and by $\mathcal{P}_{\mathrm{G}_\kappa}$ the pro-p-Sylow subgroup of $\mathcal{I}_{\mathrm{G}_\kappa}$. We have $\mathrm{H}^1(\mathrm{G}_\kappa,\mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{G}_\kappa,\mathbb{Z}/p^m\mathbb{Z})$ (cf. [Ser62, Ch.X, §3]). The situation is then expressed by the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/p^{m+1}\mathbb{Z} \arrow[r] \arrow[d, "\imath"] \arrow[dr, phantom, "\odot"] & \mathbf{W}_m(\kappa) \arrow[r, "\bar{\mathrm{F}}-1"] \arrow[d, "\mathrm{V}"] \arrow[dr, phantom, "\odot"] & \mathbf{W}_m(\kappa) \arrow[r, "\delta"] \arrow[d, "\mathrm{V}"] \arrow[dr, phantom, "\odot"] & \mathrm{Hom}(\mathrm{G}_\kappa,\mathbb{Z}/p^{m+1}\mathbb{Z}) \arrow[r] \arrow[d, "\jmath"] & 0 \\
0 \arrow[r] & \mathbb{Z}/p^{m+2}\mathbb{Z} \arrow[r] & \mathbf{W}_{m+1}(\kappa) \arrow[r, "\bar{\mathrm{F}}-1"] & \mathbf{W}_{m+1}(\kappa) \arrow[r, "\delta"] & \mathrm{Hom}(\mathrm{G}_\kappa,\mathbb{Z}/p^{m+2}\mathbb{Z}) \arrow[r] & 0
\end{tikzcd} \tag{1.3.2}$$

where $\imath : 1\mapsto p$ is the usual inclusion, and $\jmath$ is the composition with $\imath$. For $\boldsymbol{\lambda}\in\mathbf{W}_m(\kappa)$, the character $\alpha=\delta(\boldsymbol{\lambda})\in\mathrm{Hom}(\mathrm{G}_\kappa,\mathbb{Z}/p^{m+1}\mathbb{Z})$ satisfies $\alpha(\gamma)=\gamma(\boldsymbol{\nu})-\boldsymbol{\nu}\ \in\ \mathbb{Z}/p^{m+1}\mathbb{Z}$, for all $\gamma\in\mathrm{G}_\kappa$, where $\boldsymbol{\nu}\in\mathbf{W}_m(\kappa^{\mathrm{sep}})$ is a solution of the equation $\bar{\mathrm{F}}(\boldsymbol{\nu})-\boldsymbol{\nu}=\boldsymbol{\lambda}$. Passing to the inductive limit, we obtain the exact sequence:

$$0\to\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathbf{CW}(\kappa) \xrightarrow{\bar{\mathrm{F}}-1} \mathbf{CW}(\kappa) \xrightarrow{\delta} \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\kappa,\mathbb{Q}_p/\mathbb{Z}_p)\to 0\,, \tag{1.3.3}$$

where $\mathbb{Q}_p/\mathbb{Z}_p$ is considered with the discrete topology, in order that the word "cont" means that all characters $\mathrm{G}_\kappa\to\mathbb{Q}_p/\mathbb{Z}_p$ has finite image. Indeed $\varinjlim_m \mathrm{Hom}(\mathrm{G}_\kappa,\mathbb{Z}/p^m\mathbb{Z})$ can be viewed as the subset of $\mathrm{Hom}(\mathrm{G}_\kappa,\mathbb{Q}_p/\mathbb{Z}_p)$ formed by the elements killed by a power of p (cf. Remark 4.1.2).

Remark 1.2. *If the vertical arrows* V *are replaced by* $\mathrm{V}\bar{\mathrm{F}}$ *in the diagram* (1.3.2)*, then the morphisms $\imath$ and $\jmath$ remain the same. Indeed* $\delta(\boldsymbol{\lambda})=\delta(\bar{\mathrm{F}}(\boldsymbol{\lambda}))$*, because* $\bar{\mathrm{F}}(\boldsymbol{\lambda})=\boldsymbol{\lambda}+(\bar{\mathrm{F}}-1)(\boldsymbol{\lambda})$*, for all $\boldsymbol{\lambda}\in\mathbf{W}_m(\kappa)$. Hence we have also a sequence as follows:*

$$0\to\mathbb{Q}_p/\mathbb{Z}_p\to\widetilde{\mathbf{CW}}(\kappa) \xrightarrow{\bar{\mathrm{F}}-1} \widetilde{\mathbf{CW}}(\kappa) \xrightarrow{\tilde{\delta}} \mathrm{Hom}^{cont}(\mathrm{G}_\kappa,\mathbb{Q}_p/\mathbb{Z}_p)\to 0\,. \tag{1.3.4}$$

1.3.1 Artin-Schreier-Witt extensions.

Let $\boldsymbol{\lambda}=(\lambda_0,\ldots,\lambda_m)\in\mathbf{W}_m(\kappa)$. Let $\alpha:=\delta(\boldsymbol{\lambda})$, then

$$(\kappa^{\mathrm{sep}})^{\mathrm{Ker}(\alpha)} = \kappa(\{\nu_0,\ldots,\nu_m\}) \tag{1.3.5}$$

(i.e. the smallest subfield of κ^{sep} containing k and the set $\{\nu_0,\ldots,\nu_m\}$), where $\boldsymbol{\nu}=(\nu_0,\ldots,\nu_m)$ $\in\mathbf{W}_m(\kappa^{\mathrm{sep}})$ is a solution of $\bar{\mathrm{F}}(\boldsymbol{\nu})-\boldsymbol{\nu}=\boldsymbol{\lambda}$. All cyclic (separable) extensions of κ, whose degree is a power of p, are of this form for a suitable $m\geq 0$, and $\boldsymbol{\lambda}\in\mathbf{W}_m(\kappa)$.

1.4 Witt (co-)vectors of filtered and graded rings

Let A be a ring (not necessarily with unit element). We denote $\mathbb{N} := \mathbb{Z}_{\geq 0}$. A *filtration* on A indexed by $\mathbb{N}$ is a family of subgroups of $(A,+)$

$$0 = \mathrm{Fil}_{-1}(A) \subseteq \mathrm{Fil}_0(A) \subseteq \mathrm{Fil}_1(A) \subseteq \mathrm{Fil}_2(A) \subseteq \cdots \tag{1.4.1}$$

such that

$$\mathrm{Fil}_{d_1}(A) \cdot \mathrm{Fil}_{d_2}(A) \subseteq \mathrm{Fil}_{d_1+d_2}(A) , \tag{1.4.2}$$

for all $d_1, d_2 \geq 0$. The pair $(A, \{\mathrm{Fil}_d(A)\}_{d\geq 0})$ is called *filtered ring.* We say that a ring A is *graded* if $A = \oplus_{d\geq 0}\mathrm{Gr}_d(A)$ (as additive groups), where $\mathrm{Gr}_d(A)$, $d \geq 0$ are subgroups of A, with the property that $\mathrm{Gr}_{d_1}(A) \cdot \mathrm{Gr}_{d_2}(A) \subseteq \mathrm{Gr}_{d_1+d_2}(A)$. To a graded ring $A = \oplus_{d\geq 0}\mathrm{Gr}_d(A)$ it is associated a natural filtration given by $\mathrm{Fil}_n(A) = \oplus_{0\leq d\leq n}\mathrm{Gr}_d(A)$, which makes A a filtered ring.

1.4.1 Filtration associated to a valuation.

A *valuation* on A with values in $\mathbb{Z}$ is a map $v : A \longrightarrow \mathbb{Z}\cup\{\infty\}$ such that $v(a) = \infty$ if and only if $a = 0$, and for all $a_1, a_2 \in A$ one has $v(a_1+a_2) \geq \min(v(a_1), v(a_2))$, and $v(a_1\cdot a_2) = v(a_1)+v(a_2)$. If A is a valued ring, then A is a domain. By setting

$$\mathrm{Fil}_d(A) := \{a \in A \mid v(a) \geq -d\} , \quad \text{for all } d \geq 0, \tag{1.4.3}$$

one obtains a filtration on A. In particular, this applies if A is equal to $R[T]$, $TR[T]$, $R[T, T^{-1}]$, $R[[T]][T^{-1}]$, $T^{-1}R[T^{-1}]$ (where R is a ring), or if A is a valued field of any characteristic.

1.4.2 Witt (co-)vectors of a filtered ring.

Definition 1.3. *For all integers $d, i \geq 0$ we set*

$$\{d : p^i\} := \begin{cases} d/p^i & \text{if} \quad p^i|d \\ -1 & \text{if} \quad i > v_p(d) \end{cases} \tag{1.4.4}$$

We then have

Definition 1.4. *Let $(A, \{\mathrm{Fil}_d(A)\}_{d\geq 0})$ be a filtered ring. We define for all $m, d \geq 0$*

$$\mathrm{Fil}_d(\mathbf{W}_m(A)) := \{(\lambda_0, \dots, \lambda_m) \in \mathbf{W}_m(A) \mid \lambda_{m-i} \in \mathrm{Fil}_{\{d:p^i\}}(A)\} . \tag{1.4.5}$$

Lemma 1.5. *The ring $\mathbf{W}_m(A)$ together with $\{\mathrm{Fil}_d(\mathbf{W}_m(A))\}_{d\geq 0}$ is a filtered ring.*

Proof. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{Fil}_d(\mathbf{W}_m(A))$, and let $\boldsymbol{\nu} := \boldsymbol{\lambda} + \boldsymbol{\mu} = (\nu_0, \dots, \nu_m)$. Then, for all $0 \leq n \leq m$, ν_n is an isobaric polynomial in $\lambda_0, \dots, \lambda_n, \mu_0, \dots, \mu_n$, more precisely

$$\nu_n = \sum_{i_0,\dots,i_n,j_0,\dots,j_n\geq 0} k_{i_0,\dots,i_n,j_0,\dots,j_n}\lambda_0^{i_0}\cdots\lambda_n^{i_n}\mu_0^{j_0}\cdots\mu_n^{j_n} , \tag{1.4.6}$$

where, $k_{0,\dots,0} = 0$, $k_{i_0,\dots,i_n,j_0,\dots,j_n} \in \mathbb{N}$, and, for all $i_0, \dots, i_n, j_0, \dots, j_n \geq 0$, one has

$$(i_0 + i_1p + \cdots + i_np^n) + (j_0 + j_1p^1 + \cdots + j_np^n) = p^n . \tag{1.4.7}$$

Hence the condition $\lambda_{m-i}, \mu_{m-i} \in \mathrm{Fil}_{\{d:p^i\}}(A)$, implies $\nu_{m-i} \in \mathrm{Fil}_{\{d:p^i\}}(A)$.

On the other hand, if $\boldsymbol{\lambda} \in \mathrm{Fil}_{d_1}(\mathbf{W}_m(A))$, $\boldsymbol{\mu} \in \mathrm{Fil}_{d_2}(\mathbf{W}_m(A))$, let $\boldsymbol{\eta} := \boldsymbol{\lambda}\cdot\boldsymbol{\mu} = (\eta_0, \dots, \eta_m)$. Then for all $0 \leq n \leq m$, one has

$$\eta_n = \sum_{i_0,\dots,i_n,j_0,\dots,j_n\geq 0} k'_{i_0,\dots,i_n,j_0,\dots,j_n}\lambda_0^{i_0}\cdots\lambda_n^{i_n}\mu_0^{j_0}\cdots\mu_n^{j_n} \tag{1.4.8}$$

where, $k'_{0,\dots,0} = 0$, $k'_{i_0,\dots,i_n,j_0,\dots,j_n} \in \mathbb{N}$, and, for all $i_0,\dots,i_n,j_0,\dots,j_n \geq 0$, one has

$$(i_0 + i_1 p + \cdots + i_n p^n) \;=\; p^n \;=\; (j_0 + j_1 p + \cdots + j_n p^n)\ . \tag{1.4.9}$$

Then the conditions $\lambda_{m-i} \in \mathrm{Fil}_{\{d_1 : p^i\}}(A)$, $\mu_{m-i} \in \mathrm{Fil}_{\{d_2 : p^i\}}(A)$ imply $\eta_{m-i} \in \mathrm{Fil}_{\{(d_1+d_2):p^i\}}(A)$. □

1.4.3 Witt (co-)vectors of a graded ring.

Let now $A = \oplus_{d\geq 0}\mathrm{Gr}_d(A)$ be a *graded* ring. We denote by

$$\mathbf{W}_m^{(d)}(A)\ , \qquad (\text{resp. } \mathbf{CW}^{(d)}(A)) \tag{1.4.10}$$

the subset of $\mathbf{W}_m(A)$ (resp. $\mathbf{CW}(A)$) formed by vectors $(\lambda_0,\dots,\lambda_m) \in \mathbf{W}_m(A)$ (resp. co-vectors $(\cdots,0,0,\lambda_0,\dots,\lambda_m) \in \mathbf{CW}(A)$) satisfying, for all $i = 0,\dots,m$:

$$\lambda_{m-i} \in \mathrm{Gr}_{\{d:p^i\}}(A)\ , \tag{1.4.11}$$

where by definition we set $\mathrm{Gr}_{-1}(A) = 0$. It follows from the equation (1.4.6), and from the fact that $\mathrm{Gr}_{d_1}(A)\cdot\mathrm{Gr}_{d_2}(A) \subseteq \mathrm{Gr}_{d_1+d_2}(A)$, that $\mathbf{W}_m^{(d)}(A)$ (resp. $\mathbf{CW}^{(d)}(A)$) is closed under the sum in $\mathbf{W}_m(A)$ (resp. $\mathbf{CW}(A)$). Moreover

$$\mathrm{V}(\mathbf{W}_m^{(d)}(A)) \;\subseteq\; \mathbf{W}_{m+1}^{(d)}(A)\ , \tag{1.4.12}$$

and for all $d_1, d_2 \geq 0$ one has

$$\mathbf{W}_m^{(d_1)}(A)\cdot\mathbf{W}_m^{(d_2)}(A) \;\subseteq\; \mathbf{W}_m^{(d_1+d_2)}(A)\ . \tag{1.4.13}$$

We notice that $\mathrm{V}(\mathbf{W}_m^{(d)}(A)) = \mathbf{W}_{m+1}^{(d)}(A)$ if $d = np^v$, with $(n,p) = 1$, and $v \leq m$. This proves that

$$\mathbf{CW}^{(d)}(A) \;=\; \varinjlim_m \mathbf{W}_m^{(d)}(A)\ . \tag{1.4.14}$$

Lemma 1.6. *If $A = \oplus_{d\geq 0}\mathrm{Gr}_d(A)$ is graded, then the filtration* (1.4.5) *on $\mathbf{W}_m(A)$ is a grading and*

$$\mathrm{Gr}_d(\mathbf{W}_m(A)) \;=\; \mathbf{W}_m^{(d)}(A). \tag{1.4.15}$$

The decomposition $\mathbf{W}_m(A) = \oplus_{d\geq 0}\mathbf{W}_m^{(d)}(A) = \oplus_{d\geq 0}\mathrm{Gr}_d(\mathbf{W}_m(A))$ passes to the limit and defines a decomposition

$$\mathbf{CW}(A) \;=\; \oplus_{d\geq 0}\mathbf{CW}^{(d)}(A) \;=\; \oplus_{d\geq 0}\mathrm{Gr}_d(\mathbf{CW}(A))\ . \tag{1.4.16}$$

Proof. We prove first that $\mathbf{W}_m(A) \cong \oplus_{d\geq 0}\mathbf{W}_m^{(d)}(A)$. If $m = 0$, then $\mathbf{W}_0(A) = A$, and the lemma holds. Consider now a vector $(\lambda_0,\dots,\lambda_m) \in \mathbf{W}_m(A)$. If $\lambda_0 = \sum_{d\geq 0}\lambda_0^{(d)} \in A$, with $\lambda_0^{(d)} \in \mathrm{Gr}_d(A)$, then

$$(\lambda_0,\lambda_1,\dots,\lambda_m) = (\sum_{d\geq 0}\lambda_0^{(d)},0,\dots,0) + (0,\lambda_1,\dots,\lambda_m) = \sum_{d\geq 0}(\lambda_0^{(d)},0,\dots,0) + (0,\mu_1,\dots,\mu_m), \tag{1.4.17}$$

for some $\mu_1,\dots,\mu_m \in A$. The process can be iterated for $(0,\mu_1,\dots,\mu_m)$. This proves that $\mathbf{W}_m(A)$ is generated by the elements of the form $(0,\dots,0,x,0,\dots,0)$, with $x \in \mathrm{Gr}_d(A)$, placed in the i-th position, where i (resp. d) runs on the set $\{0,\dots,m\}$ (resp. $\mathbb{Z}_{\geq 0}$). Hence $\mathbf{W}_m(A) = \sum_{d\geq 0}\mathbf{W}_m^{(d)}(A)$.

We prove now that the sum is direct by induction on $m \geq 0$. For $m = 0$ this reduces to $A \cong \oplus_{d\geq 0}\mathrm{Gr}_d(A)$. Let now $m \geq 1$. If $\sum_{d\geq 0}\boldsymbol{\lambda}^{(d)} = 0$, with $\boldsymbol{\lambda}^{(d)} \in \mathbf{W}_m^{(d)}(A)$, for all $d \geq 0$, then $0 = \sum_{d\geq 0}\lambda_0^{(d)}$, hence $\lambda_0^{(d)} = 0$ for all $d \geq 0$, because $A \cong \oplus_{d\geq 0}\mathrm{Gr}_d(A)$. Then every $\boldsymbol{\lambda}^{(d)}$ belongs to $\mathrm{V}(\mathbf{W}_{m-1}^{(d)}(A))$. By induction we have $\boldsymbol{\lambda}^{(d)} = 0$ for all $d \geq 0$. Finally the decomposition passes to the limit by (1.4.14) and because every Witt co-vector belongs to $\mathbf{W}_m(A)$, for some $m \geq 0$. $\square$

1.5 Settings

Hypothesis 1.7. *Let $p > 0$, and let $q = p^h$, $h > 0$. Let k be a field of characteristic p containing $\mathbb{F}_q$. We assume that k admits a finite p-basis $\{\bar{u}_1, \ldots, \bar{u}_r\}$.*

Notation 1.8. For all field κ of characteristic $p > 0$, we denote by C_κ a Cohen ring of κ (i.e. a complete discrete valued ring of characteristic 0 with maximal ideal $p\mathrm{C}_\kappa$ and with residue field κ.).

Let E be a complete discrete valuation field of characteristic p, with residue field k. Let t be a uniformizer of $\mathcal{O}_{\mathrm{E}}$. If a section $k \subset \mathcal{O}_{\mathrm{E}}$ of the projection $\mathcal{O}_{\mathrm{E}} \to k$ is chosen, then one has an isomorphism (cf. [Bou83, Ch.9,§3,N°3,Th.1])

$$\mathrm{E} \cong k(\!(t)\!) \ , \quad \mathcal{O}_{\mathrm{E}} \cong k[\![t]\!] \ . \tag{1.5.1}$$

Lemma 1.9. *Let κ_1, κ_2 be two fields, and let $\bar{f} : \kappa_1 \to \kappa_2$ be a morphism of fields. Let $\{\bar{x}_i\}_{i\in I}$ be a p-basis of κ_1, and let $\bar{y}_i := \bar{f}(\bar{x}_i)$. Let $\mathrm{C}_{\kappa_1}, \mathrm{C}_{\kappa_2}$ be two Cohen rings with residue field κ_1, κ_2 respectively, and let $\{x_i\}_{i\in I} \subset \mathrm{C}_{\kappa_1}$, $\{y_i\}_{i\in I} \subset \mathrm{C}_{\kappa_2}$ be arbitrary liftings of $\{\bar{x}_i\}_{i\in I}$ and $\{\bar{y}_i\}_{i\in I}$. Then there exists a unique ring morphism $f : \mathrm{C}_{\kappa_1} \to \mathrm{C}_{\kappa_2}$ of $\bar{f}$ sending x_i into y_i.*

Proof. See [Bou83, §1, N°1, Prop.2; Ex.4 of §2, p.70]. See also [Whi02]. $\square$

Let C_k be a Cohen ring of k, and let L_0 be its field of fractions. Since $\mathcal{O}_{\mathcal{E}_{L_0}}$ is a complete discrete valued ring with maximal ideal generated by p, and with residue field isomorphic to E, then

$$\mathrm{C}_k \cong \mathcal{O}_{L_0} \ , \qquad \mathrm{C}_{\mathrm{E}} \cong \mathcal{O}_{\mathcal{E}_{L_0}} \ . \tag{1.5.2}$$

Let K be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. We set $L := K \otimes_{\mathbf{W}(\mathbb{F}_q)} \mathcal{O}_{L_0}$ and hence

$$\mathcal{O}_L := \mathcal{O}_K \otimes_{\mathbf{W}(\mathbb{F}_q)} \mathcal{O}_{L_0} \ . \tag{1.5.3}$$

One sees then that $\mathcal{O}_{\mathcal{E}_L} = \mathcal{O}_{\mathcal{E}_{L_0}} \otimes_{\mathbf{W}(\mathbb{F}_q)} \mathcal{O}_K$.

Definition 1.10. *Let $\{u_1, \ldots, u_r\} \subset \mathcal{O}_{L_0}$ be an arbitrary lifting of the p-basis $\{\bar{u}_1, \ldots, \bar{u}_r\}$ of k (cf. Hypothesis 1.7). By Lemma 1.9, we fix a Frobenius φ of $\mathcal{O}_{L_0}$ sending $u_1, \ldots, u_r$ into $u_1^q, \ldots, u_r^q$. We denote again by φ the Frobenius $\mathrm{Id}_{\mathcal{O}_K} \otimes \varphi$ on $\mathcal{O}_L$. We extend to $\mathcal{O}_{\mathcal{E}_L}$, $\mathcal{O}_{\mathcal{E}_L^\dagger}$, and $\mathcal{R}_L$ this Frobenius by sending $T, u_1, \ldots, u_r$ into $T^q, u_1^q, \ldots, u_r^q$.*

The situation is exposed in the following diagram:

$$\begin{tikzcd}
 & \mathcal{O}_K \arrow[r] & \mathcal{O}_L \arrow[r] & \mathcal{O}_{\mathcal{E}_L} \\
\mathbb{Z}_p \arrow[r] \arrow[d] & \mathbf{W}(\mathbb{F}_q) \arrow[r] \arrow[u] \arrow[d] & \mathcal{O}_{L_0} \arrow[r] \arrow[u] \arrow[d] & \mathcal{O}_{\mathcal{E}_{L_0}} \arrow[u] \arrow[d] \\
\mathbb{F}_p \arrow[r] & \mathbb{F}_q \arrow[r] & k \arrow[r] & \mathrm{E}
\end{tikzcd} \ . \tag{1.5.4}$$

We denote by K, L_0, L, $\mathcal{E}_L$, $\mathcal{E}_{L_0}$, the fractions fields of $\mathcal{O}_K$, $\mathcal{O}_{L_0}$, $\mathcal{O}_L$, $\mathcal{O}_{\mathcal{E}_L}$, and $\mathcal{O}_{\mathcal{E}_{L_0}}$ respectively.

1.6 The Cohen rings of $k_0(\bar{u}_1,\ldots,\bar{u}_r)$ and $k_0(\!(\bar{u}_1)\!)\ldots(\!(\bar{u}_r)\!)$

Let k_0 be a perfect field, and let $\{\bar{u}_1,\ldots,\bar{u}_r\}$ be an algebraically independent family over k_0. In this subsection we give a description of the Cohen ring $\mathrm{C}_k = \mathcal{O}_{L_0}$ in the case in which $k = k_0(\bar{u}_1,\ldots,\bar{u}_r)$ or $k = k_0(\!(\bar{u}_1)\!)\ldots(\!(\bar{u}_r)\!)$.

Let $\{u_1,\ldots,u_r\} \subset \mathcal{O}_{L_0}$ be an algebraically independent family over $F_0 := \mathrm{Frac}(\mathbf{W}(k_0))$, lifting $\{\bar{u}_1,\ldots,\bar{u}_r\}$. We will define the Cohen ring of $k = k_0(\!(\bar{u}_1)\!)\ldots(\!(\bar{u}_r)\!)$, $\mathcal{E}_{F_0,(u_1,\ldots,u_r)}$ inductively on r. For $r = 1$ we define $\mathcal{E}_{F_0,(u_1)}$ as the F_0-vector space whose elements are series $f(u_1) = \sum_{i\in\mathbb{Z}} a_i u_1^i$, with $a_i \in F_0$, satisfying

$$\sup_i |a_i|_{F_0} \quad < \quad +\infty \ , \tag{1.6.1}$$

$$\lim_{i\to-\infty} |a_i|_{F_0} \quad = \quad 0 \ . \tag{1.6.2}$$

This is a complete field under the absolute value

$$|f(u_1)|_{\mathcal{E}_{F_0,(u_1)}} = \max_i |a_i|_{F_0} ,$$

it is also unramified over $\mathbb{Q}_p$ and its residue field is $k_0(\!(\bar{u}_1)\!)$. Hence its valuation ring is a Cohen ring of $k_0(\!(\bar{u}_1)\!)$. For $r = 2$ using the case $r = 1$ we can define $\mathcal{E}_{\mathcal{E}_{F_0,(u_1)},(u_2)}$. This is a $\mathcal{E}_{F_0,(u_1)}$ vector space whose elements are Laurent series in u_2 with coefficients in $\mathcal{E}_{F_0,(u_1)}$ satisfying the previous conditions (1.6.1) and (1.6.2). It is a complete field under the extension of the previous norm. In fact, if $f(u_2) = \sum_{i\in\mathbb{Z}} a_i u_2^i$, with $a_i \in \mathcal{E}_{F_0,(u_1)}$ we define

$$|f(u_2)|_{\mathcal{E}_{\mathcal{E}_{F_0,(u_1)},(u_2)}} = \max_i |a_i|_{\mathcal{E}_{F_0,(u_1)}} ,$$

it is again non ramified and its residue field is $k_0(\!(\bar{u}_1)\!)(\!(\bar{u}_2)\!)$. This norm is an extension of the Gauss norm on rational functions. We may then continue for a general r by defining $\mathcal{E}_{F_0,(u_1,\ldots,u_r)} = \mathcal{E}_{\mathcal{E}_{F_0,(u_1,\ldots,u_{r-1})},(u_r)}$ a Cohen ring for $k_0(\!(\bar{u}_1)\!)\ldots(\!(\bar{u}_r)\!)$.

Remark 1.11. *If $k = k_0(\bar{u}_1,\ldots,\bar{u}_r)$, then the Cohen ring $\mathrm{C}_k = \mathcal{O}_{L_0}$ is the ring of integers of the completion of $F_0(u_1,\ldots,u_r)$, with respect to the Gauss norm.*

1.7 Differentials

We fix an arbitrary lifting $\{u_1,\ldots,u_r\} \subset \mathrm{C}_k = \mathcal{O}_{L_0}$ of the *finite* p-basis $\{\bar{u}_1,\ldots,\bar{u}_r\} \subset k$ of k.

Lemma 1.12. *Let $\widehat{\Omega}^1$ denote the module of continuous differentials. The following assertions hold:*

1. *$\widehat{\Omega}^1_{\mathcal{O}_L/\mathcal{O}_K}$ is a free $\mathcal{O}_L$-module of rank r, generated by $du_1,\ldots,du_r$,*

2. *$\dim_L \widehat{\Omega}^1_{L/K} = r$, and $\widehat{\Omega}^1_{L/K}$ is generated by $du_1,\ldots,du_r$.*

Proof. The proof can be done by using [Gro64, $\mathbf{0}_{IV}$, Prop.19.8.2, Th.20.4.9, Cor.20.4.10], [Gro60, $\mathbf{0}_I$, Prop.7.2.9], and [Gro61, $\mathbf{0}_{III}$, 10.1.3], □

The elements $\bar{u}_1,\ldots,\bar{u}_r \in k$ are algebraically independent over $\mathbb{F}_q$ (cf. [Bou59, Ch.V, §8 Ex.9]). So the elements $u_1,\ldots,u_r$ are algebraically independent over $\mathbb{Q}_p$ and over K, since a polynomial relation $P(u_1,\ldots,u_r) = 0$ will imply, by reduction, a relation on $\{\bar{u}_1,\ldots,\bar{u}_r\}$. Since L has the p-adic topology, hence the topology induced by L on $K(u_1,\ldots,u_r)$ is the p-adic one, which coincides with that defined by the Gauss norm $|\sum a_{i_1,\ldots,i_r} u_1^{i_1}\cdots u_r^{i_r}|_{\mathrm{Gauss}} :=$

$\sup |a_{i_1,\dots,i_r}|_K$. In particular L contains the field $\mathrm{C}_{\mathbb{F}_q(\bar{u}_1,\dots,\bar{u}_r)} \otimes_{\mathbf{W}(\mathbb{F}_q)} K = (K(u_1,\dots,u_r), |\cdot|_{\mathrm{Gauss}})\widehat{\ }$, where $(\cdot)\widehat{\ }$ means the *completion with respect to the Gauss norm.*

Let $I \subseteq \mathbb{R}_{\geq 0}$. Let $\widehat{\Omega}^1_{\mathcal{A}_L(I)/K}$ denote *continuous* differentials, with respect to the topology of $\mathcal{A}_L(I)$ (cf. section 1.1), then

$$\widehat{\Omega}^1_{\mathcal{A}_L(I)/K} \;\cong\; \mathcal{A}_L(I)\cdot dT \oplus \Big(\bigoplus_{i=1}^r \mathcal{A}_L(I)\cdot du_i\Big) . \tag{1.7.1}$$

Consequently, also $\widehat{\Omega}^1_{\mathcal{E}_L/K}$, $\widehat{\Omega}^1_{\mathcal{E}^\dagger_L/K}$, $\widehat{\Omega}^1_{\mathcal{R}_L/K}$ are freely generated by $dT, du_1, \dots, du_r$.

1.8 Spectral norms

For all complete valued ring $(H, |.|_H)/(\mathbb{Z}_p, |.|)$, and all $\mathbb{Z}_p$-derivation $\partial : H \to H$, we set

$$|\partial^n|_H := \sup_{h\in H, h\neq 0} |\partial^n(h)|_H/|h|_H \;, \qquad |\partial|_{H,\mathrm{Sp}} := \lim_{n\to\infty} |\partial^n|_H^{1/n} \;, \tag{1.8.1}$$

This section is devote to proving the following

Lemma 1.13 ([Ked07, Remark 2.1.5])**.** *Let L be as in section 1.5. Let*

$$\omega \;:=\; |p|^{\frac{1}{p-1}} \;. \tag{1.8.2}$$

Then one has

$$\begin{array}{lcllcl} |d/dT|_{\mathcal{F}_{L,\rho}} &=& \rho^{-1} \;, & |d/du_i|_{\mathcal{F}_{L,\rho}} &=& 1 \;, \\ |d/dT|_{\mathcal{F}_{L,\rho},\mathrm{Sp}} &=& \omega\cdot\rho^{-1} \;, & |d/du_i|_{\mathcal{F}_{L,\rho},\mathrm{Sp}} &=& \omega \;. \end{array} \tag{1.8.3}$$

Proof. The assertions on d/dT are well known (cf. [Chr83]). We study now only d/du_i. We split the proof in three lemmas:

Lemma 1.14. *For all $i = 1,\dots,r$, one has $|d/du_i|_{\mathcal{F}_{L,\rho}} = |d/du_i|_L$, and $|d/du_i|_{\mathcal{F}_{L,\rho},\mathrm{Sp}} = |d/du_i|_{L,\mathrm{Sp}}$.*

Proof. It is enough to show that, for all $n \geq 0$, one has $|(d/du_i)^n|_{\mathcal{F}_{L,\rho}} = |(d/du_i)^n|_L$. Since $L \subset \mathcal{F}_{L,\rho}$, then $|(d/du_i)^n|_{\mathcal{F}_{L,\rho}} \geq |(d/du_i)^n|_L$. Conversely, for all $n \geq 0$, one has $|(d/du_i)^n|_{\mathcal{F}_{L,\rho}} = \sup_{f\in\mathcal{F}_{L,\rho}\,,\,|f|_\rho=1} |(d/du_i)^n(f)|_\rho$. We observe that since $L[T,T^{-1}]$ is dense in $\mathcal{F}_{L,\rho}$, we can consider the sup on the set $\{f = \sum a_j T^j \in L[T,T^{-1}]\;,\; |f|_\rho = 1\}$. For all such $f = \sum a_j T^j$, one has $|(d/du_i)^n(\sum a_j T^j)|_\rho = |\sum (d/du_i)^n(a_j)T^j|_\rho = \sup_j |(d/du_i)^n(a_j)|\rho^j \leq \sup_j |(d/du_i)^n|_L |a_j|\rho^j = |(d/du_i)^n|_L |f|_\rho$, hence $|(d/du_i)^n|_{\mathcal{F}_{L,\rho}} \leq |(d/du_i)^n|_L$. □

Lemma 1.15. *One has $|d/du_i|_L = 1$.*

Proof. Since $d/du_i(\mathcal{O}_L) \subset \mathcal{O}_L$, one has $|d/du_i(a)|_L \leq 1 = |a|$, for all $a \in \mathcal{O}_L$, $|a| = 1$. Hence $|d/du_i|_L = \sup_{|a|=1, a\in L} |d/du_i(a)|_L \leq 1$. Conversely, we have that $K(u_1,\dots,u_r) \subset L$, and the valuation induced by L is the Gauss norm. If B denotes the completion of $K(u_1,\dots,u_r)$, then it is easy to prove explicitly, using the description given in Section 1.6, that $|d/du_i|_{\mathrm{B}} = 1$, and $|d/du_i|_{\mathrm{B},\mathrm{Sp}} = \omega$. The computations are analogous to the classical ones for d/dT (cf. [Chr83]). Since $\mathrm{B} \subseteq L$, we have the easy inequality

$$1 = |d/du_i|_{\mathrm{B}} := \sup_{b\in\mathrm{B}\,,\,|b|=1} |d/du_i(b)|_{\mathrm{B}} \leq \sup_{b\in L\,,\,|b|=1} |d/du_i(b)|_L = |d/du_i|_L \leq 1 \;. \tag{1.8.4}$$

□

Analogously one proves that $|n!| = |(d/du_i)^n|_{\mathrm{B}} \leq |(d/du_i)^n|_L$, $\omega = |d/du_i|_{\mathrm{B,Sp}} \leq |d/du_i|_{L,\mathrm{Sp}}$. It is now sufficient to prove that $|d/du_i|_{L,\mathrm{Sp}} \leq \omega$. We will prove this inequality first over $\mathrm{Frac}(\mathrm{C}_k)$, and then over L.

Lemma 1.16. *One has* $|(d/du_i)^n|_{\mathrm{Frac}(\mathrm{C}_k)} = |n!|$.

Proof. Let $A := \mathbb{F}_q(\bar{u}_1, \ldots, \bar{u}_r) \subseteq k$. Let C_k and C_A be the Cohen rings attached to k and A respectively. Fix an inclusion $\mathrm{C}_A \subset \mathrm{C}_k$. Let $\mathrm{B} := \mathrm{C}_k^p + p\mathrm{C}_k$, then B is a closed sub-ring of C_k, and hence complete. Since $k = A \otimes_{A^p} k^p$, and since $(\mathrm{B}/p\mathrm{C}_k) = k^p$, then C_k is generated by C_A and B. By the description given in Section 1.6, one sees that $|(d/du_i)^n|_A = |n!|$. On the other hand, $d/du_i(\mathrm{B}) \subseteq p\mathrm{C}_k \subseteq \mathrm{B}$, hence $|d/du_i|_{F_{\mathrm{B}}} \leq |p|$, where F_{B} is the completion of $\mathrm{Frac}(\mathrm{B})$. Hence $|(d/du_i)^n|_{F_{\mathrm{B}}} \leq |p|^n \leq |n!|$. Since every element $x \in \mathrm{C}_k$ can be written as $x = \sum_{i=1}^n a_i b_i$, with $a_i \in \mathrm{C}_A$, and $b_i \in \mathrm{B}$, then $|(d/du_i)^n|_{\mathrm{Frac}(\mathrm{C}_k)} = |n!|$. $\square$

End of the proof of Lemma 1.13 : One has $L = K \otimes_{\mathbf{W}(\mathbb{F}_q)} \mathrm{C}_k$. Every element $x \in L$ can be written as $x = \sum_{i=1}^n a_i b_i$, $a_i \in K$, $b_i \in \mathrm{C}_k$. Since $|(d/du_i)|_K = 0$, one sees that $|(d/du_i)^n|_L = |n!|$. Hence $|d/du_i|_{L,\mathrm{Sp}} = \omega$. $\square$

2 Radius of convergence and irregularities: the perfect residue field case

This section is introductory. The main goal is to connect the fundamental work of G.Christol and Z.Mebkhout with other results, definitions, and notations introduced by B.Dwork, L.Garnier, B.Malgrange, K.Kedlaya, P.Robba, N.Tsuzuki, for example. In all this section we assume that L has a perfect residual field k.

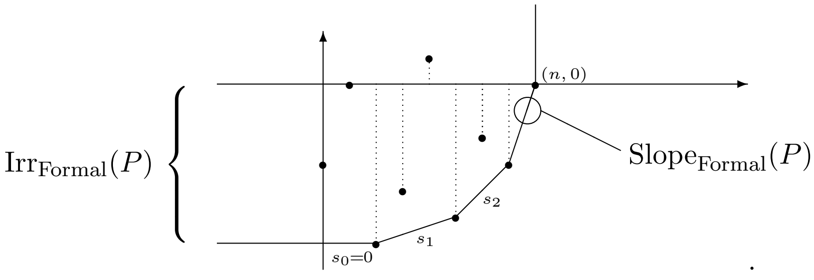
2.1 Formal Irregularity

The notion of irregularity of a differential equation finds its genesis in [Mal74], in which the definition has been introduced for the first time for a differential operator with coefficients in $\mathrm{C}((T))$, here C is a field of characteristic 0. We recall briefly the setting. If $P(T, \frac{d}{dT}) := \sum_{k=0}^n g_k(T)(\frac{d}{dT})^k$, with $g_k(T) \in \mathrm{C}((T))$, the *Formal Irregularity*, and the *Formal Slope* of P are defined as

$$\mathrm{Irr}_{\mathrm{Formal}}(P) := \max_{0\leq k\leq n} \{k - v_T(g_k)\} - (n - v_T(g_n)) \,, \tag{2.1.1}$$

$$\mathrm{Slope}_{\mathrm{Formal}}(P) = \max\Big(0 \,,\, \max_{k=0,\ldots,n} \Big(\frac{v_T(g_n) - v_T(g_k)}{n-k} - 1 \Big) \Big) \,, \tag{2.1.2}$$

where v_T is the T-adic valuation. One defines the *Formal Newton polygon* of P, denoted by $NP(P)$, as the convex hull in $\mathbb{R}^2$ of the set $\{(\ k\ ,\ (v_T(g_k) - k) - (v_T(g_n) - n)\ )\}_{k=0,\ldots,n}$ together with the extra points $\{(-\infty, 0)\}$ and $\{(0, +\infty)\}$. The formal slope is then the largest slope of the formal Newton polygon of P, and the *irregularity* of P is the height of the Newton polygon:



. (2.1.3)

Let $\mathcal{D} := \mathrm{C}(\!(T)\!)[\frac{d}{dT}]$. These definitions are actually attached to the differential module $\mathrm{M} = \mathcal{D}/\mathcal{D}\cdot P$ over $\mathrm{C}(\!(T)\!)$ defined by P, and are independent of the particular *cyclic* basis of M. We hence use the notation $NP(\mathrm{M})$, $\mathrm{Irr}_{\mathrm{Formal}}(\mathrm{M})$, $\mathrm{Slope}_{\mathrm{Formal}}(\mathrm{M})$. The differential module M admits a so called *break decomposition*

$$\mathrm{M} = \oplus_{s\geq 0}\mathrm{M}(s)\ , \tag{2.1.4}$$

into $\mathcal{D}$-submodules, in which $\mathrm{M}(s)$ is characterized by the fact that it is the unique submodule of M whose Newton polygon consists in a single slope s (counted with multiplicity) of the Newton polygon of M. Hence the Formal irregularity can be written as

$$\mathrm{Irr}_{\mathrm{Formal}}(\mathrm{M}) = \sum_{s\geq 0} s\cdot \dim_{\mathrm{C}(\!(T)\!)}\mathrm{M}(s)\ . \tag{2.1.5}$$

2.1.1 Formal indices.

We preserve the previous notations. It can be shown (cf. [Mal74, 1.3.1]), that, if $L_P : \mathrm{C}[\![T]\!] \to \mathrm{C}[\![T]\!]$ denotes the C-linear map $f(T) \mapsto P(f(T))$, then

$$\mathrm{Irr}_{\mathrm{Formal}}(\mathrm{M}) \ = \ \chi(\ L_P\ ;\ \mathrm{C}[\![T]\!]\ ) - (n - v_T(g_n))\ , \tag{2.1.6}$$

where $\chi(L_P;\mathrm{C}[\![T]\!]) = \dim_{\mathrm{C}}\mathrm{Ker}(L_P) - \dim_{\mathrm{C}}\mathrm{Coker}(L_P)$.

Assume moreover that $\mathrm{C} = \mathbb{C}$ is the field of complex numbers, and that M is a differential module over $\mathbb{C}(\{T\}) := \mathrm{Frac}(\mathbb{C}\{T\})$, the fraction field of convergent power series. B.Malgrange proved (cf. [Mal74, 3.3]) that if $G(T) \in M_n(\mathbb{C}\{T\})$ is the matrix of $T\frac{d}{dT}$ acting on M, and if $T\frac{d}{dT} + {}^tG(T) : \mathbb{C}(\{T\})^n \to \mathbb{C}(\{T\})^n$ is the differential operator attached to M in this basis, then one has also

$$\mathrm{Irr}_{\mathrm{Formal}}(\mathrm{M}) \ = \ -\chi\Big(\ T\frac{d}{dT} + {}^tG\ \ ;\ \ \mathbb{C}(\{T\})^n\ \Big)\ , \tag{2.1.7}$$

where $\chi(T\frac{d}{dT} + {}^tG;\mathbb{C}(\{T\})^n) := \dim_{\mathbb{C}}\mathrm{Ker}(T\frac{d}{dT} + {}^tG) - \dim_{\mathbb{C}}\mathrm{Coker}(T\frac{d}{dT} + {}^tG)$. Moreover the quantity $(n - v_T(g_n))$ can be related to the characteristic variety (at 0) of the $\mathcal{D}$-module M.

2.2 p-adic framework, the irregularity of Robba and Christol-Mebkhout

Assume now that C is a complete ultrametric field. Denote by $\mathrm{C}(\{T\})$ the subfield of $\mathrm{C}(\!(T)\!)$ of convergent (p-adically) power series around 0 with meromorphic singularities in 0. It has been showed by F.Baldassarri (cf. [Bal82]), that if M is a $\mathrm{C}(\{T\})$-differential module, the above Break decomposition of $\mathrm{M}\otimes_{\mathrm{C}(\{T\})}\mathrm{C}(\!(T)\!)$ *descends* to a break decomposition of M, over $\mathrm{C}(\{T\})$. In this sense, from the point of view of the Irregularity, the "*convergent theory*" in the ultrametric setting over a germ of punctured disk, with meromorphic singularities, offers nothing more than the "*formal theory*".

2.2.1 The Robba ring as the completion of the generic point of a curve.

A more interesting class of rings are those arising from Rigid Geometry. Let L be the field of Section 1.5. Let X be a projective, connected, non singular curve over $\mathcal{O}_L$, with special fiber X_k and generic fiber X_L. Let X_L^{an} be the rigid analytic curve over L defined by the generic fiber of X (because X was projective this rigid analytic space also coincides with that obtained by completion along the special fiber). For every closed point $x_0 \in X_k$ we denote by $]x_0[\ \subset X_L^{\mathrm{an}}$ the tube of x_0 in X_L^{an}. Let $x_0,\ldots,x_n \in X_k$ be a family of closed points, and let $U_k := X_k \setminus \{x_0,\ldots,x_n\}$ and denote j the open immersion of U_k in X_k . Let $]U_k[:= X_L^{\mathrm{an}} \setminus \cup_{i=0}^n]x_i[$

be the inverse image of $]U_k[$ by the specialization map. We denote by $j^\dagger\mathcal{O}_{X_L^{\mathrm{an}}}$ the sheaf on X_L^{an} of functions overconvergent along $X_k\setminus U_k$. Let $\mathcal{M}$ be a (locally free) $j^\dagger\mathcal{O}_{X_L^{\mathrm{an}}}$-sheaf of differential modules admitting a Frobenius structure. By Frobenius structure we mean the existence of an isomorphism $(\varphi^*)^h(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$, for some $h\geq 0$, where φ^* is associated with the absolute Frobenius and $(\varphi^*)^h := \varphi^*\circ\cdots\circ\varphi^*$, h-times. For all $i=0,\ldots,n$ we fix an isomorphism $\alpha_i :]x_i[\to \mathrm{D}^-(0,1)$. We define the Robba ring $\mathcal{R}_{x_i}$ as the pull back of $\mathcal{R}_L$ via the morphism α_i. Let V be a strict neighborhood of $]U_k[$. Every (overconvergent) function of $\Gamma(V,\mathcal{O}_{X_L^{\mathrm{an}}})$ defines, via the isomorphism α, a germ of analytic function in an annulus $1-\varepsilon<|T|<1$ of $\mathrm{D}^-(0,1)$. For all $i=0,\ldots,n$, and for all such V, we have then a natural restriction morphism $\Gamma(V,\mathcal{O}_{X_L^{\mathrm{an}}})\to\mathcal{R}_{x_i}$ compatible with the derivation, and with restrictions maps of the sheaf $j^\dagger\mathcal{O}_{X_L^{\mathrm{an}}}$. Then it makes sense to consider the restriction $\mathcal{M}_{x_i}$ of $\mathcal{M}$ to $]x_i[$ as a differential module over $\mathcal{R}_{x_i}$. Here $\mathcal{M}_{x_i}$ is defined as the inductive limit of group of sections of $\mathcal{M}$ on $]x_i[\cap V$ with respect to all strict neighborhoods V.

The Robba ring $\mathcal{R}_L$ is actually isomorphic to $\varinjlim_V \Gamma(V\cap]x_i[,\mathcal{O}_{X_L^{\mathrm{an}}})$ where V runs on the set of strict neighborhoods of $]U_k[$ in X_L^{an} (cf. [Cre92], see also [Mat02, After Cor.3.3]). In this sense $\mathcal{R}_{x_i}$ is the analogous of the ring $\mathrm{C}(\!(T-x_i)\!)$ of the previous section viewed as the completion at x_i of the field of functions of a nonsingular projective curve over C.

2.2.2 p-adic irregularity of differential modules over the Robba ring.

Let $\mathcal{A}_L(0,1)$ be the ring of analytic function on the open unit disk $\mathrm{D}^-(0,1)$. With the notation of the previous section, let $d/dT+G(T)$, $G(T)\in M_n(\mathcal{R}_L)$, be an operator attached to $\mathcal{M}_{x_i}$ in some basis. Following P.Robba, G.Christol and Z.Mebkhout [CM00, 8.3-8] we set:

$$\mathrm{Irr}_{x_i}(\mathcal{M}_{x_i}) := \widetilde{\chi}(\mathcal{M}_{x_i},\mathcal{A}_L(0,1)) \tag{2.2.1}$$

where $\widetilde{\chi}$ is the *generalized index* of $Td/dT+TG(T)$ on $\mathcal{A}_L(0,1)^n$ (cf. [CM00, 8.2-1]): such a generalized index exists under some assumptions on the exponents, assumptions which are fulfilled in case of Frobenius structure. Notice that $Td/dT+TG(T)$ does not act on $\mathcal{A}_L(0,1)^n$ since G has coefficients in $\mathcal{R}_L$, this is the reason for which one introduces the *generalized* index. Following [CM00, after Corollary 8.3-9] and [CM93, Def. 6.2.3], if now $L=\mathbb{C}_p$, and if M is a holonomic differential module on X_L and $\mathcal{M}$ its associated differential modules in X_L^{an}: one can consider $\mathcal{M}_{x_i}$, its localizations, as a $\mathcal{A}_L(0,1)[\frac{d}{dT}]$-module. Then if $\chi(\mathcal{M}_{x_i},\mathcal{A}_L(0,1)) = dim_{\mathbb{C}_p}\mathrm{Hom}_{\mathcal{A}_L(0,1)[\frac{d}{dT}]}(\mathcal{M}_{x_i},\mathcal{A}_L(0,1)) - dim_{\mathbb{C}_p}\mathrm{Ext}^1_{\mathcal{A}_L(0,1)[\frac{d}{dT}]}(\mathcal{M}_{x_i},\mathcal{A}_L(0,1))$ exists (again, this is the case if we have Frobenius structure) then one has

$$\mathrm{Irr}_{x_i}(\mathcal{M}_{x_i}) = \chi(\mathcal{M}_{x_i},\mathcal{A}_L(0,1)) - (n-\mathrm{ord}^-_{x_i}(\mathcal{M})) \tag{2.2.2}$$

where now $\mathrm{ord}^-_{x_i}(\mathcal{M})$ is the sum of all the multiplicities of the vertical components of the characteristic variety of $\mathcal{M}$ at every singular point contained in $]x_i[\,\cong\mathrm{D}^-(0,1)$.

2.2.3 Geometric interpretation: formal and p-adic irregularities as slopes of the generic radius of convergence.

We concentrate now our attention on differential modules over the Robba ring. Robba first and then Christol-Mebkhout have indicated that the behaviour of the *generic Radius of convergence* of the solutions of a finite free $\mathcal{R}_L$-differential module is strictly connected to its irregularity. Let M be a finite free $\mathcal{R}_L$-differential module. Let $\mathbf{e}$ be a fixed basis of M, and let $G(T)\in M_n(\mathcal{R}_L)$ be the matrix of the connection $\nabla_T:\mathrm{M}\to\mathrm{M}$ in this basis. Let $\varepsilon>0$ be such that

$G(T) \in M_n(\mathcal{A}_L(1-\varepsilon,1))$. Consider the generic Taylor solution

$$Y_G(x,y) \ := \ \sum_{n\geq 0} G_n(y)\frac{(x-y)^n}{n!} \tag{2.2.3}$$

where the matrices $\{G_n(T)\}_{n\geq 0}$ are defined inductively by the rule $G_0 = \mathrm{Id}$, $G_1 = G$, $G_{n+1} = d/dT(G_n) + G_nG_1$. The matrix Y_G verifies $(d/dx)^n(Y_G(x,y)) = G_n(x)Y_G(x,y)$, for all $n \geq 0$. The generic radius of convergence of Y at ρ is defined as $Ray(Y_G,\rho) := \liminf_n(|G_n|_\rho/|n!|)^{-1/n}$. Since $|G_n|_\rho = \max_{|y|=\rho}|G_n(y)|$, $Ray(Y_G,\rho)$ is the minimum among all possible radii (with respect to T) of $Y_G(T,y)$ with $|y| = \rho$. The quantity $Ray(Y_G,\rho)$ is not invariant under base changes. Indeed in the basis $\mathrm{B}\cdot\mathbf{e}$, $\mathrm{B} \in GL_n(\mathcal{A}_L(1-\varepsilon,1))$, the new solution is $\mathrm{B}\cdot Y_G$ which is possibly not convergent outside the disk $\mathrm{D}^-(y,\rho)$ (where $|y| = \rho$). We set then

$$Ray(\mathrm{M},\rho) := \min(Ray(Y,\rho),\rho)\ , \tag{2.2.4}$$

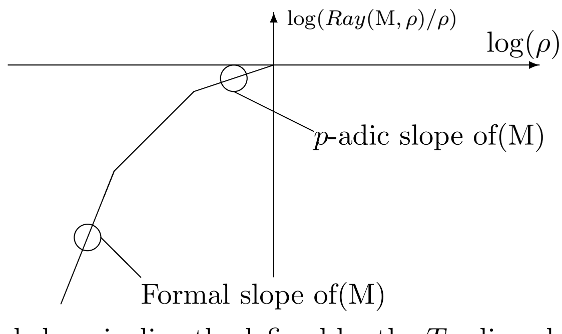
which is independent of the choice of basis. The function $\rho \mapsto Ray(\mathrm{M},\rho) :]1-\varepsilon,1[\to \mathbb{R}_+$ is continuous, piecewise of the type $a\rho^b$, for convenable $a,b \in \mathbb{R}$, and log-concave (cf. Section 3.3.1, [CD94]). We refer to the fact that M has $\lim_{\rho\to 1^-} Ray(\mathrm{M},\rho) = 1$ (cf. [CD94]),as the *solvability* of M. This is the case, for example, if M admits a Frobenius structure.

The Formal slope has a meaning in term of radius of convergence:

Proposition 2.1. *Let* $(\mathrm{M},\nabla_T^{\mathrm{M}})$ *be a solvable differential module over* $L(\!(T)\!)\cap\mathcal{R}_L \subset \mathcal{A}_K(]0,1[)$*. Then the Formal Slope of* M *as differential module over* $L(\!(T)\!)$ *coincides with the log-slope of the function* $\rho \mapsto Ray(\mathrm{M},\rho)/\rho$*, for* ρ *sufficiently close to* 0.

Proof. We fix a cyclic basis for which M is represented by an operator $\sum_{i=0}^n g_i(T)(d/dT)^i$, $g_n = 1$. For all $f(T) \in L(\!(T)\!)\cap\mathcal{R}_L$, if ρ is close to zero, then the log-slope of $\rho \mapsto |f(T)|_\rho$ is equal to $v_T(f(T))$. Since $\rho \mapsto Ray(\mathrm{M},\rho)$ is log-concave, and since $\lim_{\rho\to1^-} Ray(\mathrm{M},\rho) = 1$, then we have two possibilities $Ray(\mathrm{M},\rho) = \rho$ for all $\rho \in]0,1[$, or there exists an interval $I =]0,\delta[$, such that, for all $\rho \in I$ one has $Ray(\mathrm{M},\rho) < |\omega|\rho$, $\omega = |p|^{1/(p-1)}$. We can then apply [CM02, 6.2]. $Ray(\mathrm{M},\rho) = \rho$ if and only if $v_T(g_i) \geq i-n$ for all $i < n$ i.e. if and only if the formal slope is 0, the second case arises if and only if $v_T(g_i) < i-n$, for some i, and in this case one has the formula $Ray(\mathrm{M},\rho) = \rho\cdot|p|^{1/(p-1)}\cdot\min_{0\leq i\leq}|g_i(T)|_\rho^{-1/(n-i)}$. This prove the proposition since for ρ close to zero $|g_i(T)|_\rho = a_i\rho^{v_T(g_i)}$, for some $a_i \in \mathbb{R}_+$. □

If $(\mathrm{M},\nabla_T^{\mathrm{M}})$ is solvable, one proves that there exist $\beta \geq 0$ and $\varepsilon' > 0$, such that $Ray(\mathrm{M},\rho) = \rho^\beta$, for all $1-\varepsilon' < \rho < 1$. One defines then the *p-adic slope* of M as $\beta - 1$.



Notice that the Formal slope is directly defined by the T-adic valuations of the g_i's whereas the p-adic slope is implicitly defined by the coefficients. No explicit formulas expressing the p-adic slope as a function of the valuations of the g_i's is known. For this reason Christol and Mebkhout provide then a break decomposition theorem for these p-adic slopes, reflecting the analogous decomposition in the formal framework, and then define the Newton polygon by means of the p-adic analogue of the formula (2.1.5) given by the following:

Theorem 2.2 ([CM00]). *Let* $(\mathrm{M}, \nabla_T^{\mathrm{M}})$ *be an* $\mathcal{R}_L$*-differential module endowed with a Frobenius structure (hence solvable). Then* M *admits a break decomposition* $\mathrm{M} = \oplus_{\beta \geq 0} \mathrm{M}(\beta)$*, where* $\mathrm{M}(\beta)$ *is characterized by the following properties: there exists* $\varepsilon > 0$ *such that*

1. *For all* $\rho \in]1-\varepsilon, 1[$*,* $\mathrm{M}(\beta)$ *is the biggest submodule of* M *trivialized by every ring* $\mathcal{A}_L(y, \rho^{\beta+1})$*, for all* Ω/L*, and all* $y \in \Omega$*, with* $|y| = \rho$*;*

2. *For all* $\rho \in]1-\varepsilon, 1[$*, for all* $|y| = \rho$*,* $y \in \Omega$*, for all* Ω/L*, and for all* $\beta' < \beta$*,* $\mathrm{M}(\beta)$ *has no solutions in* $\mathcal{A}_L(y, \rho^{\beta'+1})$*.*

The number $\mathrm{Irr}(\mathrm{M}) := \sum_{\beta \geq 0} \beta \cdot rank_{\mathcal{R}_K}(\mathrm{M}(\beta))$ *is called* p-adic irregularity *of* M*, and lies in* $\mathbb{N}$*.*

Theorem 2.3. *[CM00, 8.3.7] Let* $(\mathrm{M}, \nabla_T^{\mathrm{M}})$ *be an* $\mathcal{R}_L$*-differential module endowed with a Frobenius structure (hence solvable). Then* $\mathrm{Irr}(\mathrm{M})$*, above, coincides with* $\widetilde{\chi}(\mathrm{M}, \mathcal{A}_L(0,1))$ *defined in subsection 2.2.2*

Remark 2.4. *In practice the explicit computation of the p-adic slope is possible only if it is equal to the formal slope, i.e. if the the log-graphic of the function* $\rho \mapsto Ray(\mathrm{M}, \rho)$ *has no breaks and if, for* $\rho \in]1-\varepsilon, 1[$ *sufficiently close to* $1-\varepsilon$*, the* $Ray(\mathrm{M}, \rho)$ *is* sufficiently small *to be computed explicitly via [CM02, 6.2] as in the proof of Proposition 2.1. This is often done by considering Frobenius antecedents of* M*.*

2.2.4 Radius of convergence and spectral norm of the connection.

The generic radius of convergence admits the following description. For $\rho \in]1-\varepsilon, 1[$ set $\mathrm{M}_\rho := \mathrm{M} \otimes_{\mathcal{A}_L(]1-\varepsilon,1[)} \mathcal{F}_{L,\rho}$. Let $|.|_{\mathrm{M}_\rho}$ be a norm on M_ρ compatible with $|.|_\rho$. Denote by $|\nabla_T^{\mathrm{M}_\rho}|_{\mathrm{M}_\rho}$ and $|\nabla_T^{\mathrm{M}_\rho}|_{\mathrm{M}_\rho,\mathrm{Sp}}$ the norm and the spectral norm of $\nabla_T^{\mathrm{M}_\rho}$ respectively as operator on M_ρ (cf. def. 3.6). Making these definitions explicit with respect to a basis of M for which the matrix of the connection is G, one finds that $|\nabla_T^{\mathrm{M}_\rho}|_{\mathrm{M}_\rho,\mathrm{Sp}} = \max(\ \omega \cdot \rho^{-1}\ ,\ \limsup_{n\to\infty} |G_n|_\rho^{1/n}\ )$, where $\omega = |p|^{1/(p-1)}$ and where G_n is the matrix of $(\nabla_T^{\mathrm{M}})^n$ (cf. Lemma 3.8). Hence by Lemma 1.13 one has

$$Ray(\mathrm{M}, \rho) = \rho \cdot \frac{|d/dT|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|\nabla_T^{\mathrm{M}_\rho}|_{\mathrm{M}_\rho,\mathrm{Sp}}}\ . \tag{2.2.5}$$

The quantity $\frac{|d/dT|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|\nabla_T^{\mathrm{M}_\rho}|_{\mathrm{M}_\rho,\mathrm{Sp}}}$ is more intrinsic, and actually the p-adic slope is defined as the slope, for ρ close to 1^-, of the function $\rho \mapsto \frac{|d/dT|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|\nabla_T^{\mathrm{M}_\rho}|_{\mathrm{M}_\rho,\mathrm{Sp}}}$.

3 Differential Swan conductor in the non perfect residue field case

In this section we recall some basic definitions given in [Ked04], generalizing to the non perfect residual case the more classical analogous notions in [Fon90] and [Tsu98a].

3.1 (ϕ, ∇)-modules and differential modules

Definition 3.1. *Let* B_L *be one of the rings* $\mathcal{E}_L$*,* $\mathcal{E}_L^\dagger$*,* $\mathcal{R}_L$*,* $\mathcal{A}_L(I)$ *(resp.* $\mathcal{O}_{\mathcal{E}_L}$*,* $\mathcal{O}_{\mathcal{E}_L^\dagger}$*). A* differential module M *over* B_L *is a finite free* B_L*-module, together with an integrable connection*

$$\mathrm{M} \to \mathrm{M} \otimes \widehat{\Omega}^1_{\mathrm{B}_L/K} \tag{3.1.1}$$

(resp. $\mathrm{M} \to \mathrm{M} \otimes \widehat{\Omega}^1_{\mathrm{B}_L/\mathcal{O}_K}$*). Morphisms between differential modules commute with the connections. The category of differential modules will be called* $\nabla-\mathrm{Mod}(\mathrm{B}_L/K)$ *(resp.* $\nabla-\mathrm{Mod}(\mathrm{B}_L/\mathcal{O}_K)$*).*

Definition 3.2. *Let* B_L *be one of the rings* $\mathcal{E}_L$, $\mathcal{E}^\dagger_L$, $\mathcal{R}_L$, $\mathcal{O}_{\mathcal{E}_L}$, $\mathcal{O}_{\mathcal{E}^\dagger_L}$. *A* ϕ-module *(resp. a* (φ,∇)-module*) over* B_L *is a finite free* B_L*-module (resp.* ∇*-module)* D*, together with an isomorphism*

$$\phi^{\mathrm{D}} : \varphi^*(\mathrm{D}) \xrightarrow{\sim} \mathrm{D} \tag{3.1.2}$$

of B_L*-modules (resp. of* ∇*-modules). We interpret* ϕ^{D} *as a semi-linear action of* φ *on* D*. Morphisms between* ϕ*-modules (resp.* (φ,∇)*-modules) commute with the Frobenius (resp. with the Frobenius and the connection). The category of* ϕ*-modules (resp.* (φ,∇)*-modules) over* B_L *will be denoted by* $\varphi-\mathrm{Mod}(\mathrm{B}_L)$ *(resp.* $(\varphi,\nabla)-\mathrm{Mod}(\mathrm{B}_L/K)$ *or* $(\varphi,\nabla)-\mathrm{Mod}(\mathrm{B}_L/\mathcal{O}_K)$ *if* $\mathrm{B}_L = \mathcal{O}_{\mathcal{E}_L}, \mathcal{O}_{\mathcal{E}^\dagger_L}$*).*

Notation 3.3. *We denote* (φ,∇)*-modules with the letter* D*, and* ∇*-modules with the letter* M*.*

3.2 Representations with finite local monodromy and (φ,∇)-modules

We set

$$\begin{aligned} \mathrm{G_E} &:= \mathrm{Gal}(\mathrm{E}^{\mathrm{sep}}/\mathrm{E})\ , && (3.2.1)\\ \mathcal{I}_{\mathrm{G_E}} &:= \text{Inertia of } \mathrm{G_E}\ , && (3.2.2)\\ \mathcal{P}_{\mathrm{G_E}} &:= \text{Wild inertia of } \mathrm{G_E}\ . && (3.2.3) \end{aligned}$$

Definition 3.4. *We say that a representation* $\alpha : \mathrm{G_E} \to GL_n(\mathcal{O}_K)$ *has* finite local monodromy *if the image of* $\mathcal{I}_{\mathrm{G_E}}$ *under* α *is finite. We denote by* $\mathrm{V}(\alpha)$ *the representation defined by* α*, and by* $\mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E})$ *the category of representations with finite local monodromy.*

Following [Ked07] (and [Tsu98a]) we denote by $\widetilde{\mathcal{E}^\dagger_L}$ be the quotient field of the ring

$$\mathcal{O}_{\widetilde{\mathcal{E}^\dagger_L}} := \widehat{\mathcal{O}^{\mathrm{unr}}_L} \otimes_{\mathcal{O}^{\mathrm{unr}}_L} \mathcal{O}^{\mathrm{unr}}_{\mathcal{E}^\dagger_L}\ . \tag{3.2.4}$$

We denote again by φ, d/dT, $d/du_1,\ldots,d/du_r$ the unique extension to $\widetilde{\mathcal{E}^\dagger_L}$ of φ, d/dT, $d/du_1,\ldots$, d/du_r on $\mathcal{E}^\dagger_L$. If $\mathrm{V} \in \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E})$, we consider $\mathrm{V} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widetilde{\mathcal{E}^\dagger_L}}$ together with the action of $\mathrm{G_E}$ given by $\gamma(v \otimes x) := \gamma(v) \otimes \gamma(x)$, where $\gamma \in \mathrm{G_E}$, $v \in \mathrm{V}$, $x \in \widetilde{\mathcal{E}^\dagger_L}$. We define

$$\mathrm{D}^\dagger(\mathrm{V}) = (\mathrm{V} \otimes_{\mathcal{O}_K} \mathcal{O}_{\widetilde{\mathcal{E}^\dagger_L}})^{\mathrm{G_E}}\ , \tag{3.2.5}$$

and we consider it as an object of $(\varphi,\nabla)-\mathrm{Mod}(\mathcal{O}_{\mathcal{E}^\dagger_L}/\mathcal{O}_K)$ with the action of $\phi^{\mathrm{D}^\dagger(\mathrm{V})} := 1 \otimes \varphi$, $\nabla_T^{\mathrm{D}^\dagger(\mathrm{V})} := 1 \otimes \frac{d}{dT}$, $\nabla_{u_1}^{\mathrm{D}^\dagger(\mathrm{V})} := 1 \otimes \frac{d}{du_1}$, $\ldots$, $\nabla_{u_r}^{\mathrm{D}^\dagger(\mathrm{V})} := 1 \otimes \frac{d}{du_r}$. Reciprocally, for every $(\mathrm{D},\phi^{\mathrm{D}},\nabla^{\mathrm{D}}) \in (\varphi,\nabla)-\mathrm{Mod}(\mathcal{O}_{\mathcal{E}^\dagger_L}/\mathcal{O}_K)$ we consider $\mathrm{D} \otimes_{\mathcal{O}_{\mathcal{E}^\dagger_L}} \mathcal{O}_{\widetilde{\mathcal{E}^\dagger_L}}$ together with the action of $\phi^{\mathrm{D}} \otimes \varphi$, $(\nabla^{\mathrm{D}}_T \otimes 1 + 1 \otimes d/dT)$, $(\nabla^{\mathrm{D}}_{u_i} \otimes 1 + 1 \otimes d/du_i)$, $i = 1,\ldots,r$. We set then

$$\mathrm{V}^\dagger(\mathrm{D}) := (\mathrm{D} \otimes_{\mathcal{O}_{\mathcal{E}^\dagger_L}} \mathcal{O}_{\widetilde{\mathcal{E}^\dagger_L}})^{(\phi^{\mathrm{D}}\otimes\varphi)=1}\ , \tag{3.2.6}$$

and we consider it as an object of $\mathrm{Rep}_{\mathcal{O}_K}(\mathrm{G_E})$, with the action of $\mathrm{G_E}$ given by $\gamma(x \otimes y) := x \otimes \gamma(y)$, with $\gamma \in \mathrm{G_E}$, $x \in \mathrm{D}$, $y \in \mathcal{O}_{\widetilde{\mathcal{E}^\dagger}_L}$.

Proposition 3.5 ([Ked07, 3.3.6] and [Tsu98a]). *The above representation* $\mathrm{V}^\dagger(\mathrm{D})$ *has finite local monodromy. Moreover the functor*

$$\mathrm{V} \mapsto \mathrm{D}^\dagger(\mathrm{V}) : \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E}) \longrightarrow (\varphi,\nabla)-\mathrm{Mod}(\mathcal{O}_{\mathcal{E}^\dagger_L}/\mathcal{O}_K)\;, \tag{3.2.7}$$

is an equivalence of categories with quasi inverse $\mathrm{D} \mapsto \mathrm{V}^\dagger(\mathrm{D})$. □

3.3 The differential Swan conductor.

Let $\mathrm{M} \in \nabla-\mathrm{Mod}(\mathcal{A}_L(I))$. For all $\rho \in I$ we set (cf. (1.1.6))

$$\mathrm{M}_\rho := \mathrm{M} \otimes_{\mathcal{A}_L(I)} \mathcal{F}_{L,\rho}\;. \tag{3.3.1}$$

Definition 3.6. *Let* M_ρ *be a* ∇*-module over* $\mathcal{F}_{L,\rho}$*. Let* $|\cdot|_{\mathrm{M}_\rho}$ *be a norm on* M_ρ *compatible with the norm* $|\cdot|_\rho$ *of* $\mathcal{F}_{L,\rho}$*. For all* $\nabla^{\mathrm{M}_\rho} \in \{\nabla^{\mathrm{M}_\rho}_T, \nabla^{\mathrm{M}_\rho}_{u_1}, \ldots, \nabla^{\mathrm{M}_\rho}_{u_r}\}$*, we define*

$$\begin{aligned} |\nabla^{\mathrm{M}_\rho}|_{\mathrm{M}_\rho} &:= \sup_{m\in \mathrm{M}_\rho\setminus\{0\}} \frac{|\nabla^{\mathrm{M}_\rho}(m)|_{\mathrm{M}_\rho}}{|m|_{\mathrm{M}_\rho}}\;,\\ |\nabla^{\mathrm{M}_\rho}|_{\mathrm{M}_\rho,\mathrm{Sp}} &:= \limsup_{n\to\infty} |(\nabla^{\mathrm{M}_\rho})^n|^{1/n}_{\mathrm{M}_\rho}\;. \end{aligned} \tag{3.3.2}$$

The definition of $|\nabla^{\mathrm{M}_\rho}|_{\mathrm{M}_\rho,\mathrm{Sp}}$ *does not depend on the chosen norm* $|\cdot|_{\mathrm{M}_\rho}$*, but only on the norm* $|\cdot|_\rho$ *of* $\mathcal{F}_{L,\rho}$ *(cf. [Ked07, 1.1.7]).*

Definition 3.7 (([Ked07, 2.4.6])). *Let* M *be a* ∇*-module over* $\mathcal{A}_L(I)$*. We define the* (toric) generic radius of convergence *of* M *at* ρ *as (cf. Lemma 1.13):*

$$T(\mathrm{M},\rho) := \min\Big(\frac{|d/dT|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|\nabla^{\mathrm{M}_\rho}_T|_{\mathrm{M}_\rho,\mathrm{Sp}}}\;,\;\frac{|d/du_1|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|\nabla^{\mathrm{M}_\rho}_{u_1}|_{\mathrm{M}_\rho,\mathrm{Sp}}}\;,\;\ldots\;,\;\frac{|d/du_r|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|\nabla^{\mathrm{M}_\rho}_{u_r}|_{\mathrm{M}_\rho,\mathrm{Sp}}}\Big)\;. \tag{3.3.3}$$

This definition is a generalization to the case of non perfect residual field of the notion of radius of convergence as given at the end of section 2.

Lemma 3.8. *Let* M_ρ *be a* ∇*-module over* $\mathcal{F}_{L,\rho}$*. For* $i = 0,\ldots,r$ *let* $G^i_n \in M_d(\mathcal{F}_{L,\rho})$ *be the matrix of* $(\nabla^{\mathrm{M}_\rho}_{u_i})^n$ *(resp. if* $i=0$*,* G^0_n *is the matrix of* $(\nabla^{\mathrm{M}_\rho}_T)^n$*). Then*

$$|\nabla^{\mathrm{M}_\rho}_{u_i}|_{\mathrm{M}_\rho,\mathrm{Sp}} = \max(\,\omega\;,\;\limsup_{n\to\infty} |G^i_n|^{1/n}_\rho\,)\;, \tag{3.3.4}$$

where $\omega := |p|^{\frac{1}{p-1}}$ *(cf. Lemma 1.13), and*

$$|\nabla^{\mathrm{M}_\rho}_T|_{\mathrm{M}_\rho,\mathrm{Sp}} = \max(\,\omega\cdot\rho^{-1}\;,\;\limsup_{n\to\infty} |G^0_n|^{1/n}_\rho\,)\;. \tag{3.3.5}$$

Hence, by Definition 3.7, one has

$$T(\mathrm{M}_\rho,\rho) = \min_{i=1,\ldots,r}\Big(\,1\;,\;\omega\cdot\rho^{-1}\cdot\Big[\liminf_{n\to\infty}|G^0_n|^{-1/n}_\rho\Big]\;,\;\omega\cdot\Big[\liminf_{n\to\infty}|G^i_n|^{-1/n}_\rho\Big]\,\Big)\;. \tag{3.3.6}$$

Proof. This follows directly from the definition 3.7 (cf. [Ked07, (1.1.7.1)]). □

The following definition generalizes that one given in section 2.2.4 to the case of general residue field with finite p-basis.

Definition 3.9 (([Ked07, 2.5.1])). *Let* M *be a* ∇*-module over* $\mathcal{A}_L(]\varepsilon,1[)$*, with* $0 \le \varepsilon < 1$*. We will say that* M *is* solvable *if*

$$\lim_{\rho\to 1^-} T(\mathrm{M},\rho) = 1\;. \tag{3.3.7}$$

This is the case if the module has a Frobenius structure.

3.3.1

We recall that if $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a function, the log-function $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ attached to f is defined as $\widetilde{f}(r) := \log(f(\exp(r)))$. Note that if $f(\rho) = a\rho^b$ (for $\rho \in]\rho_1, \rho_2[$), then $\widetilde{f}(r) = \log(a) + br$ (for $r \in]\log(\rho_1), \log(\rho_2)[$).

Notation 3.10. *We will say that f has a given property* logarithmically *if $\widetilde{f}$ has that property.*

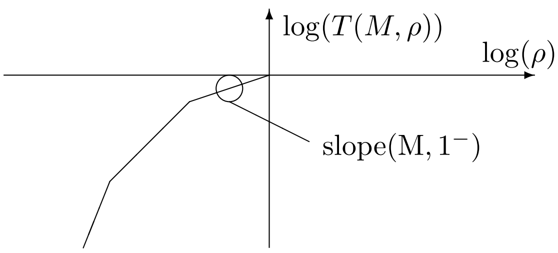
Definition 3.11 ([Ked07, 2.5.6])**.** *Let $\mathrm{M} \in \nabla - \mathrm{Mod}(\mathcal{R}_L/K)$ be a solvable ∇-module. The* slope *of* M *(at 1^-) is the log-slope of the function $\rho \mapsto T(\mathrm{M}, \rho)$, for $\rho < 1$ sufficiently close to 1^-. We denoted it by*

$$\mathrm{slope}(\mathrm{M}, 1^-)\ . \tag{3.3.8}$$

Let $\mathrm{M} \in \nabla - \mathrm{Mod}(\mathcal{A}_L(I)/K)$. In [Ked07] K.Kedlaya proved that the function $\rho \mapsto T(\mathrm{M}, \rho)$ is *continuous*, log-*concave*, *piecewise* log-*affine* (i.e. the function $\rho \mapsto T(\mathrm{M}, \rho)$ is locally of the type $a\rho^b$ in a partition of I.) and moreover, if M is solvable, that the Definition 3.11 has a meaning, since there exist a $\varepsilon < 1$, and a number β such that $T(\mathrm{M}, \rho) = \rho^\beta$, for all $\rho \in]\varepsilon, 1[$. In this case the slope of M is (cf. [Ked07, 2.5.5]):

$$\mathrm{slope}(\mathrm{M}, 1^-) = \beta. \tag{3.3.9}$$

The log-function $\log(\rho) \mapsto \log(T(\mathrm{M}, \rho))$ draws a graphic as the following:



(3.3.10)

Following [Ked07, Definitions 2.4.6, 1.2.3, 2.7.1] let M be a *solvable* ∇-module over $\mathcal{O}_{\mathcal{E}_L^\dagger}$. Assume that M is defined over $]\epsilon, 1[$. Let $\mathrm{M}_{\rho,1}, \mathrm{M}_{\rho,2}, \ldots, \mathrm{M}_{\rho,n}$ be the Jordan-Hölder factors of M_ρ, for $\rho \in]\epsilon, 1[$. We define $S(\mathrm{M}, \rho)$ as the multi-set whose elements are $T(\mathrm{M}_{\rho,1}, \rho), \ldots, T(\mathrm{M}_{\rho,n}, \rho)$ with multiplicity $\dim_{\mathcal{F}_\rho}\mathrm{M}_{\rho,1}, \ldots, \dim_{\mathcal{F}_\rho}\mathrm{M}_{\rho,n}$ respectively. We will say that M has *uniform break* β if there exists $\epsilon \leq \epsilon' < 1$ such that, for all $\rho \in]\epsilon', 1[$, $S(\mathrm{M}, \rho)$ consists in a single element ρ^β with multiplicity $\mathrm{rank}_{\mathcal{O}_{\mathcal{E}_L^\dagger}} \mathrm{M}$.

Theorem 3.12 ([Ked07, 2.7.2])**.** *We maintain the notation of section 3.3.1. Every indecomposable solvable ∇-module over $\mathcal{O}_{\mathcal{E}_L^\dagger}$ has a uniform break. In general, for M a solvable ∇-module we have a decomposition*

$$\mathrm{M} = \bigoplus_{\beta \in \mathbb{Q}_{\geq 0}} \mathrm{M}_\beta\ , \tag{3.3.11}$$

where M_β is a solvable ∇-module over $\mathcal{O}_{\mathcal{E}_L^\dagger}$-module, with uniform break β. □

Definition 3.13 (differential Swan conductor [Ked07, 2.8.1])**.** *Let* M *be a* solvable ∇*-module over $\mathcal{O}_{\mathcal{E}_L^\dagger}$. Let $\mathrm{M} = \oplus_{\beta \in \mathbb{Q}_{\geq 0}} \mathrm{M}_\beta$, be the decomposition of Theorem 3.12. We define the* differential Swan conductor *of* M *as*

$$\mathrm{sw}^\nabla(\mathrm{M}) \ :=\ \sum_{\beta \in \mathbb{Q}_{\geq 0}} \beta \cdot \mathrm{rank}_{\mathcal{O}_{\mathcal{E}_L^\dagger}} \mathrm{M}_\beta\ . \tag{3.3.12}$$

Moreover, for all $\mathrm{V} \in \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E})$, we set:

$$\mathrm{sw}^\nabla(\mathrm{V}) \ :=\ \mathrm{sw}^\nabla(\mathrm{D}^\dagger(\mathrm{V}))\ . \tag{3.3.13}$$

Remark 3.14. *If* $\mathrm{rank}_{\mathcal{O}_{\mathcal{E}_L^\dagger}} \mathrm{M} = 1$*, then* $\mathrm{sw}^\nabla(\mathrm{M}) = \mathrm{slope}(\mathrm{M}, 1^-)$.

3.3.2

The Toric Generic Radius of convergence $T(\mathrm{M}, \rho)$ can be seen as a *concrete* Radius of convergence of certain Taylor solutions (cf. [Ked07, Section 2.2]). Hence, by the usual properties of the Radius of convergence of solutions of a differential equation one has $\mathrm{sw}^\nabla(\mathrm{M}_1 \otimes \mathrm{M}_2) \leq \max(\mathrm{sw}^\nabla(\mathrm{M}_1), \mathrm{sw}^\nabla(\mathrm{M}_2))$ (resp. $T(\mathrm{M}_1 \otimes \mathrm{M}_2, \rho) \geq \min(T(\mathrm{M}_1, \rho), T(\mathrm{M}_2, \rho))$), and moreover equality holds if $\mathrm{sw}^\nabla(\mathrm{M}_1) \neq \mathrm{sw}^\nabla(\mathrm{M}_2)$ (resp. $T(\mathrm{M}_1, \rho) \neq T(\mathrm{M}_2, \rho)$).

3.3.3 Ramification filtration.

The group $\mathrm{G_E}$ is canonically imbedded into the Tannakian group of the category $\mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E})$. The definition of the differential Swan conductor, together with Theorem 3.12, defines a *ramification filtration* on $\mathrm{G_E}$. Indeed, by Theorem 3.12, we apply the formalism introduced in [And02] to define a filtration on the Tannakian group of $\mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E})$. Hence $\mathrm{G_E}$ inherits the filtration.

4 Arithmetic Swan conductor for rank one representations

In rank one case the arithmetic Swan conductor as defined by K.Kato (cf. [Kat89] and Def. 4.4) coincides with that one of A.Abbes and T.Saito (cf. [AS02], [AS06, 9.10]). In this section we recall Kato's definition of the Swan conductor of a rank one representation $\alpha : \mathrm{G_E} \to \mathbb{Q}/\mathbb{Z}$ and we describe completely Kato's filtration on $\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}/\mathbb{Z})$: we will use this later in our study of rank one p-adic representations with finite local monodromy. Indeed, in section 4.3, we obtain the decomposition $\mathrm{G}^{\mathrm{ab}}_{\mathrm{E}} = \mathcal{I}_{\mathrm{G}^{\mathrm{ab}}_{\mathrm{E}}} \oplus \mathrm{G}^{\mathrm{ab}}_k$. This fact will allow us to *define* the arithmetic Swan conductor for rank one representations with finite local monodromy (cf. Def. 4.17).

Remark 4.1. *In all the section 4, k is an arbitrary field of characteristic p (we do not assume that k has a finite p-basis).*

4.1 Kato's arithmetic Swan conductor for rank one representations

In [Kat89] K. Kato defined the Swan conductor of a character in $\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G_E}, \mathbb{Q}/\mathbb{Z})$. He gave this definition by introducing a filtration on $\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p)$ (cf. section 4.1.2). This filtration is defined as the image (via the Artin-Schreier-Witt sequence) of a filtration of $\mathbf{CW}(\mathrm{E})$. The filtration on $\mathbf{CW}(\mathrm{E})$ is nothing but that introduced in Section 1.4, corresponding to the canonical filtration of E (actually a graduation) induced by the valuation (cf. Section 1.4.1). In this section we describe completely these filtrations, without any assumption on k (no finiteness of the p-base is required), and without using Kato's logarithmic differentials and differential Swan conductors. Almost all material of this section comes from [Pul07], we reproduce it here for the convenience of the reader. What is new here is the relations of the results of [Pul07] with Kato's definitions. In section 5 one finds a link with the constructions of K.Kato.

4.1.1 Kato's Filtration on $\mathbf{W}_m(\mathrm{E})$ and $\mathbf{CW}(\mathrm{E})$.

Let R be a ring, and let $v : R \to \mathbb{R} \cup \{\infty\}$ be a valuation on R (i.e. satisfying $v(0) = \infty$, $v(\lambda_1 + \lambda_2) \geq \min(v(\lambda_1), v(\lambda_2))$, and $v(\lambda_1 \cdot \lambda_2) \geq v(\lambda_1) + v(\lambda_2)$). The valuation v of R extends to a valuation, denoted again by v, on the ring of Witt vectors $\mathbf{W}_s(R)$ as follows:

$$v(\lambda_0, \ldots, \lambda_s) \ := \ \min(\ p^s v(\lambda_0)\ ,\ p^{s-1} v(\lambda_1)\ ,\ \ldots\ ,\ v(\lambda_s)\ )\ . \tag{4.1.1}$$

The function $v : \mathbf{W}_s(R) \to \mathbb{R}$ verifies $v(\mathbf{0}) = +\infty$, $v(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2) \geq \min(v(\boldsymbol{\lambda}_1), v(\boldsymbol{\lambda}_2))$, and $v(\boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2) \geq v(\boldsymbol{\lambda}_1) + v(\boldsymbol{\lambda}_2)$, for all $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbf{W}_s(R)$. One has also $v(\mathrm{V}(\boldsymbol{\lambda})) = v(\boldsymbol{\lambda})$, and, if R is an $\mathbb{F}_p$-ring, then $v(\bar{\mathrm{F}}(\boldsymbol{\lambda})) = p \cdot v(\boldsymbol{\lambda})$. Hence v extends to a valuation

$$v : \mathbf{CW}(R) \to \mathbb{R} \cup \{+\infty\} \; . \tag{4.1.2}$$

Definition 4.2 (Kato's filtration on $\mathbf{W}_s(\mathrm{E})$)**.** *Let as usual* $\mathrm{E} := k(\!(t)\!)$*, and let* $v = v_t$ *be the* t*-adic valuation. The* Kato's filtration on $\mathbf{W}_s(\mathrm{E})$ *is defined as* $\mathrm{Fil}_{-1}(\mathbf{W}_s(\mathrm{E})) := 0$*, and, for all* $d \geq 0$*, as:*

$$\mathrm{Fil}_d(\mathbf{W}_s(\mathrm{E})) := \{\boldsymbol{\lambda} \in \mathbf{W}_s(\mathrm{E}) \mid v(\boldsymbol{\lambda}) \geq -d\} \; . \tag{4.1.3}$$

One has

$$\mathrm{V}(\ \mathrm{Fil}_d(\mathbf{W}_s(\mathrm{E}))\ ) \ \subset\ \mathrm{Fil}_d(\mathbf{W}_{s+1}(\mathrm{E}))\ , \qquad \bar{\mathrm{F}}(\ \mathrm{Fil}_d(\mathbf{W}_s(\mathrm{E}))\ ) \ \subseteq\ \mathrm{Fil}_{pd}(\mathbf{W}_s(\mathrm{E}))\ . \tag{4.1.4}$$

Hence Kato's filtration on $\mathbf{W}_s(\mathrm{E})$ passes to the limit and defines a filtration on $\mathbf{CW}(\mathrm{E})$. One has $\mathrm{Fil}_{-1}(\mathbf{CW}(\mathrm{E})) := 0$ and, for all $d \geq 0$, one has

$$\mathrm{Fil}_d(\mathbf{CW}(\mathrm{E})) \ :=\ \{\boldsymbol{\lambda} \in \mathbf{CW}(\mathrm{E}) \mid v(\boldsymbol{\lambda}) \geq -d\} \ =\ \bigcup_{s\geq 0} \mathrm{Fil}_d(\mathbf{W}_s(\mathrm{E}))\ . \tag{4.1.5}$$

4.1.2 Kato's Filtration on $\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p)$ and $\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}/\mathbb{Z})$.

Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. The *Pontriagyn dual* of G_E is by definition the discrete abelian group $\mathrm{Hom}^{\mathrm{cont}}$ $(\mathrm{G}_\mathrm{E}, \mathbb{T})$. Every proper closed subgroup of $\mathbb{T}$ is finite (cf. [RZ00, 2.9.1]). The image of a continuous morphism $\alpha : \mathrm{G}_\mathrm{E} \to \mathbb{T}$ is then finite, and hence

$$\mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\mathrm{E}, \mathbb{T}) = \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\mathrm{E}, \mathbb{Q}/\mathbb{Z}) = \bigcup_{n\geq 1} \mathrm{Hom}(\mathrm{G}_\mathrm{E}, (n^{-1}\mathbb{Z})/\mathbb{Z})\ , \tag{4.1.6}$$

where $\mathbb{Q}/\mathbb{Z} \subseteq \mathbb{T}$ is considered with the discrete topology, so that every continuous character $\alpha \in \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\mathrm{E}, \mathbb{Q}/\mathbb{Z})$ has finite image in $\mathbb{Q}/\mathbb{Z}$. On the other hand, we recall that if A is a finite abelian group with trivial action of G_E, then $\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathrm{A}) = \mathrm{Hom}(\mathrm{G}_\mathrm{E}, \mathrm{A})$ (cf. [Ser62, VII, §3]), so we have $\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\mathrm{E}, \mathbb{Q}/\mathbb{Z})$.

Definition 4.3. *The* Kato's filtration on $\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p)$ *is defined as the image, under the morphism* δ *of the Sequence* (1.3.3)*, of the filtration on* $\mathbf{CW}(\mathrm{E})$*:*

$$\mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p)) \ :=\ \delta(\mathrm{Fil}_d(\mathbf{CW}(\mathrm{E})))\ . \tag{4.1.7}$$

4.1.3 Arithmetic Swan conductor.

Definition 4.4 (Arithmetic Swan conductor)**.** *Let* $\alpha \in \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\mathrm{E}, \mathbb{Q}/\mathbb{Z})$*, the* arithmetic Swan conductor *of* α *(with respect to* $v = v_t$*) is defined as*

$$\mathrm{sw}(\alpha) = \min\{\ d \geq 0 \ \mid\ \alpha_p \in \mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p))\ \}\ , \tag{4.1.8}$$

where α_p *is the image of* α *under the projection*

$$\mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\mathrm{E}, \mathbb{Q}/\mathbb{Z}) = \bigoplus_{\ell = prime} \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\mathrm{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p)\ , \tag{4.1.9}$$

*where the word "*cont*" means, as usual, that the images of the homomorphisms are finite (cf. Section 4.1.2). We then define* Kato's filtration on $\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\mathrm{E}, \mathbb{Q}/\mathbb{Z})$*, by taking* $\mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}/\mathbb{Z}))$ *to be the inverse image of* $\mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p))$*, via the morphism* (4.1.9)*.*

Remark 4.5. *In section 4.3 we will generalize this definition to all rank one representations with* finite local monodromy *of* G_E*, i.e. characters* $\alpha : \mathrm{G}_\mathrm{E} \to \mathcal{O}_K^\times$ *such that* $\alpha(\mathcal{I}_{\mathrm{G}_\mathrm{E}})$ *is finite.*

4.2 Description of $\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p)$ and computation of $\mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}/\mathbb{Z}))$

In this section we recall an explicit description of $\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p)$ that was obtained in [Pul07, Section 3.2, and Lemma 4.1]. The proofs of the statements are in [Pul07]. In order to study Swan conductor, we are interested in $\mathrm{Hom}^{\mathrm{cont}}(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p)$. The Artin-Schreier-Witt sequence (1.3.3) describes the latter as

$$\mathrm{Hom}^{\mathrm{cont}}(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p) = \frac{\mathbf{CW}(\mathrm{E})}{(\bar{\mathrm{F}}-1)\mathbf{CW}(\mathrm{E})} \ . \tag{4.2.1}$$

Now the $\bar{\mathrm{F}}$-module $\mathbf{CW}(\mathrm{E})$ admits the following decomposition in $\bar{\mathrm{F}}$-sub-$\mathbb{Z}$-modules (cf. [Pul07, Lemma 3.4], cf. Section 1.2.1):

$$\mathbf{CW}(k(\!(t)\!)) = \mathbf{CW}(t^{-1}k[t^{-1}]) \oplus \mathbf{CW}(k) \oplus \mathbf{CW}(tk[[t]]) \ . \tag{4.2.2}$$

Moreover, one proves that $\mathbf{CW}(tk[[t]])/(\bar{\mathrm{F}}-1)(\mathbf{CW}(tk[[t]])) = 0$ (cf. [Pul07, Prop.3.1]). Hence

$$\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p) = \frac{\mathbf{CW}(t^{-1}k[t^{-1}])}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(t^{-1}k[t^{-1}]))} \oplus \frac{\mathbf{CW}(k)}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(k))} \ . \tag{4.2.3}$$

We will see that this decomposition corresponds (via the Pontriagyn duality) to a decomposition of the p-primary part of $\mathrm{G}_\mathrm{E}^{\mathrm{ab}}$ into the wild inertia $\mathcal{P}_{\mathrm{G}_\mathrm{E}^{\mathrm{ab}}}$ and the p-primary part of $\mathrm{G}_k^{\mathrm{ab}} = \mathrm{Gal}(k^{\mathrm{sep}}/k)^{\mathrm{ab}}$ (cf. Proof of Proposition 4.11).

Since the Swan conductor of an element of $\frac{\mathbf{CW}(k)}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(k))}$ is 0, we are led to study $\mathbf{CW}(t^{-1}k[t^{-1}])$ (corresponding to the wild inertia of $\mathrm{G}_\mathrm{E}^{\mathrm{ab}}$, cf. Formula (4.3.5)).

4.2.1 Description of $\mathbf{CW}(t^{-1}R[t^{-1}])$.

In this subsection R denotes an arbitrary ring, $v_p(d)$ denotes the p-adic valuation of d, and $v = v_t$ denotes the t-adic valuation on $R[[t]][t^{-1}]$. The decomposition (4.2.2) holds also for $R[[t]][t^{-1}]$.

Definition 4.6 (cf. [Pul07, Def.3.3]). *Let $d, n, m \in \mathbb{N}$ be such that $d = np^m > 0$, with $(n,p) = 1$. Let $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(R)$. We set*

$$\boldsymbol{\lambda} \cdot t^{-d} := (\cdots, 0, 0, 0, \lambda_0 t^{-n}, \lambda_1 t^{-np}, \ldots, \lambda_m t^{-d}) \ \in \ \mathbf{CW}(t^{-1}R[t^{-1}]) \ . \tag{4.2.4}$$

We call $\boldsymbol{\lambda} t^{-d}$ the co-monomial of degree $-d$ relative to $\boldsymbol{\lambda}$. *We denote by $\mathbf{CW}^{(-d)}(R)$ the subgroup of $\mathbf{CW}(t^{-1}R[t^{-1}])$ formed by co-monomials of degree $-d$.*

With the notation of Sections 1.4.3 and 1.4.1, if $A = R[T^{-1}]$, then one has for all $d > 0$

$$\mathbf{CW}^{(-d)}(R) \ = \ \mathbf{CW}^{(d)}(A) \ . \tag{4.2.5}$$

Proposition 4.7 ([Pul07, Remark 3.3, Lemma 3.4]). *Let $d = np^m > 0$, $(n,p) = 1$. The map*

$$\mathbf{W}_m(R) \xrightarrow{\ \sim\ } \mathbf{CW}^{(-d)}(R) \tag{4.2.6}$$

sending $\boldsymbol{\lambda} \in \mathbf{W}_m(R)$ into $\boldsymbol{\lambda} t^{-d} \in \mathbf{CW}(R)^{(-d)}$ is an isomorphism of groups. Moreover, one has

$$\mathbf{CW}(t^{-1}R[t^{-1}]) = \oplus_{d>0} \mathbf{CW}^{(-d)}(R) \ \ \cong \ \ \oplus_{d>0} \mathbf{W}_{v_p(d)}(R) \ , \tag{4.2.7}$$

where $v_p(d)$ is the p-adic valuation of d. In other words, every co-vector $\boldsymbol{f}^-(t) \in \mathbf{CW}(t^{-1}R[t^{-1}])$ can be uniquely written as a finite sum

$$\boldsymbol{f}^-(t) = \sum_{d>0} \boldsymbol{\lambda}_{-d} t^{-d} \ , \tag{4.2.8}$$

with $\boldsymbol{\lambda}_{-d} \in \mathbf{W}_{v_p(d)}(R)$, for all $d \geq 0$. □

This decomposition extends to $\mathbf{CW}(t^{-1}R[t^{-1}])$ the trivial decomposition $t^{-1}R[t^{-1}] = \oplus_{d>0} R\cdot t^{-d}$. Moreover a co-vector $\boldsymbol{f}(t) \in \mathbf{CW}(R[[t]][t^{-1}])$ can be uniquely written as a finite sum

$$\boldsymbol{f}(t) = \sum_{d>0} \boldsymbol{\lambda}_{-d} t^{-d} + \boldsymbol{f}_0 + \boldsymbol{f}^+(t) \;, \tag{4.2.9}$$

with $\boldsymbol{f}^+(t) \in \mathbf{CW}(tR[[t]])$, $\boldsymbol{f}_0 \in \mathbf{CW}(R)$, $\boldsymbol{\lambda}_{-d} \in \mathbf{W}_{v_p(d)}(R)$, for all $d > 0$. We denote by $\boldsymbol{f}^-(t)$ the co-vector $\sum_{d>0} \boldsymbol{\lambda}_{-d} t^{-d} \in \mathbf{CW}(t^{-1}R[t^{-1}])$.

Corollary 4.8. *The module $\mathbf{CW}(R[[t]][t^{-1}])$ is graded. Moreover for all $d \geq 0$, one has:*

$$\begin{aligned}
\mathrm{Fil}_d(\mathbf{CW}(R[[t]][t^{-1}])) \quad &= \quad \oplus_{1\leq d' \leq d} \mathbf{CW}^{(-d')}(R) \oplus \mathbf{CW}(R) \oplus \mathbf{CW}(tR[[t]]) \;, \\
\mathrm{Gr}_d(\mathbf{CW}(R[[t]][t^{-1}])) \quad &= \quad \mathrm{Fil}_d/\mathrm{Fil}_{d-1} = \begin{cases} \mathbf{CW}^{(-d)}(R) & \textit{if} \quad d > 0 \;, \\ \mathbf{CW}(R[[t]]) & \textit{if} \quad d = 0 \;. \end{cases}
\end{aligned}$$

Proof. The proof results from Lemma 1.6. For all $\boldsymbol{\lambda} \in \mathbf{W}_{v_p(d)}(R)$, $\boldsymbol{\lambda} \neq \mathbf{0}$, one has (cf. Def. (4.1.1)) $v_t(\boldsymbol{\lambda} t^{-d}) = -d$. Hence a co-vector $\boldsymbol{f}(t) = \sum_{d>0} \boldsymbol{\lambda}_{-d} t^{-d} + \boldsymbol{f}_0 + \boldsymbol{f}^+(t) \in \mathbf{CW}(R[[t]][t^{-1}])$ lies in $\mathrm{Fil}_d(\mathbf{CW}(R[[t]][t^{-1}]))$, $d \geq 0$, if and only if $\boldsymbol{\lambda}_{-d'} = \mathbf{0}$, for all $d' < -d$. □

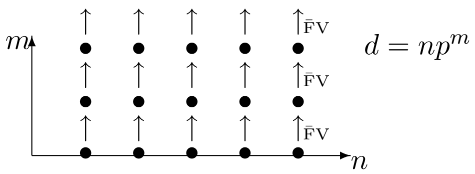
4.2.2 Action of Frobenius, and description of $\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p)$.

We want now to study the action of $\bar{\mathrm{F}}$ on $\mathbf{CW}(\mathrm{E})$, in order to describe $\mathbf{CW}(\mathrm{E})/(\bar{\mathrm{F}}-1)(\mathbf{CW}(\mathrm{E}))$.

One sees that $\bar{\mathrm{F}}(\boldsymbol{\lambda} \cdot t^{-d}) = \mathrm{V}\bar{\mathrm{F}}(\boldsymbol{\lambda}) \cdot t^{-pd}$, then one has the following commutative diagram, where the horizontal arrows are the isomorphisms (4.2.6):

$$\begin{tikzcd}
\mathbf{W}_{v_p(d)}(k) \arrow[r, "\sim"] \arrow[d, "p=\mathrm{V}\bar{\mathrm{F}}"'] \arrow[dr, phantom, "\odot"] & \mathbf{CW}^{(-d)}(k) \arrow[d, "\bar{\mathrm{F}}"] \arrow[r, phantom, "\subset"] \arrow[dr, phantom, "\odot"] & \mathbf{CW}(t^{-1}k[t^{-1}]) \arrow[d, "\bar{\mathrm{F}}"] \\
\mathbf{W}_{v_p(d)+1}(k) \arrow[r, "\sim"] & \mathbf{CW}^{(-pd)}(k) \arrow[r, phantom, "\subset"] & \mathbf{CW}(t^{-1}k[t^{-1}]) \;.
\end{tikzcd} \tag{4.2.10}$$

The Frobenius $\bar{\mathrm{F}}$ acts then on the family $\{\mathbf{CW}^{(-d)}(k)\}_{d\geq 1}$, as indicated in the following picture:



(4.2.11)

where $d = np^m \geq 1$, with $(n,p) = 1$, and $m = v_p(d)$. Hence for all $n \geq 1$, $(n,p) = 1$, the subgroup

$$\mathbf{C}_n(k) := \oplus_{m\geq 0} \mathbf{CW}^{(-np^m)}(k) \tag{4.2.12}$$

is a sub-$\bar{\mathrm{F}}$-module of $\mathbf{CW}(\mathrm{E})$, and

$$\frac{\mathbf{CW}(t^{-1}k[t^{-1}])}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(t^{-1}k[t^{-1}]))} = \bigoplus_{n\in \mathrm{J}_p} \frac{\mathbf{C}_n(k)}{(\bar{\mathrm{F}}-1)(\mathbf{C}_n(k))} \;, \tag{4.2.13}$$

where

$$\mathrm{J}_p := \{n \in \mathbb{N} \mid (n,p) = 1, n \geq 1\} \;. \tag{4.2.14}$$

One sees that

$$\frac{\mathbf{C}_n(k)}{(\bar{\mathrm{F}}-1)(\mathbf{C}_n(k))} = \varinjlim_{m\geq 0}(\mathbf{CW}^{(-np^m)}(k) \xrightarrow{\bar{\mathrm{F}}} \mathbf{CW}^{(-np^{m+1})}(k) \xrightarrow{\bar{\mathrm{F}}} \cdots) \tag{4.2.15}$$

$$\stackrel{(*)}{\cong} \varinjlim_{m\geq 0}(\mathbf{W}_m(k) \xrightarrow{p} \mathbf{W}_{m+1}(k) \xrightarrow{p} \cdots) = \widetilde{\mathbf{CW}}(k)\ , \tag{4.2.16}$$

where the isomorphism $(*)$ is deduced by the isomorphism (4.2.6).

Theorem 4.9. *The following statements hold:*

1. *One has:*

$$\mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Q}_p/\mathbb{Z}_p) \cong \frac{\mathbf{CW}(\mathrm{E})}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(\mathrm{E}))} \cong \widetilde{\mathbf{CW}}(k)^{(\mathrm{J}_p)} \oplus \frac{\mathbf{CW}(k)}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(k))}\ , \tag{4.2.17}$$

where $\widetilde{\mathbf{CW}}(k)^{(\mathrm{J}_p)}$ *is the direct sum of copies of* $\widetilde{\mathbf{CW}}(k)$*, indexed by* J_p*.*

2. *For* $d=0$ *one has* $\mathrm{Fil}_0(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Q}_p/\mathbb{Z}_p)) = \mathbf{CW}(k)/(\bar{\mathrm{F}}-1)\mathbf{CW}(k)$*, and for* $d\geq 1$ *one has (cf.* (1.2.1)*)*

$$\mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Q}_p/\mathbb{Z}_p)) = \oplus_{n\in \mathrm{J}_p}\Big(\mathrm{Fil}_{m_{n,d}}(\widetilde{\mathbf{CW}}(k))\Big) \oplus \frac{\mathbf{CW}(k)}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(k))}\ ,$$

$$\mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Q}_p/\mathbb{Z}_p)) = \begin{cases} \mathbf{W}_{v_p(d)}(k)/p\mathbf{W}_{v_p(d)}(k) & \text{if} \quad d>0\ ,\\ \frac{\mathbf{CW}(k)}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(k))} & \text{if} \quad d=0\ ,\end{cases}$$

where $m_{n,d} := \max\{m\geq 0 \mid np^m \leq d\}$*, and* $v_p(d)$ *is the* p*-adic valuation of* d *(we recall that* $\mathrm{Fil}_m(\widetilde{\mathbf{CW}}(k)) \cong \mathbf{W}_m(k)$ *(cf. section 1.2.3)).*

3. *The epimorphism* $\mathrm{Proj}_d : \mathrm{Gr}_d(\mathbf{CW}(\mathrm{E})) \to \mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Q}_p/\mathbb{Z}_p))$ *corresponds via the isomorphism* (4.2.6) *and Corollary 4.8 to the following:*

$$\mathrm{Proj}_d = \begin{cases} \mathbf{W}_{v_p(d)}(k) \longrightarrow \mathbf{W}_{v_p(d)}(k)/p\mathbf{W}_{v_p(d)}(k) & \text{if} \quad d>0\ ,\\ \mathbf{CW}(k[[t]]) \xrightarrow[t\mapsto 0]{} \frac{\mathbf{CW}(k)}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(k))} & \text{if} \quad d=0\ .\end{cases} \tag{4.2.18}$$

4. *One has*

$$\mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Q}/\mathbb{Z})) = \mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Q}_p/\mathbb{Z}_p)) \oplus \Big(\oplus_{\ell\neq p}\ \mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Q}_\ell/\mathbb{Z}_\ell)\Big)\ , \tag{4.2.19}$$

and

$$\mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Q}/\mathbb{Z})) = \begin{cases} \mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Q}_p/\mathbb{Z}_p)) & \text{if} \quad d>0\ ,\\ \mathrm{Fil}_0(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Q}_p/\mathbb{Z}_p)) \oplus \Big(\oplus_{\ell\neq p}\mathrm{H}^1(\mathrm{G}_{\mathrm{E}},\mathbb{Q}_\ell/\mathbb{Z}_\ell)\Big) & \text{if} \quad d=0\ .\end{cases} \tag{4.2.20}$$

Proof. The theorem follows immediately from the above computations. In particular from Corollary 4.8, Expression (1.2.1), Section 4.2.2, Equation (4.1.7), and Definition 4.3. We observe that $m_{n,d} = m_{n,d-1}+1$ if and only if $d = np^{m_{n,d}}$, $n\in \mathrm{J}_p$. □

Corollary 4.10. *Let* $\boldsymbol{\lambda} \in \mathbf{W}_m(k)$*, and let* $n\in\mathrm{J}_p$*. Let* $\alpha^- := \delta(\boldsymbol{\lambda}\cdot t^{-np^m})$*, where* δ *is the Artin-Schreier-Witt morphism (cf.* (1.3.3)*). If* $\boldsymbol{\lambda} \in p^k\mathbf{W}_m(k) - p^{k+1}\mathbf{W}_m(k)$*, then* $\mathrm{sw}(\alpha^-) = np^{m-k}$*.*
□

4.2.3 Minimal Lifting.

Let $\alpha \in \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p)$. Let $\alpha := \alpha^- \cdot \alpha_0$ be the decomposition given by Theorem 4.9. Then one can choose a lifting $\boldsymbol{f}(t)$ of α in $\mathbf{CW}(k[t^{-1}])$ (i.e. $\delta(\boldsymbol{f}) = \alpha$, cf. Sequence (1.3.3)) of the form ([Pul07, Def. 3.6]):

$$\boldsymbol{f}(t) = \sum_{n \in \mathrm{J}_p} \boldsymbol{\lambda}_{-n} t^{-np^{m(n)}} + \boldsymbol{f}_0 \ , \tag{4.2.21}$$

with $\boldsymbol{f}_0 \in \mathbf{CW}(k)$, and with $\boldsymbol{\lambda}_{-n} \in \mathbf{W}_{m(n)}(k) - p\mathbf{W}_{m(n)}(k)$, for every $n \in \mathrm{J}_p$ such that $\boldsymbol{\lambda}_{-n} \neq 0$. Such a lifting of α will be called a *minimal lifting*, and a Witt co-vector of the form (4.2.21) will be called *pure*. In this case one has

$$\mathrm{sw}(\alpha) = \max(\ 0\ ,\ \max\{\ np^{m(n)} > 0 \mid \boldsymbol{\lambda}_{-n} \neq 0\ \}\ )\ . \tag{4.2.22}$$

4.2.4 Representation of $\mathbf{W}_m(k)/p\mathbf{W}_m(k)$.

Let $\{\bar{u}_\gamma\}_{\gamma \in \Gamma}$ be a (not necessarily finite) p-basis of k (over k^p). We can write every element $\lambda \in k$ as $\lambda = \sum_{\underline{s} \in I_\Gamma} \lambda_{\underline{s}} \bar{u}^{\underline{s}}$, where $I_\Gamma := \{\underline{s} = (s_\gamma)_\gamma \in [0, p-1]^\Gamma$ such that $s_\gamma \neq 0$ for finitely many values of $\gamma\}$. Let $I'_\Gamma := I_\Gamma - \{(0, \ldots, 0)\}$. We denote by k' the sub-k^p-vector space of k with basis $\{\bar{u}^{\underline{s}}\}_{\underline{s} \in I'_\Gamma}$, that is the set of elements of k satisfying $\lambda_{\underline{0}} = 0$. Then an element of $\mathbf{W}_m(k)/p\mathbf{W}_m(k)$ admits a unique lifting in $\mathbf{W}_m(k)$ of the form $(\lambda_0, \lambda_1, \ldots, \lambda_m)$, satisfying $\lambda_0 \in k$, $\lambda_1, \ldots, \lambda_m \in k'$. Hence an element of $\mathbf{W}_m(k)/p\mathbf{W}_m(k)$ can be uniquely represented by a Witt vector in $\mathbf{W}_m(k)$ of this form.

4.3 Decomposition of $\mathrm{G_E^{ab}}$

The object of this section is the proof of Proposition 4.11 below. This fact was proved in [MS89]. The result is stated also in [Pul07, Remark 4.18] without any proof: here we will give a new proof using the framework we have just introduced. The reader must be cautious in reading [Pul07] to the fact that there is sometime a confusion between $\mathcal{I}^{\mathrm{ab}}_{\mathrm{G}_E}$ and $\mathcal{I}_{\mathrm{G_E^{ab}}}$. Every statement of [Pul07] is correct, if one consider $\mathcal{I}_{\mathrm{G_E^{ab}}}$ and $\mathcal{P}_{\mathrm{G_E^{ab}}}$, instead of $\mathcal{I}^{\mathrm{ab}}_{\mathrm{G_E}}$ and $\mathcal{P}^{\mathrm{ab}}_{\mathrm{G_E}}$ (cf. Remark 4.12). The main tool to prove Proposition 4.11 is Theorem 4.9 and its proof.

Proposition 4.11. *Let as usual $\mathrm{E} \cong k(\!(t)\!)$. One has:*

$$\mathrm{G_E^{ab}} = \mathcal{I}_{\mathrm{G_E^{ab}}} \oplus \mathrm{G}_k^{\mathrm{ab}}\ . \tag{4.3.1}$$

Proof. We can replace k with k^{perf}. Indeed we have a canonical isomorphism

$$\mathrm{Gal}(k^{\mathrm{perf}}(\!(t)\!)^{\mathrm{sep}}/k^{\mathrm{perf}}(\!(t)\!)) \xrightarrow{\ \sim\ } \mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!)), \tag{4.3.2}$$

identifying $\mathcal{I}_{\mathrm{G}_{k((t))}}$ and $\mathcal{P}_{\mathrm{G}_{k((t))}}$ with $\mathcal{I}_{\mathrm{G}_{k^{\mathrm{perf}}((t))}}$ and $\mathcal{P}_{\mathrm{G}_{k^{\mathrm{perf}}((t))}}$ respectively. This is proved by including both $k(\!(t)\!)^{\mathrm{sep}}$ and $k^{\mathrm{perf}}(\!(t)\!)^{\mathrm{sep}}$ in the t-adic completion $\widehat{k(\!(t)\!)^{\mathrm{alg}}}$ of $k(\!(t)\!)^{\mathrm{alg}}$. On the right hand side of (4.3.2) there is the logarithmic Abbes-Sato's filtration, while on the left hand side we have the classical filtration, as presented in [Ser62]. *These two filtrations are not preserved by the above isomorphism,* as proved by Theorem 4.9. We need now some lemmas (and we make a remark).

Remark 4.12. *We recall that if $K_1 \subset K_2 \subset K_3$ are Galois extensions of ultrametric complete valued fields with* perfect *residue fields $k_1 \subset k_2 \subset k_3$, then the map $\mathcal{I}_{\mathrm{Gal}(K_3/K_1)} \to \mathcal{I}_{\mathrm{Gal}(K_2/K_1)}$ is always surjective. We recall also that if K_1^{ab} (resp k_1^{ab}) is the maximal abelian extension of K_1 in K_2 (resp. of k_1 in k_2), then the residual field of K_1^{ab} is k_1^{ab}. Hence one has a surjective map $\mathcal{I}^{\mathrm{ab}}_{\mathrm{G_E}} \longrightarrow \mathcal{I}_{\mathrm{G_E^{ab}}}$, which is usually not an isomorphism.*

Lemma 4.13. *Let $\mathcal{I}, H$ be two subgroups of a given group* G*. Assume that $\mathcal{I}$ is normal in* G*, and that* G *is a semidirect product of $\mathcal{I}$ and H. Then* G^{ab} *is a direct product of a quotient $\mathcal{I}^{\mathrm{ab}}/N$ of $\mathcal{I}^{\mathrm{ab}}$ with H^{ab}.*

Proof. Straightforward. □

Lemma 4.14. *Assume that k is perfect. Let* $\mathrm{E}^{\mathrm{tame}}$ *be the maximal tamely ramified extension of* E *in* $\mathrm{E}^{\mathrm{sep}}$*. Then* $\mathrm{Gal}(\mathrm{E}^{\mathrm{tame}}/\mathrm{E})$ *is a semi-direct product of* $\mathcal{I}_{\mathrm{Gal}(\mathrm{E}^{\mathrm{tame}}/\mathrm{E})}$*, with* $\mathrm{G}_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$.

Proof. Let $\mathrm{E}^{\mathrm{unr}} \subset \mathrm{E}^{\mathrm{tame}}$ be the maximal unramified extension of E in $\mathrm{E}^{\mathrm{sep}}$. It is known that $\mathrm{E}^{\mathrm{tame}} = \cup_{(N,p)=1}\mathrm{E}^{\mathrm{unr}}(t^{1/N})$. In other words, every element $x \in \mathrm{E}^{\mathrm{tame}}$ can be written as $x = \sum_{i=0}^{N-1} a_i t^{i/N}$, for some N-th root $t^{1/N}$ of t, with $a_1, \ldots, a_{N-1} \in \mathrm{E}'$, where E' is the unramified extension of E associated to some finite Galois extension k' of k. Then via the isomorphism $\mathrm{Gal}(\mathrm{E}^{\mathrm{unr}}/\mathrm{E}) \xrightarrow{\sim} \mathrm{G}_k$, G_k acts on $\mathrm{E}^{\mathrm{tame}}$ by $\sigma(x) := \sum_{i=0}^{n} \sigma(a_i) t^{i/N}$. Hence the sequence

$$1 \to \mathcal{I}_{\mathrm{Gal}(\mathrm{E}^{\mathrm{tame}}/\mathrm{E})} \to \mathrm{Gal}(\mathrm{E}^{\mathrm{tame}}/\mathrm{E}) \to \mathrm{G}_k \to 1 \tag{4.3.3}$$

is splitting. □

Corollary 4.15. *Assume that k is perfect. Let* $\mathrm{E}^{\mathrm{tame,ab}} := \mathrm{E}^{\mathrm{tame}} \cap \mathrm{E}^{\mathrm{ab}}$*. Then*

$$\mathrm{Gal}(\mathrm{E}^{\mathrm{tame,ab}}/\mathrm{E}) = \mathcal{I}_{\mathrm{Gal}(\mathrm{E}^{\mathrm{tame,ab}}/\mathrm{E})} \times \mathrm{G}_k^{\mathrm{ab}} \ . \tag{4.3.4}$$

Proof. Apply Lemmas 4.13 and 4.14. □

Remark 4.16. *The decomposition of Corollary 4.15 is not unique (e.g. [FV02, IV.6, Ex.6]) and depends on the choice of a compatible system of N-th roots of t in the proof of Lemma 4.14.*

Continuation of the proof of Proposition 4.11: Let P (resp. P_k) be the p-primary subgroup of $\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}$ (resp. $\mathrm{G}_k^{\mathrm{ab}}$). By the classical properties of p-primary subgroups, P and P_k are direct factors of $\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}$ and $\mathrm{G}_k^{\mathrm{ab}}$. One has the exact sequence:

$$0 \to \mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}} \to \mathrm{P} \to \mathrm{P}_k \to 0 \ . \tag{4.3.5}$$

By the Artin-Schreier-Witt theory, one has canonical identifications of $\mathrm{Hom}^{\mathrm{cont}}(\mathrm{P}, \mathbb{Q}_p/\mathbb{Z}_p)$ with $\mathbf{CW}(\mathrm{E})/(\bar{\mathrm{F}}-1)(\mathbf{CW}(\mathrm{E}))$, and $\mathrm{Hom}^{\mathrm{cont}}(\mathrm{P}_k, \mathbb{Q}_p/\mathbb{Z}_p) = \mathbf{CW}(k)/(\bar{\mathrm{F}}-1)(\mathbf{CW}(k))$. On the other hand, one has the exact sequence

$$0 \leftarrow \frac{\mathbf{CW}(t^{-1}k[t^{-1}])}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(t^{-1}k[t^{-1}]))} \leftarrow \frac{\mathbf{CW}(\mathrm{E})}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(\mathrm{E}))} \leftarrow \frac{\mathbf{CW}(k)}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(k))} \leftarrow 0 \ . \tag{4.3.6}$$

Hence, by Pontriagyn duality, $\mathrm{Hom}^{\mathrm{cont}}(\mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}, \mathbb{Q}_p/\mathbb{Z}_p)$ is canonically identified to

$$\frac{\mathbf{CW}(t^{-1}k[t^{-1}])}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(t^{-1}k[t^{-1}]))}.$$

Since the sequence (4.3.6) splits, then the sequence (4.3.5) splits too.

Since P is the p-primary part of $\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}$, then $\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}} \cong \mathrm{P} \oplus (\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}/\mathrm{P})$. By Corollary 4.15 one finds $\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}/\mathrm{P} = \mathrm{Gal}(\mathrm{E}^{\mathrm{tame,ab}}/\mathrm{E})/(\mathrm{P}/\mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}) \cong (\mathcal{I}_{\mathrm{Gal}(\mathrm{E}^{\mathrm{tame,ab}}/\mathrm{E})} \times \mathrm{G}_k^{\mathrm{ab}})/(\mathrm{P}/\mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}) \cong \mathcal{I}_{\mathrm{Gal}(\mathrm{E}^{\mathrm{tame,ab}}/\mathrm{E})} \times (\mathrm{G}_k^{\mathrm{ab}}/\mathrm{P}_k)$. Indeed $\mathcal{I}_{\mathrm{Gal}(\mathrm{E}^{\mathrm{tame,ab}}/\mathrm{E})} = (\mathcal{I}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}/\mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}})$. This shows that $\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}} \cong \mathrm{P}_k \oplus \mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}} \oplus (\mathcal{I}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}/\mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}) \oplus (\mathrm{G}_k^{\mathrm{ab}}/\mathrm{P}_k)$. □

4.4 Definition of arithmetic Swan conductor for rank one representations with finite local monodromy

Since

$$\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}} \;\cong\; \mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}} \oplus (\mathcal{I}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}/\mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}) \oplus \mathrm{G}_k^{\mathrm{ab}} \;, \tag{4.4.1}$$

then every rank one representation with finite local monodromy $\alpha : \mathrm{G}_{\mathrm{E}} \to \mathcal{O}_K^{\times}$, $\mathrm{V}(\alpha) \in \mathrm{Rep}_{\mathcal{O}_K}^{\mathrm{fin}}(\mathrm{G}_{\mathrm{E}})$ is a product of three characters:

$$\alpha = \alpha_{\mathrm{wild}} \cdot \alpha_{\mathrm{tame}} \cdot \alpha_k \;, \tag{4.4.2}$$

where α_k (resp. α_{tame}, α_{wild}) is equal to 1 on $\mathcal{I}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}$ (resp. $\mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}} \oplus \mathrm{G}_k^{\mathrm{ab}}$, $(\mathcal{I}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}/\mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}) \oplus \mathrm{G}_k^{\mathrm{ab}}$). In term of representations, this is equivalent to the expression of $\mathrm{V}(\alpha)$ as a tensor product:

$$\mathrm{V}(\alpha) = \mathrm{V}(\alpha_{\mathrm{wild}}) \otimes \mathrm{V}(\alpha_{\mathrm{tame}}) \otimes \mathrm{V}(\alpha_k) \;. \tag{4.4.3}$$

Definition 4.17 (Swan conductor of a rank one representation with finite local monodromy)**.** *Let $\mathrm{V}(\alpha) \in \mathrm{Rep}_{\mathcal{O}_K}^{\mathrm{fin}}(\mathrm{G}_{\mathrm{E}})$ be a rank one representation with finite local monodromy. Let n be the greatest number such that $\boldsymbol{\mu}_n(\mathcal{O}_K) = \boldsymbol{\mu}_n(K^{\mathrm{alg}})$, and let*

$$\psi : \mathbb{Z}/n\mathbb{Z} \xrightarrow{\;\sim\;} \boldsymbol{\mu}_n(\mathcal{O}_K) \tag{4.4.4}$$

be a fixed identification. We define the Swan conductor*, $\mathrm{sw}(\mathrm{V}(\alpha))$, of α as*

$$\mathrm{sw}(\mathrm{V}(\alpha)) = \mathrm{sw}(\psi^{-1} \circ (\alpha_{\mathrm{wild}} \cdot \alpha_{\mathrm{tame}})) \;, \tag{4.4.5}$$

where, in the right hand side, we mean the Kato's definition of the character $\psi^{-1} \circ (\alpha_{\mathrm{wild}} \cdot \alpha_{\mathrm{tame}}) : \mathrm{G}_{\mathrm{E}} \longrightarrow \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ (cf. Def. 4.4).

The above definition does not depend on the choice of ψ. Indeed if $\psi' : \mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_n(\mathcal{O}_K)$ is another choice, there exists $N \in \mathbb{Z}$, with $(N, n) = 1$, such that $\psi' = (N\cdot) \circ \psi$, where $(N\cdot) : \mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ is the multiplication by N. Let $\beta := \psi^{-1} \circ (\alpha_{\mathrm{wild}} \cdot \alpha_{\mathrm{tame}})$, and $\beta' := (N\cdot) \circ \beta = \psi'^{-1} \circ (\alpha_{\mathrm{wild}} \cdot \alpha_{\mathrm{tame}})$. If n is prime to p, then $\beta(\mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}) = \beta'(\mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}) = \{0\}$ in $\mathbb{Z}/n\mathbb{Z}$. Hence the Swan conductor is equal to 0 in both cases (cf. Section 4.4.1 below). If n is not prime to p, let $\mathbb{Z}/p^{m+1}\mathbb{Z}$ be the p-primary part of $\mathbb{Z}/n\mathbb{Z}$. In this case N is prime to p: then the multiplication by N preserves Kato's filtration, it implies that the Swan conductors of β and $\beta' = (N\cdot) \circ \beta$ are equal.

4.4.1 Vanishing of residual and tame arithmetic Swan conductors.

Definition 4.17 agrees with Definition 4.4, and more precisely the Swan conductors of α_k and α_{tame} are always equal to 0. Indeed Definition 4.17 and Definition 4.4 coincide for α_{tame}, and, by Theorem 4.9, $\mathrm{sw}(\alpha_{\mathrm{tame}}) = 0$. Moreover, if the image of α_k is finite in $\mathcal{O}_K^{\times}$, then, by Theorem 4.9, one has $\mathrm{sw}(\alpha_k) = 0$, and this agrees with Definition 4.17. For all characters $\alpha : \mathrm{G}_{\mathrm{E}} \to \mathcal{O}_K^{\times}$, with $\mathrm{V}(\alpha) \in \mathrm{Rep}_{\mathcal{O}_K}^{\mathrm{fin}}(\mathrm{G}_{\mathrm{E}})$, one has

$$\mathrm{sw}(\alpha) = \mathrm{sw}(\alpha_{\mathrm{wild}}) \;. \tag{4.4.6}$$

Remark 4.18. *One sees that* $\mathrm{sw}(\mathrm{V}_1 \otimes \mathrm{V}_2) \leq \max(\mathrm{sw}(\mathrm{V}_1), \mathrm{sw}(\mathrm{V}_2))$*, for all* $\mathrm{V}_1, \mathrm{V}_2 \in \mathrm{Rep}_{\mathcal{O}_K}^{\mathrm{fin}}(\mathrm{G}_{\mathrm{E}})$*. Moreover equality holds if* $\mathrm{sw}(\mathrm{V}_1) \neq \mathrm{sw}(\mathrm{V}_2)$

5 Kato's refined Swan conductor

Let E be, as usual, a complete discrete valued field of characteristic p, with residue field k. We fix a uniformizer $t \in \mathcal{O}_{\mathrm{E}}$ and an isomorphism $\mathcal{O}_{\mathrm{E}} \cong k[\![t]\!]$. We identify k with its image in $\mathcal{O}_{\mathrm{E}}$ via that isomorphism. As usual for all $\mathbb{F}_p$-ring R we set $\Omega^1_R := \Omega^1_{R/R^p}$.

In [Kat89] K.Kato was able to introduce a filtration on Ω^1_{E} and then a family of submodules of the d-th graded $\mathrm{BGr}_d\Omega^1_{\mathrm{E}} \subset \mathrm{Gr}_d\Omega^1_{\mathrm{E}}$, $d \geq 0$. In such a way he was able to define an isomorphism, for all $d \geq 0$:

$$\psi_d \;:\; \mathrm{Gr}_d\mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\;\sim\;} \mathrm{BGr}_d\Omega^1_{\mathrm{E}} \;\subset\; \mathrm{Gr}_d\Omega^1_{\mathrm{E}}\;, \tag{5.0.7}$$

associating in this way to a character a 1-differential class. Whereas the arithmetic Swan conductor $\mathrm{sw}(\alpha)$ of a character $\alpha \in \mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Q}/\mathbb{Z})$ is the smallest integer $d \geq 0$, such that $\alpha \in \mathrm{Fil}_d\mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Q}/\mathbb{Z})$ (cf. Definition 4.4), the so called "*refined Swan conductor*" of α is the image of the class of α by the morphism:

$$\psi_{\mathrm{sw}(\alpha)} : \mathrm{Gr}_{\mathrm{sw}(\alpha)}\mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Gr}_{\mathrm{sw}(\alpha)}\Omega^1_{\mathrm{E}}\;. \tag{5.0.8}$$

The refined Swan conductor of α is defined only if $\mathrm{sw}(\alpha) > 0$, we denote it by $\mathrm{rsw}(\alpha)$.

In this section we interpret the refined Swan conductor, and the isomorphisms ψ_d's, in term of our isomorphism $\mathrm{Gr}_d\mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Q}/\mathbb{Z}) \cong \mathbf{W}_{v_p(d)}(k)/p \cdot \mathbf{W}_{v_p(d)}(k)$, if $d > 0$, (cf. Theorem 4.9): hence we explicitly associate a differential to a Witt vector on $\mathbf{W}_{v_p(d)}(k)$.

We improve the explicit description recently obtained by A.Abbes and T.Saito. We first recall the definition of ψ_d and the work of Abbes-Saito for the convenience of the reader (cf. Section 5.1). Then we apply our description to that context (cf. Section 5.2). The notation and settings come from [AS06]. In this section, according to [AS06], we assume that k has a finite p-basis $\{\bar{u}_1, \ldots, \bar{u}_r\}$.

5.1 Definition of Kato's refined Swan conductor and Abbes-Saito's computations.

5.1.1 Kato's Filtration of Ω^1_{E}.

We refer to [AS06, 5.4] for the formal definition of $\Omega^1_{k[\![t]\!]}(\log)$. Considering the trivialization $\mathrm{E} \cong k(\!(t)\!)$, this is nothing but

$$\Omega^1_{k[\![t]\!]}(\log) \;\cong\; \Big(\oplus_{i=1}^r k[\![t]\!] \cdot d\bar{u}_i\Big) \oplus k[\![t]\!] \cdot d\log(t) \;\;\subset\;\; \Big(\oplus_{i=1}^r \mathrm{E} \cdot d\bar{u}_i\Big) \oplus \mathrm{E} \cdot d\log(t) \;\cong\; \Omega^1_{\mathrm{E}}\;, \tag{5.1.1}$$

where $d\log(t) := dt/t \in \Omega^1_{\mathrm{E}}$. For all $d \geq 0$, one sets

$$\mathrm{Fil}_d\Omega^1_{\mathrm{E}} := t^{-d} \cdot \Omega^1_{k[\![t]\!]}(\log) \quad , \qquad \Omega^1_k(\log) := \Omega^1_{k[\![t]\!]}(\log) \otimes_{k[\![t]\!]} k\;. \tag{5.1.2}$$

For $d > 0$, the graded admits then the following trivialization

$$\mathrm{Gr}_d\Omega^1_{\mathrm{E}} \cong \Big(\oplus_{i=1}^r k \cdot t^{-d} \cdot d\bar{u}_i\Big) \oplus k \cdot t^{-d} \cdot d\log(t) \;=\; t^{-d} \cdot \Omega^1_k(\log)\;. \tag{5.1.3}$$

In particular Ω^1_{E} is graded: $\Omega^1_{\mathrm{E}} = \oplus_{d \geq 0}\mathrm{Gr}_d(\Omega^1_{\mathrm{E}})$.

5.1.2 Kato's isomorphism ψ_d.

We recall that, by [AS06, 10.7], for all $s \geq 0$, $d > 0$, there exists a unique group morphism $\psi_{s,d}$ making the following diagram commutative

$$\begin{tikzcd} \mathrm{Gr}_d \mathbf{W}_s(\mathrm{E}) \arrow[r, "-\mathrm{gr}_d(\mathrm{F}^s d)"] \arrow[d, "\mathrm{gr}_d(\delta)"'] & \mathrm{Gr}_d(\Omega^1_{\mathrm{E}}) \\ \mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Z}/p^{s+1}\mathbb{Z})) \arrow[ur, "\psi_{s,d}"'] & \end{tikzcd} \tag{5.1.4}$$

where $\delta : \mathbf{W}_s(\mathrm{E}) \to \mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Z}/p^{s+1}\mathbb{Z})$ is the Artin-Schreier-Witt morphism (cf. (1.3.2)), and $\mathrm{F}^s d : \mathbf{W}_s(\mathrm{E}) \to \Omega^1_{\mathrm{E}}$ is given by

$$\mathrm{F}^s d(\bar{f}_0, \ldots, \bar{f}_s) = \sum_{i=0}^{s} \bar{f}_i^{p^{s-i}} d\log(f_i) \ . \tag{5.1.5}$$

By [AS06, 10.8], the family of maps $\{\psi_{s,d}\}_{s \geq 0}$ is compatible with the inclusions $\jmath : \mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Z}/p^s\mathbb{Z}) \to \mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Z}/p^{s+1}\mathbb{Z})$ (cf. (1.3.2)). We have hence a map:

$$\psi_d : \mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Q}_p/\mathbb{Z}_p)) \to \mathrm{Gr}_d(\Omega^1_{\mathrm{E}}) \ . \tag{5.1.6}$$

5.1.3 The groups $\mathrm{B}_m\Omega^1_k$ and $\mathrm{BGr}_d(\Omega^1_{\mathrm{E}})$.

For each $q \geq 0$ we denote by

$$\mathrm{Z}^q\Omega^\bullet_k = \mathrm{Ker}(d : \Omega^q_k \to \Omega^{q+1}_k) \quad , \quad \mathrm{B}^q\Omega^\bullet_k = \mathrm{Im}(d : \Omega^{q-1}_k \to \Omega^q_k) \quad , \quad \mathrm{H}^q(\Omega^\bullet_k) = \mathrm{Z}^q\Omega^\bullet_k/\mathrm{B}^q\Omega^\bullet_k. \tag{5.1.7}$$

We denote the inverse Cartier isomorphism (cf. [Car58, Ch2, Section 6]) by

$$\mathrm{C}^{-1} \ : \ \Omega^q_k \ \xrightarrow{\sim} \ \mathrm{H}^q(\Omega^\bullet_k) \tag{5.1.8}$$

where $\mathrm{C} : \mathrm{Z}^q\Omega^\bullet_k \to \Omega^q_k$ is the Cartier operation. For $r \geq 0$ we introduce subgroups (cf. [Ill79, 2.2.2])

$$\mathrm{B}_m\Omega^q_k \ \subset \ \mathrm{Z}_m\Omega^q_k \ \subset \ \Omega^q_k \tag{5.1.9}$$

where $\mathrm{B}_0\Omega^q_k = 0$, $\mathrm{B}_1\Omega^q_k = \mathrm{B}^q\Omega^\bullet_k$ (resp. $\mathrm{Z}_0\Omega^q_k = \Omega^q_k$, $\mathrm{Z}_1\Omega^q_k = \mathrm{Z}^q\Omega^\bullet_k$), and inductively $\mathrm{B}_{m+1}\Omega^q_k$ (resp. $\mathrm{Z}_{m+1}\Omega^q_k$) is the inverse image in $\mathrm{Z}^q\Omega^\bullet_k$, under the canonical projection $\mathrm{Z}^q\Omega^\bullet_k \to \mathrm{H}^q(\Omega^\bullet_k)$, of $\mathrm{C}^{-1}(\mathrm{B}_m\Omega^q_k) \subset \mathrm{H}^q(\Omega^\bullet_k)$ (resp. $\mathrm{C}^{-1}(\mathrm{Z}_m\Omega^q_k) \subset \mathrm{H}^q(\Omega^\bullet_k)$). They respect the following inclusions

$$0 = \mathrm{B}_0\Omega^q_k \subset \cdots \subset \mathrm{B}_m\Omega^q_k \subset \mathrm{B}_{m+1}\Omega^q_k \subset \cdots \subset \mathrm{Z}_{m+1}\Omega^q_k \subset \mathrm{Z}_m\Omega^q_k \subset \cdots \subset \mathrm{Z}_0\Omega^q_k = \Omega^q_k \ . \tag{5.1.10}$$

By definition $\mathrm{C}(\mathrm{Z}_{m+1}\Omega^q_k) = \mathrm{Z}_m\Omega^q_k$ and $\mathrm{C}(\mathrm{B}_{m+1}\Omega^q_k) = \mathrm{B}_m\Omega^q_k$, we denote by $\mathrm{C}^m : \mathrm{Z}_m\Omega^q_k \to \Omega^q_k$ the m-th iteration of the Cartier operation ($\mathrm{C}^0 := \mathrm{Id}_{\Omega^q_k}$).

Definition 5.1 (([AS06, 10.11])). *Let as usual $d = np^m > 0$, $(n,p) = 1$, $m = v_p(d)$. We denote by $\mathrm{BGr}_d\Omega^1_{\mathrm{E}} \subset \mathrm{Gr}_d\Omega^1_{\mathrm{E}}$ the subgroup formed by elements of the form $t^{-d}(\alpha + \beta \cdot d\log(t))$ (cf. Formula (5.1.3)), with $\alpha \in \mathrm{B}_{m+1}\Omega^1_k$, $\beta \in \mathrm{Z}_m\Omega^0_k (= \mathrm{Z}_m k = k^{p^m})$ satisfying $n\mathrm{C}^m(\alpha) + d \circ \mathrm{C}^m(\beta) = 0$.*

Remark 5.2. *Notice that if $\beta = a^{p^m}$, $a \in k$, then $\mathrm{C}^m(\beta) = \beta^{p^{-m}} = a$. In particular, for $m = 0$, and $(n,p) = 1$, one finds $\mathrm{BGr}_n\Omega^1_{\mathrm{E}} = \{t^{-n}(d(x) - n \cdot x \cdot d\log(t)) \mid x \in k\} \subset t^{-n} \cdot \Omega^1_k(\log) \xrightarrow{\sim} \mathrm{Gr}_n\Omega^1_{\mathrm{E}}$.*

5.1.4 Abbes-Saito's explicit description of $\mathrm{BGr}_d\Omega^1_{\mathrm{E}}$.

Let $\{\bar{u}_1, \ldots, \bar{u}_r\}$ be a p-basis of k (over $\mathbb{F}_p$). Let $I_r := [0, p-1]^r \subset \mathbb{N}^r$, and $I'_r := I_r - \{(0, \ldots, 0)\}$. For $\underline{s} := (s_1, \ldots, s_r) \in I_r$, we set $\bar{u}^{\underline{s}} := \bar{u}_1^{s_1} \cdots \bar{u}_r^{s_r}$. Every λ of k can be uniquely written as $\lambda = \sum_{\underline{s} \in I_r} \lambda_{\underline{s}} \cdot \bar{u}^{\underline{s}}$, with $\lambda_{\underline{s}} \in k^p$. Hence $d(\lambda) = \sum_{\underline{s} \in I'_r} \lambda_{\underline{s}} \cdot \bar{u}^{\underline{s}} \cdot d\log(\bar{u}^{\underline{s}}) \in \Omega^1_k$, where, as usual, $d\log(\bar{u}^{\underline{s}}) = \sum_{i=1}^r s_i \cdot d\bar{u}_i / \bar{u}_i$. For $j \geq 0$, we set

$$d_{\mathrm{F}^j}(\lambda) := \sum_{\underline{s} \in I'_r} \lambda_{\underline{s}}^{p^j} \cdot (\bar{u}^{\underline{s}})^{p^j} \cdot d\log(\bar{u}^{\underline{s}}) \ . \qquad (5.1.11)$$

Since Ω^1_k is freely generated by $d\bar{u}_1/\bar{u}_1, \ldots, d\bar{u}_r/\bar{u}_r$, then, by assuming $\lambda_{\underline{0}} = 0$, $d_{\mathrm{F}^j}(\lambda)$ determines λ. We denote by $k' \subset k$, the sub-k^p-vector space of k with basis $\{\bar{u}^{\underline{s}}\}_{\underline{s} \in I'_r}$, i.e. whose elements $\lambda = \sum_{\underline{s} \in I_r} \lambda_{\underline{s}} \bar{u}^{\underline{s}}$, $\lambda_{\underline{s}} \in k^p$, satisfy $\lambda_{\underline{0}} = 0$.

Proposition 5.3. *([AS06, 10.12]) Let $d = np^m > 0$, $(n,p) = 1$, $m = v_p(d)$. Every element y_d of $\mathrm{BGr}_d(\Omega^1_{\mathrm{E}})$ can be uniquely written as*

$$y_d \ = \ t^{-d} \cdot \Big(\lambda_0^{p^{m-1}} d(\lambda_0) - n \lambda_0^{p^m} d\log(t) + \sum_{1 \leq j \leq m} d_{\mathrm{F}^{m-j}}(\lambda_j) \Big) \ \in \ t^{-d} \cdot \Omega^1_k(\log) \xrightarrow{\sim} \mathrm{BGr}_d(\Omega^1_{\mathrm{E}}) \ , \qquad (5.1.12)$$

with unique choice of $\lambda_1, \ldots, \lambda_m \in k'$, and $\lambda_0 \in k$. We will write $y_d = y_d(\lambda_0, \lambda_1, \ldots, \lambda_m)$. It is understood that, if $m = 0$, this sum reduces to $y_d = t^{-d} \cdot (d(\lambda_0) - n\lambda_0 d\log(t))$ (cf. Remark 5.2).
□

Remark 5.4. *The reader should notice the analogy with Section 4.2.4.*

In [AS06] one proves the following description of ψ_d (or better its inverse ρ_d) by means of the group $\mathrm{BGr}_d(\Omega^1_{\mathrm{E}})$ (cf. [AS06, 10.13]).

Proposition 5.5 (([AS06, 10.14])). *For all $d > 0$, $d = np^{v_p(d)}$, the image of ψ_d is $\mathrm{BGr}_d\Omega^1_{\mathrm{E}}$. Let*

$$\rho_d \ : \ \mathrm{BGr}_d(\Omega^1_{\mathrm{E}}) \xrightarrow{\sim} \mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Q}_p/\mathbb{Z}_p)) \qquad (5.1.13)$$

be the inverse of ψ_d. Let $\lambda_0 \in k$, $\lambda_1, \ldots, \lambda_{v_p(d)-1} \in k'$ and let $y_d(\lambda_0, \lambda_1, \ldots, \lambda_{v_p(d)})$ be the element (5.1.12) of $\mathrm{BGr}_d(\Omega^1_{\mathrm{E}})$. For $j = 1, \ldots, v_p(d)$ write $\lambda_j = \sum_{\underline{s} \in I'_r} \lambda_{j,\underline{s}} \bar{u}^{\underline{s}}$. Then $y_d(\lambda_0, \lambda_1, \ldots, \lambda_{v_p(d)})$ is sent, by ρ_d into

$$\rho_d(y_d(\lambda_0, \lambda_1, \ldots, \lambda_{v_p(d)})) \ = \mathrm{pr}_d \Big(\theta_{v_p(d)}(\lambda_0 t^{-n}) + \sum_{1 \leq j \leq v_p(d)} \sum_{\underline{s} \in I'_r} \theta_{v_p(d)-j}(\lambda_{j,\underline{s}} t^{-np^j}) \Big) \ , \qquad (5.1.14)$$

where $\theta_j : \mathrm{E} \to \mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Q}_p/\mathbb{Z}_p)$ is the composite of the j-th Teichmüller map $f \mapsto (f, 0, \ldots, 0) \in \mathbf{W}_j(\mathrm{E})$ with the Artin-Schreier-Witt morphism $\delta : \mathbf{W}_j(\mathrm{E}) \to \mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Z}/p^{j+1}\mathbb{Z}) \subset \mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Q}_p/\mathbb{Z}_p)$ (cf. (1.3.2)), and pr_d is the canonical projection of H^1 into $\mathrm{Gr}_d(\mathrm{H}^1)$. □

5.2 Explicit description of ψ_d in terms of Witt co-vectors.

The following Theorem improves the description given by proposition 5.5.

Theorem 5.6. *Let $d > 0$, $d = np^m$, $m = v_p(d)$. Let*

$$\overline{\delta}_d : \mathbf{W}_{v_p(d)}(k)/p\mathbf{W}_{v_p(d)}(k) \xrightarrow{\sim} \mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Q}/\mathbb{Z})) \qquad (5.2.1)$$

be the isomorphism of Theorem 4.9. Then the composite map

$$\psi_d \circ \overline{\delta}_d : \mathbf{W}_{v_p(d)}(k)/p\mathbf{W}_{v_p(d)}(k) \xrightarrow{\sim} \mathrm{BGr}_d(\Omega^1_{\mathrm{E}}) \qquad (5.2.2)$$

is given by

$$\psi_d \circ \bar{\delta}_d(\lambda_0, \dots, \lambda_{v_p(d)}) = t^{-d} \cdot \Big(-n\lambda_0^{p^{v_p(d)}} d\log(t) + \sum_{0 \le j \le v_p(d)} \lambda_j^{p^{v_p(d)-j}} \cdot d\log(\lambda_j) \Big) , \tag{5.2.3}$$

where we represent every element of $\mathbf{W}_{v_p(d)}(k)/p\mathbf{W}_{v_p(d)}(k)$ *by a unique Witt vector* $(\lambda_0, \dots, \lambda_{v_p(d)}) \in \mathbf{W}_{v_p(d)}(k)$ *satisfying* $\lambda_0 \in k$, $\lambda_1, \dots, \lambda_{v_p(d)} \in k'$, *as in Section 4.2.4. Moreover every element of* $\mathrm{BGr}_d(\Omega^1_{\mathrm{E}})$ *can be uniquely written as*

$$t^{-d} \cdot \Big(-n\lambda_0^{p^{v_p(d)}} d\log(t) + \sum_{0 \le j \le v_p(d)} \lambda_j^{p^{v_p(d)-j}} d\log(\lambda_j) \Big) , \tag{5.2.4}$$

with unique $\lambda_0 \in k$ *and* $\lambda_1, \dots, \lambda_{v_p(d)} \in k'$ *(cf. Section 4.2.4).*

Proof. Passing to the limit (with respect to s) of the Diagram (5.1.4), since $\mathrm{F}^{s+1}d \circ \mathrm{V} = \mathrm{F}^s d$, we define

$$\mathrm{F}^\infty d = \lim_{s \to \infty} \mathrm{F}^s d \ : \ \mathbf{CW}(\mathrm{E}) \to \Omega^1_K . \tag{5.2.5}$$

We obtain $\psi_s = \mathrm{gr}_d(\delta) \circ \mathrm{gr}(\mathrm{F}^\infty d)$, where $\mathrm{gr}_d(\delta) : \mathrm{Gr}_d(\mathbf{CW}(\mathrm{E})) \to \mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Q}_p/\mathbb{Z}_p))$. Since $\mathbf{CW}(\mathrm{E})$ is graded with the d-th graded $\mathbf{CW}^{(-d)}(k)$ (cf. Corollary 4.8), the map $\mathrm{gr}(\mathrm{F}^\infty d)$ (resp. $\mathrm{gr}_d(\delta)$) is nothing but the restriction of $\mathrm{F}^\infty d$ (resp. δ) to the subgroup $\mathbf{CW}^{(-d)}(k)$.

The elements of $\mathbf{CW}^{(-d)}(k)$ have the form $\boldsymbol{\lambda}_d t^{-d} = (\cdots, 0, 0, \lambda_0 t^{-n}, \dots, \lambda_m t^{-np^m})$, with $\boldsymbol{\lambda}_d := (\lambda_0, \dots, \lambda_m) \in \mathbf{W}_m(k)$. One easily sees that (cf. (5.1.5))

$$\mathrm{F}^\infty(\boldsymbol{\lambda}_d t^{-d}) = \sum_{0 \le j \le m} (\lambda_j t^{-np^j})^{p^{m-j}} d\log(\lambda_j t^{-np^j}) = t^{-d} \cdot \Big(\sum_{0 \le j \le m} \lambda_j^{p^{m-j}} d\log(\lambda_j t^{-np^j}) \Big) . \tag{5.2.6}$$

This proves the first assertion. To conclude we observe that the diagram

$$\begin{tikzcd} \mathbf{W}_m(k)/p\mathbf{W}_m(k) \arrow[r, "\mathrm{F}^m d", "\sim"'] \arrow[d, "\bar{\delta}", "\wr"'] & \mathrm{BGr}_d(\Omega^1_{\mathrm{E}}) \\ \mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Q}_p/\mathbb{Z}_p)) \arrow[ur, "\sim", "\psi_d"'] & \end{tikzcd} \tag{5.2.7}$$

implies that the map $\psi_d \circ \bar{\delta} : \mathbf{W}_m(k)/p\mathbf{W}_m(k) \to \mathrm{Gr}_d(\Omega^1_{\mathrm{E}})$ is injective with image $\mathrm{BGr}_d(\Omega^1_{\mathrm{E}})$. By section 4.2.4 every element of $\mathbf{W}_m(k)/p\mathbf{W}_m(k)$ can be uniquely represented by a Witt vector $(\lambda_0, \dots, \lambda_m) \in \mathbf{W}_m(k)$ satisfying $\lambda_0 \in k$, $\lambda_1, \dots, \lambda_m \in k'$. This concludes the proof. □

6 Explicit computation of Fontaine's functor in rank one case

In this section we compute the (ϕ, ∇)-module $\mathrm{D}^\dagger(\mathrm{V}(\alpha))$ (cf. Section 3.2) for rank one representations with finite local monodromy. Following Section 4.3 we decompose a general character $\alpha : \mathrm{G}_{\mathrm{E}} \to \mathcal{O}_K^\times$ into $\alpha = \alpha_{\mathrm{wild}} \cdot \alpha_{\mathrm{tame}} \cdot \alpha_k$. We study then separately the *residual case*, the *tame case*, and the *wild case*.

6.1 The residual case

Let $\alpha_k : \mathrm{G}_{\mathrm{E}} \to \mathcal{O}_K^\times$ be a rank one representation with finite local monodromy such that $\alpha_k|_{\mathcal{I}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}}} = 1$. One sees, directly from the definition of the functor (cf. Section 3.2), that the (φ, ∇)-module

$\mathrm{D} := \mathrm{D}^\dagger(\mathrm{V}(\alpha_k))$ is trivialized by some unramified extension of $\mathcal{O}_L$. In other words, D comes from a (φ,∇)-module over $\mathcal{O}_L$ by scalar extension. Let $\mathbf{e}_\mathrm{D}$ be basis of such a lattice, then

$$\left\{\begin{array}{rcl} \phi^{\mathrm{D}}(\mathbf{e}_\mathrm{D}) & = & \theta(\underline{u})\cdot\mathbf{e}_\mathrm{D} \ , \\ \nabla_T^{\mathrm{D}}(\mathbf{e}_\mathrm{D}) & = & 0 \ , \\ \nabla_{u_i}^{\mathrm{D}}(\mathbf{e}_\mathrm{D}) & = & g_k(\underline{u})\cdot\mathbf{e}_\mathrm{D} \ , \end{array}\right. \tag{6.1.1}$$

where $\underline{u} := (u_1,\ldots,u_r)$, and $\theta(\underline{u}), g_k(\underline{u}) \in \mathcal{O}_L$.

Remark 6.1. *If the image of α_k is finite, it is actually possible, by using the theorem [Pul07, Th.2.8], to express $\theta(\underline{u})$ as value at $T=1$ of a certain overconvergent function, but this is not necessary for our purposes.*

6.2 The tamely ramified case

Let $\alpha_{\mathrm{tame}} : \mathrm{G_E} \to \mathcal{O}_K^\times$ be a character, with $\mathrm{V}(\alpha_{\mathrm{tame}}) \in \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E})$, such that $\alpha_{\mathrm{tame}}|_{\mathcal{P}_{\mathrm{G_E^{ab}}}\oplus \mathrm{G}_k^{\mathrm{ab}}} = 1$ (in the decomposition (4.4.1)). One has $\alpha_{\mathrm{tame}}(\mathrm{G_E}) = \boldsymbol{\mu}_N \subset \mathcal{O}_K^\times$, for some $(N,p)=1$. In particular $\boldsymbol{\mu}_N(\mathbb{F}_p^{\mathrm{alg}}) \subset \mathbb{F}_q \subseteq k$. By Kummer theory, the kernel of α_{tame} defines an extension of $k(\!(t)\!)$ of the type $k(\!(t^{1/N})\!)/\mathrm{E}$ (i.e. $\mathrm{Ker}(\alpha_{\mathrm{tame}}) = \mathrm{Gal}(\mathrm{E}^{\mathrm{sep}}/k(\!(t^{1/N})\!))$). This is the smallest extension trivializing $\mathrm{V}(\alpha_{\mathrm{tame}})$, and the action of $\mathrm{G_E}$ on $t^{1/N}$ is given by

$$\gamma(t^{1/N}) = \alpha_{\mathrm{tame}}(\gamma)\cdot t^{1/N} \ , \quad \text{for all } \gamma\in\mathrm{G_E} \ . \tag{6.2.1}$$

Then the unramified extension of $\mathcal{O}_{\mathcal{E}_L^\dagger}$ corresponding to $k(\!(t^{1/N})\!)/k(\!(t)\!)$ is given by

$$\mathcal{O}_{\mathcal{E}^\dagger_{L,T^{1/N}}}/\mathcal{O}_{\mathcal{E}^\dagger_{L,T}} \ , \tag{6.2.2}$$

where the notation indicates the variable of the rings as in [Tsu98b]. The inclusion $\mathcal{O}_{\mathcal{E}^\dagger_{L,T}} \subset \mathcal{O}_{\mathcal{E}^\dagger_{L,T^{1/N}}}$ is given by sending T into $(T^{1/N})^N$. Let $\mathbf{e}_\mathrm{V} \in \mathrm{V}(\alpha)$ be a basis in which $\mathrm{G_E}$ acts as

$$\gamma(\mathbf{e}_\mathrm{V}) = \alpha_{\mathrm{tame}}(\gamma)\cdot\mathbf{e}_\mathrm{V} \ , \quad \text{for all } \gamma\in\mathrm{G_E} \ . \tag{6.2.3}$$

This shows that a basis of $\mathrm{D} := \mathrm{D}^\dagger(\mathrm{V}(\alpha)) = (\mathrm{V}(\alpha)\otimes_{\mathcal{O}_K}\mathcal{O}_{\widetilde{\mathcal{E}_L^\dagger}})^{\mathrm{G_E}}$ is given by

$$\mathbf{e}_\mathrm{D} \ := \ \mathbf{e}_\mathrm{V}\otimes T^{-1/N} \ . \tag{6.2.4}$$

Indeed, for all $\gamma\in\mathrm{G_E}$, one has $\gamma(\mathbf{e}_\mathrm{D}) = \gamma(\mathbf{e}_\mathrm{V})\otimes\gamma(T^{-1/N}) = \alpha_{\mathrm{tame}}(\gamma)\mathbf{e}_\mathrm{V}\otimes\alpha_{\mathrm{tame}}(\gamma)^{-1}T^{-1/N} = \mathbf{e}_\mathrm{D}$. In this basis the action of ϕ and ∇ are given by

$$\left\{\begin{array}{rcl} \phi^{\mathrm{D}}(\mathbf{e}_\mathrm{D}) & = & T^{(1-q)/N}\cdot\mathbf{e}_\mathrm{D} \ , \\ \nabla_T^{\mathrm{D}}(\mathbf{e}_\mathrm{D}) & = & -\frac{1}{N}T^{-1}\cdot\mathbf{e}_\mathrm{D} \ , \\ \nabla_{u_i}^{\mathrm{D}}(\mathbf{e}_\mathrm{D}) & = & 0 \ . \end{array}\right. \tag{6.2.5}$$

In particular, the solution of this differential equation is $T^{-1/N}$, which is simultaneously the Kummer generator of the smallest extension of $\mathcal{E}_L^\dagger$ trivializing $\mathrm{V}(\alpha_{\mathrm{tame}})$.

6.3 The wild ramified case

To compute $\mathrm{D}^\dagger(\mathrm{V}(\alpha_{\mathrm{wild}}))$ we need to know a Kummer generator of the smallest unramified extension of $\mathcal{E}_L^\dagger$ trivializing $\mathrm{D}^\dagger(\mathrm{V}(\alpha_{\mathrm{wild}}))$. In the *tame* case (cf. section 6.2), the Kummer generator was $T^{-1/N}$, and it was at the same time the solution of the differential equation. In the *wild* case, the good Kummer generator will be a so called $\boldsymbol{\pi}$*-exponential*, and it will be at the same time the solution of the differential equation defined by $\mathrm{D}^\dagger(\mathrm{V}(\alpha_{\mathrm{wild}}))$ too.

All results and proofs of this section come from [Pul07]. We outline briefly the contents of [Pul07, Sections 2.3.1, 3.1, 3.2], and in Section 6.3.2 we adapt [Pul07, Section 4.6] to our context.

6.3.1 $\boldsymbol{\pi}$-exponentials.

Let $\mathfrak{G}(X,Y) \in \mathbb{Z}_p[[X,Y]]$ be a Lubin-Tate formal group law isomorphic (but not necessarily equal) to $\mathbb{G}_m$. Let $\boldsymbol{\pi} := (\pi_m)_{m\geq 0}$ be a *fixed* generator of the Tate module of $\mathfrak{G}$. In other words, $\{\pi_m\}_m \subset \mathbb{Q}_p^{\mathrm{alg}}$ verifies $[p]_{\mathfrak{G}}(\pi_0) = 0$, $\pi_0 \neq 0$, $|\pi_0| < 1$, and $[p]_{\mathfrak{G}}(\pi_{m+1}) = \pi_m$, for all $m \geq 0$, where $[p]_{\mathfrak{G}}(X) \in \mathbb{Z}_p[[X]]$ is the multiplication by p in $\mathfrak{G}$.

Definition 6.2 ([Pul07, Def.3.1])**.** *Let $\boldsymbol{f}^-(T) = (\cdots, 0, 0, 0, f_0^-(T), \ldots, f_m^-(T))$ be an element of $\mathbf{CW}(T^{-1}\mathcal{O}_L[T^{-1}])$. We call $\boldsymbol{\pi}-$exponential relative to $\boldsymbol{f}^-(T)$ the following power series in T^{-1}:*

$$\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1) := \exp\Big(\pi_m\phi_0^-(T) + \pi_{m-1}\frac{\phi_1^-(T)}{p} + \cdots + \pi_0\frac{\phi_m^-(T)}{p^m}\Big)\;, \tag{6.3.1}$$

where $\phi_j^-(T) := \sum_{k=0}^{j} p^k \cdot f_k^-(T)^{p^{j-k}} \in T^{-1}\mathcal{O}_L[T^{-1}]$ is the j-th phantom component of the Witt vector $(f_0^-(T),\ldots,f_m^-(T)) \in \mathbf{W}_m(T^{-1}\mathcal{O}_L[T^{-1}])$.

Proposition 6.3. *Let $L_m := L(\pi_m)$, and let $L_\infty := \cup_{m\geq 0} L(\pi_m)$. Then:*

1. *One has*

$$\mathrm{e}_{p^\infty}((\cdots,0,0,0,f_0^-(T),\ldots,f_m^-(T)),1) \;\in\; 1 + \pi_m T^{-1}\mathcal{O}_{L_m}[[T^{-1}]]\;. \tag{6.3.2}$$

2. *The map $\boldsymbol{f}^-(T) \mapsto \mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)$ is a group homomorphism*

$$\mathbf{CW}(T^{-1}\mathcal{O}_L[T^{-1}]) \longrightarrow \bigcup_{m\geq 0} 1 + T^{-1}\mathcal{O}_{L_m}[[T^{-1}]] \;\subset\; 1 + T^{-1}\mathcal{O}_{L_\infty}[[T^{-1}]]\;. \tag{6.3.3}$$

   In particular, for every $\boldsymbol{f}^-(T)$, the power series $\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)$ converges at least for $|T| > 1$.

3. *The power series $\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)$ is over-convergent (i.e. converges for $|T| > 1-\varepsilon$, for some $\varepsilon > 0$) if and only if the reduction $\overline{\boldsymbol{f}^-(T)}$ lies in $(\bar{\mathrm{F}}-1)(\mathbf{CW}(t^{-1}k[t^{-1}]))$ (i.e. the Artin-Schreier-Witt character $\delta(\overline{\boldsymbol{f}^-(T)})$ defined by $\overline{\boldsymbol{f}^-(T)}$ via the Equation (1.3.3) is equal to 0).*

Proof. See [Pul07, Th.3.2 and Section 3.2]. □

6.3.2 The function $\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)$ as an explicit Kummer generator.

Let $\alpha_{\mathrm{wild}} : \mathrm{G_E} \to \mathcal{O}_K^\times$, $\mathrm{V}(\alpha_{\mathrm{wild}}) \in \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E})$, be a character such that $\alpha_{\mathrm{wild}}|_{(\mathcal{I}_{\mathrm{G_E^{ab}}}/\mathcal{P}_{\mathrm{G_E^{ab}}})\oplus \mathrm{G}_k^{\mathrm{ab}}} = 1$ in the decomposition (4.4.1). In particular, observe that

$$\alpha_{\mathrm{wild}}(\mathcal{P}_{\mathrm{G_E^{ab}}}) = \boldsymbol{\mu}_{p^{m+1}} \subset \mathcal{O}_K^\times \;, \tag{6.3.4}$$

for some $m \geq 0$.

Remark 6.4. *Observe that, since we assume that $\mathfrak{G} \xrightarrow{\sim} \mathbb{G}_m$, it follows by Lubin-Tate theory that $\boldsymbol{\mu}_{p^{m+1}} \subset \mathcal{O}_K^\times$ if and only if $\pi_0, \ldots, \pi_m \in \mathcal{O}_K$.*

Definition 6.5. *Let $\psi_m : \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_{p^{m+1}}$ be the isomorphism sending $\overline{1}$ into the unique primitive p^{m+1}-th root of unity ξ_{p^m} satisfying:*

$$|\pi_m - (\xi_{p^m} - 1)| < |\pi_m| = |p|^{\frac{1}{p^m(p-1)}} \;. \tag{6.3.5}$$

Let F/E be the cyclic extension of degree p^{m+1} defined by $(\psi_m^{-1} \circ \alpha_{\mathrm{wild}})$ (i.e. $\mathrm{Ker}(\psi_m^{-1} \circ \alpha_{\mathrm{wild}}) = \mathrm{Gal}(\mathrm{E^{sep}}/\mathrm{F})$, cf. Section 1.3), and let $\mathcal{F}^\dagger/\mathcal{E}_L^\dagger$ be the cyclic unramified extension whose residue field is F/E.

Proposition 6.6. *Let $\overline{\boldsymbol{f}^-}(t) = (\cdots, 0,0,0,\overline{f_0^-}(t), \ldots, \overline{f_m^-}(t)) \in \mathbf{CW}(t^{-1}k[t^{-1}])$ be a co-vector of length m defining $\psi_m^{-1} \circ \alpha_{\mathrm{wild}}$ (i.e. $\delta(\overline{\boldsymbol{f}^-}(t)) = (\psi_m^{-1} \circ \alpha_{\mathrm{wild}})$ in the sequence* (1.3.3)*, for example one can take $\overline{\boldsymbol{f}^-}(t)$ as a* minimal lifting *of α_{wild} as in Section 4.2.3). Let $\boldsymbol{f}^-(T) = (\cdots, 0,0,0, f_0^-(T), \ldots, f_m^-(T)) \in \mathbf{CW}(T^{-1}\mathcal{O}_L[T^{-1}])$ be a lifting of $\overline{\boldsymbol{f}^-}(t)$ of length m. Then:*

1. $\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)^{p^{m+1}} \in \mathcal{O}^\times_{\mathcal{E}_L^\dagger}$*;*

2. $\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)$ *is a Kummer generator of $\mathcal{F}^\dagger/\mathcal{E}_L^\dagger$:*

$$\mathcal{F}^\dagger \;=\; \mathcal{E}_L^\dagger(\; \mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)\;) \;; \tag{6.3.6}$$

3. *For all $\gamma \in \mathrm{G_E} \xrightarrow{\sim} \mathrm{Gal}(\widetilde{\mathcal{E}_L^\dagger}/\mathcal{E}_L^\dagger)$ (cf. Equation* (3.2.4)*), one has*

$$\gamma(\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)) = \alpha_{\mathrm{wild}}(\gamma)^{-1} \cdot \mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1) \;. \tag{6.3.7}$$

Proof. See [Pul07, Section 2.3]. Observe that in [Pul07, Section 2.3] the author was working in a more general context, and for this reason he used a function called "$\theta_{p^s}(\boldsymbol{\nu},1)$", which actually coincides with $\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)$ modulo a p^{m+1}-th root of unity (cf. [Pul07, Equation (3.1)]). □

We can now proceed as in the tame case (cf. Section 6.2). We preserve the notation of proposition 6.6. Let $\mathbf{e}_{\mathrm{V}} \in \mathrm{V}(\alpha_{\mathrm{wild}})$ be a basis in which $\mathrm{G_E}$ acts as

$$\gamma(\mathbf{e}_{\mathrm{V}}) = \alpha_{\mathrm{wild}}(\gamma) \cdot \mathbf{e}_{\mathrm{V}} \;, \quad \text{for all } \gamma \in \mathrm{G_E} \;. \tag{6.3.8}$$

Then a basis of $\mathrm{D}^\dagger(\mathrm{V}(\alpha_{\mathrm{wild}})) = (\mathrm{V}(\alpha_{\mathrm{wild}}) \otimes_{\mathcal{O}_K} \mathcal{O}_{\widetilde{\mathcal{E}_L^\dagger}})^{\mathrm{G_E}}$ is given by

$$\mathbf{e}_{\mathrm{D}} \;:=\; \mathbf{e}_{\mathrm{V}} \otimes \mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1) \;. \tag{6.3.9}$$

Indeed, for all $\gamma \in \mathrm{G_E}$, one has

$$\gamma(\mathbf{e}_{\mathrm{D}}) = \gamma(\mathbf{e}_{\mathrm{V}}) \otimes \gamma(\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)) = \alpha_{\mathrm{wild}}(\gamma)\mathbf{e}_{\mathrm{V}} \otimes \alpha_{\mathrm{wild}}(\gamma)^{-1}\mathrm{e}_{p^m}(\boldsymbol{f}^-(T),1) = \mathbf{e}_{\mathrm{D}} \;. \tag{6.3.10}$$

6.3.3

We compute now the action of ϕ^{D} and ∇^{D}. In the basis $\mathbf{e}_{\mathrm{D}}$ (cf. (6.3.9)), the function $\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)$ is the Taylor solution at ∞ of the ∇-module underlying $\mathrm{D}^\dagger(\mathrm{V}(\alpha_{\mathrm{wild}}))$. We recall that $\boldsymbol{f}^-(T)=\boldsymbol{f}^-(\underline{u},T)=(f_0^-(\underline{u},T),\ldots,f_m^-(\underline{u},T))$ has coefficients which depend also on $\underline{u}=(u_1,\ldots,u_r)$. In the basis $\mathbf{e}_{\mathrm{D}}$ the action of ϕ and ∇ are given by

$$\left\{\begin{array}{rcl}\phi^{\mathrm{D}}(\mathbf{e}_{\mathrm{D}}) & = & \theta_{p^m}(\boldsymbol{f}^-(\underline{u},T),1)\cdot\mathbf{e}_{\mathrm{D}}\ ,\\ \nabla_T^{\mathrm{D}}(\mathbf{e}_{\mathrm{D}}) & = & g^0_{\boldsymbol{f}^-}(u_1,\ldots,u_r,T)\cdot\mathbf{e}_{\mathrm{D}}\ ,\\ \nabla_{u_i}^{\mathrm{D}}(\mathbf{e}_{\mathrm{D}}) & = & g^i_{\boldsymbol{f}^-}(u_1,\ldots,u_r,T)\cdot\mathbf{e}_{\mathrm{D}}\ ,\end{array}\right. \tag{6.3.11}$$

where:

$$\theta_{p^m}(\boldsymbol{f}^-(\underline{u},T),1):=\mathrm{e}_{p^m}(\varphi(\boldsymbol{f}^-(\underline{u},T))-\boldsymbol{f}^-(\underline{u},T),1) \tag{6.3.12}$$

$$\begin{aligned} g^0_{\boldsymbol{f}^-}(u_1,\ldots,u_r,T) &:= \sum_{j=0}^{m}\pi_{m-j}\frac{\frac{d}{dT}(\phi_j^-(T))}{p^j}\\ &=\sum_{j=0}^{m}\pi_{m-j}\sum_{k=0}^{j}f_k^-(\underline{u},T)^{p^{j-k}}\frac{\frac{d}{dT}(f_k^-(\underline{u},T))}{f_k^-(\underline{u},T)},\end{aligned} \tag{6.3.13}$$

$$\begin{aligned} g^i_{\boldsymbol{f}^-}(u_1,\ldots,u_r,T) &:= \sum_{j=0}^{m}\pi_{m-j}\frac{\frac{d}{du_i}(\phi_j^-(T))}{p^j}\\ &=\sum_{j=0}^{m}\pi_{m-j}\sum_{k=0}^{j}f_k^-(\underline{u},T)^{p^{j-k}}\frac{\frac{d}{du_i}(f_k^-(\underline{u},T))}{f_k^-(\underline{u},T)},\end{aligned} \tag{6.3.14}$$

where φ acts on $\mathbf{CW}(T^{-1}\mathcal{O}_L[T^{-1}])$ coefficient by coefficient. Observe that we have chosen $\varphi(T)=T^p\in\mathcal{O}_L[T]$ (cf. Def. 1.10). Observe also that Equations (6.3.13) and (6.3.14) are obtained by taking the logarithmic derivative of the Definition (6.3.1) as $g^0_{\boldsymbol{f}^-}=\frac{\frac{d}{dT}(\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1))}{\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)}$, and $g^i_{\boldsymbol{f}^-}=\frac{\frac{d}{du_i}(\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1))}{\mathrm{e}_{p^\infty}(\boldsymbol{f}^-(T),1)}$. In other words the knowledge of the solution leads to recover the matrix of the connection.

Since we have chosen ψ_m satisfying Equation (6.3.5), hence this construction does not depend on the choice of $\boldsymbol{\pi}:=(\pi_m)_{m\geq 0}$.

7 Comparison between arithmetic and differential Swan conductors

The object of this section is to prove the following

Theorem 7.1. *Let* $\mathrm{V}\in\mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E})$ *be a rank one representation. Then:*

$$\mathrm{sw}(\mathrm{V})=\mathrm{sw}^\nabla(\mathrm{D}^\dagger(\mathrm{V}))\ , \tag{7.0.15}$$

where $\mathrm{sw}(\mathrm{V})$ *is the arithmetic Swan conductor (cf. Def. 4.4 and Def. 4.17), and* $\mathrm{sw}^\nabla(\mathrm{D}^\dagger(\mathrm{V}))$ *is the differential Swan conductor of* $\mathrm{D}^\dagger(\mathrm{V})\in(\phi,\nabla)-\mathrm{Mod}(\mathcal{O}_{\mathcal{E}_L^\dagger}/\mathcal{O}_K)$*, considered as an object of* $\nabla-\mathrm{Mod}(\mathcal{R}_L/K)$ *(cf. Def. 3.11).*

7.1 A Small Radius Lemma

In this section we prove that if $T(\mathrm{M},\rho)$ is "small" (see Definition 3.7), then we are able to link $T(\mathrm{M},\rho)$ to the valuation of the coefficients of the equation. The following "Small Radius Lemma" generalizes the analogous result [CM02, 6.2 and 6.4], [You92].

Lemma 7.2 (Small Radius)**.** *Let M_ρ be a* rank one *∇-module over $\mathcal{F}_{L,\rho}$. For $i=0,\ldots,r$ let $g_n^i \in \mathcal{F}_{L,\rho}$ be the matrix of $(\nabla_{u_i}^{\mathrm{M}})^n$ (resp. if $i=0$, g_n^0 is the matrix of $(\nabla_T^{\mathrm{M}})^n$). Write (cf. Lemma 1.13):*

$$\begin{array}{rcl} \omega & \overset{Lemma 1.13}{=} & \dfrac{|d/dT|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|d/dT|_{\mathcal{F}_{L,\rho}}} = \dfrac{|d/du_1|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|d/du_1|_{\mathcal{F}_{L,\rho}}} = \ldots = \dfrac{|d/du_r|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|d/du_r|_{\mathcal{F}_{L,\rho}}}\ , \\ \omega_{\mathrm{M}}(\rho) & \overset{\mathrm{def}}{:=} & \min\Big(\dfrac{|d/dT|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|g_1^0|_\rho}\ ,\ \dfrac{|d/du_1|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|g_1^1|_\rho}\ ,\ \ldots\ ,\ \dfrac{|d/du_r|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|g_1^r|_\rho} \Big)\ . \end{array}$$

Then:

- **(1)** $T(\mathrm{M},\rho)\ \geq\ \min(\ \omega\ ,\ \omega_{\mathrm{M}}(\rho)\ )$.
- **(2)** *The following conditions are equivalent:*
  - **(*a*)** $T(\mathrm{M},\rho)\ <\ \omega$;
  - **(*b*)** $\omega_{\mathrm{M}}(\rho)\ <\ \omega$;
  - **(*c*)** $|g_1^0|_\rho > \rho^{-1}$, *or* $|g_1^i|_\rho > 1$ *for some* $i\in\{1,\ldots,r\}$.
- **(3)** *If one of the equivalent conditions of point* **(2)** *is verified, then one has*

$$T(\mathrm{M},\rho) = \min\Big(\frac{\omega\cdot\rho^{-1}}{|g_1^0|_\rho}\ ,\ \frac{\omega}{|g_1^1|_\rho}\ ,\ \ldots\ ,\ \frac{\omega}{|g_1^r|_\rho} \Big)\ . \tag{7.1.1}$$

Proof. By Lemma 1.13, one has $\omega = \frac{|d/dT|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|d/dT|_{\mathcal{F}_{L,\rho}}} = \frac{|d/du_i|_{\mathcal{F}_{L,\rho},\mathrm{Sp}}}{|d/du_i|_{\mathcal{F}_{L,\rho}}}$, for all $i=1,\ldots,r$. This proves that **(*b*)** $\Leftrightarrow$ **(*c*)**. We prove now **(1)**. The matrices g_n^i verify the inductive relations $g_{n+1}^0 = d/dT(g_n^0) + g_n^0\cdot g_1^0$, and $g_{n+1}^i = d/du_i(g_n^i) + g_n^i\cdot g_1^i$. By induction, one has $|g_n^0|_\rho \leq \max(|d/dT|_{\mathcal{F}_{L,\rho}}, |g_1^0|_\rho)^n$, and $|g_n^i|_\rho \leq \max(|d/du_i|_{\mathcal{F}_{L,\rho}}, |g_1^i|_\rho)^n$. This proves that

$$[\liminf_n |g_n^0|_\rho^{-1/n}]\ \geq\ \min\Big(\frac{1}{|d/dT|_{\mathcal{F}_{L,\rho}}}, \frac{1}{|g_1^0|_\rho}\Big) = \min\Big(\ \rho\ ,\ |g_1^0|_\rho^{-1}\ \Big)\ , \tag{7.1.2}$$

$$[\liminf_n |g_n^i|_\rho^{-1/n}]\ \geq\ \min\Big(\frac{1}{|d/du_i|_{\mathcal{F}_{L,\rho}}}, \frac{1}{|g_1^i|_\rho}\Big) = \min\Big(\ 1\ ,\ |g_1^i|_\rho^{-1}\ \Big)\ . \tag{7.1.3}$$

Hence, by formula (3.3.6), the point **(1)** holds. Moreover, the same computation proves the following sub-lemma:

Lemma 7.3. *The following conditions are equivalent :*

- **(*c'*)** $|g_1^0|_\rho > \rho^{-1}$ *(resp.* $|g_1^i|_\rho > 1$*)* ,
- **(*a'*)** $[\liminf_n |g_n^0|_\rho^{-1/n}] < \rho$ *(resp.* $[\liminf_n |g_n^i|_\rho^{-1/n}] < 1$*)* .

□

Continuation of the proof of Lemma 7.2: By using again Formula (3.3.6), Lemma 7.3 proves that **(*c*)** implies **(*a*)**, and that the point **(3)** holds. Clearly, by assertion **(1)**, **(*a*)** implies $\omega > T(\mathrm{M},\rho) \geq \min(\omega, \omega_{\mathrm{M}}(\rho))$, hence **(*b*)** holds. □

Lemma 7.4. *Let* M *be a* solvable *rank one* ∇*-module over* $\mathcal{A}_L(]0,1[)$*. For* $i=1,\ldots,r$*, let* $g_1^i \in \mathcal{A}_L(]0,1[)$ *be the matrix of* $\nabla_{u_i}^{\mathrm{M}}$ *(resp. if* $i=0$*,* g_1^0 *is the matrix of* ∇_T^{M}*). Assume that, for all* $i=0,1,\ldots,r$*, the matrix* g_1^i *is of the form*

$$g_1^i = \sum_{j\geq -n_i} a_j^{(i)} T^j \in \mathcal{A}_L(]0,1[)\;, \qquad \text{with } a_{-n_i}^{(i)} \neq 0. \tag{7.1.4}$$

Assume moreover that $n_0,n_1,\ldots,n_r < +\infty$ *satisfy* $n_0 \geq 2$*, or* $n_i \geq 1$*, for some* $i=1,\ldots,r$*. Then:*

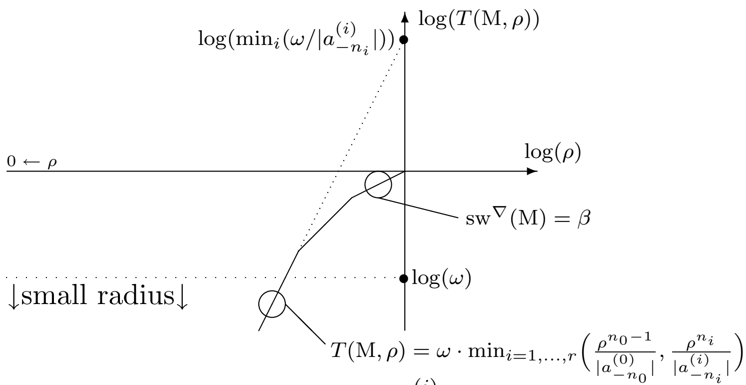
1. $|a_{-n_i}^{(i)}| \leq \omega$*, for all* $i=0,\ldots,r$*;*
2. *For* ρ *sufficiently close to* 0*, one has*
$$T(\mathrm{M},\rho) \;=\; \omega\cdot\min_{i=1,\ldots,r}\Big(\,|a_{-n_0}^{(0)}|^{-1}\cdot\rho^{n_0-1}\;,\;|a_{-n_i}^{(i)}|^{-1}\cdot\rho^{n_i}\,\Big)\;; \tag{7.1.5}$$
3. *If* $|a_{-n_i}^{(i)}| = \omega$ *for some* $i=0,\ldots,r$*, then Equation* (7.1.5) *holds for all* $\rho \in]0,1[$*, and*
$$\mathrm{sw}^{\nabla}(\mathrm{M}) = \max\{\,\epsilon_0(n_0-1)\;,\;\epsilon_1 n_1\;,\;\ldots\;,\;\epsilon_r n_r\,\} \tag{7.1.6}$$
where $\epsilon_i = 0$ *if* $|a_{-n_i}^{(i)}| < \omega$*, and* $\epsilon_i = 1$ *if* $|a_{-n_i}^{(i)}| = \omega$*.*

Proof. By assumption, $\lim_{\rho\to 1^-} T(\mathrm{M},\rho) = 1$. Moreover we know that $T(\mathrm{M},\rho)$ is continuous and log-concave. We have then only two possibilities: $T(\mathrm{M},\rho)=1$ for all $\rho\in]0,1[$, or there exists $\beta>0$ such that $T(\mathrm{M},\rho)\leq\rho^{\beta}$, for all $\rho\in]0,1[$. In the first case, $T(\mathrm{M},\rho)=1>\omega$, for all $\rho<1$. Hence by the Small Radius Lemma 7.2, we have $|g_1^0|_\rho \leq \rho^{-1}$, and $|g_1^i|_\rho \leq 1$, for all $i=1,\ldots,r$, and for all $\rho\in]0,1[$. This contradicts our assumptions. Indeed if ρ is close to zero, one has $|g_1^i|_\rho = |a_{-n_i}^{(i)}|\rho^{-n_i}$, for all $i=0,\ldots,r$, and the assumption $\max(\frac{n_0}{2},n_1,\ldots,n_r)\geq 1$ implies that, for ρ close to 0, one has $|g_1^0|_\rho > \rho^{-1} > 1$, or that $|g_1^i|_\rho > 1$, for some $i=1,\ldots,r$.

Hence we are in the second case: $T(\mathrm{M},\rho)\leq\rho^{\beta}$, for all $\rho<1$, where $\beta=\mathrm{sw}^{\nabla}(\mathrm{M})$ (This follows from Definition 3.14 and the fact that $T(\mathrm{M},\rho)$ is log-concave). Since $\beta>0$, if ρ is sufficiently close to 0, one has $T(\mathrm{M},\rho)\leq\rho^{\beta}<\omega$. So Small Radius Lemma 7.2 applies, and $T(\mathrm{M},\rho)=\omega\cdot\min_{i=1,\ldots,r}\Big(\frac{\rho^{n_0-1}}{|a_{-n_0}^{(0)}|},\frac{\rho^{n_i}}{|a_{-n_i}^{(i)}|}\Big)$. Now, since $\lim_{\rho\to1^-}T(\mathrm{M},\rho)=1$, and since the function $\rho\mapsto T(\mathrm{M},\rho)$ is log-concave, its log-slope at 0^+ is greater than its log-slope at 1^- (i.e. $\beta=\mathrm{sw}^{\nabla}(\mathrm{M})$). Hence one has the inequality

$$\min_{i=0,\ldots,r}\frac{\omega}{|a_{-n_i}^{(i)}|} = \lim_{\rho\to1^-}\omega\cdot\min_{i=1,\ldots,r}\Big(\frac{\rho^{n_0-1}}{|a_{-n_0}^{(0)}|}\;,\;\frac{\rho^{n_i}}{|a_{-n_i}^{(i)}|}\Big) \geq \lim_{\rho\to1^-}T(\mathrm{M},\rho)=1\;, \tag{7.1.7}$$

as in the following picture:



This implies $|a_{-n_i}^{(i)}|\leq\omega$, for all $i=0,\ldots,r$. Moreover if $|a_{-n_i}^{(i)}|=\omega$, for some $i=0,\ldots,r$, then this graphic is a line, and the assertion iii) holds. $\square$

7.2 Proof of theorem 7.1

Let $\mathrm{V}(\alpha) \in \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E})$. As usual, we decompose $\alpha = \alpha_{\mathrm{wild}} \cdot \alpha_{\mathrm{tame}} \cdot \alpha_k$ (cf. Equation (4.4.2)). By Section 4.4.1, in both residual and tame cases the arithmetic Swan conductor is equal to zero. On the other hand, we have the following:

Lemma 7.5. *The differential Swan conductors of* $\mathrm{D}^\dagger(\mathrm{V}(\alpha_k))$ *and* $\mathrm{D}^\dagger(\mathrm{V}(\alpha_{\mathrm{tame}}))$ *are equal to* 0. *More precisely there exist bases of* $\mathrm{D}^\dagger(\mathrm{V}(\alpha_k))$ *and* $\mathrm{D}^\dagger(\mathrm{V}(\alpha_{\mathrm{tame}}))$ *in which the connections are defined over* $\mathcal{A}_L(]0,1[)$ *and* $T(\mathrm{D}^\dagger(\mathrm{V}(\alpha_k)),\rho) = T(\mathrm{D}^\dagger(\mathrm{V}(\alpha_{\mathrm{tame}})),\rho) = 1$*, for all* $\rho \in]0,1[$.

Proof. Let $\mathrm{D}_k := \mathrm{D}^\dagger(\mathrm{V}(\alpha_k))$ and $\mathrm{D}_{\mathrm{tame}} := \mathrm{D}^\dagger(\mathrm{V}(\alpha_{\mathrm{tame}}))$. By equations (6.1.1), since the matrices of $\nabla^{\mathrm{D}_k}_T$ and $\nabla^{\mathrm{D}_k}_{u_i}$ belongs to $\mathcal{O}_L$, hence $T(\mathrm{D}_k,\rho)$ does not depend on ρ, and its log-slope is equal to 0 (one has actually $T(\mathrm{D}_k,\rho) = 1$, for all $\rho < 1$). On the other hand, in the notation of Lemma 3.8, since $g^0_0 \in \mathbb{Z}_p \cdot T^{-1}$ (cf. Equations 6.2.5), the function $\rho \mapsto \liminf_n |g^0_n|_\rho^{-1/n}$ is a constant function (cf. [Pul07, Lemma 1.4]). Moreover, the matrix of $\nabla^{\mathrm{D_{tame}}}_{u_i}$ is equal to 0, hence $\rho \mapsto \liminf_n |g^i_n|_\rho^{-1/n}$ is the constant function equal to ∞. So $T(\mathrm{D}_{\mathrm{tame}},\rho) = 1$, for all $\rho < 1$, and its log-slope is equal to 0. □

7.2.1

Since in both residual and tamely ramified cases the arithmetic and the differential Swan conductors are equal to zero, then Remark 4.18 and Section 3.3.2, allow us to reduce to prove the theorem for α_{wild}. The proof of Theorem 7.1 is then carried out in two steps: first we prove the theorem for the case in which $\alpha_{\mathrm{wild}} = \psi_m \circ \delta(\overline{\boldsymbol{\lambda}}_{-d} t^{-d})$, with $\boldsymbol{\lambda}_{-d} \in \mathbf{W}_{v_p(d)}(k) - p\mathbf{W}_{v_p(d)}(k)$, where $\psi_m : \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_{p^{m+1}} \subset \mathcal{O}_K$ is the identification of Definition 6.5, and δ is the Artin-Schreier-Witt morphism of Sequence (1.3.3) (cf. Lemma 7.6 below). The second step consists of extending the theorem to every character using Section 4.2.3.

Lemma 7.6. *Let* $\overline{\boldsymbol{\lambda}} \cdot t^{-np^m}$ *be a co-monomial satisfying* $\overline{\boldsymbol{\lambda}} \in \mathbf{W}_m(k) - p\mathbf{W}_m(k)$ *(cf. Section 4.2.3), let* $\alpha_{\mathrm{wild}} := \psi_m \circ \delta(\overline{\boldsymbol{\lambda}} t^{-np^m})$. *We denote by* M *the* ∇*-module over* $\mathcal{R}_L$ *defined by* $\mathrm{D}^\dagger(\mathrm{V}(\alpha_{\mathrm{wild}}))$. *Then*

$$\mathrm{sw}^\nabla(\mathrm{M}) = np^m = \mathrm{sw}(\alpha_{\mathrm{wild}})\ . \tag{7.2.1}$$

More precisely, one has $T(\mathrm{M},\rho) = \rho^{np^m}$*, for all* $\rho \in]0,1[$.

Proof. By Corollary 4.10 one has $\mathrm{sw}(\alpha_{\mathrm{wild}}) = np^m$, hence it is enough to prove the last assertion. To prove this, we now use the assertion iii) of Lemma 7.4. We verify that all assumptions of Lemma 7.4 are verified.

Let $\boldsymbol{\lambda} = (\lambda_0,\ldots,\lambda_m) \in \mathbf{W}_m(\mathcal{O}_{L_0})$ be an arbitrary lifting of $\overline{\boldsymbol{\lambda}} = (\overline{\lambda}_0,\ldots,\overline{\lambda}_m)$. Then the solution of M in this basis is the $\boldsymbol{\pi}$-exponential (cf. Section 6.3.3)

$$\mathrm{e}_{p^\infty}(\boldsymbol{\lambda}T^{-np^m},1) = \exp\Big(\pi_m\phi_0 T^{-n} + \pi_{m-1}\phi_1\frac{T^{-np}}{p} + \cdots + \pi_0\phi_m\frac{T^{-np^m}}{p^m}\Big)\ , \tag{7.2.2}$$

where $\langle\phi_0,\ldots,\phi_m\rangle \in \mathcal{O}_{L_0}^{m+1}$ is the phantom vector of $\boldsymbol{\lambda} \in \mathbf{W}_m(\mathcal{O}_{L_0})$. Indeed the phantom vector of $\boldsymbol{\lambda}T^{-np^m}$ is $\langle\phi_0T^{-n},\phi_1T^{-np},\ldots,\phi_mT^{-np^m}\rangle$. Writing $\boldsymbol{\lambda}$ as function of $\underline{u} := (u_1,\ldots,u_r)$, the matrices of the connections are explicitly given by (cf. Equation (6.3.12))

$$\begin{aligned} g^0_{\boldsymbol{f}^-}(u_1,\ldots,u_r,T) &= -n\ (\ \pi_m\phi_0T^{-n-1} + \pi_{m-1}\phi_1T^{-np-1} + \cdots + \pi_0\phi_mT^{-np^m-1}\ )\ , \\ g^i_{\boldsymbol{f}^-}(u_1,\ldots,u_r,T) &= \sum_{j=0}^m \pi_{m-j}T^{-np^j} \cdot \sum_{k=0}^j \lambda_k(\underline{u})^{p^{j-k}-1}\frac{d}{du_i}(\lambda_k(\underline{u}))\ . \end{aligned}$$

The coefficients of lowest degree (with respect to T) in the matrix $g^i_{\boldsymbol{f}^-}(u_1,\ldots,u_r,T)$ are respectively

$$a^{(0)}_{-np^m-1} \;:=\; -n\pi_0\phi_m\ , \tag{7.2.3}$$

$$a^{(i)}_{-np^m} \;:=\; \pi_0\cdot p^{-m}\cdot\frac{d}{du_i}(\phi_m(u_1,\ldots,u_m))\ ,\quad \text{for all } i=1,\ldots,r\ . \tag{7.2.4}$$

More explicitly, for $i=1,\ldots,r$, one has

$$a^{(i)}_{-np^m} \;:=\; \pi_0\cdot\sum_{k=0}^{j}\lambda_k(\underline{u})^{p^{j-k}-1}\frac{d}{du_i}(\lambda_k(\underline{u}))\;\in\;\mathcal{O}_L\ . \tag{7.2.5}$$

Since $\boldsymbol{\lambda}\in\mathbf{W}_m(\mathcal{O}_{L_0})$ has coefficients in $\mathcal{O}_{L_0}$ (which is a Cohen ring), one sees that

$$|\phi_m| = |p|^{s(\overline{\boldsymbol{\lambda}})}\ , \tag{7.2.6}$$

where $s(\overline{\lambda}_0,\ldots,\overline{\lambda}_m) := \min\{s\geq 0 \mid \overline{\lambda}_s\neq 0\}$. We study separately two cases: $\overline{\lambda}_0\neq 0$, and $\overline{\lambda}_0=0$. Assume first that $\overline{\lambda}_0\neq 0$ (i.e. $s(\overline{\boldsymbol{\lambda}})=0$). Since $n>0$, then the T-adic valuations of $g^0_{\boldsymbol{f}^-}$ and $g^i_{\boldsymbol{f}^-}$ satisfy conditions of Lemma 7.4. Since $|\pi_0|=\omega$ and $|\phi_m|=1$, then $|a^{(0)}_{-np^m-1}|=\omega$. Hence the point iii) of Lemma 7.4 applies, and one has $T(\mathrm{M},\rho)=\rho^{np^m}$, for all $\rho\in]0,1[$.

Assume now that $\overline{\lambda}_0=0$, i.e. $s(\overline{\boldsymbol{\lambda}})\geq 1$. By assumption, $\overline{\boldsymbol{\lambda}}\in\mathbf{W}_m(k)-p\mathbf{W}_m(k)$, but since $\overline{\lambda}_0=0$, this is equivalent to (cf. Equation (1.2.3))

$$\overline{\boldsymbol{\lambda}}\notin\mathbf{W}_m(k^p)\ . \tag{7.2.7}$$

The following Lemma 7.7 proves that

$$\max_{i=1,\ldots,r}|a^{(i)}_{-np^m}|<\omega \quad\Longrightarrow\quad \overline{\boldsymbol{\lambda}}\in\mathbf{W}_m(k^p)\ , \tag{7.2.8}$$

which contradicts our assumption. Hence $|a^{(i)}_{-np^m}|=\omega$, for some $i=1,\ldots,r$. By applying point iii) of Lemma 7.4, as above, we find $T(\mathrm{M},\rho)=\rho^{np^m}$, for all $\rho\in]0,1[$. □

Lemma 7.7. *Let k be a field of characteristic $p>0$. Let $\overline{\boldsymbol{\lambda}}=(\overline{\lambda}_0,\overline{\lambda}_1,\ldots,\overline{\lambda}_m)\in\mathbf{W}_m(k)$ be a Witt vector. The following assertions are equivalent:*

(1) *For all $i=1,\ldots,r$ one has*

$$\overline{\lambda}_0^{p^m-1}\frac{d}{d\bar{u}_i}(\overline{\lambda}_0)+\overline{\lambda}_1^{p^{m-1}-1}\frac{d}{d\bar{u}_i}(\overline{\lambda}_1)+\cdots+\overline{\lambda}_{m-1}^{p-1}\frac{d}{d\bar{u}_i}(\overline{\lambda}_{m-1})+\frac{d}{d\bar{u}_i}(\overline{\lambda}_m)=0\ . \tag{7.2.9}$$

(2) $\overline{\boldsymbol{\lambda}}\in\mathbf{W}_m(k^p)$.

Proof. Clearly **(2)** implies **(1)**. Assume then that **(1)** holds. The condition **(1)** is equivalent to

$$\overline{\lambda}_0^{p^m-1}d(\overline{\lambda}_0)+\overline{\lambda}_1^{p^{m-1}-1}d(\overline{\lambda}_1)+\cdots+\overline{\lambda}_{m-1}^{p-1}d(\overline{\lambda}_{m-1})+d(\overline{\lambda}_m)=0\ , \tag{7.2.10}$$

in $\Omega^1_{k/\mathbb{F}_p}$, where $d:k\to\Omega^1_{k/\mathbb{F}_p}$ is the canonical derivation. We proceed by induction on $m\geq 0$. The case $m=0$ is evident, since $d(\overline{\lambda}_0)=0$ in $\Omega^1_{k/\mathbb{F}_p}$ implies $\overline{\lambda}_0\in k^p$ (cf. [Gro64, Ch.0, 21.4.6]). Let now $m\geq 1$. Assume Equation (7.2.10) holds. Following [Car58, II.6], let $Z^1(k):=\mathrm{Ker}(d:\Omega^1_{k/\mathbb{F}_p}\to\Omega^2_{k/\mathbb{F}_p})$, and let

$$C^{-1}\;:\;\Omega^1_{k/\mathbb{F}_p}\;\xrightarrow{\ \sim\ }\;Z^1(k)/d(k) \tag{7.2.11}$$

be the Cartier isomorphism (cf. [Car58, II.6]). Here $d(k)$ is the image of the map $d : k \to \Omega^1_{k/\mathbb{F}_p}$. We recall that $C^{-1}(ad(b)) = \overline{a^p b^{p-1} d(b)}$, where $\overline{\omega}$ is the class of $\omega \in Z(k)$ modulo $d(k)$.

Let $E_m(\overline{\lambda}_0, \ldots, \overline{\lambda}_m) := \overline{\lambda}_0^{p^m-1} d(\overline{\lambda}_0) + \overline{\lambda}_1^{p^{m-1}-1} d(\overline{\lambda}_1) + \cdots + \overline{\lambda}_{m-1}^{p-1} d(\overline{\lambda}_{m-1}) + d(\overline{\lambda}_m) \in \Omega^1_{k/\mathbb{F}_p}$. Since $E_m(\overline{\lambda}_0, \ldots, \overline{\lambda}_m) = 0$, then $E_m(\overline{\lambda}_0, \ldots, \overline{\lambda}_m) \in Z(k)$, and the class of $\overline{E_m(\overline{\lambda}_0, \ldots, \overline{\lambda}_m)} \in Z(k)/d(k)$ is equal to

$$0 = \overline{E_m(\overline{\lambda}_0, \ldots, \overline{\lambda}_m)} = \overline{\overline{\lambda}_0^{p^m-1} d(\overline{\lambda}_0) + \overline{\lambda}_1^{p^{m-1}-1} d(\overline{\lambda}_1) + \cdots + \overline{\lambda}_{m-1}^{p-1} d(\overline{\lambda}_{m-1})} \ . \tag{7.2.12}$$

By definition, one has $C^{-1}\Big(\overline{\lambda}_i^{p^{m-1-i}-1} d(\overline{\lambda}_i)\Big) = \overline{\overline{\lambda}_i^{p^{m-i}-1} d(\overline{\lambda}_i)}$, hence we find

$$0 = \overline{E_m(\overline{\lambda}_0, \ldots, \overline{\lambda}_m)} = C^{-1}(E_{m-1}(\overline{\lambda}_0, \ldots, \overline{\lambda}_{m-1})) \ . \tag{7.2.13}$$

Since C^{-1} is an isomorphism (cf. [Car58, II.6]), then the equation (7.2.13) implies that $E_{m-1}(\overline{\lambda}_0, \ldots \ldots, \overline{\lambda}_{m-1}) = 0$. By induction, one finds then $\overline{\lambda}_0, \ldots, \overline{\lambda}_{m-1} \in k^p$, but this implies $E_m(\overline{\lambda}_0, \ldots, \overline{\lambda}_m) = d(\overline{\lambda}_m) = 0$, hence one has also $\overline{\lambda}_m \in k^p$ (cf. [Gro64, Ch.0, 21.4.6]). □

End of the Proof of Theorem 7.1 : Let $\overline{\boldsymbol{f}^-}(t) = \sum_{n \in \mathrm{J}_p} \overline{\boldsymbol{\lambda}}_{-n} t^{-np^{m(n)}}$ be a minimal lifting of $\psi_m^{-1} \circ \alpha_{\mathrm{wild}}$ in $\mathbf{CW}(\mathrm{E})$ (cf. Section 4.2.3). Again by Section 4.2.3, one has

$$\mathrm{sw}(\delta(\overline{\boldsymbol{f}^-}(t))) = \max_{n \in \mathrm{J}_p} \mathrm{sw}(\delta(\overline{\boldsymbol{\lambda}}_{-n} t^{-np^{m(n)}})) \ , \tag{7.2.14}$$

where δ is the Artin-Schreier-Witt morphism (cf. Equation (1.3.3)). Now we recall that $\mathrm{V}(\alpha_{\mathrm{wild}}) = \otimes_{n \in \mathrm{J}_p} \mathrm{V}(\delta(\overline{\boldsymbol{\lambda}}_{-n} T^{-np^{m(n)}}))$, and

$$\mathrm{D}^\dagger(\mathrm{V}(\alpha_{\mathrm{wild}})) = \otimes_{n \in \mathrm{J}_p} \mathrm{D}^\dagger(\mathrm{V}(\delta(\overline{\boldsymbol{\lambda}}_{-n} T^{-np^{m(n)}}))) \ . \tag{7.2.15}$$

By Lemma 7.6, one has

$$\mathrm{sw}^\nabla(\mathrm{D}^\dagger(\mathrm{V}(\delta(\overline{\boldsymbol{\lambda}}_{-n} T^{-np^{m(n)}})))) = np^{m(n)} \ . \tag{7.2.16}$$

Now if $n_1 \neq n_2$ then $n_1 \cdot p^{m(n_1)} \neq n_2 \cdot p^{m(n_2)}$. Hence, by Section 3.3.2, one has

$$\mathrm{sw}^\nabla(\mathrm{D}^\dagger(\mathrm{V}(\alpha_{\mathrm{wild}}))) \ = \ \max_{n \in \mathrm{J}_p} np^{m(n)} \ . \tag{7.2.17}$$

This proves Theorem 7.1. □

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