The convergence Newton polygon of a p-adic differential equation III : global decomposition and controlling graphs

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Abstract

We deal with locally free $\mathscr{O}_X$-modules with connection over a Berkovich curve X. As a main result we prove local and global decomposition theorems of such objects by the radii of convergence of their solutions. We also derive a bound of the number of edges of the controlling graph, in terms of the geometry of the curve and the rank of the equation. As an application we provide a classification result of such equations over elliptic curves.

This is a first draft, containing a maximum number of details. We plan to reduce its volume in a next version.

Version of August 6, 2013

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2000 Mathematics Subject Classification Primary 12h25; Secondary 14G22

Keywords: p-adic differential equations, Berkovich spaces, Radius of convergence, Newton polygon, spectral radius, Skeleton, finiteness

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INTRODUCTION

Following an original idea of Dwork, if a differential equation has two solutions with different radii of convergence, then it should correspond to a decomposition of the equation. Decomposition theorems are the central tool in the classification of p-adic differential equations. As an example they are the first step to obtain the p-adic local monodromy theorem [And02], [Ked04], [Meb02].

The main contributions to the decomposition results are due to Robba [Rob75a], [Rob75b], [Rob80], Dwork-Robba [DR77], Christol-Mebkhout [CM00], [CM01], Kedlaya [Ked10], [Ked13]. We also recall [DMR07, p. 97-107], and [Ram78] for the decomposition by the *formal* slopes of a differential equation over the field of power series $K((T))$ (cf. Section 5.7).

There are very few examples of global nature of such decomposition theorems by the radii (mainly on annuli or disks), and all with restrictive assumptions (as an example see [Ked10, Ch.12], [Ked13, 5.4.2]). As a matter of fact, a large part of the literature is devoted to the following two cases: differential equations defined over a germ of punctured disk, or over the Robba ring. In the language of Berkovich curves this corresponds respectively to a germ of segment out of a rational point, or a germ of segment out of a point of type 2 or 3. Except in those two situations, there is a lack of results.

We present here a general decomposition theorem of a global nature in the framework of Berkovich smooth curves, that works without any technical assumption (solvability, exponents, Frobenius, harmonicity, ...).

A theoretical point of a crucial importance is the definition itself of the radii of convergence. The former definition of radii relates them to the spectral norm of the connection. This (partially) fails in Berkovich geometry because there are solutions converging more than the natural bound prescribed by the spectral norm. Here we deal with a more geometrical definition of the radii due to F. Baldassarri in [Bal10], following the ideas of [BV07]. He improves the former definition by introducing *over-solvable radii*, i.e. radii that are larger than the spectral bound (cf. (2.26)). These radii are not intelligible in terms of spectral norm even if the point is of type 2, 3, or 4. Moreover he "*normalizes*" the usual spectral radii of convergence, with respect to a semi-stable formal model of the curve. He is also able to prove the continuity of the smallest radius. In [PP12b] we introduce in that picture the notion of *weak triangulation* as a substitute of Baldassarri's semi-stable model. In fact such a semi-stable model produces a (non weak) triangulation (see [Duc] for instance). This permits to generalize the definition to a larger class of curves, and it has the advantage for us of being completely within the framework of Berkovich curves. Exploiting this point of view, we have proved in [Pul12] and [PP12b] that there exists a locally finite graph outside which the radii are all locally constant, as firstly conjectured by Baldassarri. This proves that there are relatively few numerical invariants of the equation encoded in the radii of convergence. The finiteness theorem have been proved in [PP12b] by recreating the notion of *generic disk* in the framework of Berkovich curves, as in the very original point of view of Bernard Dwork and Philippe Robba. The global decomposition theorem presented here enlarges the picture, it makes evident that Baldassarri's idea for the definition of the radii is the good one, and gives to it a more operative meaning.

We come now come more specifically into the content of the paper. Let $(K, |.|)$ be a complete ultrametric valued field of characteristic 0. Let X be a quasi-smooth K-analytic curve, in the sense of Berkovich theory[1]. We assume without loss of generality that X is connected. Let $\mathscr{F}$ be a locally free $\mathscr{O}_X$-module of finite rank r endowed with an integrable connection ∇.

In [PP12b] we explained how to associate to each point $x \in X$ the so called *convergence Newton polygon of* $\mathscr{F}$ *at* x. Its slopes are the logarithms of the radii of convergence $\mathcal{R}_{S,1}(x, \mathscr{F}) \leqslant \cdots \leqslant \mathcal{R}_{S,r}(x, \mathscr{F})$ of a conveniently chosen basis of solutions of $\mathscr{F}$ at x. Here S is a weak triangulation.

Following [Pul12], we then define for all $i \in \{1, \dots, r = \mathrm{rank}(\mathscr{F})\}$ a *locally finite* graph $\Gamma_{S,i}(\mathscr{F})$, as the locus of points that do not admit a virtual open disk in $X - S$ as a neighborhood on which $\mathcal{R}_{S,i}(-, \mathscr{F})$ is constant.

We say that the index i separates the radii (globally over X) if for all $x \in X$ one has

$$\mathcal{R}_{S,i-1}(x, \mathscr{F}) \ < \ \mathcal{R}_{S,i}(x, \mathscr{F}) \, . \tag{0.1}$$

THEOREM 1 (cf. Theorem 5.3.1 and Proposition 5.3.3)**.** *If the index i separates the radii of $\mathscr{F}$, then there exists a sub-differential equation $(\mathscr{F}_{\geqslant i}, \nabla_{\geqslant i}) \subseteq (\mathscr{F}, \nabla)$ of rank $r - i + 1$ together with an exact sequence*

$$0 \to \mathscr{F}_{\geqslant i} \to \mathscr{F} \to \mathscr{F}_{<i} \to 0 \tag{0.2}$$

such that for all $x \in X$ one has

$$\mathcal{R}_{S,j}(x, \mathscr{F}) \ = \ \begin{cases} \mathcal{R}_{S,j}(x, \mathscr{F}_{<i}) & \text{if} \quad j = 1, \dots, i-1 \\ \mathcal{R}_{S,j-i+1}(x, \mathscr{F}_{\geqslant i}) & \text{if} \quad j = i, \dots, r \, . \end{cases} \tag{0.3}$$

Moreover $\mathscr{F}_{\geqslant i}$ is independent on S, in the sense that if i separates the radii of $\mathscr{F}$ with respect to another weak triangulation S', then the resulting sub-object $\mathscr{F}_{\geqslant i}$ is the same.

In section 8 we provide an explicit example where (0.2) does not split. In section 5.4 we provide criteria to guarantee that $\mathscr{F}_{\geqslant i}$ is a direct summand of $\mathscr{F}$. More precisely we have the following

THEOREM 2 (cf. Theorems 5.4.3 and 5.4.10)**.** *In each one of the following two situations $\mathscr{F}_{\geqslant i}$ is a direct summand of $\mathscr{F}$:*

i) *The index i separates the radii of $\mathscr{F}^*$ and of $\mathscr{F}$, and (X, S) is either different from a virtual open disk with empty triangulation, or, if X is a virtual open disk D with empty triangulation, there exists a point $x \in D$ such that one of $\mathcal{R}_{\emptyset,i-1}(x, \mathscr{F})$ or $\mathcal{R}_{\emptyset,i-1}(x, \mathscr{F}^*)$ is spectral at x.*

ii) *One has $\Big(\Gamma_{S,1}(\mathscr{F}) \cup \cdots \cup \Gamma_{S,i-1}(\mathscr{F})\Big) \subseteq \Gamma_{S,i}(\mathscr{F})$.*

In both cases $(\mathscr{F}^*)_{\geqslant i}$ is isomorphic to $(\mathscr{F}_{\geqslant i})^*$, and it is also a direct summand of $\mathscr{F}^*$.

In section 6 we provide conditions to describe the controlling graphs, and in particular to fulfill the assumptions of Thm. 2.

[1] Quasi-smooth means that Ω_X is locally free, see [Duc, 2.1.8]. This corresponds to the notion called "rig-smooth" in the rigid analytic setting.

Condition ii) of Thm. 2 implies i), and it has the advantage that it involves only the radii of $\mathscr{F}$. Nevertheless i) is more natural, and general, by the following reason. If one is allowed to choose arbitrarily the weak triangulation, then to guarantee the existence of $\mathscr{F}_{\geqslant i}$ one has to choose it as small as possible, and the same is true for condition i). On the other hand to fulfill ii) one is induced to choose it quite large. For more precise statements see Remarks 5.3.4 and 5.4.11.

As a corollary we obtain the following:

Corollary 3 (cf. Cor. 5.6.8)**.** *Let $\mathscr{F}$ be a differential equation over X. There exists a locally finite subset $\mathfrak{F}$ of X such that, if Y is a connected component of $X - \mathfrak{F}$, then:*

i) *For all $i < j$ one has either $\mathcal{R}_{S,i}(y,\mathscr{F}) = \mathcal{R}_{S,j}(y,\mathscr{F})$ for all $y \in Y$, or $\mathcal{R}_{S,i}(y,\mathscr{F}) < \mathcal{R}_{S,j}(y,\mathscr{F})$ for all $y \in Y$.*

ii) *Let $1 = i_1 < i_2 < \ldots < i_h$ be the indexes separating the global radii $\{\mathcal{R}_{S,i}(-,\mathscr{F})\}_i$ over Y. Then one has a filtration*
$$0 \neq (\mathscr{F}_{|Y})_{\geqslant i_h} \subset (\mathscr{F}_{|Y})_{\geqslant i_{h-1}} \subset \cdots \subset (\mathscr{F}_{|Y})_{\geqslant i_1} = \mathscr{F}_{|Y} , \tag{0.4}$$
such that the rank of $(\mathscr{F}_{|Y})_{\geqslant i_k}$ is $r - i_k + 1$ and its solutions at each point of Y are the solutions of $\mathscr{F}$ with radius larger than $\mathcal{R}_{S,i_k}(-,\mathscr{F})$.

Note that we did not endow Y with a weak triangulation, and that the radii of the Corollary 3 are those of $\mathscr{F}$ viewed as an equation over X. In section 5.6 we also provide conditions making (0.4) a graduation.

A direct corollary of the above results is the Christol-Mebkhout decomposition over the Robba ring [CM00] (cf. Section 5.5). Another corollary is the following classification result:

Theorem 4 (cf. Cor. 7.3.1)**.** *Assume that X is either a Tate curve, or that $p \neq 2$ and X is an elliptic curve with good reduction. Consider a triangulation S of X formed by an individual point. Let $\mathscr{F}$ be a differential equation over X of rank r. Then*

i) *For all $i = 1,\ldots,r$ the radius $\mathcal{R}_{S,i}(-,\mathscr{F})$ is a constant function on X, and $\Gamma_{S,i}(\mathscr{F}) = \Gamma_S$;*

ii) *One has a direct sum decomposition as*
$$\mathscr{F} = \bigoplus_{0<\rho\leqslant 1} \mathscr{F}^\rho , \tag{0.5}$$
where $\mathcal{R}_{S,j}(-,\mathscr{F}^\rho) = \rho$ for all $j = 1,\ldots,\mathrm{rank}\,\mathscr{F}^\rho$.

The proof of the existence of $\mathscr{F}_{\geqslant i}$ (cf. Theorem 5.3.1) is obtained as follows. Firstly, in section 3, we prove a local decomposition theorem by the spectral radii for differential modules over the field $\mathscr{H}(x)$, of a point x of type 2, 3, or 4 of a Berkovich curve. This generalizes to curves the classical Robba's classical decomposition theorem [Rob75a], originally proved for points of type 2 or 3 of the affine line. Then, in section 4, we descends that decomposition to $\mathscr{O}_{X,x} \subseteq \mathscr{H}(x)$ (i.e. to a neighborhood of x in X). This is a generalization to curves of Dwork-Robba's decomposition result [DR77], originally proved for points of type 2 or 3 of the affine line. The language of generic disks introduced in [PP12b] permits to extend smoothly these proofs to curves, up to minor implementations.

Such local decompositions are also present in [Ked13], where one makes everywhere a systematic use of the spectral norm of the connection, as in [Ked10]. Methods involving spectral norms work thank to the Hensel factorization theorem of [Rob80], and [CD94, Lemme 1.4]. We presents here the same local decomposition results as a consequence of the original, and more geometric, techniques of [Rob75a] and [DR77].

Robba's and Dwork-Robba's local decompositions take in account only spectral radii, because over-solvable radii are not invariant by localization. For this reason Dwork-Robba's local decompositions (at the points of type 2, 3, and 4) do not glue, and they do not give the global decomposition. We then *augment* the Dwork-Robba decomposition of the stalk $\mathscr{F}_x$ by taking in account the decomposition of the trivial submodule of $\mathscr{F}_x$ coming from the existence of solutions converging in a disk containing x, i.e. taking in account over-solvable radii. This augmented decomposition of $\mathscr{F}_x$ glues without any obstructions, and it provides the existence of $\mathscr{F}_{\geqslant i}$.

In Theorem 1 we actually use the continuity of the radii $\mathcal{R}_{S,i}(-,\mathscr{F})$ (cf. proof of Proposition 2.9.7). Implicitly we also use the local finiteness of $\Gamma_{S,i}(\mathscr{F})$, since for $i \geqslant 2$ the continuity is an indirect consequence of the finiteness (cf. [Pul12] and [PP12b]).

In the second part of the paper, we provide an operative description of the controlling graphs, that is essential, for instance, to fulfill the assumptions of the above theorems.

As a consequence we obtain, in section 7, a bound on the number of edges of $\Gamma_{S,i}(\mathscr{F})$. We prove that this number is controlled by the knowledge of the slopes of the radii at a certain *locally finite* family of points, that are roughly speaking those where the super-harmonicty fails.

If X is a smooth geometrically connected projective curve, then we find *unconditional* bounds, in the sense that they depend only on the geometry of the curve and the rank of $\mathscr{F}$, not on the equation itself.

In particular, when $i = 1$, we prove the following neat result:

Theorem 5 (cf. Cor. 7.2.3)**.** *Let X be a smooth geometrically connected projective curve of genus $g \geqslant 1$. Let E_S be the number of edges of the skeleton Γ_S of the weak triangulation. Then the number of edges of $\Gamma_{S,1}(\mathscr{F})$ is at most*

$$E_S + 4r(g-1)\ . \tag{0.6}$$

Similar bounds are derived if $i \geqslant 2$. The key point in the proof of Theorem 5 is the control of the *locus of failure* of the super-harmonicity property of the partial heights of the convergence Newton polygon, as in [Pul12]. In particular, in section 6.2, we use the local part of the decomposition theorem to reprove a formula of [Ked10, Thm. 5.3.6] describing the Laplacian of the partial heights of the convergence Newton polygon (cf. Thm. 6.2.26). We extend it by taking into account solvable and over-solvable radii.

NOTE. This is a first draft, containing a maximum number of details. We plan to reduce its volume in a next version. We shall also improve the bounds of section 7, with further developments.

Structure of the paper.

In Section 1 we recall some notations and basic results. In Section 2 we define the radii and their graphs together with their elementary properties. In Section 3 we give Robba's local decomposition by the spectral radii over $\mathscr{H}(x)$, for a point of type 2, 3, or 4. In Section 4 we descend that decomposition to the local ring $\mathscr{O}_{X,x} \subseteq \mathscr{H}(x)$ following Dwork-Robba's original techniques. In Section 5 we obtain the global decomposition Theorem 5.3.1, and the criteria to have a direct sum decomposition (cf. Theorems 5.4.3 and 5.4.10). In Section 6 we provide an operative description of the graphs $\Gamma_{S,i}(\mathscr{F})$, together with the control of the failure of super-harmonicity using Christol-Mebkhout index theorems. In section 7 we obtain the bound on the number of their edges, and the classification results for the elliptic curves (cf. Corollary 7.3.1). In Section 8 we provide explicit counterexamples of the basic pathologies of over-solvable radii (incompatibility with duality, and exact sequence, a link between super-harmonicity property and presence of Liouville numbers, by means of the Grothendieck-Ogg-Shafarevich formula). In Appendix A we discuss the definition of the radius.

Acknowledgments. Thanks to Yves André, Francesco Baldassarri, Gilles Christol, Kiran S. Kedlaya, Adriano Marmora, and Zoghman Mebkhout for helpful comments.

1. Definitions and notations

In this section we give definitions and notations that are used in the sequel of the paper.

Let K be an ultrametric complete valued field of characteristic 0. Let p be the characteristic of its residue field k (either 0 or a prime number). We denote by $\widehat{K^{\mathrm{alg}}}$ the completion of an algebraic closure K^{alg} of K.

Setting 1.0.1. *Let X be a quasi-smooth K-analytic curve endowed with a weak triangulation S as in [PP12b]. Without loss of generality from now on we assume that X is connected.*

By a *differential equation* or *differential module* over X we mean a coherent $\mathscr{O}_X$-module $\mathscr{F}$ endowed with an (integrable) connection ∇. Proposition 1.0.2 below shows that $\mathscr{F}$ is automatically a locally free $\mathscr{O}_X$-module of finite rank. This is locally a differential module as defined in section 1.2. In the sequel of the article, we shall switch freely between the different terminologies.

Proposition 1.0.2. *Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module with a connection ∇. Then $\mathscr{F}$ is locally free of finite rank over $\mathscr{O}_X$.*

Proof. If $\mathscr{O}_{X,x}$ is a field, then $\mathscr{F}_x$ is free and finite dimensional, hence we are done. Otherwise x is L-rational for a finite Galois extension L/K. By Galois descent, one easily reduces to the case $x = 0$ and $d = d/dT$. The ring $\mathscr{O}_{X,x} \subset K[[T]]$ of convergent power series at $x = 0$ is then a discrete valuation ring without non trivial ideals stable by d/dT. A classical result then says that a differential module over $\mathscr{O}_{X,x}$ has no torsion, hence it is free (cf. the proof of [Ked10, 9.1.2]). See also [And01, Cor. 2.5.2.2] and [Kat70, Prop.8.8]. □

Corollary 1.0.3. *The category of locally free of finite rank $\mathscr{O}_X$-modules with connection is an abelian category.* □

Recall that X is connected by Setting 1.0.1.

Proposition 1.0.4. *Let $\sigma : \mathscr{F} \to \mathscr{F}'$ be a morphism between two differential equations over X. If there exists $x \in X$ such that $\sigma(x) : \mathscr{F}(x) \to \mathscr{F}'(x)$ is an isomorphism (resp. monomorphism, epimorphism) over $\mathscr{H}(x)$, then σ is globally an isomorphism (resp. monomorphism, epimorphism).*

Proof. The ranks of the kernel and the cokernel of σ are locally constant functions. □

1.1 Curves

Here we introduce some definitions and notations that will be frequently used in the paper.

Notation 1.1.1. *Let $\mathbb{A}^{1,\mathrm{an}}_K$ be the Berkovich affine line with coordinate T. Let L be a complete* *valued extension of K and $c \in L$. We set*

$$\begin{aligned} D^+_L(c,R) &= \left\{x\in \mathbb{A}^{1,\mathrm{an}}_L \,\middle|\, |(T-c)(x)|\leqslant R\right\}, && R\geqslant 0 && (1.1)\\ D^-_L(c,R) &= \left\{x\in \mathbb{A}^{1,\mathrm{an}}_L \,\middle|\, |(T-c)(x)|< R\right\}, && R> 0 && (1.2)\\ C^+_L(c;R_1,R_2) &= \left\{x\in \mathbb{A}^{1,\mathrm{an}}_L \,\middle|\, R_1\leqslant|(T-c)(x)|\leqslant R_2\right\}, && R_2\geqslant R_1\geqslant 0 && (1.3)\\ C^-_L(c;R_1,R_2) &= \left\{x\in \mathbb{A}^{1,\mathrm{an}}_L \,\middle|\, R_1<|(T-c)(x)|< R_2\right\}. && R_2> R_1> 0 && (1.4)\end{aligned}$$

If $D\subseteq \mathbb{A}^{1,\mathrm{an}}_L$ is a disk, we denote by $\mathscr{O}(D)$ (resp. $\mathcal{B}(D)$) the ring of *analytic* (resp. *bounded analytic*) functions on D. If $D=D^-_L(c,R)$, then

$$\mathscr{O}(D) := \left\{\sum_{n\geqslant 0} a_n(T-c)^n,\ a_n\in L,\ \lim_n |a_n|\rho^n=0,\ \forall\,\rho<R\right\}, \tag{1.5}$$

$$\mathcal{B}(D) := \left\{\sum_{n\geqslant 0} a_n(T-c)^n,\ a_n\in L,\ \sup_n |a_n|R^n<+\infty\right\}. \tag{1.6}$$

We write $\mathscr{O}_L(D)$ and $\mathcal{B}_L(D)$ instead of $\mathscr{O}(D)$ and $\mathcal{B}(D)$ respectively when we want to specify the base field.

The ring $\mathcal{B}_L(D)$ is a Banach algebra with respect to the sup-norm $\|.\|_D$ on D. If $D=D^-_L(c,R)$, this norm will be denoted by $|.|_{c,R}$. For $f\in\mathcal{B}(D^-_L(c,R))$ one has $|f|_{c,R}=\sup_{n\geqslant 0}\left|\frac{1}{n!}(\frac{d}{dT})^n(f)(c)\right|_L\cdot R^n$. This norm is indeed multiplicative. Hence it defines a point of Berkovich line $\mathbb{A}^{1,\mathrm{an}}_K$. We will denote it by $x_{c,R}$.

More generally, for all complete valued field extensions L/K, and all pairs (t,R), with $t\in L$, $R\geqslant 0$, the formula $|f|_{t,R} := \sup_{n\geqslant 0}\left|\frac{1}{n!}(\frac{d}{dT})^n(f)(t)\right|_L\cdot R^n$, for all polynomial $f\in K[T]$, defines a point of $\mathbb{A}^{1,\mathrm{an}}_K$. We will denote it by $x_{t,R}$.

Definition 1.1.2. *A virtual open disk (resp. annulus) is a connected analytic space that becomes isomorphic to a union of open disks (resp. annuli) after base change to $\widehat{K^{\mathrm{alg}}}$.*

1.1.1 *Radius of a point.* If $x\in\mathbb{A}^{1,\mathrm{an}}_K$ we define

$$r(x) := \max\Big(r\geqslant 0 \text{ such that there exists } t\in L/K \text{ satisfying } x=x_{t,r}\Big). \tag{1.7}$$

This is the radius of the point (cf. [Ber90, 4.2]). The real number $r(x)$ is not an invariant of the point x because it depends on the coordinate T. It is not invariant by arbitrary extension of K, but it is stable under extension contained in $\widehat{K^{\mathrm{alg}}}$. We refer to [Pul12, sections 1.3.1, 1.3.2] for additional properties of $r(x)$.

1.1.2 *Modulus of an inclusion of virtual disks.* The radius of a disk is not invariant by K-linear isomorphisms. On the other hand, annuli carry an intrinsic numerical datum: their so-called *modulus*.

Assume first that K is algebraically closed. The *modulus* of an annulus is the ratio $R_2/R_1\geqslant 1$ of the external radius R_2 by the internal radius R_1 with respect to an arbitrary coordinate (cf. [PP12a, Def. 2.2.2]). The definition is extended to a punctured disk $D-\{x\}$ where x is a point of type 1.

Assume now that K is not algebraically closed. Let C be a virtual annulus (resp. $D-\{x\}$ be a disk without a point x of type 1 or 4). Since the Galois group $\mathrm{Gal}(K^{\mathrm{alg}}/K)$ acts isometrically, all the connected components of $C\widehat{\otimes}\widehat{K^{\mathrm{alg}}}$ (resp. $(D-\{x\})\widehat{\otimes}\widehat{K^{\mathrm{alg}}}$) are isomorphic annuli (resp. punctured disks if x is of type 1, open annuli if x is of type 4) having all the same modulus. We call it the *modulus of* C (resp. $D-\{x\}$).

Let $D'\subseteq D$ be an inclusion of virtual open disks. If $D\neq D'$, the modulus of the semi-open annulus $D-D'$ is called the *modulus* of the inclusion $D'\subseteq D$. If $D=D'$ we define the modulus of the inclusion to be 1.

1.1.3 *Branches, germ of segments and sections.* Let X be a curve as in Setting 1.0.1. The Berkovich space X is naturally endowed with a graph structure (cf. [Duc]). By a *closed segment* $[x,y]\subset X$ we mean the image in X of an injective continuous path $[0,1]\to X$ with initial point x and end point y. We also call segments the images $]x,y[$, $]x,y]$, $[x,y[$ of $]0,1[$, $]0,1]$, $[0,1[$ respectively. In this case we say that the segments are open, or semi-open. By convention a segment is never reduced to a point nor to the empty set.

A *germ of segment* b out of $x\in X$ is an equivalence class of open segments $]x,y[$ given by $]x,y_1[\sim]x,y_2[$ if and only if there exists z such that $\emptyset\neq]x,z[\subseteq]x,y_1[\cap]x,y_2[$. By abuse we often write $b=]x,y[$ instead of $]x,y[\in b$.

If x is a point of type 2 or 3, then a germ of segment b out of x is always represented by an open segment $]x,y[$ which is the skeleton of a virtual open annulus C. In particular x belongs to the boundary of C in X. If x is a point of type 1 or 4, then $]x,y[$ corresponds to $D-\{x\}$, where D is a virtual open disk containing x. By abuse we say that $]x,y[$ is the skeleton of $D-\{x\}$, indeed this is the set of points that do not admit a disk as a neighborhood in of $D-\{x\}$.

A *section of a germ of segment* b out of x in X is a connected open subset of U of X containing such an annulus C (resp. $D-\{x\}$), if x is of type 2 or 3 (resp. 1 or 4), and such that x belongs to the closure of U in X, but not to U itself.

We refer to [Duc] for the definition of a branch. Roughly speaking a branch out of x corresponds to a direction out of x. Germs of segments out of x correspond bijectively to branches out of x. They will often be denoted by the same symbol b. By a section of the branch b, we mean a section of the corresponding germ of segment.

1.1.4 *Slopes.* Let b be a germ of segment out of $x\in X$, let $]x,y[$ be a representative of b, and let $F:[x,y]\to\mathbb{R}$ be a continuous function. If it has a meaning, we use the symbol $\partial_bF(x)$ to indicate the log-*slope at* x *of* F *along* b. The definition is the following.

Assume firstly that x is of type 2 or 3, and that K is algebraically closed. Then $]x,y[$ is the skeleton of an open annulus $C_{x,y}$ in X. Choose a coordinate T on $C_{x,y}$ identifying $C_{x,y}$ with $C_K^-(0;R_1,R_2)$. Then we set

$$\partial_bF(x)\;:=\;\lim_{\substack{y\to x\\ ]x,y[\in b}}\frac{\log(F(y))-\log(F(x))}{\log(|T(y)|)-\log(|T(x)|)}\;. \tag{1.8}$$

If the resulting limit exists, then it is independent on the chosen coordinate, because $|T(y)|/|T(x)|$ is the modulus of $C_{x,y}$. In general the definition descends from K^{alg} to K as the modulus does. So the slope $\partial_bF(x)$ is a K-rational notion, and it only depends on b and F.

If now x is of type 1 or 4, then $]x,y[$ is the skeleton of a virtual open disk $D-\{x\}$, with x removed. We express $D-\{x\}$ as a union of annuli $C_{z,y}$, with $z\in]x,y[$. In this case we define

$$\partial_bF(x)\;:=\;\lim_{\substack{z\to x\\ z\in]x,y[\in b}}\partial_bF(z)\;. \tag{1.9}$$

We say that F *is* log-*affine over* $]x,y[$ if it is continuous on $]x,y[$ and if its slope function $z\mapsto\partial_bF(z)$ is constant on $]x,y[$.

The functions of this paper will always be piecewise log-*affine*. With the above settings, this implies that for all $z \in [x,y[$ one has $\log(F(z)) = \alpha\log(|T(z)|)+\beta$, with $\alpha,\beta\in\mathbb{R}$. Hence $\partial_b F(x) = \alpha$.

1.1.5 *Graphs.* The reader can find in [Duc, 1.3.1] the definition of a graph. A single point and the empty set are graphs. We say that a germ of segment b out of x *belongs to a graph* $\Gamma\subseteq X$ if b is represented by a segment $]x,y[\subset\Gamma$. A point $x\in\Gamma$ is a *bifurcation point* if there is more than two germs of segments out of x belonging to Γ. A point $x\in\Gamma$ is an *end point of* Γ if there is no open segments $]z,y[$ in X such that $x\in]z,y[\subset\Gamma$.

In this paper by a *locally finite graph* we mean a closed connected subset $\Gamma\subset X$ such that

i) Each point $x\in\Gamma$ admits a neighborhood U in X such that $\Gamma\cap U$ is a finite union of segments and points;

ii) $X-\Gamma$ is a disjoint union of virtual open disks.

All the graphs of this paper are locally finite with the exception of those in the Appendix A.

1.1.6 *Weak triangulations.* Following [Duc, Section 4] and [PP12b, Section 2.1], a weak triangulation of X is a locally finite subset $S\subset X$, formed by points of type 2, and 3, such that $X-S$ is a disjoint union of virtual open annuli and virtual open disks. The skeleton Γ_S of S is then the union of S and the skeletons of the open annuli that are connected components of $X-S$.

Remark that the empty set is a triangulation of a virtual open annulus or virtual open disk.

1.1.7 *Star-shaped neighborhoods.* A connected open neighborhood U of a point $x\in X$ will be called *star-shaped* if

i) U is a virtual open disk containing x, if x is of type 1 or 4;

ii) U is an open subset of X such that $\{x\}$ is a weak triangulation of U, if x is of type 2 or 3.

We define the *canonical weak triangulation* S_U of U as the empty set in the first case, and as $S_U=\{x\}$ in the second case. We define the *pointed skeleton of* U *(as a neighborhood of* x*)* as

iii) the open segment connecting x to the boundary of the disk U, if x is of type 1 or 4;

iv) the skeleton Γ_{S_U} of $S_U=\{x\}$, if x is of type 2 or 3.

Let us now consider a connected affinoid neighborhood Y of x in X. If ∂Y denotes the topological boundary of Y, we say that Y is a *star-shaped* affinoid neighborhood of x in X if the connected component U_Y of $Y-\partial Y$ containing x is a star-shaped open neighborhood of x in X, and the other connected components of $Y-\partial Y$ are all virtual open disks. We define the canonical triangulation of Y as $\partial Y\cup\{x\}$, and the pointed skeleton as the union of the pointed skeleton of U_Y with ∂Y.

1.2 Differential modules and trivial submodules

Let A be a commutative ring with a non zero derivation $d: A\to A$ is called a differential ring.

Definition 1.2.1. *A* differential module (M,∇) *over* (A,d) *is a locally free* A*-module with finite rank, together with a connection* $\nabla:\mathrm{M}\to\mathrm{M}$*, i.e. a* $\mathbb{Z}$*-linear map satisfying the Leibnitz rule* $\nabla(am)=d(a)m+a\nabla(m)$ *for all* $a\in A$*,* $m\in\mathrm{M}$*.*

Remark 1.2.2. *By [Bou98, II.5.2, Thm. 1],* M *is locally free of finite rank over* A*, if and only if* M *is projective of finite rank over* A*.*

Denote by $A\langle d\rangle$ the Weyl algebra of differential polynomials. As an additive groups one has $A\langle d\rangle=\oplus_{n\geqslant 0}A\circ d^n$, and the multiplication $\circ$ is given by $a\circ d=d\circ a+d(a)$ for all $a\in A$. A differential module is naturally an $A\langle d\rangle$-module which is locally free of finite rank over A, and where the action of d on M is given by ∇. Morphisms between differential modules are $A\langle d\rangle$-linear morphisms, or equivalently A-linear maps commuting with the connections. The linear algebra constructions on M are naturally $A\langle d\rangle$-modules, so they acquire canonically a connection. As an example the dual module $\mathrm{M}^* := \mathrm{Hom}_A(\mathrm{M}, A)$ is endowed with the connection $(\nabla^*(\alpha))(m) = \alpha \circ \nabla - d \circ \alpha$.

Let (A', d') be another ring with derivation, and let $A \to A'$ be a ring morphism commuting with the derivations. Let M be an $A\langle d\rangle$-module. The scalar extension of M is the module $\mathrm{M} \otimes_A A'$ endowed with the connection $\nabla' := \nabla \otimes \mathrm{Id}_{A'} + \mathrm{Id}_{\mathrm{M}} \otimes d'$. A *solution* of M with values in A' is an element in the kernel of ∇':

$$H^0(\mathrm{M}, A') := \omega(\mathrm{M}, A') \ := \ \mathrm{Ker}(\nabla' : \mathrm{M} \otimes_A A' \to \mathrm{M} \otimes_A A') \ . \tag{1.10}$$

The kernel of the derivation $A^{d=0} := \mathrm{Ker}(d : A \to A)$ is a sub-ring of A. The group $\omega(\mathrm{M}, A)$ is naturally an sub-$A^{d=0}$-module of M, and the rule $\mathrm{M} \mapsto \omega(\mathrm{M}, A)$ is a covariant functor.

We denote the co-kernel by

$$H^1(\mathrm{M}, A') \ := \ \mathrm{Coker}(\nabla' : \mathrm{M} \otimes_A A' \to \mathrm{M} \otimes_A A') \ . \tag{1.11}$$

We often write $H^1(\mathrm{M}) := H^1(\mathrm{M}, A)$, if no confusion is possible.

$$\chi(\mathrm{M}, A') \ := \ \mathrm{rank}\, H^0(\mathrm{M}, A') - \mathrm{rank}\, H^1(\mathrm{M}, A') \ . \tag{1.12}$$

Lemma 1.2.3 ([Bou62, II.4.2])**.** *If* M *is a differential module, then*

$$\omega(\mathrm{M}, A') \ = \ \mathrm{Hom}_{A\langle d\rangle}(\mathrm{M}^*, A') \ . \qquad \square \tag{1.13}$$

Let M be a $A\langle d\rangle$-module. The canonical arrow

$$j_{\mathrm{M}} \ : \ A \otimes_{A^{d=0}} \omega(\mathrm{M}, A) \longrightarrow \mathrm{M} \ , \tag{1.14}$$

defined by $j_{\mathrm{M}}(a \otimes v) := a \cdot v \in \mathrm{M}$, is a morphism of differential modules, if $A \otimes_{A^{d=0}} \omega(\mathrm{M}, A)$ is endowed with the connection $\nabla(a \otimes v) := d(a) \otimes v$. The collection $\{j_{\mathrm{M}}\}_{\mathrm{M}}$ is a natural transformation of functors. We denote by M_A the image of j_{M}. If $f : \mathrm{M} \to \mathrm{N}$ is a morphism, then $f(\mathrm{M}_A) \subseteq \mathrm{N}_A$, and the rule $\mathrm{M} \mapsto \mathrm{M}_A$ is a functor. If $\mathrm{N} \to \mathrm{M} \to \mathrm{P}$ is exact, then $\mathrm{N}_A \to \mathrm{M}_A \to \mathrm{P}_A$ is a complex (i.e. the image of N_A in M_A lies in the kernel of $\mathrm{M}_A \to \mathrm{P}_A$).

Definition 1.2.4. *We say that the* $A\langle d\rangle$*-module* M *is* trivial *if* j_{M} *is an isomorphism. We say that* M *is* trivialized by A' *if* $\mathrm{M} \otimes_A A'$ *is trivial as a differential module over* A'*.*

Remark 1.2.5. *If* $\omega(\mathrm{M}, A)$ *is a finite free* $A^{d=0}$*-module, then* M *is trivial if and only if it is isomorphic, as* $A\langle d\rangle$*-module, to* A^r *with the connection* $\nabla(a_1, \ldots, a_r) := (d(a_1), \ldots, d(a_r))$*.*

Remark 1.2.6. *Assume that the derivation* $d : A \to A$ *is a surjective map. Then for every trivial* $A\langle d\rangle$*-module* M *the connection* $\nabla : \mathrm{M} \to \mathrm{M}$ *is surjective too. Indeed so is the connection* $d' \otimes \mathrm{Id}$ *of* $A \otimes_{A^{d=0}} \omega(\mathrm{M}, A)$*.*

Lemma 1.2.7 ([Ked10, 5.3.3, and 5.3.4])**.** *Let* M *and* N *be two* $A\langle d\rangle$*-modules. Denote by* $\mathrm{Ext}^1(\mathrm{M}, \mathrm{N})$ *the Yoneda extension group of exact sequences* $0 \to \mathrm{N} \to \mathrm{P} \to \mathrm{M} \to 0$*. If* M *is projective as* A*-module, then* $\mathrm{Ext}^1(\mathrm{M}, \mathrm{N}) \xrightarrow{\sim} H^1(\mathrm{M}^* \otimes \mathrm{N})$*.* $\square$

Lemma 1.2.8. *Let* $E : 0 \to \mathrm{N} \to \mathrm{M} \to \mathrm{Q} \to 0$ *be an exact sequence of* $A\langle d\rangle$*-modules. Then*

i) *The sequence* $\omega(E, A)$ *is left exact, moreover* $\omega(\mathrm{N}, A) = \omega(\mathrm{M}, A) \cap \mathrm{N}$*.*

ii) *Assume the derivation* $d : A \to A$ *is surjective, and that* N *is trivial. Then* $\omega(E, A)$ *is also right exact.*

iii) *Assume that the derivation $d : A \to A$ is surjective, and that* N *and* Q *are both trivial. Then* M *is trivial.*

Proof. i) Clearly $\omega(\mathrm{N}, A) \subseteq \omega(\mathrm{M}, A)$. If $x \in \mathrm{M}$ is killed by ∇, and if its image in Q is 0, then $x \in \omega(\mathrm{M}, A) \cap \mathrm{N} = \omega(\mathrm{N}, A)$. This proves the exactness of $\omega(E, A)$.

ii) If moreover $d : A' \to A'$ is surjective, and N is trivialized by A', then the connection $\nabla' : \mathrm{N}\otimes_A A' \to \mathrm{N}\otimes_A A'$ is surjective by Remark 1.2.6. So the snake lemma of the diagram $\nabla' : E\otimes_A A' \to E \otimes_A A'$ provides the exactness of the sequence $\omega(E, A')$.

iii) Assume moreover that Q is trivialized by A' too. Let $E' := E \otimes_A A'$ and $\mathrm{M}' := \mathrm{M} \otimes_A A'$. Then the five lemma applied to the diagram $j_{E'} : A'\otimes_{(A')^{d'=0}}\omega(E', A') \to E'$ implies that the middle map $j_{\mathrm{M}'}$ is an isomorphism, and M' is trivial. □

Remark 1.2.9. *Let A' be a differential ring over A. With the notation of Lemma 1.2.8, if* $\mathrm{Tor}_1^A(\mathrm{Q}, A') = 0$*, then $E \otimes_A A'$ is exact, and hence $\omega(E, A')$ is left exact by i). Under this condition the statements of ii) and iii) holds replacing the word trivial by trivialized by A', and the condition d is surjective by d' is surjective.*

Recall that a derivation d' on an integral domain A' extends uniquely to a derivation on its fraction field $F(A')$. We will abuse notation and also denote it d'.

Lemma 1.2.10. *Assume that A is an integral domain such that $A^{d=0} = F(A)^{d=0}$. Then*

i) *If* M *is a $A\langle d\rangle$-module which is projective as A-module, then j_{M} is injective.*
ii) *Assume that* M *trivial. Let $0 \to \mathrm{N} \to \mathrm{M} \to \mathrm{Q} \to 0$ be an exact sequence of $A\langle d\rangle$-modules such that* Q *is projective as A-module. Then* N *and* Q *are both trivial (in particular* M, N *and* Q *are free A-modules).*
iii) *With the assumptions of ii), if moreover the derivation $d : A \to A$ is a surjective map, then the sequence $0 \to \mathrm{N} \to \mathrm{M} \to \mathrm{Q} \to 0$ splits.*

Proof. i) If $A = F(A)$, then it is classical (cf. [Ked10, Lemma 5.1.5]). If $A \neq F(A)$, we deduce the injectivity of j_{M} from the injectivity of $j_{\mathrm{M}\otimes_A F(A)}$. For this it is enough to prove that the induced map $\omega(\mathrm{M}, A) \otimes_{A^{d=0}} A \to \omega(\mathrm{M} \otimes_A F(A), F(A)) \otimes_{A^{d=0}} F(A)$ is injective. The inclusions $\omega(\mathrm{M}, A) \subseteq \omega(\mathrm{M}, F(A)) := \omega(\mathrm{M} \otimes_A F(A), F(A))$ follows by left exactness of $\omega(\mathrm{M}, -)$, using (1.13). This is an inclusion of vector spaces over the same field $A^{d=0} = F(A)^{d=0}$. We deduce that the maps deduced by scalar extension $\omega(\mathrm{M}, A) \otimes_{A^{d=0}} A \to \omega(\mathrm{M}, A) \otimes_{A^{d=0}} F(A)$ and $\omega(\mathrm{M}, A) \otimes_{A^{d=0}} F(A) \to \omega(\mathrm{M} \otimes_A F(A), F(A)) \otimes_{A^{d=0}} F(A)$ are injective. So the composite map $\omega(\mathrm{M}, A) \otimes_{A^{d=0}} A \to \omega(\mathrm{M} \otimes_A F(A), F(A)) \otimes_{A^{d=0}} F(A)$ is injective.

ii) If M is trivial, and if $E : 0 \to \mathrm{N} \xrightarrow{i} \mathrm{M} \xrightarrow{p} \mathrm{Q} \to 0$ is an exact sequence, we consider the morphism of sequences $j_E : \omega(E, A) \otimes_{A^{d=0}} A \to E$. From the surjectivity of $p \circ j_{\mathrm{M}}$ we deduce the surjectivity of j_{Q}. By i) one has its injectivity, hence Q is trivial. Moreover the map $\omega(p, A) \otimes \mathrm{Id}_A$ coincides with p, so it is surjective. Now $A^{d=0}$ is a field, hence A is flat over $A^{d=0}$, and $\omega(E, A) \otimes_{A^{d=0}} A$ is left exact. Then we can apply the snake lemma, and N is trivial.

iii) Now $\mathrm{Ext}^1(\mathrm{Q}, \mathrm{N}) = H^1(\mathrm{Q}^* \otimes \mathrm{N})$, by Lemma 1.2.7. Since all modules are trivial, $\mathrm{Q}^* \otimes \mathrm{N}$ is trivial too. In this situation if $d : A \to A$ is surjective, then $H^1(\mathrm{Q}^* \otimes \mathrm{N}) = 0$ by Remark 1.2.6. □

Remark 1.2.11. *The trivial sub-module M_A is rarely a direct factor of* M*. We provide explicit examples of this in section 8.*

Remark 1.2.12 (Duality and trivial sub-modules)**.** *The duality endo-functor $\mathrm{M} \mapsto \mathrm{M}^*$ is an additive and exact equivalence of category of differential modules. Unfortunately it is not compatible with the* *functor* $\mathrm{M} \mapsto \mathrm{M}_A$. *Namely there exists a canonical composite morphism*

$$(\mathrm{M}^*)_A \;\to\; \mathrm{M}^* \;\to\; (\mathrm{M}_A)^* \tag{1.15}$$

which is often not an isomorphism. The dimensions of $(\mathrm{M}_A)^*$ *and* $(\mathrm{M}^*)_A$ *can be different (see the counterexamples of section 8). But even when the two dimensions are equal, there is no reasons to have an isomorphism. This means that* $\omega(\mathrm{M}^*, A)$ *can not be identified to the dual of* $\omega(\mathrm{M}, A)$*. This is closely related to the assumption of Theorem 5.4.3.*

Remark 1.2.13. *The dual convention, which is often used in literature*[2]*, consists in defining the solutions of* M *with values in* A' *as the morphisms* $\mathrm{Hom}_{A\langle d\rangle}(\mathrm{M}, A') = \mathrm{Hom}_{A'\langle d\rangle}(\mathrm{M} \otimes_A A', A')$*. With respect to the convention (1.10) these are the solutions of the dual module* M^**. Duality is an equivalence of the category of differential modules, so each statement admits a dual statement.*

1.2.1 *Change of derivation.* Let $d_1, d_2 : A \to A$ be two derivations satisfying $d_2 = ad_1$ with a invertible in A. The two categories of differential modules with respect to d_1 and to d_2 are isomorphic as follows: to a d_1-module (M, ∇_1) one associates the d_2-module (M, ∇_2) with $\nabla_2 = a \cdot \nabla_1$. Then an A-linear morphism commutes with ∇_1 if and only if it commutes with ∇_2. In other words the functor is the identity on the morphisms. A differential module is d_1-trivial if and only if it is d_2-trivial.

1.2.2 *Filtrations of cyclic modules, and factorization of operators.* We say that a differential module M is *cyclic* if it is of the form $A\langle d\rangle / A\langle d\rangle P$, for some unitary differential polynomial P. The latter module will be denoted by M_P. If A is a field, any differential module is cyclic (cf. [Del70, Ch.II, Lemme 1.3], [Kat87]).

If P factorizes in $A\langle d\rangle$ as $P = P_1 \cdot P_2$, then right multiplication $L \mapsto L \cdot P_2$ by P_2 identifies $A\langle d\rangle P_1$ to $A\langle d\rangle P$ and one has an exact sequence

$$0 \to \mathrm{M}_{P_1} \to \mathrm{M}_P \to \mathrm{M}_{P_2} \to 0\,. \tag{1.16}$$

If A is a field, then the converse is also true: given an exact sequence of differential modules $0 \to \mathrm{N} \to \mathrm{M}_P \to \mathrm{Q} \to 0$, there exists a factorization $P = P_1 \cdot P_2$ such that $\mathrm{N} \cong \mathrm{M}_{P_1}$ and $\mathrm{Q} \cong \mathrm{M}_{P_2}$ (cf. [Chr83, 3.5.6] for more details).

2. Radii and filtrations by the radii

In this section we introduce the radii of convergence of a differential equation over X, and we recall some results of [Pul12] and [PP12b]. Without explicit mention of the contrary, we assume everywhere that the curve X is endowed with a weak triangulation S.

2.1 Generic disks

We begin by recreating the framework of Dwork's generic disks, in the context of Berkovich analytic curves.

Definition 2.1.1. *Let* $x \in X$*. The map* $\mathscr{M}(\mathscr{H}(x)) \to X$ *lifts canonically to a map* $\mathscr{M}(\mathscr{H}(x)) \to X_{\mathscr{H}(x)}$ *by the universal property of the Cartesian diagram* $X_{\mathscr{H}(x)}/\mathscr{H}(x) \to X/K$*. We denote by* $t_x \in X_{\mathscr{H}(x)}$ *the* $\mathscr{H}(x)$*-rational point so obtained.*

There is another way of lifting the points of X:

[2] This is indeed the convention adopted in all the papers by Christol, Dwork, Mebkhout, Robba,. . .

Proposition 2.1.2 ([PP12b]). *Assume that K is algebraically closed. Let Ω be a complete valued field extension of $\mathscr{H}(x)$, and let $X_\Omega := X\widehat{\otimes}_K\Omega$. Denote by $\pi_\Omega : X_\Omega \to X$ the canonical projection. For all $x \in X$ there exists a unique point $\sigma_\Omega(x) \in \pi_\Omega^{-1}(x)$ such that $\pi_\Omega^{-1}(x) - \{\sigma_\Omega(x)\}$ is a union of virtual open disks $D \subseteq X_\Omega$ such that their topological closure $\overline{D}$ in X_Ω is $D \cup \{\sigma_\Omega(x)\}$.* □

Definition 2.1.3. *Assume that K is algebraically closed. For all field extensions L/K one has a canonical section*

$$\sigma_L \ : \ X \to X_L \tag{2.1}$$

associating to x the image of the point $\sigma_\Omega(x) \in X_\Omega$ of Prop. 2.1.2 by the canonical projection $X_\Omega \to X_L$, where Ω/K contains L and $\mathscr{H}(x)$.

The map σ_L is well defined and independent on the choice of Ω (cf. [PP12b, Def. 2.1.10]).

Remark 2.1.4. *The map σ_L is defined in [Ber90, p.98], see also [Poi12].*

2.1.1 *Generic disks.* If now K is arbitrary, then $X \cong X_{\widehat{K^{\mathrm{alg}}}}/\mathrm{Gal}(K^{\mathrm{alg}}/K)$. This allows to describe $\pi_\Omega^{-1}(x)$ as follows. Consider the composite arrow $\pi_\Omega : X_\Omega \to X_{\widehat{K^{\mathrm{alg}}}} \to X$, where the first projection $\pi_{\Omega/\widehat{K^{\mathrm{alg}}}}$ is that of Proposition 2.1.2. If $\{x_1, \ldots, x_n\}$ is the set of points in $X_{\widehat{K^{\mathrm{alg}}}}$ lifting $x \in X$, then one has a disjoint union

$$\pi_\Omega^{-1}(x) \ = \ \bigcup_{i=1}^{n} \pi^{-1}_{\Omega/\widehat{K^{\mathrm{alg}}}}(x_i) \ . \tag{2.2}$$

By Proposition 2.1.2, $\pi_\Omega^{-1}(x) - \{\sigma_\Omega(x_1), \ldots, \sigma_\Omega(x_n)\}$ is a disjoint union of virtual open disks.

Now if x is of type 1, then these disks are empty for all Ω/K. Otherwise, if x is of type 2, 3, or 4, then they are not empty as soon as $\mathscr{H}(x)$ embeds into Ω. In this case we call them *Dwork generic disks for x*. The group $\mathrm{Gal}^{\mathrm{cont}}(\Omega/K)$ acts transitively on these disks (cf. [PP12b, Cor. 2.1.14]). So we often say "*the*" generic disk to indicate one of them, and we denote it by

$$D(x) \ \subset \ X_\Omega \ . \tag{2.3}$$

By construction $D(x)$ exists if and only if x is not of type 1, and it always comes by scalar extension from a $\mathscr{H}(x)$-rational disk, still denoted by $D(x)$.

Notation 2.1.5. *In the sequel we choose t_x as a canonical center of $D(x)$.*

2.1.2 *Existence of a big field.* The field Ω appears

i) in Prop. 2.1.2 to describe the structure of $\pi_\Omega^{-1}(x) - \{\sigma_\Omega(x)\}$;
ii) to prove the independence of the radii on the center t_x (cf. Remark 2.3.2);
iii) when one needs a spherically complete field to make differential modules free over $D(x)$ by Lazard's theorem [Laz62] (cf. also [Chr12, Ch.II]).

By convenience of notations, from now on we fix a field spherically complete and algebraically closed field Ω/K containing the family of fields $\{\mathscr{H}(x)\}_{x\in X}$. This is possible thanks to Proposition 2.1.7 below. The field Ω does not play any essential role, and it will not be specified anymore.

Remark 2.1.6. *The point t_x plays no particular role too, since two centers of two disks $D(x)$ are always conjugated by $\mathrm{Gal}^{\mathrm{cont}}(\Omega/K)$. We fix t_x and Ω merely by convenience of notation.*

Proposition 2.1.7. *Let L be a valued field. Let I be a set and $(L_i)_{i\in I}$ be a family of valued extensions of L. Then, there exists a valued extension M of L such, for every $i \in I$, there exists an L-linear* *isometric embedding $j_i\colon L_i \hookrightarrow M$.*

Proof. For every $i \in I$, consider an isometric embedding $h_i\colon L \hookrightarrow L_i$. Fix a set $\mathcal{W}$ of cardinality at least $\sum_{i\in I}\mathrm{Card}(L_i) + \aleph_0$.

Consider the collection $\mathscr{T}$ of tuples $(M, \sigma_M, h_M, J, (j_i)_{i\in J})$ such that

i) M is a subset of $\mathcal{W}$;

ii) σ_M is a valued field structure on M (which will be implicit in what follows);

iii) h_M is an isometric embedding of L into M;

iv) J is a subset of I;

v) for every $i \in J$, j_i is an L-linear isometric embedding of L_i into M;

vi) the subfield generated by the union of the images of the j_i's is dense in M.

By the first condition, $\mathscr{T}$ is a set.

Remark that, if M is a valued field that satisfies all the conditions but the first one, then there exists a valued field M', that is isometrically isomorphic to M, and satisfy all the condition. Indeed, the last condition implies that the cardinality of M is at most $\sum_{i\in I}\mathrm{Card}(L_i) + \aleph_0$, hence there exists a bijection between M and a subset M' of $\mathcal{W}$. Transporting the structures, we are done. For this reason, from now on, we will forget this first condition.

We endow $\mathscr{T}$ with the following order relation: $(M, \sigma_M, h_M, J, (j_i)_{i\in J}) \leqslant (M', \sigma_{M'}, h_{M'}, J', (j'_i)_{i\in J'})$ if

i) there exists an L-linear isometric embedding $j_{M',M}$ of M into M';

ii) J is a subset of J' ;

iii) for every $i \in J$, we have $j'_i = j_{M',M} \circ j_i$.

By an inductive limit construction, it is easy to check that every totally ordered subset of $\mathscr{T}$ has an upper-bound. Hence, by Zorn's lemma, there exists a maximal element $(M, J, (j_i)_{i\in J})$ in $\mathscr{T}$.

We claim that $J = I$, which proves the lemma. By contradiction, assume that $J \subset I$ and choose $k \in I \setminus J$. It is known that there exists a valued field M' that is both an extension of M and L_k: we have isometric embeddings $j_{M',M}\colon M \to M'$ and $j'_k\colon L_k \to M'$. Moreover, we may assume that $j_{M',M}\circ h_M = j'_k \circ h_k$, and that the subfield generated by $j_{M',M}(M) \cup j_k(L_k)$ is dense in M'. Set $h_{M'} = j_{M,M'} \circ h_M$ and, for every $i \in J$, $j'_i = j_{M',M} \circ j_i$. Then the tuple $(M', \sigma_{M'}, h_{M'}, J \cup \{k\}, (j'_i)_{i\in J\cup\{k\}})$ is an element of $\mathscr{T}$ that is greater than $(M, \sigma_M, h_M, J, (j_i)_{i\in J})$. □

2.2 Maximal disks

Definition 2.2.1. *Let $Z \subseteq X$ be any subset such that $X - Z$ is a disjoint union of virtual open disks or annuli. Let $x \in X$, and let $\Omega/\mathscr{H}(x)$ be any complete valued field extension. We define*

$$D(x, Z) \ \subseteq \ X_\Omega - \sigma_\Omega(Z) \tag{2.4}$$

as the largest open disk centered at t_x and contained in $X_\Omega - \sigma_\Omega(Z)$.

Remark 2.2.2. *If $x \in Z$, then $D(x, Z) = D(x)$. As a consequence if $Z = X$, then one has $D(x, Z) = D(x)$ for all $x \in X$. Note however that the field Ω depends on x.*

Definition 2.2.3 (Maximal disks)**.** *Let $x \in X$. We call* maximal disk of x *(with respect to the weak triangulation S) the disk*

$$D(x, S) \ = \ D(x, \Gamma_S) \ . \tag{2.5}$$

If Ω/K is algebraically closed and spherically complete, all maximal disks in X_Ω are isomorphic under the action of $\mathrm{Gal}^{\mathrm{cont}}(\Omega/K)$.

Remark 2.2.4. *There are two possibilities for $D(x,S)$. If x belongs to the skeleton Γ_S, then $D(x,S)$ is contained in $\pi_\Omega^{-1}(x)$, and $D(x,S)=D(x)$ (cf. Remark 2.2.2). Otherwise, if $x\notin\Gamma_S$, then $D(x,S)$ contains $\pi_\Omega^{-1}(x)$, and it is strictly larger than the generic disk. In this case $D(x,S)$ has a K^{alg}-rational center, and its image in X is an open virtual disk containing x (cf. [Pul12, Lemma 1.5]).*

2.3 Multiradius.

Assume that Ω/K is algebraically closed and spherically complete. Choose an isomorphism $D(x,S)\xrightarrow{\sim} D^-_\Omega(0,R)$ sending t_x at 0. By a result of M. Lazard (cf. [Laz62] and [Chr12, Ch.II, Section 4.4]) the restriction $\widetilde{\mathscr{F}}$ of $\mathscr{F}$ to $D^-_\Omega(0,R)$ is free of rank $r=\mathrm{rank}(\mathscr{F}_x)$, and it is hence given by a differential module over $\mathscr{O}(D^-_\Omega(0,R))$. Denote by

$$\mathcal{R}^{\widetilde{\mathscr{F}}}_{S,i}(x)\ >\ 0 \tag{2.6}$$

the radius of the maximal open disk centered at 0 and contained in $D^-_\Omega(0,R)$ on which the connection of $\widetilde{\mathscr{F}}$ admits at least $r-i+1$ horizontal sections that are linearly independent over Ω.

Definition 2.3.1 (Multiradius)**.** *We call multiradius of $\mathscr{F}$ at x the tuple*

$$\boldsymbol{\mathcal{R}}_S(x,\mathscr{F})\ :=\ (\mathcal{R}_{S,1}(x,\mathscr{F}),\ldots,\mathcal{R}_{S,r}(x,\mathscr{F})) \tag{2.7}$$

where, for every i, one has

$$\mathcal{R}_{S,i}(x,\mathscr{F})\ :=\ \mathcal{R}^{\widetilde{\mathscr{F}}}_{S,i}(x)/R\ \in\,]0,1]\ . \tag{2.8}$$

The definition only depends on x and $(\mathscr{F},\nabla)$. Each $\mathcal{R}_{S,i}(x,\mathscr{F})$ is the inverse of the *modulus* (cf. 1.1.2) of a well defined sub-disk $D_{S,i}(x,\mathscr{F})\subseteq D(x,S)$, centered at t_x:

$$\emptyset\neq D_{S,1}(x,\mathscr{F})\ \subseteq\ D_{S,2}(x,\mathscr{F})\ \subseteq\ \cdots\ \subseteq\ D_{S,r}(x,\mathscr{F})\ \subseteq\ D(x,S)\ . \tag{2.9}$$

For every $0<R\leqslant 1$, we denote respectively by

$$D_S(x,R)\ \subseteq\ D(x,S) \tag{2.10}$$

$$D(x,R)\ \subseteq\ D(x) \tag{2.11}$$

the open sub-disks centered at t_x of modulus equal to $1/R$. With this convention one has

$$D_{S,i}(x,\mathscr{F})\ :=\ D_S(x,\mathcal{R}_{S,i}(x,\mathscr{F}))\ . \tag{2.12}$$

Remark 2.3.2. *If t_x is replaced by another center of $D(x)$, the radius $\mathcal{R}^{\widetilde{\mathscr{F}}}_{S,i}(x)$ does not change. This is because all centers of $D(x)$ are permuted by the Galois group $\mathrm{Gal}^{\mathrm{cont}}(\Omega/K)$ that acts isometrically (cf. [Pul12, Section 4.3]).*

Remark 2.3.3. *Let S,S' be two weak triangulations of X. If $\Gamma_S=\Gamma_{S'}$, then $\boldsymbol{\mathcal{R}}_S(-,\mathscr{F})=\boldsymbol{\mathcal{R}}_{S'}(-,\mathscr{F})$. Indeed the disk $D(x,S)$ only depends on Γ_S.*

Definition 2.3.4 (Convergence Newton polygon)**.** *We call* convergence Newton polygon *of $\mathscr{F}$ at $x\in X$ the epigraph of the unique continuous convex function $h_x:[-\infty,r[\to\mathbb{R}_{\geqslant 0}$ satisfying*

i) *$h_x(0)=0$, and $h_x(i)-h_x(i-1)=-\log(\mathcal{R}_{S,r-i+1}(x,\mathscr{F}))$, for all $i=1,\ldots,r$;*
ii) *For all $i=1,\ldots,r$ the function h_x is affine over $[i-1,i]$, and constant on $]-\infty,0]$.*

In other words it is the polygon whose slopes are $-\log\mathcal{R}_{S,r}(x,\mathscr{F})\leqslant\cdots\leqslant-\log\mathcal{R}_{S,1}(x,\mathscr{F})$. For all $i=1,\ldots,r$ the numbers $h_x(i)=\sum_{j=r-i+1}^r-\log\mathcal{R}_{S,j}(x,\mathscr{F})$ are called the partial heights of the *polygon.*

The following definition is convenient for technical reasons concerning the super-harmonicity properties (cf. section 6.2):

Definition 2.3.5 (Partial heights of $\mathscr{F}$)**.** *Let $i \leqslant r = \mathrm{rank}(\mathscr{F})$. We call i-th partial height of $\mathscr{F}$ the function*

$$H_{S,i}(x,\mathscr{F}) \;:=\; \prod_{j=1,\ldots,i} \mathcal{R}_{S,j}(x,\mathscr{F}) \;. \tag{2.13}$$

With the notations of Def. 2.3.4 one has

$$\ln(H_{S,i}(x,\mathscr{F})) \;=\; h_x(r-i+1) - h_x(r) \;. \tag{2.14}$$

2.4 Controlling graphs

In [Pul12] and [PP12b] we obtained the following result.

Theorem 2.4.1 ([Pul12],[PP12b])**.** *For all $i = 1,\ldots,r$ the functions $x \mapsto \mathcal{R}_{S,i}(x,\mathscr{F})$ are continuous. Moreover there exists a locally finite graph $\Gamma \subseteq X$ such that for all i the radius $\mathcal{R}_{S,i}(-,\mathscr{F})$ is constant on every connected components of $X - \Gamma$.*

If $S = \emptyset$, then, by definition X is a virtual open disk or annulus (recall that X is connected by Setting 1.0.1). On the other hand $\Gamma_S = \emptyset$ if and only if X is a virtual open disk with empty (weak) triangulation. This is the unique case in which there is no retraction $X \to \Gamma_S$. Any other connected curve X admits a canonical retraction

$$\delta_{\Gamma_S} \;:\; X \to \Gamma_S \;. \tag{2.15}$$

The map δ_{Γ_S} is the identity on Γ_S, and it associates to $x \notin \Gamma_S$ the boundary in Γ_S of the maximal disk $D(x,\Gamma_S) = D(x,S)$. This makes sense since $X - \Gamma_S$ is disjoint union of virtual open disks.

Definition 2.4.2. *Let $\mathcal{T}$ be a set, and let $f : X \to \mathcal{T}$ be a function. We call S-*controlling graph *(or S-*skeleton*) of f the set $\Gamma_S(f)$ of points $x \in X$ that admit no neighborhoods[3] D in X such that*

i) *D is a virtual disk;*

ii) *f is constant on D;*

iii) *$D \cap \Gamma_S = \emptyset$ (or equivalently $D \cap S = \emptyset$).*

In particular $\Gamma_S \subseteq \Gamma_S(f)$.

Remark 2.4.3. *The graph $\Gamma_S(f)$ is different from the locus defined as the complement of the union of the open subsets of X on which f is constant. Indeed f can be constant along some segments in $\Gamma_S(f)$, and hence on the corresponding annulus in X. This is because the definition involves only disks on which f is constant, and not arbitrary subsets.*

We denote by $\Gamma_{S,i}(\mathscr{F})$ the controlling graph of the function $\mathcal{R}_{S,i}(-,\mathscr{F})$. By definition

$$\Gamma_S \;\subseteq\; \Gamma_{S,i}(\mathscr{F}) \;. \tag{2.16}$$

Hence $X - \Gamma_{S,i}(\mathscr{F})$ is a disjoint union of virtual open disks. If $X = D$ is a virtual open disk with empty weak triangulation, and if $\mathcal{R}_{S,i}(-,\mathscr{F})$ is constant on D, then $\Gamma_{S,i}(\mathscr{F}) = \Gamma_S = \emptyset$. In all other

[3] Note that $D(x)$ is not a neighborhood of x in X.

cases $\Gamma_{S,i}(\mathscr{F})$ is not empty, and there is a canonical continuous retraction

$$\delta_{\Gamma_{S,i}(\mathscr{F})} \;:\; X \to \Gamma_{S,i}(\mathscr{F})\;. \tag{2.17}$$

The *controlling graph* $\Gamma_S(\mathscr{F})$ of $(\mathscr{F},\nabla)$ is by definition the union of all the $\Gamma_{S,i}(\mathscr{F})$:

$$\Gamma_S(\mathscr{F}) \;:=\; \bigcup_{i=1}^{r}\Gamma_{S,i}(\mathscr{F})\;. \tag{2.18}$$

One has $\Gamma_S(\mathscr{F}) = \emptyset$ if and only if $X = D$ is a virtual disk with empty weak triangulation, and $\boldsymbol{\mathcal{R}}_S(-,\mathscr{F})$ is a constant function on D. In all other cases $\Gamma_S(\mathscr{F}) \neq \emptyset$ is the smallest graph containing Γ_S on which $\boldsymbol{\mathcal{R}}_S(-,\mathscr{F})$ factorizes by the canonical continuous retraction

$$\delta_{\Gamma_S(\mathscr{F})} \;:\; X \to \Gamma_S(\mathscr{F})\;. \tag{2.19}$$

An operative description of the controlling graphs is given in section 6.

2.5 Filtered space of solutions

We now define the space of solutions of $\mathscr{F}$ at a point $x \in X$, and its filtration by the radii.

Definition 2.5.1. *We say that a tuple $(d_1,\dots,d_r)$ is a* scale *if*

i) *for all $i \in \{1,\dots,r\}$ one has $d_i \in \{1,\dots,r\}$;*

ii) *$d_1 = r$ and $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_r$;*

iii) *If $d_i \neq d_{i-1}$, then $d_i = r-i+1$.*

It follows from the definition that $r-i+1 \leqslant d_i \leqslant r$, for all $i = 1,\dots,r$.

Definition 2.5.2. *Let V be a vector space of dimension r over a field Ω. A* filtration by the radii *of V is a totally ordered family of r sub-spaces*

$$V_r \;\subseteq\; V_{r-1} \;\subseteq\; \cdots \;\subseteq\; V_1 \;=\; V \tag{2.20}$$

such that the sequence $(d_1,\dots,d_r) := (\dim V_1,\dots,\dim V_r)$ is a scale. Note that for $i \in \{2,\dots,r\}$

$$V_i \;\neq\; V_{i-1} \quad \textit{if and only if} \quad \dim V_i \;=\; r-i+1\;. \tag{2.21}$$

If $i = 1$, or if i is an index such that (2.21) holds, we say that the index i separates the filtration.

2.5.1 *Filtration by the radii of the solutions.* Let $x \in X$, and let t_x and $\Omega/\mathscr{H}(x)$ be as in Section 2.1.1. We call *solution of $\mathscr{F}$ at x* any element in the stalk $\mathscr{F}_{t_x} := \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X_\Omega,t_x}$ which is killed by the connection ∇. The kernel of ∇ acting on $\mathscr{F}_{t_x}$ will be denoted by

$$\omega(x,\mathscr{F})\;. \tag{2.22}$$

Let $D \subseteq D(x,S)$ be an open disk containing t_x. We define

$$\omega(D,\mathscr{F}) \tag{2.23}$$

as the image in $\omega(x,\mathscr{F})$ of the kernel of the connection of the $\mathscr{O}(D)$-differential module $\mathscr{F}(D)$. The rule $\mathscr{F} \mapsto \omega(D,\mathscr{F})$ is a left exact functor (cf. Lemma 1.2.8). We then define (cf. (2.9))

$$\omega_{S,i}(x,\mathscr{F}) \;:=\; \omega(D_{S,i}(x,\mathscr{F}),\mathscr{F})\;. \tag{2.24}$$

The Ω-vector space $\omega(x,\mathscr{F})$ admits a filtration:

$$0 \;\neq\; \omega_{S,r}(x,\mathscr{F}) \;\subseteq\; \omega_{S,r-1}(x,\mathscr{F}) \;\subseteq\; \cdots \;\subseteq\; \omega_{S,1}(x,\mathscr{F}) \;=\; \omega(x,\mathscr{F})\;. \tag{2.25}$$

Lemma 2.5.3. *The filtration (2.25) is a filtration by the radii.* □

Condition (2.21) is then expressed by the following

Definition 2.5.4. *Let $i \in \{1, \ldots, r\}$. We say that the index i* separates the radii of $\mathscr{F}$ at $x \in X$ *if either $i = 1$ or if one of the following equivalent conditions holds:*

i) $\mathcal{R}_{S,i-1}(x, \mathscr{F}) < \mathcal{R}_{S,i}(x, \mathscr{F})$;

ii) $\omega_{S,i-1}(x, \mathscr{F}) \supset \omega_{S,i}(x, \mathscr{F})$.

We say that i separates the radii of $\mathscr{F}$ *if it separates the radii of $\mathscr{F}$ at all $x \in X$.*

Remark 2.5.5. *The index i separates the radii of $\mathscr{F}$ (at all point) if and only if the function $x \mapsto \dim_\Omega \omega_{S,i}(x, \mathscr{F})$ is constant on X of value $r - i + 1$.*

2.6 Spectral, solvable, and over-solvable radii.

We say that the i-th radius $\mathcal{R}_{S,i}(x, \mathscr{F})$ is

$$\begin{cases} \textit{spectral} & \text{if} \quad D_{S,i}(x,\mathscr{F}) \subseteq D(x) , \\ \textit{solvable} & \text{if} \quad D_{S,i}(x,\mathscr{F}) = D(x) , \\ \textit{over-solvable} & \text{if} \quad D_{S,i}(x,\mathscr{F}) \supset D(x) . \end{cases} \tag{2.26}$$

Solvable radii are spectral by definition. We also say that *the index i*, or the *i-th step of the filtration* $\omega_{S,i}(x, \mathscr{F})$, or *the disk $D_{S,i}(x, \mathscr{F})$* is spectral, solvable, over-solvable.

Definition 2.6.1. *We denote by $0 \leqslant i_x^{\mathrm{sp}} \leqslant i_x^{\mathrm{sol}} \leqslant r$ the indexes such that*

i) *$\mathcal{R}_{S,i}(x, \mathscr{F})$ is spectral non solvable for $i \leqslant i_x^{\mathrm{sp}}$,*

ii) *$\mathcal{R}_{S,i}(x, \mathscr{F})$ is solvable for $i_x^{\mathrm{sp}} < i \leqslant i_x^{\mathrm{sol}}$,*

iii) *$\mathcal{R}_{S,i}(x, \mathscr{F})$ is over-solvable for $i_x^{\mathrm{sol}} < i$.*

We call i_x^{sp} and i_x^{sol} the spectral *and* over-solvable cutoffs *respectively.*

If $i_x^{\mathrm{sol}} = 0$ (resp. $i_x^{\mathrm{sol}} = r$), then all the radii are over-solvable (resp. spectral). If $i_x^{\mathrm{sp}} = 0$ (resp. $i_x^{\mathrm{sp}} = r$), then all the radii are solvable or over-solvable (resp. spectral non solvable). If $i_x^{\mathrm{sp}} = i_x^{\mathrm{sol}}$, then $\mathscr{F}$ has no solvable radii.

Remark 2.6.2. *If $x \in \Gamma_{S,i}(\mathscr{F})$, then the indexes $1, \ldots, i$ are all spectral at x. Indeed if i is over-solvable at x, then $D_{S,i}(x, S)$ is an open neighborhood of x in X on which $\mathcal{R}_{S,i}(-, \mathscr{F})$ is constant, so $x \notin \Gamma_{S,i}(\mathscr{F})$. The case where i is solvable at x is also particular. It will be discussed in Lemma 6.1.4.*

2.6.1 We now provide a criterion to test whether a point lies in $\Gamma_{S,i}(\mathscr{F})$. Define the *constancy disk* of $\mathscr{F}$ at x as the maximal open disk $D^{\mathrm{c}}_{S,i}(x, \mathscr{F})$ centered at t_x and contained in $D(x, S)$ on which $\mathcal{R}_{S,i}(-, \mathscr{F})$ is constant. With the notations of Definition 2.2.1 one has $D^{\mathrm{c}}_{S,i}(x, \mathscr{F}) = D(x, \Gamma_{S,i}(\mathscr{F}))$. Then $D(x)$ and $D_{S,i}(x, \mathscr{F})$ are both contained in $D^{\mathrm{c}}_{S,i}(x, \mathscr{F})$ (cf. [Pul12, Eq. (4.9)]):

$$D(x) \cup D_{S,i}(x, \mathscr{F}) \subseteq D^{\mathrm{c}}_{S,i}(x, \mathscr{F}) \subseteq D(x, S) \tag{2.27}$$

The following proposition follows immediately from Definition 2.4.2 (cf. [Pul12, Prop. 2.2, iv)]):

Proposition 2.6.3. *A point $x \in X$ lies in $\Gamma_{S,i}(\mathscr{F})$ if and only if $D(x) = D^{\mathrm{c}}_{S,i}(x, \mathscr{F})$. Moreover*

$$\Gamma_{S,i}(\mathscr{F}) = X - \bigcup_{x \in X_{[1]}} D^{\mathrm{c}}_{S,i}(x, \mathscr{F}) , \tag{2.28}$$

where $X_{[1]} \subset X$ is the subset of points of type 1. □

2.7 Change of triangulation

Here, we discuss how the radii depend on the triangulation. Let S and S' be two triangulations of X. From Definition 2.3.1 for all $x \in X$ one has

$$D_{S',i}(x,\mathscr{F}) \cap D(x,S) \;=\; D_{S,i}(x,\mathscr{F}) \cap D(x,S')\;. \tag{2.29}$$

Note that either $D(x,S) \subseteq D(x,S')$, or $D(x,S') \subseteq D(x,S)$, because they are both disks centered at t_x. So either the left hand side of (2.29) is reduced to $D_{S',i}(x,\mathscr{F})$, or the right hand side is reduced to $D_{S,i}(x,\mathscr{F})$. If $\Gamma_S \subseteq \Gamma_{S'}$, then $D(x,S') \subseteq D(x,S)$ for all $x \in X$, and we have the following

Proposition 2.7.1 ([PP12b, (2.3.1)]). *Let S, S' be two triangulations such that $\Gamma_S \subseteq \Gamma_{S'}$. Then for all $i = 1,\ldots,r$ one has*

$$D_{S',i}(x,\mathscr{F}) \;=\; D_{S,i}(x,\mathscr{F}) \cap D(x,S')\;. \tag{2.30}$$

In particular $D_{S',i}(x,\mathscr{F}) = D_{S,i}(x,\mathscr{F})$ if $D(x,S) = D(x,S')$. Hence for all $i = 1,\ldots,r$ one has

$$\mathcal{R}_{S',i}(x,\mathscr{F}) \;=\; \min\Big(\,1\;,\; f_{S,S'}(x)\cdot\mathcal{R}_{S,i}(x,\mathscr{F})\,\Big)\;, \tag{2.31}$$

where $f_{S,S'} : X \to [1,+\infty[$ is the function associating to x the modulus $f_{S,S'}(x) \geqslant 1$ of the inclusion of disks $D(x,S') \subseteq D(x,S)$ (cf. (1.1.2)). □

Proposition 2.7.2 ([PP12b, 3.3.1]). *Let S, S' be two triangulations such that $\Gamma_S \subseteq \Gamma_{S'}$. Then*

$$\Gamma_{S',i}(\mathscr{F}) \;=\; \Gamma_{S'} \cup \Gamma_{S,i}(\mathscr{F})\;. \tag{2.32}$$

Proof. Indeed $f_{S,S'}$ is determined by the following properties:

i) $f_{S,S'}$ is constant on each connected component of $X - \Gamma_{S'}$ (which is necessarily a virtual open disks);
ii) $f_{S,S'}(x) = 1$ for all $x \in \Gamma_S \subseteq \Gamma_{S'}$;
iii) Let $x \in \Gamma_{S'} - \Gamma_S$. Let R be the radius of $D(x,S)$ in a given coordinate. Then $f_{S,S'}(x) = R/r(x)$, where $r(x) \leqslant R$ is the radius of the point x with respect to the chosen coordinate of $D(x,S)$.

Let D be a virtual disk such that $D \cap (\Gamma_{S'} \cup \Gamma_{S,i}(\mathscr{F})) = \emptyset$. Since $D \cap \Gamma_{S,i}(\mathscr{F}) = \emptyset$, $\mathcal{R}_{S,i}(-,\mathscr{F})$ is constant on D. Since $D \cap \Gamma_{S'} = \emptyset$ then $f_{S,S'}$ is constant on D. So by (2.31) the function $\mathcal{R}_{S',i}(-,\mathscr{F})$ is constant on D. This proves that $\Gamma_{S',i}(\mathscr{F}) \subseteq (\Gamma_{S'} \cup \Gamma_{S,i}(\mathscr{F}))$.

Conversely $\Gamma_S \subseteq \Gamma_{S'} \subseteq \Gamma_{S',i}(\mathscr{F})$. So it is enough to prove that $\Gamma_{S,i}(\mathscr{F}) \subseteq \Gamma_{S',i}(\mathscr{F})$. By Prop. 2.6.3 this amounts to prove that for all point x of type 1 one has $D^{\mathrm{c}}_{S',i}(x,\mathscr{F}) \subseteq D^{\mathrm{c}}_{S,i}(x,\mathscr{F})$. By definition $D^{\mathrm{c}}_{S',i}(x,\mathscr{F}) \subseteq D(x,S')$, so we have to prove that if $\mathcal{R}_{S,i}(-,\mathscr{F})$ is constant on a disk $D \subseteq D(x,S')$, then so does $\mathcal{R}_{S',i}(-,\mathscr{F})$. This follows from (2.31) since $f_{S,S'}$ is constant on D. □

2.8 Localization

One of the major differences between spectral and over-solvable cases is that the spectral terms of the filtration are preserved by localization, while over-solvable ones result truncated.

The pre-image $\pi_\Omega^{-1}(x)$ is independent on S and X (cf. Proof of 2.1.2), as well as the generic disk $D(x)$, the separating index i_x^{sol}, and all the disks $D_{S,i}(x,\mathscr{F})$ for $i \leqslant i_x^{\mathrm{sol}}$. Hence we obtain

Proposition 2.8.1. *Let $U \subset X$ be an analytic domain. And let S_U be a weak-triangulation of U. For all $i = 1,\ldots,r$ and all $x \in U$ one has*

$$D_{S_U,i}(x,\mathscr{F}_{|U}) \cap D(x,S) \;=\; D_{S,i}(x,\mathscr{F}) \cap D(x,S_U)\;. \tag{2.33}$$

In particular $D_{S_U,i}(x,\mathscr{F}_{|U}) = D_{S,i}(x,\mathscr{F})$ if $D(x,S_U) = D(x,S)$, or if $i \leqslant i_x^{\mathrm{sol}}$. Then for all $x \in U$

and all $i = 1, \ldots, i_x^{\mathrm{sol}}$ one has

$$\omega_{S,i}(x, \mathscr{F}) \;=\; \omega_{S_U,i}(x, \mathscr{F}_{|U}) \,. \tag{2.34}$$

For all $i = i_x^{\mathrm{sol}} + 1, \ldots, r$ one has

$$\omega_{S_U,i}(x, \mathscr{F}_U) \;=\; \begin{cases} \omega_{S,i}(x,\mathscr{F}) & \textit{if} \quad D_{S,i}(x,\mathscr{F}) \subseteq D(x,S_U) \\ \omega(D(x,S_U),\mathscr{F}) & \textit{if} \quad D_{S,i}(x,\mathscr{F}) \supset D(x,S_U) \,. \end{cases} \qquad \square \tag{2.35}$$

The proof of the following proposition is similar to those of section 2.7

Proposition 2.8.2. *Let $Y \subseteq X$ be an analytic domain. Let S_X and S_Y be triangulations of X and Y respectively such that $(\Gamma_{S_X} \cap Y) \subseteq \Gamma_{S_Y}$. If $y \in Y$, then for all $i = 1, \ldots, r$ we have*

$$D_{S_Y,i}(x, \mathscr{F}_{|Y}) \;=\; D_{S_X,i}(x,\mathscr{F}) \cap D(x, S_Y) \,. \tag{2.36}$$

and

$$\mathcal{R}_{S_Y,i}(y, \mathscr{F}_{|Y}) \;=\; \min\Big(\, 1 \,,\; f_{S_X,S_Y}(y) \cdot \mathcal{R}_{S_X,i}(y,\mathscr{F}) \,\Big) \,, \tag{2.37}$$

where $f_{S_X,S_Y} : Y \to [1, +\infty[$ is the function associating to $y \in Y$ the modulus $f_{S_X,S_Y}(y) \geqslant 1$ of the inclusion $D(y, S_Y) \subseteq D(y, S_X)$. Hence

$$\Gamma_{S_Y,i}(\mathscr{F}_{|Y}) \;=\; (\Gamma_{S_X,i}(\mathscr{F}) \cap Y) \cup \Gamma_{S_Y} \,. \qquad \square \tag{2.38}$$

2.8.1 *Local nature of spectral radii.* Let $x \in X$ be a point of type 2, 3 or 4. By section 2.1.1 the pull-back of an element $f \in \mathscr{H}(x)$ is a (constant) bounded function on $D(x)$, and $|f(x)|$ coincides with the sup-norm $\|f\|_{D(x)}$ on $D(x)$. The inclusion

$$\mathscr{H}(x) \;\subset\; \mathcal{B}_\Omega(D(x)) \tag{2.39}$$

obtained in this way is isometric.

Let M be a differential module over $\mathscr{O}_{X,x}$ or $\mathscr{H}(x)$. Let $\mathrm{M}_{D(x)}$ be the pull-back on $D(x)$ of M, considered as a free $\mathscr{O}(D(x))$-module with connection. More precisely one can consider $D(x)$ as an Ω-analytic curve with empty triangulation and $\mathrm{M}_{D(x)}$ as an equation on $D(x)$. Definition 2.3.1 then applies, and it make sense to attribute to $\mathrm{M}_{D(x)}$ the multiradius

$$\boldsymbol{\mathcal{R}}_\emptyset(t_x, \mathrm{M}_{D(x)}) \,. \tag{2.40}$$

The vector space of solution $\omega(t_x, \mathrm{M}_{D(x)})$ is then filtered by the sub-spaces $\omega_{\emptyset,i}(t_x, \mathrm{M}_{D(x)})$, corresponding to the disks $D_{\emptyset,i}(t_x, \mathrm{M}_{D(x)}) \subseteq D(t_x, \emptyset) = D(x)$. The i-th radius $\mathcal{R}_{\emptyset,i}(t_x, \mathrm{M}_{D(x)})$ is the inverse of the *modulus* of the inclusion $D_{\emptyset,i}(t_x, \mathrm{M}_{D(x)}) \subseteq D(x)$.[4]

We now come back to our global sheaf $\mathscr{F}$ on X. Since $\mathscr{F}_{D(x)} = \mathscr{F}(x) \otimes_{\mathscr{H}(x)} \mathscr{O}(D(x))$, the space of convergent solutions around t_x only depends on $\mathscr{F}(x)$ (cf. (2.22)):

$$\omega(t_x, \mathscr{F}_{D(x)}) \;=\; \omega(x, \mathscr{F}) \,. \tag{2.41}$$

By Proposition 2.8.1, the *spectral* steps of the filtration $\omega_{S,i}(x,\mathscr{F})$, $i \leqslant i_x^{\mathrm{sol}}$, are intrinsically attached to x, so that $\omega(t_x, \mathscr{F}_{D(x)})$ carries the *spectral part* of the filtration coming from the global definition of the multiradius. More precisely

$$D_{\emptyset,i}(t_x, \mathscr{F}_{D(x)}) \;=\; D_{S,i}(x,\mathscr{F}) \cap D(x) \,, \tag{2.42}$$

so that the corresponding i-th radius results truncated (cf. (2.12)). The vector space $\omega(t_x, \mathscr{F}_{D(x)})$ then results filtered by

$$\omega_{\emptyset,i}(t_x, \mathscr{F}_{D(x)}) \;:=\; \begin{cases} \omega_{S,i}(x,\mathscr{F}) & \text{if} \quad i \leqslant i_x^{\mathrm{sol}} \\ \omega_{S,i_x^{\mathrm{sol}}}(x,\mathscr{F}) & \text{if} \quad i > i_x^{\mathrm{sol}} \,. \end{cases} \tag{2.43}$$

[4]Notice that the maximal disk here is $D(t_x, \emptyset) = D(x)$.

2.9 Some results about morphisms and duality

Theorem 2.9.1. *Let $E : 0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of differential modules. Let D be an open disk contained in the generic disk $D(x)$. Then the sequence (cf. (2.23))*

$$\omega(D,E) \;:\; 0 \to \omega(D,\mathscr{F}') \to \omega(D,\mathscr{F}) \to \omega(D,\mathscr{F}'') \to 0 \tag{2.44}$$

is exact.

Proof. The theorem is placed here for expository reasons, see Prop. 3.6.1 for the proof. □

Remark 2.9.2. *Thm. 2.9.1 may fails if $D(x) \subset D$ (over-solvable case). See Section 8.*

Lemma 2.9.3. *Let $\mathscr{F}' \to \mathscr{F}$ be an injective morphism of differential modules. Then the radii of $\mathscr{F}'$ appears among the radii of $\mathscr{F}$ at least with the same multiplicity.* □

Lemma 2.9.4. *Let $\mathscr{F}' \to \mathscr{F}$ be an injective morphism of differential modules of ranks r' and r respectively. Assume that for some j', j one has $\omega_{S,j'}(x,\mathscr{F}') = \omega_{S,j}(x,\mathscr{F})$. Let i' and i be the largest indexes separating the filtrations that are smaller than or equal to j' and j respectively. Then $r - i = r' - i'$ and for all $k = 1, \ldots, r-i$ one has*

$$\mathcal{R}_{S,i'+k}(x,\mathscr{F}') \;=\; \mathcal{R}_{S,i+k}(x,\mathscr{F})\;, \qquad \omega_{S,i'+k}(x,\mathscr{F}') \;=\; \omega_{S,i+k}(x,\mathscr{F})\;. \tag{2.45}$$

Proof. One has $\omega_{S,i'}(x,\mathscr{F}') = \omega_{S,j'}(x,\mathscr{F}') = \omega_{S,j}(x,\mathscr{F}) = \omega_{S,i}(x,\mathscr{F})$. By (2.21) they satisfy $r'-i' = r-i$. A solution in this space has a well defined radius of convergence which is independent on the differential equation of which it is the solution. So the two filtrations of $\omega_{S,i'}(x,\mathscr{F}')$ coincide. □

Proposition 2.9.5. *Let $E : 0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of differential equations. Let r', r, r'' be their respective ranks. The following conditions are equivalent:*

i) $\mathcal{R}_{S,r''}(x,\mathscr{F}) < \mathcal{R}_{S,1}(x,\mathscr{F}')$;

ii) $\mathcal{R}_{S,1}(x,\mathscr{F}') = \mathcal{R}_{S,r''+1}(x,\mathscr{F})$ *and* $\mathcal{R}_{S,r''}(x,\mathscr{F}) < \mathcal{R}_{S,r''+1}(x,\mathscr{F})$;

iii) $\omega(x,\mathscr{F}') = \omega_{S,r''+1}(x,\mathscr{F})$;

iv) $\mathcal{R}_{S,r''}(x,\mathscr{F}'') < \mathcal{R}_{S,1}(x,\mathscr{F}')$;

v) $\mathcal{R}_{S,j}(x,\mathscr{F}) = \mathcal{R}_{S,j}(x,\mathscr{F}'')$ *for all* $j = 1, \ldots, r''$, *and* $\mathcal{R}_{S,r''}(x,\mathscr{F}) < \mathcal{R}_{S,r''+1}(x,\mathscr{F})$.

If one of them holds, the multi-radius of $\mathscr{F}$ is given by:

$$\boldsymbol{\mathcal{R}}_S(x,\mathscr{F}) \;=\; (\mathcal{R}_{S,1}(x,\mathscr{F}''), \ldots, \mathcal{R}_{S,r''}(x,\mathscr{F}''), \mathcal{R}_{S,1}(x,\mathscr{F}'), \ldots, \mathcal{R}_{S,r'}(x,\mathscr{F}'))\;. \tag{2.46}$$

Moreover for all $x \in X$ and all disk $D \subseteq D(x,S)$ containing t_x, the sequence $\omega(D,E)$ is exact.

Proof. i) $\Rightarrow$ ii). The radii of $\mathscr{F}'$ are larger than $\mathcal{R}_{S,r''+1}(x,\mathscr{F})$, so by Lemma 2.9.3 one has $\mathcal{R}_{S,j}(x,\mathscr{F}') = \mathcal{R}_{S,r''+j}(x,\mathscr{F})$, for all $j = 1, \ldots, r'$. Moreover $r''+1$ separates the radii of $\mathscr{F}$.

ii) $\Rightarrow$ iii). The index $r''+1$ separates the radii of $\mathscr{F}$. So $\dim \omega_{S,r''+1}(x,\mathscr{F}) = r - (r''+1) + 1 = r'$ by (2.21). Let $D := D_{S,1}(x,\mathscr{F}') = D_{S,r''+1}(x,\mathscr{F})$. Then $\omega(x,\mathscr{F}') = \omega(D,\mathscr{F}') \subseteq \omega(D,\mathscr{F}) = \omega_{S,r''+1}(x,\mathscr{F})$. Since they have the same dimension, they coincide.

iii) $\Rightarrow$ iv). One has $\dim\, \omega_{S,r''+1}(x,\mathscr{F}) = r' = r - (r''+1) + 1$, so the index $r''+1$ separates the radii of $\mathscr{F}$, since the dimensions of the terms of the filtration by the radii form a scale (cf. Def. 2.5.1). So $\mathcal{R}_{S,r''}(x,\mathscr{F}) < \mathcal{R}_{S,r''+1}(x,\mathscr{F}) = \mathcal{R}_{S,1}(x,\mathscr{F}')$. Now $\mathscr{F}'$ is trivialized by $D := D_{S,1}(x,\mathscr{F}')$, and $\omega(D,\mathscr{F}') = \omega(D,\mathscr{F})$. Then by Lemma 1.2.8 one has $\omega(D,\mathscr{F}'') = 0$, hence $\mathcal{R}_{S,r''}(x,\mathscr{F}'') < \mathcal{R}_{S,1}(x,\mathscr{F}')$.

iv) $\Rightarrow$ v) and i). iv) implies that the sequence $\omega(D,E)$ is exact for all $D \subseteq D(x,S)$ containing t_x. Indeed if D strictly contains $D_{S,r''}(x,\mathscr{F}'')$ one has $\omega(D,\mathscr{F}'') = 0$, and hence $\omega(D,\mathscr{F}') = \omega(D,\mathscr{F})$

by left exactness. On the other hand, for all $D \subseteq D_{S,1}(x,\mathscr{F}')$, the equation $\mathscr{F}'$ is trivialized by $\mathscr{O}(D)$, hence the exactness follows in this case from point iii) of Lemma 1.2.8. This shows that iv) implies (2.46), hence i) and v).

v) $\Rightarrow$ i). Let D be an open disk such that $D_{S,r''}(x,\mathscr{F}) \subset D \subset D_{S,r''+1}(x,\mathscr{F})$. The sequence $0 \to \omega(D,\mathscr{F}') \to \omega(D,\mathscr{F}) \to \omega(D,\mathscr{F}'')$ verifies $\omega(D,\mathscr{F}'')=0$, hence $\omega(D,\mathscr{F}')=\omega(D,\mathscr{F})$. Since $r''+1$ separates the radii of $\mathscr{F}$ at x, then $\dim_\Omega \omega(D,\mathscr{F}) = r-(r''+1)+1 = r'$ because the dimensions of the terms of the filtration by the radii form a scale (cf. Def. 2.5.1). This proves that $\omega(D,\mathscr{F}) = \omega(x,\mathscr{F})$, so $D \subseteq D_{S,1}(x,\mathscr{F})$ and i) holds. □

Proposition 2.9.6. *Let $\mathscr{F} = \mathscr{F}' \oplus \mathscr{F}''$. Then for all $x \in X$ the set of radii $\mathcal{R}_{S,i}(x,\mathscr{F})$ of $\mathscr{F}$ at x with multiplicities is the union with multiplicities of the radii of $\mathscr{F}'$ and $\mathscr{F}''$ at x. In other words if the value R appears m'-times in $\boldsymbol{\mathcal{R}}_S(x,\mathscr{F}')$ and m''-times in $\boldsymbol{\mathcal{R}}_S(x,\mathscr{F}'')$, then R appears $(m'+m'')$-times in $\boldsymbol{\mathcal{R}}_S(x,\mathscr{F})$.*

Proof. One may assume $X = D(x,S)$ with empty weak triangulation. Then the proposition follows from [Pul12, Prop. 5.5] □

Recall that X is connected (cf. Setting 1.0.1).

Proposition 2.9.7. *Assume that the index i separates the radii of $\mathscr{F}$ (at all point of X, cf. Def. 2.5.4). Let $(\mathscr{F}',\nabla) \subseteq (\mathscr{F},\nabla)$ be a sub-object such that there exists a point $x \in X$ satisfying*

$$\omega_{S,1}(x,\mathscr{F}') \;=\; \omega_{S,i}(x,\mathscr{F})\;. \tag{2.47}$$

Then the rank of $\mathscr{F}'$ is $r-i+1$ and

$$\omega_{S,j}(y,\mathscr{F}') \;=\; \omega_{S,j+i-1}(y,\mathscr{F})\;,\ \textit{for all } y \in X \textit{ and all } j=1,\ldots,r-i+1, \tag{2.48}$$

$$\mathcal{R}_{S,j}(y,\mathscr{F}') \;=\; \mathcal{R}_{S,j+i-1}(y,\mathscr{F})\;,\ \textit{for all } y \in X \textit{ and all } j=1,\ldots,r-i+1. \tag{2.49}$$

Proof. From (2.21) it follows that the rank of the locally free sheaf $\mathscr{F}'$ is $r-i+1$. By Lemma 2.9.4 it is enough to prove that $\omega_{S,1}(y,\mathscr{F}') = \omega_{S,i}(y,\mathscr{F})$ for all $y \in X$. By Proposition 2.9.5 this is equivalent to $\mathcal{R}_{S,1}(y,\mathscr{F}') = \mathcal{R}_{S,i}(y,\mathscr{F})$ for all $y \in X$. Let $\mathcal{L} \subseteq X$ be the locus on which the equality holds. By assumption $\mathcal{L}$ is not empty since $x \in \mathcal{L}$. By continuity of the radii it is a closed subset of X. By Proposition 2.9.5 the condition $\mathcal{R}_{S,1}(y,\mathscr{F}') = \mathcal{R}_{S,i}(y,\mathscr{F})$ is equivalent to $\mathcal{R}_{S,1}(y,\mathscr{F}') > \mathcal{R}_{S,i-1}(y,\mathscr{F})$, hence $\mathcal{L}$ is open. Since X is connected, we deduce that $\mathcal{L} = X$. □

Proposition 2.9.8. *For all $x \in X$ one has $\mathcal{R}_{S,1}(x,\mathscr{F}) = \mathcal{R}_{S,1}(x,\mathscr{F}^*)$ (this holds even if $\mathcal{R}_{S,1}(x,\mathscr{F})$ is over-solvable). Moreover for all $i = 1,\ldots,i_x^{\mathrm{sol}}$ (spectral case) one has $\mathcal{R}_{S,i}(x,\mathscr{F}) = \mathcal{R}_{S,i}(x,\mathscr{F}^*)$.*

Proof. The assertion about $\mathcal{R}_{S,1}$ is equivalent to "*$\mathscr{F}$ is trivial over a disk if and only if $\mathscr{F}^*$ is*", which is clearly true. The second assertion is well known if x is of type 2, or 3, and it will follow from Section 3 in the general case. □

Remark 2.9.9. *The statement of Prop. 2.9.8 may fail for over-solvable radii. In section 8 we give an example of equation whose over-solvable radii are not stable by duality.*

2.10 Notes

If $i=1$, the radius $\mathcal{R}_{S,1}(x,\mathscr{F})$ admits the following classical description. Let $D \subseteq X_\Omega$ be a maximal disk. Let M be the restriction of $\mathscr{F}$ to D. Fix an isomorphism $D \xrightarrow{\sim} D^-_\Omega(0,R)$, and consider the empty triangulation on it. The disk D is the maximal disk of all its points, and for all $x \in D$ one has $\mathcal{R}_{S,i}(x,\mathscr{F}) = \mathcal{R}_{\emptyset,i}(x,\mathrm{M})$. By (2.8) the function $\mathcal{R}_{\emptyset,i}(-,\mathrm{M})$ is determined by the function $\mathcal{R}^{\mathrm{M}}_{\emptyset,i}(x)$ appearing in (2.6). If $i=1$ this last has the following interpretation involving Taylor solutions. Since

Ω is spherically complete, M is free over $\mathscr{O}(D)$. Let $G \in \mathrm{M}_n(\mathscr{O}(D))$ be the matrix of $\nabla : \mathrm{M} \to \mathrm{M}$ with respect to some basis of M. Namely the columns of G are the images of the elements of a basis of M by ∇. If $x \in D$, then a basis of solutions of M at t_x is given by

$$Y(T, t_x) \ := \ \sum_{n \geqslant 0} G_n(t_x) \frac{(T - t_x)^n}{n!} \ . \tag{2.50}$$

where $G_0 = \mathrm{Id}$, $G_1 = G$, and inductively $G_{n+1} = G_n' + G_n G$. For all $x \in D$ we set $\mathcal{R}^Y(x) := \liminf_n (|G_n|(x)/|n!|)^{-1/n}$, then

$$\mathcal{R}^{\mathrm{M}}_{\emptyset,1}(x) \ = \ \min(R, \mathcal{R}^Y(x)) \ , \qquad \mathcal{R}_{\emptyset,1}(x, \mathrm{M}) = \mathcal{R}_{S,1}(x, \mathscr{F}) = \min(1, \mathcal{R}^Y(x)/R) \ . \tag{2.51}$$

Exploiting this formula one can prove the continuity of the individual radius $\mathcal{R}_{S,1}(-, \mathscr{F})$. This process is used in [BV07] for affinoid domains of the affine line, in [Bal10] (compact curves), [PP12a] (curves without boundary).

For $i \geqslant 2$ such an explicit expression of $\mathcal{R}_{S,i}(-, \mathscr{F})$ is missing. In the practice $\mathcal{R}_{S,i}(-, \mathscr{F})$ behave as a first radius only outside $\Gamma_{S,1}(\mathscr{F}) \cup \Gamma_{S,2}(\mathscr{F}) \cup \cdots \cup \Gamma_{S,i-1}(\mathscr{F})$. The situation is described in Remark 6.1.3.

3. Robba's decomposition by the spectral radii over $\mathscr{H}(x)$

In Sections 3 and 4, we generalize to curves Robba's [Rob75a], and Dwork-Robba's [DR77] theorem of decomposition by spectral radii. Such decompositions are proved with different methods in [Ked13]. Methods of [Ked13] and [Ked10] make a systematic use of spectral norm of the connection, which permits to separate the radii thank to the Hensel factorization [Rob80] and [CD94, Lemme 1.4].

We show here that the original proofs of [Rob75a] and [DR77] can be generalized quite smoothly to curves, up to minor implementations. The reason of our choice is that the point of view of Dwork's generic disks is more adapted to our global definition of radii. Working with spectral norms would oblige us to a translate the terminologies, and loosing the evocative image of generic disks.

Hypothesis 3.0.1. *In sections 3 and 4 we assume that K is algebraically closed.*

In Lemma 5.1.2 we will descend the obtained decomposition to K. The hypothesis is due to our use of Theorem 3.1.1 below, and it is unnecessary if x has a neighborhood which is isomorphic to an affinoid domain of the affine line.

The statements as well as the proofs are similar to Dwork and Robba's original ones (cf. [Dwo73], [Rob75a], and [DR77]), as improved by Christol (cf. [Chr12, Section 5], [Chr83, Section 5.3]). For the convenience of the reader, and to permit a complete understanding of section 4, we provide a complete set of proofs. This makes the paper self contained.

In Sections 3 and 4, $x \in X$ is a point of type 2, 3 or 4, M_x (resp. M) is a differential module over $\mathscr{O}_{X,x}$ (resp. $\mathscr{H}(x)$).

3.1 Étale maps.

We will construct nice étale maps from the curve X to the affine line, at least locally. To achieve this, in the following sections, we assume that K is algebraically closed (cf. Hypothesis 3.0.1).

Theorem 3.1.1 ([PP12b, 3.2.1], cf. also [Duc])**.** *Let x be a point of X of type 2. Let $b_1, \ldots, b_t$ be distinct branches out of x. There exists an affinoid neighbourhood Y of x in X, an affinoid domain W of $\mathbb{P}^{1,\mathrm{an}}_K$ and a finite étale map $\psi \colon Y \to W$ such that*

i) *the degree* $[\mathscr{H}(x) : \mathscr{H}(\psi(x))]$ *is prime to* p*;*

ii) $\psi^{-1}(\psi(x)) = \{x\}$*;*

iii) *almost every connected component of* $Y \setminus \{x\}$ *is an open unit disk with boundary* $\{x\}$*;*

iv) *almost every connected component of* $W \setminus \{\psi(x)\}$ *is an open unit disk with boundary* $\{\psi(x)\}$*;*

v) *for almost every connected component* C *of* $Y \setminus \{x\}$*, the induced morphism* $C \to \psi(C)$ *is an isomorphism;*

vi) *for every* $i = 1, \dots, t$*, the morphism* ψ *induces an isomorphism between a section of* b_i *and a section of* $\psi(b_i)$*.* □

Let $\mathcal{C}$ be the set of connected components C of $Y \setminus \{x\}$ such that ψ induces an isomorphism $C \xrightarrow{\sim} \psi(C)$. Let $\mathscr{B}$ be the set of branches out of x that have no representative in $\mathcal{C}$. Let $b \in \mathscr{B}$. If there exists a section of b such that ψ induces an isomorphism between it and its image, choose such a section S_b that is a semi-open annulus. Otherwise, set $S_b = \emptyset$. Now, define

$$V = \bigcup_{C \in \mathcal{C}} C \cup \bigcup_{b \in \mathscr{B}} S_b \cup \{x\}. \tag{3.1}$$

It is an affinoid domain of Y that contains almost every branch out of x.

By property vi), there exists a finite family $Y_1, \dots, Y_n$ of affinoid neighborhoods of x, $V_1, \dots, V_n$ as in (3.1), and étale maps

$$\psi_j \colon Y_j \to W_j \subset \mathbb{P}^{1,\mathrm{an}}_K \tag{3.2}$$

as above such that

a) $V = \bigcup_{j=1}^n V_j$ is an affinoid neighborhood of x;

b) the map ψ_j induces an isomorphism between every connected component of $V_j \setminus \{x\}$ and its image in W_j.

Lemma 3.1.2. *Let* $x \in X$*. There exists an integer* n *and, for every* $j = 1, \dots, n$*, an affinoid domain* V_j *of* X *containing* x*, an affinoid domain* W_j *of* $\mathbb{P}^{1,\mathrm{an}}_K$*, and a finite étale map* $\psi_j \colon V_j \to W_j$ *such that*

i) $V = \bigcup_{j=1}^n V_j$ *is a star-shaped affinoid neighborhood of* x *in* X *(cf. 1.1.7);*

ii) *for every* $y \in V$*, there exists* j *such that* $(\psi_j)_\Omega$ *induces an isomorphism between generic disks*

$$(\psi_j)_\Omega \colon D(y) \xrightarrow{\sim} D(\psi_j(y)). \tag{3.3}$$

iii) *in case* $y = x$*, we have such an isomorphism for every* $j = 1, \dots, n$*.*

Proof. If x is of type 1, 3 or 4, then it has a neighborhood that is isomorphic to an affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_K$ and the result is obvious.

Let us assume that x is of type 2 and proceed as we did at the beginning of the section. Property i) holds by construction. By property b) after (3.2), property ii) holds for every $y \in V \setminus \{x\}$. For the point x itself, use property i) of Theorem 3.1.1 and conclude (cf. also [PP12b, Lemma 3.4.1]). □

Remark 3.1.3. *For technical reasons in some proof we need to work with the derivation* d/dT*, where* T *is a coordinate of the generic disk* $D(y)$*. We need to choose* T *in order that* $d/dT : \mathscr{O}_\Omega(D(y)) \to \mathscr{O}_\Omega(D(y))$ *stabilizes the sub-rings* $\mathscr{O}_{X,y}$*, and* $\mathscr{H}(y)$*. Such a particular coordinate is given by the pull-back by* ψ_j *of a* K*-rational coordinate on* W_j*.*

More precisely fix a coordinate T_j *on* W_j *and let* d_j *be the derivation on* $\mathscr{O}_{Y_j}$ *corresponding to* $1 \otimes d/dT_j$ *by the isomorphism*

$$\mathscr{O}_{Y_j} \widehat{\otimes}_{\mathscr{O}_{W_j}} \widehat{\Omega}^1_{W_j/K} \xrightarrow{\sim} \widehat{\Omega}^1_{Y_j/K}. \tag{3.4}$$

In the situation of property ii) of Lemma 3.1.2, if $y \in V_j$, the rings $\mathscr{O}(D(y))$, $\mathcal{B}(D(y))$, $\mathscr{O}_{X,y}$, and $\mathscr{H}(y)$ are stable under d_j.

In the situation of property i) of Lemma 3.1.2, let $Y \subseteq V$ be an affinoid neighborhood of x in X. Then d_j is a generator of $\Omega^1_{Y/K}$, for all $j = 1, \ldots, n$. In particular the rings $\mathscr{O}(D(x))$, $\mathcal{B}(D(x))$, $\mathscr{O}_{X,x}$, and $\mathscr{H}(x)$ are stable under all the derivations $d_1, \ldots, d_n$.

Definition 3.1.4. *We say that an affinoid neighborhood Y of x in X is* elementary *if $Y \subseteq V$.*

3.2 Norms on differential operators.

Definition 3.2.1. *Let $(G, \|.\|)$ be an abelian ultrametric normed group. If $\varphi : G \to G$ is a linear map we set $\|\varphi\|_{op,G} := \sup_{g \in G - \{0\}} \frac{\|\varphi(g)\|}{\|g\|}$.*

Recall that we denote by $D(x, \rho)$ the sub-disk of $D(x)$ with modulus ρ^{-1}, where $\rho \in]0, 1]$ (cf. (2.11)).

Notation 3.2.2. *Let P be a differential operator with coefficients in $\mathcal{B}_\Omega(D(x))$. For all $\rho \in]0, 1]$ we denote the norm of P as an operator on $\mathcal{B}_\Omega(D(x, \rho))$ by*

$$\|P\|_{op,D(x,\rho)} \;:=\; \|P\|_{op,\mathcal{B}_\Omega(D(x,\rho))} \;:=\; \sup_{0 \neq f \in \mathcal{B}_\Omega(D(x,\rho))} \frac{\|P(f)\|_{D(x,\rho)}}{\|f\|_{D(x,\rho)}} \,. \tag{3.5}$$

Remark 3.2.3. *By section 3.1, there exists a coordinate T on $D(x)$ such that d/dT stabilizes $\mathscr{H}(x) \subset \mathcal{B}_\Omega(D(x))$. Let $r(x)$ be the radius of $D(x)$ with respect to the coordinate T. Then the radius of $D(x, \rho)$ is $\rho \cdot r(x)$, and the norm of $(d/dT)^k$ is given by (cf. [Chr83, Prop.4.3.1])*

$$\|(d/dT)^k\|_{op,D(x,\rho)} \;=\; |k!|(\rho \cdot r(x))^{-k} \;=\; |k!| \cdot \|d/dT\|^k_{op,D(x,\rho)} \,. \tag{3.6}$$

More generally if $P = \sum_{k=0}^n f_k \cdot (d/dT)^k$, with $f_k \in \mathcal{B}_\Omega(D(x, \rho))$, then

$$\Big\|\sum_{k=0}^n f_k \cdot (d/dT)^k\Big\|_{op,D(x,\rho)} \;=\; \max_{k=0,\ldots,n} |k!| \cdot \|f_k\|_{D(x,\rho)} \cdot (\rho \cdot r(x))^{-k} \,. \tag{3.7}$$

3.3 Topologies on differential modules

Consider the topology $\mathcal{T}_\rho$ on $\mathscr{H}(x)\langle d\rangle$ induced by $\|.\|_{op,D(x,\rho)}$. For all $q \geqslant 1$ we consider $\mathscr{H}(x)\langle d\rangle^q$ as endowed with the norm : $\|(P_1, \ldots, P_q)\|_{op,D(x,\rho)} := \max_{i=1,\ldots,q} \|P_i\|_{op,D(x,\rho)}$. Each differential module M over $\mathscr{H}(x)$ then acquires automatically a canonical topology $\mathcal{T}_\rho(\mathrm{M})$ as follows. If

$$\Psi : \mathscr{H}(x)\langle d\rangle^q \to \mathrm{M} \to 0 \tag{3.8}$$

is a presentation of M (i.e. a surjective morphism of $\mathscr{H}(x)\langle d\rangle$-modules), then we endow M with the quotient semi-norm $\|m\|_\Psi := \inf_{\Psi(P_1,\ldots,P_q)=m} \|(P_1, \ldots, P_q)\|_{op,D(x,\rho)}$. The semi-norms on M relative to two presentations are always equivalent (cf. [Chr12, Prop. 7.7]), so that the topology $\mathcal{T}_\rho(\mathrm{M})$ induced by $\|.\|_\Psi$ is independent on Ψ.

3.4 Decomposition over $\mathscr{H}(x)$ by the bounded solutions on $D(x, \rho)$

The topology $\mathcal{T}_\rho(\mathrm{M})$ on M is not necessarily Hausdorff. Denote by $\overline{\mathrm{M}}$ (resp. $\mathrm{M}^{[1]}$) the Hausdorff quotient of M (resp. the closure of $0 \in \mathrm{M}$). The sub-module $\mathrm{M}^{[1]} \subseteq \mathrm{M}$ is a differential sub-module because $\|P \cdot m\|_\Psi \leqslant \|P\|_{op,D(x,\rho)} \cdot \|m\|_\Psi$, for all $P \in \mathscr{H}(x)\langle d\rangle$, $m \in \mathrm{M}$. One has a exact sequence of differential modules

$$0 \to \mathrm{M}^{[1]} \to \mathrm{M} \xrightarrow{\gamma} \overline{\mathrm{M}} \to 0 \,. \tag{3.9}$$

Remark 3.4.1. *Let $\|\overline{m}\|^{-}_{\Psi} := \min_{\gamma(m)=\overline{m}} \|m\|_{\Psi}$ be the norm on $\overline{\mathrm{M}}$ induced by $\|.\|_{\Psi}$. The corresponding topology $\overline{\mathcal{T}_\rho(\mathrm{M})}$ on $\overline{\mathrm{M}}$ is the canonical one : $\overline{\mathcal{T}_\rho(\mathrm{M})} = \mathcal{T}_\rho(\overline{\mathrm{M}})$. This follows from the independence of the presentation by choosing $\gamma \circ \Psi$ as a presentation of $\overline{\mathrm{M}}$. Conversely the canonical topology $\mathcal{T}_\rho(\mathrm{M}^{[1]})$ of $\mathrm{M}^{[1]}$ is often* non trivial, *so it differs from the (trivial) topology induced by $\mathcal{T}_\rho(\mathrm{M})$ and $\mathrm{M}^{[1]}$ can be different from $(\mathrm{M}^{[1]})^{[1]}$ (cf. section 3.5). This is a consequence of the fact that the functor $\mathrm{M} \mapsto \mathrm{M}^{[1]}$ is not exact.*

Let M^* be the dual module of M. Define

$$\mathrm{M}^b \;:=\; (\overline{\mathrm{M}^*})^* \;, \qquad\qquad \mathrm{M}_{[1]} \;:=\; ((\mathrm{M}^*)^{[1]})^* \;, \tag{3.10}$$

in order to have an exact sequence

$$0 \to \mathrm{M}^b \to \mathrm{M} \to \mathrm{M}_{[1]} \to 0 \;. \tag{3.11}$$

Proposition 3.4.2. *The sub-module M^b of M controls the solutions of M with values in $\mathcal{B}_\Omega(D(x,\rho))$ (bounded solutions with radius $\geqslant \rho$). More precisely* M^b and $\overline{\mathrm{M}}$ are trivialized by $\mathcal{B}_\Omega(D(x,\rho))$ *and*

$$\omega(\mathrm{M}, \mathcal{B}_\Omega(D(x,\rho))) \;=\; \omega(\mathrm{M}^b, \mathcal{B}_\Omega(D(x,\rho))) \;, \tag{3.12}$$

$$\mathrm{Hom}_{\mathscr{H}(x)\langle d\rangle}(\mathrm{M}, \mathcal{B}_\Omega(D(x,\rho))) \;=\; \mathrm{Hom}_{\mathscr{H}(x)\langle d\rangle}(\overline{\mathrm{M}}, \mathcal{B}_\Omega(D(x,\rho))) \;. \tag{3.13}$$

Proof. The assertions (3.12) and (3.13) are equivalent by duality (1.13). We prove (3.13). Let $e_1, \ldots, e_q$ denote the canonical basis of $\mathscr{H}(x)\langle d\rangle^q$. If $m = \Psi(\sum_i P_i(d)e_i) \in \mathrm{M}^{[1]}$ and if $s \in \mathrm{Hom}_{\mathscr{H}(x)\langle d\rangle}(\mathrm{M}, \mathcal{B}_\Omega(D(x,\rho)))$ one has

$$\begin{aligned}\|s(m)\|_{D(x,\rho)} \;&=\; \|s\circ\Psi(\sum P_i(d)e_i)\|_{D(x,\rho)} \;=\; \|\sum P_i(d)s\circ\Psi(e_i)\|_{D(x,\rho)} \\ &\leqslant\; (\max_i \|P_i\|_{op,D(x,\rho)})(\max_i \|s\circ\Psi(e_i)\|_{D(x,\rho)}).\end{aligned} \tag{3.14}$$

Since $0 = \|m\|_\Psi = \min_{\Psi(\sum_i P_i e_i)=m} \max_i \|P_i\|_{op,D(x,\rho)}$ one obtains $\|s(m)\|_{D(x,\rho)} = 0$, hence $s(m) = 0$. So (3.13) holds.

Now we prove that $\overline{\mathrm{M}}$ is trivialized by $\mathcal{B}_\Omega(D(x,\rho))$. For this choose a presentation Ψ such that $\{\Psi(e_i)\}_{i=1,\ldots,q}$ is a basis of M as $\mathscr{H}(x)$-vector space and the image $\overline{\boldsymbol{m}} := \{\overline{m}_1, \ldots, \overline{m}_r\}$ in $\overline{\mathrm{M}}$ of the first r vectors $\{\Psi(e_1), \ldots, \Psi(e_r)\}$ is a basis of $\overline{\mathrm{M}}$. Since $\overline{\mathrm{M}}$ is a vector space over the complete valued field $\mathscr{H}(x)$, all norms on $\overline{\mathrm{M}}$ are equivalent. So the sup-norm

$$\|\sum_i f_i\overline{m}_i\|_{\overline{\boldsymbol{m}}} \;:=\; \max_i |f_i(x)| \tag{3.15}$$

satisfies

$$\|\cdot\|_{\overline{\boldsymbol{m}}} \;\leqslant\; C\|\cdot\|^{-}_{\Psi} \;. \tag{3.16}$$

Then by (3.7) one finds

$$\begin{aligned}\Big\|\frac{1}{n!}(\frac{d}{dT})^n(\overline{m}_i)\Big\|_{\overline{\boldsymbol{m}}} \;&\leqslant\; C\cdot\Big\|\frac{1}{n!}(\frac{d}{dT})^n(\overline{m}_i)\Big\|^{-}_{\Psi} &(3.17)\\ &=\; C\cdot\Big(\min_{\gamma(\Psi(\sum_j P_je_j))=\frac{1}{n!}(\frac{d}{dT})^n(\overline{m}_i)} \max_j \|P_j\|_{op,D(x,\rho)}\Big) &(3.18)\\ &\leqslant\; C\cdot\Big\|\frac{1}{n!}(\frac{d}{dT})^n\Big\|_{op,D(x,\rho)} \;=\; C\cdot(\rho\cdot r(x))^{-n} \;, &(3.19)\end{aligned}$$

where the last inequality follows by choosing $P_i = \frac{1}{n!}(\frac{d}{dT})^n$ and $P_j = 0$ for $j \neq i$.

Let $\nabla = d/dT : \mathrm{M} \to \mathrm{M}$. Let G_n be the matrix whose columns are the images by ∇^n of the basis

$\overline{m}_1,\dots,\overline{m}_r$. If $G_n=(g_{n;i,j})_{i,j}$, then $\|G_n\|_{D(x,\rho)}=\max_{i,j}\|g_{n;i,j}\|_{D(x,\rho)}$. The bound (3.19) means that

$$\frac{\|G_n\|_{D(x,\rho)}}{|n!|}=\max_{i=1,\dots,r}\|\frac{\nabla^n}{n!}(\overline{m}_i)\|_{\overline{\boldsymbol{m}}}=\max_{i=1,\dots,r}\|\frac{1}{n!}(\frac{d}{dT})^n(\overline{m}_i)\|_{\overline{\boldsymbol{m}}}\leqslant C(\rho\cdot r(x))^{-n}\;. \tag{3.20}$$

Since the entries $g_{n;i,j}$ of G_n lie in $\mathscr{H}(x)$ one has $\|g_{n;i,j}\|_{D(x,\rho)}=|g_{n;i,j}(x)|=|g_{n;i,j}(t_x)|_\Omega$. This proves that the Taylor solution $Y(T):=\sum_{n\geqslant 0}G_n(t_x)(T-t_x)^n/n!\in\Omega[\![T-t_x]\!]$ of the dual module $\overline{\mathrm{M}}^*$ belongs to $\mathcal{B}_\Omega(D(x,\rho))$. Equivalently $\overline{\mathrm{M}}^*$ is trivialized by $\mathcal{B}_\Omega(D(x,\rho))$, and then so is $\overline{\mathrm{M}}$. □

3.5 Decomposition over $\mathscr{H}(x)$ by the analytic solutions

As observed in Remark 3.4.1 the canonical topology of $\mathrm{M}^{[1]}$ is often non-trivial, so that one can repeat the construction and define inductively $\mathrm{M}^{[i+1]}:=(\mathrm{M}^{[i]})^{[1]}$ and $\mathrm{M}_{[i+1]}:=(\mathrm{M}_{[i]})_{[1]}$. Then $\mathrm{M}_{[i]}=((\mathrm{M}^*)^{[i]})^*$. The process ends because M is finite dimensional. One obtains a finite sequence of surjective maps

$$\mathrm{M}:=\mathrm{M}_{[0]}\to\mathrm{M}_{[1]}\to\cdots\to\mathrm{M}_{[k-1]}\to\mathrm{M}_{[k]}\;, \tag{3.21}$$

where $\mathrm{M}_{[k+1]}=\mathrm{M}_{[k]}$, and $\mathrm{M}_{[i+1]}\neq\mathrm{M}_{[i]}$ for all $i=0,\dots,k-1$. [5]

Definition 3.5.1. *Denote by $\mathrm{M}^{<\rho}$ the module $\mathrm{M}_{[k]}$, and set $\mathrm{M}^{\geqslant\rho}:=\mathrm{Ker}(\mathrm{M}\to\mathrm{M}_{[k]})$. We have*

$$0\to\mathrm{M}^{\geqslant\rho}\to\mathrm{M}\to\mathrm{M}^{<\rho}\to 0\;. \tag{3.22}$$

Remark 3.5.2. *Since $\mathrm{M}_{[k]}=\mathrm{M}_{[k+1]}$, we have $\mathrm{M}^b_{[k]}=0$, hence $\omega(\mathrm{M}_{[k]},\mathcal{B}_\Omega(D(x,\rho)))=\omega(\mathrm{M}^b_{[k]},\mathcal{B}_\Omega(D(x,\rho)))=0$. In other words the module $\mathrm{M}^{<\rho}$ has no non trivial solutions with values in $\mathcal{B}_\Omega(D(x,\rho))$.*

The following proposition shows that the module $\mathrm{M}^{\geqslant\rho}$ takes into account the solutions of M with values in $\mathscr{O}_\Omega(D(x,\rho))$:

Proposition 3.5.3. *We have*

i) $\omega(\mathrm{M},\mathscr{O}_\Omega(D(x,\rho)))=\omega(\mathrm{M}^{\geqslant\rho},\mathscr{O}_\Omega(D(x,\rho)))$;

ii) $\omega(\mathrm{M}^{<\rho},\mathscr{O}_\Omega(D(x,\rho)))=0$;

iii) $\mathrm{M}^{\geqslant\rho}$ *is trivialized by* $\mathscr{O}_\Omega(D(x,\rho))$.

Proof. iii) The kernels $K_i:=\mathrm{Ker}(\mathrm{M}\to\mathrm{M}_{[i]})$ of the sequence (3.21) define a filtration $0\subset K_1\subset K_2\subset\cdots\subset K_k=\mathrm{M}^{\geqslant\rho}$ of M where every sub-quotient K_i/K_{i-1} is trivialized by $\mathcal{B}_\Omega(D(x,\rho))$, hence also by $\mathscr{O}_\Omega(D(x,\rho))$. Point iv) of Lemma 1.2.8 then implies that $\mathrm{M}^{\geqslant\rho}$ is trivialized by $\mathscr{O}_\Omega(D(x,\rho))$. Indeed we can apply Lemma 1.2.8 because d is surjective on $\mathscr{O}_\Omega(D(x,\rho))$ since $d=f_jd_j$, with f_j invertible in $\mathscr{O}_{X,x}$ (cf. Remark 3.1.3).

ii) By Remark 3.5.2, $\mathrm{M}^{<\rho}$ has no nontrivial solutions in $\mathcal{B}_\Omega(D(x,\rho))$. Then ii) now follows from Proposition 3.5.5 below.

i) now follows from ii) using also iii) together with points i) and ii) of Lemma 1.2.8. □

Remark 3.5.4. *The following proposition asserts that* a differential module having some non trivial analytic solutions in $\mathscr{O}_\Omega(D(x,\rho))$ must have at least a non trivial bounded solution in $\mathcal{B}_\Omega(D(x,\rho))$. *This is a crucial point of the theory, and it is originally due to Dwork [Dwo73].*

Proposition 3.5.5. *The following statements hold:*

i) *If $\mathrm{M}_{[1]}=\mathrm{M}$, then $\omega(\mathrm{M},\mathscr{O}_\Omega(D(x,\rho)))=0$.*

[5] It can be shown that $\mathrm{M}_{[i]}$ takes into account the solutions with *logarithmic growth of order* $i-1$, so that the solutions of M have at most logarithmic growth of order $r=\dim\mathrm{M}$ (cf. [Chr83]).

ii) *If* $\mathrm{M}^{[1]}=\mathrm{M}$, *then* $\mathrm{Hom}_{\mathscr{H}(x)\langle d\rangle}(\mathrm{M},\mathscr{O}_\Omega(D(x,\rho)))=0$.

Proof. The two assertions are equivalent by duality (see (1.13)). We prove ii). Let $m\in\mathrm{M}$, and $s\in\mathrm{Hom}_{\mathscr{H}(x)\langle d\rangle}(\mathrm{M},\mathscr{O}_\Omega(D(x,\rho)))$. The assumption implies that $0=\|m\|_\Psi=\inf_{\Psi(\sum_i P_ie_i)=m}\max_i\|P_i\|_{op,D(x,\rho)}$. Let $P_1,\ldots,P_q$ be such that $\max_i\|P_i\|_{op,D(x,\rho)}<1$. Now for all $\rho'<\rho$ the restriction of s belongs to $\mathrm{Hom}_{\mathscr{H}(x)\langle d\rangle}(\mathrm{M},\mathcal{B}_\Omega(D(x,\rho')))$, so

$$\begin{aligned}\|s(m)\|_{D(x,\rho')} &= \|\sum_i P_is(\Psi(e_i))\|_{D(x,\rho')}\leqslant \max_i\|P_i\|_{op,D(x,\rho')}\|s(\Psi(e_i))\|_{D(x,\rho')} &(3.23)\\ &\leqslant \max_i\|P_i\|_{op,D(x,\rho')}\cdot\max_i\|s(\Psi(e_i))\|_{D(x,\rho')} &(3.24)\end{aligned}$$

By (3.7) each map $\rho'\mapsto\|P_i\|_{op,D(x,\rho')}$ is continuous, hence there exists $\rho'<\rho$ such that $\max_i\|P_i\|_{op,D(x,\rho')}<1$. For all i inequality (3.23) applied to $m=\Psi(e_i)$ gives

$$\|s(\Psi(e_i))\|_{D(x,\rho')} < \max_i\|s(\Psi(e_i))\|_{D(x,\rho')}\,. \tag{3.25}$$

Hence $s=0$. □

3.6 Exactness, compatibility with dual, and direct sum decomposition

Proposition 3.6.1 ([Rob75a]). *The functors associating to an $\mathscr{H}(x)$-differential module M the Ω-vector spaces $\mathrm{Hom}_{\mathscr{H}(x)\langle d\rangle}(\mathrm{M},\mathscr{O}_\Omega(D(x,\rho)))$ and $\omega(D(x,\rho),\mathrm{M})$ respectively are exact for all $\rho\in]0,1]$.*

Proof. The two assertions are equivalent by duality (1.13). To prove the first one, it is enough to show that, for all $\mathscr{H}(x)$-differential modules M, one has $\mathrm{Ext}^1_{\mathscr{H}(x)\langle d\rangle}(\mathrm{M},\mathscr{O}_\Omega(D(x,\rho)))=0$. Since $\mathrm{M}^{\geqslant\rho}$ is trivialized by $\mathscr{O}_\Omega(D(x,\rho))$ one has (cf. [Chr12, Section 6.7])

$$\begin{aligned}\mathrm{Ext}^1_{\mathscr{H}(x)\langle d\rangle}(\mathrm{M}^{\geqslant\rho},\mathscr{O}_\Omega(D(x,\rho))) &= \mathrm{Ext}^1_{\mathscr{O}_\Omega(D(x,\rho))\langle d\rangle}(\mathscr{O}_\Omega(D(x,\rho))\otimes_{\mathscr{H}(x)}\mathrm{M}^{\geqslant\rho},\mathscr{O}_\Omega(D(x,\rho))) &(3.26)\\ &= \mathrm{Ext}^1_{\mathscr{O}_\Omega(D(x,\rho))\langle d\rangle}(\mathscr{O}_\Omega(D(x,\rho)),\mathscr{O}_\Omega(D(x,\rho)))^{\dim\mathrm{M}^{\geqslant\rho}} = 0\,, &(3.27)\end{aligned}$$

where the last equality follows from Lemmas 1.2.7 and 1.2.8. Writing $0\to\mathrm{M}^{\geqslant\rho}\to\mathrm{M}\to\mathrm{M}^{<\rho}\to0$ we are reduced to proving that

$$\mathrm{Ext}^1_{\mathscr{H}(x)\langle d\rangle}(\mathrm{M}^{<\rho},\mathscr{O}_\Omega(D(x,\rho))) = 0\,. \tag{3.28}$$

So we may assume $\mathrm{M}=\mathrm{M}^{<\rho}$. Let $L\in\mathscr{H}(x)\langle d\rangle$ be such that $\mathrm{M}\cong\mathrm{M}_L$ (cf. section 1.2.2). By [Chr12, Sections 6.6, 6.7], in order to prove (3.28) it is enough to prove that L is surjective as an operator on $\mathscr{O}_\Omega(D(x,\rho))$. This follows from Lemma 3.6.3 below. □

Remark 3.6.2. *For all $\rho'<\rho$ close enough to ρ one has $\mathrm{M}^{\geqslant\rho'}=\mathrm{M}^{\geqslant\rho}$ and $\mathrm{M}^{<\rho'}=\mathrm{M}^{<\rho}$. Indeed M is a finite dimensional vector space so the filtration $\{\mathrm{M}^{\geqslant\rho}\}_\rho$ has a finite number of steps, whose dimensions equal those of the family $\{\omega(\mathrm{M},\mathscr{O}_\Omega(D(x,\rho)))\}_\rho$ by Prop. 3.5.3.*

Lemma 3.6.3. *Let $L\in\mathscr{H}(x)\langle d\rangle$, and let M_L be the corresponding cyclic differential module over $\mathscr{H}(x)$. Assume that $\mathrm{M}_L=\mathrm{M}_L^{<\rho}$ or, equivalently, that L is injective on $\mathscr{O}_\Omega(D(x,\rho))$. Then for all $\varepsilon>0$ there exists $Q_\varepsilon\in\mathscr{O}_{X,x}\langle d\rangle$ such that*

i) $\|Q_\varepsilon L-1\|_{op,\mathcal{B}_\Omega(D(x,\rho))}<\varepsilon$;

ii) *Q_ε and L are bijective as endomorphisms of $\mathscr{H}(x)$, of $\mathcal{B}_\Omega(D(x,\rho))$ and of $\mathscr{O}_\Omega(D(x,\rho))$.*

The same statements hold replacing ρ by a $\rho'<\rho$ close enough to ρ, with the same operator Q_ε.

Proof. Let $\varepsilon>0$, we may assume that $\varepsilon<1$. Consider the presentation $\Psi:\mathscr{H}(x)\langle d\rangle\to\mathrm{M}_L\to0$ with kernel $\mathscr{H}(x)\langle d\rangle L$. Since $\mathrm{M}_L=\mathrm{M}_L^{[1]}$, there exists $\widetilde{P}\in\mathscr{H}(x)\langle d\rangle$ such that $\|\widetilde{P}\|_{op,\mathcal{B}_\Omega(D(x,\rho))}<\varepsilon$, and $\Psi(\widetilde{P})=\Psi(-1)$, that is $\widetilde{P}=-1+\widetilde{Q}_\varepsilon L$ for some $\widetilde{Q}_\varepsilon\in\mathscr{H}(x)\langle d\rangle$. Let n be the order of $\widetilde{Q}_\varepsilon$, and let $\mathscr{H}(x)\langle d\rangle^{\leqslant n}$ be the $\mathscr{H}(x)$-vector space of differential polynomials of order $\leqslant n$. Since all norms on $\mathscr{H}(x)\langle d\rangle^{\leqslant n}$ are equivalent, $\|.\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$ is equivalent to the sup-norm with respect to the basis $1, d, \dots, d^n$: $\|\sum_{i=0}^n g_i d^i\|_{\mathscr{H}(x),d} := \max_i |g_i|(x)$. We deduce that $\mathscr{O}_{X,x}\langle d\rangle^{\leqslant n}$ is dense in $\mathscr{H}(x)\langle d\rangle^{\leqslant n}$ with respect to $\|.\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$. Hence there exists $Q_\varepsilon \in \mathscr{O}_{X,x}\langle d\rangle^{\leqslant n}$ such that $\|Q_\varepsilon L - 1\|_{op,\mathcal{B}_\Omega(D(x,\rho))} < \varepsilon$. Since $\mathcal{B}_\Omega(D(x,\rho))$ is complete, $Q_\varepsilon L = 1 + P$ is invertible as an endomorphism of $\mathcal{B}_\Omega(D(x,\rho))$ with inverse $U := \sum_{i\geqslant 0}(-1)^i P^i$. It follows that Q_ε (resp. L) is surjective (resp. injective) as an operator on $\mathcal{B}_\Omega(D(x,\rho))$. By Lemma 3.6.4 below, Q_ε (and hence also L) is invertible as operators on $\mathcal{B}_\Omega(D(x,\rho))$.

To deduce that L is also invertible on $\mathscr{O}_\Omega(D(x,\rho))$ one considers $\rho' < \rho$ such that $\mathrm{M}_L^{<\rho'} = \mathrm{M}_L$ (cf. Remark 3.6.2). Then L is bijective on $\mathcal{B}_\Omega(D(x,\rho''))$ for all $\rho' \leqslant \rho'' < \rho$, and on $\mathscr{O}_\Omega(D(x,\rho))$ since $\mathscr{O}_\Omega(D(x,\rho)) = \bigcup_{\rho''<\rho} \mathcal{B}_\Omega(D(x,\rho''))$.

Now the inclusion $\mathscr{H}(x) \subset \mathcal{B}_\Omega(D(x,\rho))$ is isometric, so $\|.\|_{op,\mathscr{H}(x)} \leqslant \|.\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$. Hence, as above, $Q_\varepsilon L$ is invertible as an endomorphism of $\mathscr{H}(x)$ with inverse U. Moreover Q_ε is injective on $\mathscr{H}(x)$ (because it is injective on $\mathcal{B}_\Omega(D(x,\rho))$). So L is bijective on $\mathscr{H}(x)$.

Now by Lemma 3.6.4 below Q_ε is injective as an operator on $\mathscr{O}(D(x,\rho))$, so we can reproduce the above proof replacing L with Q_ε to prove the bijectivity of Q_ε on the three rings.

Now to prove that the same holds for $\rho' < \rho$ close to ρ, we observe that L is then injective in $\mathscr{O}(D(x,\rho'))$ by Remark 3.6.2, and we can reproduce the proof for ρ'. Note that $\rho \mapsto \|Q_\varepsilon L - 1\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$ is a continuous function of ρ (cf. (3.7)), so for $\rho' < \rho$ close enough to ρ the inequality i) is preserved, and the rest of the proof works identically. □

Lemma 3.6.4 (cf. [Chr12, 15.4])**.** *If $Q \in \mathscr{H}(x)\langle d\rangle$ is surjective as an endomorphism of $\mathcal{B}_\Omega(D(x,\rho))$, then it is injective as an endomorphism of $\mathscr{O}_\Omega(D(x,\rho))$, hence also of $\mathcal{B}_\Omega(D(x,\rho))$ and of $\mathscr{H}(x)$.*

Proof. By contrapositive if Q is not injective on $\mathscr{O}_\Omega(D(x,\rho))$, then, by Prop. 3.5.5 (cf. Remark 3.5.4), it is not on $\mathcal{B}_\Omega(D(x,\rho))$ either. Let $u \in \mathcal{B}_\Omega(D(x,\rho))$ be such that $Q(u) = 0$. The primitive of a bounded function is often not bounded. If T is a coordinate on $D(x)$, as in Remark 3.2.3, then $d = f \cdot d/dT$, with f invertible. Hence d has an infinite dimensional cokernel as an operator on $\mathcal{B}_\Omega(D(x,\rho))$. More precisely one may find an infinite dimensional Ω-sub-vector space $V \subseteq \mathscr{O}(D(x,\rho))$ such that $V \cap \mathcal{B}_\Omega(D(x,\rho)) = 0$ and $d(V) \subset \mathcal{B}_\Omega(D(x,\rho))$, this is proven in [Chr12, 15.1] for $d = d/dT$, since $d = fd/dT$, the same holds for d. The vector space uV satisfies

$$uV \cap \mathcal{B}_\Omega(D(x,\rho)) \;=\; 0\;, \qquad Q(uV) \;\subset\; \mathcal{B}_\Omega(D(x,\rho))\;. \tag{3.29}$$

Indeed if $uf \in \mathcal{B}_\Omega(D(x,\rho))$, then $f \in \mathcal{B}_\Omega(D(x,\rho))$ since for all $\rho_0 \leqslant \rho' < \rho$ one has $\|f\|_{D(x,\rho')} = \frac{\|uf\|_{D(x,\rho')}}{\|u\|_{D(x,\rho')}} \leqslant \frac{\|uf\|_{D(x,\rho)}}{\|u\|_{D(x,\rho_0)}}$, hence $uV \cap \mathcal{B}_\Omega(D(x,\rho)) = 0$. The inclusion $Q(uV) \subset \mathcal{B}_\Omega(D(x,\rho))$ follows from the fact that $d^n(uf) = d^n(u)f + \sum_{i=1}^n b_i d^i(f)$ with $b_i = \binom{n}{i} d^{n-i}(u)$, then $Q(uf) = Q(u)f + Pd(f) = Pd(f)$, with $P \in \mathcal{B}_\Omega(D(x,\rho))\langle d\rangle$. So if $f \in V$, then $Q(uf) \in \mathcal{B}_\Omega(D(x,\rho))$. By (3.29) the dimension of $\mathrm{coker}(Q, \mathcal{B}_\Omega(D(x,\rho)))$ is infinite, which contradicts the fact that Q is surjective. □

Corollary 3.6.5. *$\mathrm{M} \mapsto \mathrm{M}^{\geqslant\rho}$ is an additive exact functor.*

Proof. Additivity is clear. Let $\mathcal{F}(D(x,\rho))$ be the fraction field of $\mathscr{O}_\Omega(D(x,\rho))$. Let $E : 0 \to \mathrm{N} \to \mathrm{M} \to \mathrm{P} \to 0$ be an exact sequence. By faithfully flatness of $\mathcal{F}(D(x,\rho))/\mathscr{H}(x)$, it is enough to prove that $E^{\geqslant\rho} \otimes_{\mathscr{H}(x)} \mathcal{F}(D(x,\rho))$ is exact. This now follows from the exactness of the sequence $E^{\geqslant\rho} \otimes_{\mathscr{H}(x)} \mathscr{O}_\Omega(D(x,\rho))$. Indeed $E^{\geqslant\rho}$ is constituted by modules trivialized by $\mathscr{O}_\Omega(D(x,\rho))$, so the sequence $E^{\geqslant\rho} \otimes_{\mathscr{H}(x)} \mathscr{O}_\Omega(D(x,\rho))$ is exact as soon as the sequence

$$\omega(E^{\geqslant\rho} \otimes_{\mathscr{H}(x)} \mathscr{O}_\Omega(D(x,\rho)), \mathscr{O}_\Omega(D(x,\rho))) \;=\; \omega(E^{\geqslant\rho}, \mathscr{O}_\Omega(D(x,\rho))) \;=\; \omega(E, \mathscr{O}_\Omega(D(x,\rho))) \tag{3.30}$$

is exact (cf. Lemmas 1.2.8 and 1.2.10). The exactness of $\omega(E, \mathscr{O}_\Omega(D(x,\rho)))$ follows from Prop.

3.6.1. □

Lemma 3.6.6. *All sub-quotients* S *of* $\mathrm{M}^{<\rho}$ *satisfy* $\mathrm{S}=\mathrm{S}^{<\rho}$ *(i.e.* $\mathrm{S}^{\geqslant\rho}=0$*). All sub-quotients of* $\mathrm{M}^{\geqslant\rho}$ *are trivialized by* $\mathscr{O}_\Omega(D(x,\rho))$*.*

Proof. By exactness each sub-module N and each quotient Q of $\mathrm{M}^{<\rho}$ satisfy $\mathrm{N}^{\geqslant\rho}=\mathrm{Q}^{\geqslant\rho}=0$. So $\mathrm{N}=\mathrm{N}^{<\rho}$ and $\mathrm{Q}=\mathrm{Q}^{<\rho}$. The second assertion follows from Lemma 1.2.10. □

Proposition 3.6.7 (Compatibility with duals)**.** *The composite map*

$$c:(\mathrm{M}^*)^{\geqslant\rho}\to\mathrm{M}^*\to(\mathrm{M}^{\geqslant\rho})^* \tag{3.31}$$

is an isomorphism.

Proof. Since $\mathrm{M}^{\geqslant\rho}$ is trivialized by $\mathscr{O}_\Omega(D(x,\rho))$ then so does its dual, hence $(\mathrm{M}^{\geqslant\rho})^*=((\mathrm{M}^{\geqslant\rho})^*)^{\geqslant\rho}$. The exact sequence $E:0\to(\mathrm{M}^{<\rho})^*\to\mathrm{M}^*\to(\mathrm{M}^{\geqslant\rho})^*\to 0$ then gives $E^{\geqslant\rho}:0\to((\mathrm{M}^{<\rho})^*)^{\geqslant\rho}\to(\mathrm{M}^*)^{\geqslant\rho}\xrightarrow{c}(\mathrm{M}^{\geqslant\rho})^*\to 0$. In order to prove that c is an isomorphism it is enough to prove that $((\mathrm{M}^{<\rho})^*)^{\geqslant\rho}=0$. Its dual is a quotient S of $\mathrm{M}^{<\rho}$ satisfying $S=S^{\geqslant\rho}$, so $S=0$ by Lemma 3.6.6. □

Proposition 3.6.8. *For all* $\rho\in]0,1]$ *one has* $\mathrm{M}=\mathrm{M}^{<\rho}\oplus\mathrm{M}^{\geqslant\rho}$*.*

Proof. By Proposition 3.6.7 the composite map $\mathrm{M}^{\geqslant\rho}\subseteq\mathrm{M}\to((\mathrm{M}^*)^{\geqslant\rho})^*$ is an isomorphism, and by Lemma 3.6.6 one has $((\mathrm{M}^*)^{\geqslant\rho})^*\cap\mathrm{M}^{<\rho}=0$ since $((\mathrm{M}^*)^{\geqslant\rho})^*$ is trivialized by $\mathscr{O}_\Omega(D(x,\rho))$. □

Corollary 3.6.9 (Robba)**.** *The module* $\mathrm{M}^{\geqslant\rho}$ *is the union of all differential sub- modules of* M *trivialized by* $\mathscr{O}_\Omega(D(x,\rho))$*. Moreover one has a unique decomposition*

$$\mathrm{M}\;:=\;\bigoplus_{0<\rho\leqslant 1}\mathrm{M}^\rho \tag{3.32}$$

with the following properties:

i) M^ρ *is trivialized by* $\mathscr{O}_\Omega(D(x,\rho))$*,*

ii) $(\mathrm{M}^\rho)^{\geqslant\rho'}=0$*, for all* $\rho<\rho'\leqslant 1$*.*

One has moreover the following properties:

iii) *For all* $\rho\in]0,1]$ *one has*

$$(\mathrm{M}^\rho)^*\;\cong\;(\mathrm{M}^*)^\rho\;. \tag{3.33}$$

iv) *The module* M^ρ *satisfies*

$$\mathrm{Hom}_{\mathscr{H}(x)\langle d\rangle}(\mathrm{M}^\rho,\mathscr{O}_\Omega(D(x,\rho')))\;\cong\;\begin{cases}\omega(x,\mathrm{M}^\rho)^* & \textit{if}\quad \rho'\leqslant\rho\\ 0 & \textit{if}\quad \rho'>\rho.\end{cases} \tag{3.34}$$

v) *If* M,N *are differential modules over* $\mathscr{H}(x)$*, then for all* $\rho\neq\rho'$*, one has*

$$\mathrm{Hom}_{\mathscr{H}(x)\langle d\rangle}(\mathrm{M}^\rho,\mathrm{N}^{\rho'})\;=\;0\;. \tag{3.35}$$

Proof. If $\mathrm{N}\subseteq\mathrm{M}$ is trivialized by $\mathscr{O}_\Omega(D(x,\rho))$, then so is $\mathrm{N}+\mathrm{M}^{\geqslant\rho}$ by Lemma 1.2.8. Hence $(\mathrm{N}+\mathrm{M}^{\geqslant\rho})\cap\mathrm{M}^{<\rho}=0$ by Lemma 3.6.6, and $\mathrm{N}\subseteq\mathrm{M}^{\geqslant\rho}$ by a dimension's argument. This proves that $\mathrm{M}^{\geqslant\rho}$ is the union of all differential sub-modules of M trivialized by $\mathscr{O}_\Omega(D(x,\rho))$.

The existence of decomposition (3.32) follows from Prop. 3.6.8 by setting $\mathrm{M}^\rho:=\mathrm{M}^{\geqslant\rho}/(\cup_{\rho'>\rho}\mathrm{M}^{\geqslant\rho'})$.

Compatibility with duals (3.33) then follows from Prop. 3.6.7.

Property (3.34) is equivalent, by (1.13), to

$$\omega((\mathrm{M}^\rho)^*,\mathscr{O}_\Omega(D(x,\rho')))\;\cong\;\begin{cases}\omega(x,\mathrm{M}^\rho)^* & \text{if}\quad \rho'\leqslant\rho\\ 0 & \text{if}\quad \rho'>\rho.\end{cases} \tag{3.36}$$

By (3.33) one has $\omega((\mathrm{M}^\rho)^*,\mathscr{O}_\Omega(D(x,\rho'))) = \omega((\mathrm{M}^*)^\rho,\mathscr{O}_\Omega(D(x,\rho')))$. So (3.36) follows from Prop. 3.5.3, since $\omega(x,\mathrm{M}^\rho)^*$ may be identified to $\omega(x,(\mathrm{M}^\rho)^*)$, and hence to $\omega(x,(\mathrm{M}^*)^\rho)$.

We now prove (3.35). Let $\alpha : \mathrm{M}^\rho \to \mathrm{N}^{\rho'}$. By (3.34), if $\rho < \rho'$, then $s\circ\alpha = 0$ for all $s \in \mathrm{Hom}_{\mathscr{H}(x)\langle d\rangle}(\mathrm{N}^{\rho'},\mathscr{O}_\Omega(D(x,\rho')))$. Since $\mathrm{N}^{\rho'}$ is trivialized by $\mathscr{O}_\Omega(D(x,\rho'))$, then $\bigcap_s \mathrm{Ker}(s) = 0$, where s runs in $\mathrm{Hom}_{\mathscr{H}(x)\langle d\rangle}(\mathrm{N}^{\rho'},\mathscr{O}_\Omega(D(x,\rho')))$. Hence $\alpha = 0$. If $\rho > \rho'$ the same argument proves, by duality (3.33), that the dual of α is zero, and hence also $\alpha = 0$.

Assume now that $\mathrm{M} = \oplus_{\eta\leqslant 1}\widetilde{\mathrm{M}}^\eta$ is another decomposition satisfying i) and ii). By the fact that this graduation has a finite number of non zero terms, and by ii), one has

$$\mathrm{M}^\rho \;=\; \frac{\mathrm{M}^{\geqslant\rho}}{\cup_{\rho'>\rho}\mathrm{M}^{\geqslant\rho'}} \;=\; \frac{(\oplus_{\eta\leqslant 1}\widetilde{\mathrm{M}}^\eta)^{\geqslant\rho}}{\cup_{\rho'>\rho}(\oplus_{\eta\leqslant 1}\widetilde{\mathrm{M}}^\eta)^{\geqslant\rho'}} \;=\; \frac{\oplus_{\eta\in[\rho,1]}\widetilde{\mathrm{M}}^\eta}{\cup_{\rho'>\rho}(\oplus_{\eta\in[\rho',1]}\widetilde{\mathrm{M}}^\eta)} \;=\; \widetilde{\mathrm{M}}^\rho\,. \tag{3.37}$$

□

4. Dwork-Robba's decomposition by the spectral radii over $\mathscr{O}_{X,x}$

In this section we prove that, if M_x is an differential module over $\mathscr{O}_{X,x}$, then Robba's decomposition of $\mathrm{M}_x\otimes_{\mathscr{O}_{X,x}}\mathscr{H}(x)$ by the spectral radii descends to a decomposition over the ring $\mathscr{O}_{X,x}$. Such a decomposition was obtained by Dwork and Robba in [DR77, First Thm. of section 4] for a point $x\in\mathbb{A}^{1,\mathrm{an}}_K$ of type 2.

In this section, as in section 3, K is algebraically closed (cf. Hypothesis 3.0.1).

4.1 Statement of Dwork-Robba's decomposition

Let $x\in X$ be a point of type 2, 3, or 4. Let M_x be a differential module over $\mathscr{O}_{X,x}$.

Theorem 4.1.1 ([DR77, 4.1])**.** *There exists a unique decomposition*

$$\mathrm{M}_x \;=\; \bigoplus_{0<\rho\leqslant 1}\mathrm{M}_x^\rho \tag{4.1}$$

such that for all $0<\rho\leqslant 1$ one has $\mathrm{M}_x^\rho\widehat{\otimes}_{\mathscr{O}_{X,x}}\mathscr{H}(x) = (\mathrm{M}_x\widehat{\otimes}_{\mathscr{O}_{X,x}}\mathscr{H}(x))^\rho$. Hence (3.34) holds for M_x^ρ. Moreover if $\mathrm{M}_x^{\geqslant\rho} := \oplus_{\rho\leqslant\rho'}\mathrm{M}_x^{\rho'}$ then

i) *The canonical composite map $(\mathrm{M}_x^*)^{\geqslant\rho}\to\mathrm{M}_x^*\to(\mathrm{M}_x^{\geqslant\rho})^*$ is an isomorphism, in particular $(\mathrm{M}_x^\rho)^*\cong(\mathrm{M}_x^*)^\rho$.*

ii) *If M_x and N_x are differential modules over $\mathscr{O}_{X,x}$, and if $\rho\neq\rho'$, then $\mathrm{Hom}_{\mathscr{O}_{X,x}\langle d\rangle}(\mathrm{M}_x^\rho,\mathrm{N}_x^{\rho'}) = 0$.*

Proof. We firstly prove the uniqueness. Let $\mathrm{M}_x = \oplus_{\eta\in]0,1]}\widetilde{\mathrm{M}}_x^\eta$ be another decomposition. One sees that $\mathrm{M}_x^\rho\otimes_{\mathscr{O}_{X,x}}\mathscr{H}(x) = \widetilde{\mathrm{M}}_x^\rho\otimes_{\mathscr{O}_{X,x}}\mathscr{H}(x)$. Then, by (3.35), for all $\rho\neq\rho'$ the composite morphism $\alpha : \mathrm{M}_x^\rho\subseteq\mathrm{M}_x\to\widetilde{\mathrm{M}}_x^{\rho'}$ is zero after scalar extension to $\mathscr{H}(x)$. So $\alpha=0$ since $\mathscr{O}_{X,x}$ is a field. This proves that $\mathrm{M}^\rho\subseteq\widetilde{\mathrm{M}}^\rho$, and the reverse of that argument gives the equality.

To prove the existence of (4.1) we claim that it is enough to show the existence, for all differential module M over $\mathscr{H}(x)$, and all $0<\rho\leqslant 1$, of an $\mathscr{O}_{X,x}$-lattice $\mathrm{M}_x^{\geqslant\rho}$ of $\mathrm{M}^{\geqslant\rho}$ such that :

$$\text{The inclusion } \mathrm{M}^{\geqslant\rho}\subseteq\mathrm{M} \text{ is the scalar extension of an inclusion of } \mathscr{O}_{X,x}\text{-lattices } \mathrm{M}_x^{\geqslant\rho}\subseteq\mathrm{M}_x. \tag{4.2}$$

Indeed, from (4.2) one shows that the scalar extension of the map $(\mathrm{M}_x^*)^{\geqslant\rho}\to\mathrm{M}_x^*\to(\mathrm{M}_x^{\geqslant\rho})^*$ to $\mathscr{H}(x)$ has a non zero determinant, so the map itself has a non zero determinant. This proves the compatibility with duals, and hence also that $\mathrm{M}_x^{\geqslant\rho}$ is a direct summand of M_x, as in Prop. 3.6.8. We then define $\mathrm{M}_x^\rho := \mathrm{M}_x^{\geqslant\rho}/(\cup_{\rho'>\rho}\mathrm{M}_x^{\geqslant\rho'})$, and one sees that $\mathrm{M}_x = \oplus_{\rho\in]0,1]}\mathrm{M}_x^\rho$, and that M_x^ρ is a lattice of M^{ρ}.

Now, to prove ii), we consider a non zero map $\mathrm{Hom}_{\mathscr{O}_{X,x}\langle d\rangle}(\mathrm{M}^{\rho}_x,\mathrm{N}^{\rho'}_x)$ exists, this produces a non zero map in $\mathrm{Hom}_{\mathscr{H}(x)\langle d\rangle}(\mathrm{M}^{\rho},\mathrm{N}^{\rho'})$ which contradicts (3.35).

Claim (4.2) is proved in Theorem 4.3.1 below. □

Remark 4.1.2. *It is possible to derive the proof of Dwork-Robba's decomposition 4.1.1 over X from the knowledge of the decomposition over the affine line as follows. Let Y be an affinoid neighborhood of x in X, and let $f : Y \to \mathbb{P}^{1,\mathrm{an}}_K$ be an étale map such that $[\mathscr{H}(x) : \mathscr{H}(f(x))]$ is prime to p. Let M_x be a differential module over $\mathscr{O}_{X,x}$, and let $\mathrm{M} := \mathrm{M}_x\widehat{\otimes}_{\mathscr{O}_{X,x}}\mathscr{H}(x)$. Then the radii of $f^*f_*\mathrm{M}_x$ can be easily computed by Lemma 6.2.13. By Corollary 3.6.9, v), the maps $\mathrm{M}_x \to f^*f_*\mathrm{M}_x$ and $\mathrm{M} \to f^*f_*\mathrm{M}$ preserve the radii. From this it is possible to deduce the decomposition of M_x by expressing it as $\mathrm{M}_x = f^*f_*\mathrm{M}_x \cap \mathrm{M}$ inside $f^*f_*\mathrm{M}_x$. Then one shows that the decomposition of M and of $f^*f_*\mathrm{M}$ are compatible, and gives the Dwork-Robba's decomposition of M_x.*

The fact is that the original proof of Dwork-Raobba [DR77] works on both the affine line, and on X (up to minor implementations). In order to provide a complete set of proofs, in this section we provide the entire proof of Dwork-Robba's theorem directly on X.

4.2 Norms on differential operators

Let x be a point of type 2, 3, or 4 of X. Let $\{\psi_j\}_{j=1,\dots,n}$ be the maps around x of Lemma 3.1.2. A *star* around x is a subset $\mathbf{U}$ of X such that

i) Each germ of segment b out of x is represented by a segment $[x, y]$ contained in $\mathbf{U}$;

ii) There exists $\varepsilon > 0$ such that for all $z \in \mathbf{U}$, and all $j = 1,\dots,n$, one has $r(\psi_j(z)) > \varepsilon$.

Let Y be an affinoid neighborhood of x. Let $\mathbf{U}_Y \subset Y$ be a star around x such that the Shilov boundary of Y is contained in $\mathbf{U}_Y$.

Define the product of Banach algebras $\mathcal{P}(\mathbf{U}_Y,\rho) := \prod_{y\in\mathbf{U}_Y}\mathcal{B}_\Omega(D(y,\rho))$ as the set of tuples $(f_y)_{y\in\mathbf{U}_Y}$, with $f_y \in \mathcal{B}_\Omega(D(y,\rho))$ such that $\|(f_y)_y\|_{\mathcal{P}(\mathbf{U}_Y,\rho)} := \sup_{y\in\mathbf{U}_Y}\|f_y\|_{D(y,\rho)} < +\infty$. Since $\mathbf{U}_Y$ contains the Shilov boundary of Y, the natural maps

$$\mathscr{O}(Y) \;\to\; \prod_{y\in\mathbf{U}_Y}\mathscr{H}(y) \;\to\; \mathcal{P}(\mathbf{U}_Y,\rho) \tag{4.3}$$

associating to $f \in \mathscr{O}(Y)$ the tuples $(f(y))_{y\in\mathbf{U}_Y}$ and $(f_{|D(y,\rho)})_{y\in\mathbf{U}_Y}$ are isometric.

A differential operator with coefficients in $\mathcal{P}(\mathbf{U}_Y,\rho)$ is a tuple $(P_y)_{y\in\mathbf{U}_Y}$ of differential operators $P_y \in \mathcal{B}_\Omega(D(y,\rho))\langle d\rangle$. One easily proves that

$$\|(P_y)_{y\in\mathbf{U}_Y}\|_{op,\mathcal{P}(\mathbf{U}_Y,\rho)} \;=\; \sup_{y\in\mathbf{U}_Y}\|P_y\|_{op,\mathcal{B}_\Omega(D(y,\rho))}\;. \tag{4.4}$$

As a consequence if $P \in \mathscr{O}(Y)\langle d\rangle$ then the isometric inclusions of (4.3) imply

$$\|P\|_{op,\mathscr{O}(Y)} \;\leqslant\; \|P\|_{op,\prod_{y\in\mathbf{U}_Y}\mathscr{H}(y)} \;\leqslant\; \|P\|_{op,\mathcal{P}(\mathbf{U}_Y,\rho)}\;. \tag{4.5}$$

Recall that each point $x \in X$ admits a basis of elementary affinoids neighborhoods (cf. Def. 3.1.4).

Proposition 4.2.1. *Let Y be an affinoid neighborhood of x in X such that $\widehat{\Omega}^1_{Y/K}$ is free, and let $d : \mathscr{O}(Y) \to \mathscr{O}(Y)$ be a derivation corresponding to a generator of $\widehat{\Omega}^1_{Y/K}$. For all $P \in \mathscr{O}(Y)\langle d\rangle$ and $\varepsilon > 0$, there exists a basis of elementary affinoid neighborhoods $Y_{P,\varepsilon}$ of x such that for all $\rho \in]0,1]$ one has*

$$\|P\|_{op,\mathscr{O}(Y_{P,\varepsilon})} \;\leqslant\; \|P\|_{op,\mathcal{B}_\Omega(D(x,\rho))} + \varepsilon\;. \tag{4.6}$$

More precisely for all $\rho \in]0,1]$ there exists an elementary affinoid neighborhood $Y_{P,\varepsilon,\rho}$ such that

$$\|P\|_{op,\mathcal{P}(\mathbf{U}_{Y_{P,\varepsilon,\rho}},\rho)} \;\leqslant\; \|P\|_{op,\mathcal{B}_\Omega(D(x,\rho))} + \varepsilon \;, \tag{4.7}$$

where $\mathbf{U}_{Y_{P,\varepsilon,\rho}} \subset Y_{P,\varepsilon,\rho}$ is a star around x containing the Shilov boundary of $Y_{P,\varepsilon,\rho}$.

Proof. The bound (4.6) follows from (4.7), with $\rho = 1$, since the function $\rho \mapsto \|P\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$ is non increasing by (3.7). Assume that Y is an elementary neighborhood of x (cf. Def. 3.1.4). Let $\mathbf{U}' \subset Y$ be a star around x containing the Shilov boundary of Y.

Let $[x,y] \subset \mathbf{U}'$ be a segment. By Lemma 3.1.2, the segment $[x,y]$ is contained in some V_j, and the map $\psi_j : Y \to W_j$ gives the identification $\psi_j : D(z) \xrightarrow{\sim} D(\psi_j(z))$ for all $z \in Y \cap V_j$. Let $r_j(z)$ be the radius of $D(\psi_j(z))$ with respect to a coordinate T_j on W_j. The radius of $\psi_j(D(z,\rho))$ is then $\rho \cdot r_j(z)$. Let $d_j = 1 \otimes d/dT_j$ be the corresponding derivation of $\mathscr{O}(Y)$ (cf. Remark 3.1.3). Then for all $z \in \mathbf{U}' \cap V_j$ one has (cf. Remark 3.2.3)

$$\|d_j\|_{op,\mathcal{B}_\Omega(D(z,\rho))} \;=\; (\rho \cdot r_j(z))^{-1} \;. \tag{4.8}$$

As explained in Remark 3.1.3, P can be written as $P = \sum f_i d_j^i$, with $f_i \in \mathscr{O}(Y)$. Together with (3.7), this proves that $z \mapsto \|P\|_{op,\mathcal{B}_\Omega(D(z,\rho))}$ is a continuous function of z along $[x,y]$. Up to restrict $[x,y]$ we may assume that $\|P\|_{op,\mathcal{B}_\Omega(D(z,\rho))} \leqslant \|P\|_{op,\mathcal{B}_\Omega(D(x,\rho))} + \varepsilon$ for all $z \in [x,y]$. This bound is independent on the chosen maps ψ_j.

Now for all germ of segment b out of x chose $[x,y] \in b$ with this property, and define the star $\mathbf{U}$ as the union of all those segments. Then $\sup_{y\in\mathbf{U}} \|P\|_{op,\mathcal{B}_\Omega(D(y,\rho))} \leqslant \|P\|_{op,\mathcal{B}_\Omega(D(x,\rho))} + \varepsilon$. Now for all star-shaped neighborhood $Y_{P,\varepsilon} \subseteq Y$ of x, with Shilov boundary in $\mathbf{U}$, equation (4.6) follows from (4.4) and (4.5) with $\mathbf{U}_{Y_{P,\varepsilon}} := \mathbf{U} \cap Y_{P,\varepsilon}$. □

4.2.1 *Norms on $A\langle d\rangle^{\leqslant n}$.* Let $(A, \|.\|_A)$ be a normed algebra with a derivation $d : A \to A$. Let

$$\|(f_0,\ldots,f_n)\|_A \;:=\; \max_i \|f_i\|_A \tag{4.9}$$

be the sup-norm on A^n. Now let $A\langle d\rangle^{\leqslant n}$ be the subset of differential operators $\sum_{i=0}^n g_i d^i$ of order at most n. We can identify $A\langle d\rangle^{\leqslant n}$ to A^{n+1} by associating to $\sum_{i=0}^n f_i d^i$ the tuple $(f_0,\ldots,f_n)$. We denote the resulting norm on $A\langle d\rangle^{\leqslant n}$ by

$$\|\sum f_i d^i\|_{A,d} \;:=\; \|(f_0,\ldots,f_n)\|_A \;=\; \max_i \|f_i\|_A \;. \tag{4.10}$$

Lemma 4.2.2. *If d_1, d_2 are two derivations as above, then $\|.\|_{A,d_1}$ and $\|.\|_{A,d_2}$ are equivalent on $A\langle d\rangle^{\leqslant n}$.*

Proof. The A-module $A\langle d\rangle^{\leqslant n}$ is finite and free. The subsets $\{1, d_1, \ldots, d_1^n\}$ and $\{1, d_2, \ldots, d_2^n\}$ are two basis. If $U \in GL_n(A)$ is the base change matrix, then $\|U^{-1}\|_A^{-1} \cdot \|.\|_{A,d_2} \leqslant \|.\|_{A,d_1} \leqslant \|U\|_A \cdot \|.\|_{A,d_1}$, where $\|(u_{i,j})\|_A := \max_{i,j} \|u_{i,j}\|_A$. □

Proposition 4.2.3. *Let $0 < \rho \leqslant 1$. The following claims hold:*

i) *Let $d : \mathcal{B}_\Omega(D(x,\rho)) \to \mathcal{B}_\Omega(D(x,\rho))$ be a derivation generating $\widehat{\Omega}^1_{\mathcal{B}_\Omega(D(x,\rho))/\Omega}$. Let A be a normed ring,* isometrically *included in $\mathcal{B}_\Omega(D(x,\rho))$ and stable under d. Then $\|.\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$ and $\|.\|_{A,d}$ are equivalent as norms on $A\langle d\rangle^{\leqslant n}$.*

ii) *Let Y be an star-shaped elementary affinoid neighborhood of x (cf. Def. 3.1.4) such that $\widehat{\Omega}^1_{Y/K}$ is free, and let $d : \mathscr{O}(Y) \to \mathscr{O}(Y)$ be a derivation corresponding to a generator of $\widehat{\Omega}^1_{Y/K}$. Let $\mathbf{U}_Y \subset Y$ be a star around x containing the Shilov boundary of Y. Then $\|.\|_{op,\mathcal{P}(\mathbf{U}_Y,\rho)}$ and $\|.\|_{\mathscr{O}(Y),d}$ are equivalent as norms on $\mathscr{O}(Y)\langle d\rangle^{\leqslant n}$.*

Proof. In i) we can assume $A = \mathcal{B}_\Omega(D(x,\rho))$. Let T be a coordinate on $D(x,\rho)$. By Lemma 4.2.2 we can replace d with d/dT. The radius of $D(x,\rho)$ with respect to T is $\rho\cdot r(x)$, where $r(x)$ is the radius of $D(x)$ in this coordinate. By (3.7) one has

$$\Big\|\sum_{k=0}^{n} f_k\cdot(d/dT)^k\Big\|_{op,\mathcal{B}_\Omega(D(x,\rho))} \;=\; \max_{k=0,\ldots,n}|k!|\cdot|f_k|(x)\cdot(\rho\cdot r(x))^{-k}\;. \tag{4.11}$$

Let $C_1 := \min_k|k!|(\rho\cdot r(x))^{-k}$ and $C_2 := \max_k|k!|(\rho\cdot r(x))^{-k}$, then

$$C_1\|.\|_{A,d/dT} \;\leqslant\; \|.\|_{op,\mathcal{B}_\Omega(D(x,\rho))} \;\leqslant\; C_2\|.\|_{A,d/dT}\;. \tag{4.12}$$

We now prove ii). With the notations of Remark 3.1.3 one has $Y\subseteq\bigcup_j V_j$, hence

$$\|.\|_{op,\mathcal{P}(\mathbf{U}_Y,\rho)} \;=\; \max_j\|.\|_{op,\mathcal{P}(\mathbf{U}_Y\cap V_j,\rho)}\;, \tag{4.13}$$

$$\|.\|_{\mathscr{O}(Y),d} \;=\; \max_j\|.\|_{\mathscr{O}(V_j\cap Y),d}\;. \tag{4.14}$$

So it is enough to prove that $\|.\|_{op,\mathcal{P}(\mathbf{U}_Y\cap V_j,\rho)}$ is equivalent to $\|.\|_{\mathscr{O}(V_j\cap Y),d}$ for all j. Now by Lemma 4.2.2 we can replace d by the derivation $d_j = d/dT_j$ of Remark 3.1.3. The reason of this choice is that the map $\psi_j : V_j\to W_j$ identifies $D(y,\rho)$ with $D(\psi_j(y),\rho)$ for all $y\in V_j$. So the coordinate T_j on W_j is then *simultaneously* a coordinate on $D(y,\rho)$, for all $y\in\mathbf{U}_Y\cap V_j$. We then can write $P = \sum_k f_{j,k}d_j^k\in\mathcal{B}_\Omega(D(y,\rho))\langle d_j\rangle^{\leqslant n}$, with $f_{j,k}\in\mathscr{O}(V_j)$, *simultaneously* for all $y\in\mathbf{U}_Y\cap V_j$. As a consequence we have the explicit formula

$$\|\sum_{k=0}^{n} f_{k,j}d_j^k\|_{op,\mathcal{P}(\mathbf{U}_Y\cap V_j,\rho)} \;=\; \sup_{y\in\mathbf{U}_Y\cap V_j}\max_{k=0,\ldots,n}|k!|\cdot|f_{k,j}|(y)\cdot(\rho\cdot r_j(y))^{-k}\;, \tag{4.15}$$

where $r_j(y)$ is the radius of $D(\psi_j(y))$ with respect to T_j. By the definition of V_j the Shilov boundary of $V_j\cap Y$ is contained in the union of $\{x\}$ with that of Y (cf. section 3.1). Hence $\mathbf{U}_Y\cap V_j$ contains the Shilov boundary of $Y\cap V_j$. Then

$$\sup_{y\in\mathbf{U}_Y\cap V_j}\max_{k=0,\ldots,n}|k!|\cdot|f_{k,j}|(y)\cdot(\rho\cdot r_j(y))^{-k} \;\leqslant\; \max_{k=0,\ldots,n}|k!|\cdot\|f_{k,j}\|_{V_j\cap Y}\Big(\inf_{y\in\mathbf{U}_Y\cap V_j}\rho\cdot r_j(y)\Big)^{-k}\;. \tag{4.16}$$

Now by definition of star (cf. point ii) at the beginning of section 4.2) one has $\inf_{y\in\mathbf{U}_Y\cap V_j} r_j(y) > 0$. If $r_j^- := \inf_{y\in\mathbf{U}_Y\cap V_j}\rho\cdot r_j(y)$, then from (4.15) and (4.16) one gets

$$\|\sum_{k=0}^{n} f_{k,j}d_j^k\|_{op,\mathcal{P}(\mathbf{U}_Y\cap V_j,\rho)} \;\leqslant\; \max_{k=0,\ldots,n}|k!|\cdot\|f_{k,j}\|_{V_j\cap Y}\cdot(r_j^-)^{-k}\;. \tag{4.17}$$

If $C_2 := \max_k|k!|\cdot(r_j^-)^{-k}$, then $\|.\|_{op,\mathcal{P}(\mathbf{U}_Y\cap V_j,\rho)}\leqslant C_2\|.\|_{\mathscr{O}(Y\cap V_j),d_j}$.

On the other hand let $r_j^+ := \sup_{z\in W_j} r(z)$, where $r(z)$ is the radius of the point (cf. (1.7)). Then $r_j^+ < +\infty$, because W_j is an affinoid domain of $\mathbb{A}_K^{1,\mathrm{an}}$. Since $\rho\leqslant 1$, then $\sup_{y\in\mathbf{U}_Y\cap V_j}\rho\cdot r_j(y)\leqslant r_j^+$. If $C_1 := \min_k|k!|(r_j^+)^{-k}$, then as above one has $\|.\|_{op,\mathcal{P}(\mathbf{U}_Y\cap V_j,\rho)}\geqslant C_1\cdot\|.\|_{\mathscr{O}(V_j\cap Y),d_j}$ as required. □

4.3 Lifting Robba's decomposition to $\mathscr{O}_{X,x}$

Let $x\in X$ be a point of type 2, 3, or 4. Let $d : \mathscr{O}_{X,x}\to\mathscr{O}_{X,x}$ be a derivation generating the $\mathscr{O}_{X,x}$-module of K-linear derivations of $\mathscr{O}_{X,x}$. Since $\mathscr{O}_{X,x}$ is a field, any differential module M_x is cyclic (cf. section 1.2.2). Assertion (4.2) is then equivalent to the following

Theorem 4.3.1 (Dwork-Robba's decomposition)**.** *Let $\rho\leqslant 1$. Let $P\in\mathscr{O}_{X,x}\langle d\rangle$ be a monic differential polynomials corresponding to* M. *Let $P = P^{\geqslant\rho}\cdot P^{<\rho}$ be a factorization in $\mathscr{H}(x)\langle d\rangle$ corresponding to the Robba's decomposition* (3.22)

$$0\to\mathrm{M}^{\geqslant\rho}\to\mathrm{M}\to\mathrm{M}^{<\rho}\to 0\;. \tag{4.18}$$

Then $P^{\geqslant\rho}$ and $P^{<\rho}$ also belongs to $\mathscr{O}_{X,x}\langle d\rangle$.

Proof. Let Y be an elementary affinoid neighborhood of x (cf. Def. 3.1.4) such that the coefficients of P lies in $\mathscr{O}(Y)$. Assume moreover that $\widehat{\Omega}^1_{Y/K}$ is free, and choose a derivation $d : \mathscr{O}(Y) \to \mathscr{O}(Y)$ corresponding to a generator of $\widehat{\Omega}^1_{Y/K}$. Let A be one of the rings $\mathscr{O}(Y)$, $\mathscr{H}(x)$, $\mathcal{B}_\Omega(D(x,\rho))$. Consider the injective maps

$$\beta_n \ : \ A^n \ \hookrightarrow \ A\langle d\rangle^{\leqslant n} \ , \qquad\qquad \widetilde{\beta}_n \ : \ A^n \ \hookrightarrow \ A\langle d\rangle^{\leqslant n} \tag{4.19}$$

defined by $\widetilde{\beta}_n(g_0,\ldots,g_{n-1}) := \sum_{i=0}^{n-1} g_i d^i$, and $\beta_n(g_0,\ldots,g_{n-1}) := d^n + \widetilde{\beta}_n(g_0,\ldots,g_{n-1})$. Let n_1, n_2, $n = n_1+n_2$ be the orders of $P^{\geqslant\rho}$, $P^{<\rho}$, P respectively. The multiplication in $A\langle d\rangle$ provides a diagram

$$\begin{tikzcd}
 & A\langle d\rangle^{\leqslant n_1} \times A\langle d\rangle^{\leqslant n_2} \arrow[r, "\mathfrak{m}"] & A\langle d\rangle^{\leqslant n} \\
A^n \arrow[ur, "\gamma"] \arrow[r, "\sim", "\delta"'] \arrow[rr, bend right, "G_A"'] & A^{n_1} \times A^{n_2} \arrow[u, hook, "\beta_{n_1}\times\beta_{n_2}"] \arrow[r, dashed] \arrow[ur, phantom, "\circlearrowleft"] & A^n \arrow[u, hook, "\widetilde{\beta}_n"']
\end{tikzcd} \tag{4.20}$$

where

$$\mathfrak{m}(Q_1,Q_2) \ := \ Q_1Q_2 - P \ , \tag{4.21}$$

δ identifies A^n with $A^{n_1}\times A^{n_2}$ by $(v_1,\ldots,v_n) \mapsto ((v_1,\ldots,v_{n_1}),(v_{n_1+1},\ldots,v_{n_1+n_2}))$, and $\gamma := (\beta_{n_1}\times\beta_{n_2})\circ\delta$. The image of $\mathfrak{m}\circ\gamma$ is contained in that of $\widetilde{\beta}_n$. We define G_A as the map $A^n \to A^n$ obtained in this way.

Let $\boldsymbol{u} \in \mathscr{H}(x)^n$ be such that $\gamma(\boldsymbol{u}) = (P^{\geqslant\rho}, P^{<\rho}) \in \mathscr{H}(x)\langle d\rangle^{\leqslant n_1} \times \mathscr{H}(x)\langle d\rangle^{\leqslant n_2}$. Then

$$G_{\mathscr{H}(x)}(\boldsymbol{u}) \ = \ 0 \ . \tag{4.22}$$

This is a non linear system of differential equations on the entries of $\boldsymbol{u}$. We have to prove that there exists an affinoid neighborhood Y of x in X such that $\boldsymbol{u} \in \mathscr{O}(Y)^n$ (i.e. the coefficients of P_1 and P_2 lie in $\mathscr{O}(Y)$). For all $\boldsymbol{v} = (v_1,\ldots,v_n) \in A^n$, the vector $G_A(\boldsymbol{v}) \in A^n$ is a finite sum in of the form

$$G_A(\boldsymbol{v}) \ = \ \sum_{\substack{i,j=1,\ldots,n\\ k=0,\ldots,n_1}} \boldsymbol{C}_{i,j,k}\cdot v_i \cdot d^k(v_j) - \boldsymbol{\ell}\ , \tag{4.23}$$

where $\boldsymbol{C}_{i,j,k} \in \mathbb{N}^n$, and the vector $\boldsymbol{\ell}$ lies in $\mathscr{O}(Y)^n$ for some affinoid neighborhood Y of X (indeed $\boldsymbol{\ell}$ is associated to P).

We linearize the problem as follows. Let $\{X_i^{(k)}\}_{i=1,\ldots,n;\,k=0,\ldots,n_1}$ be a family of indeterminates. For all $0\leqslant k\leqslant n_1$, set $\boldsymbol{X}^{(k)} := (X_1^{(k)},\ldots,X_n^{(k)})$ and $\boldsymbol{X} = (X_1^{(0)},\ldots,X_n^{(0)},X_1^{(1)},\ldots,X_n^{(1)},\ldots,X_1^{(n_1)},\ldots,X_n^{(n_1)})$. With the notations of (4.23) let

$$F(\boldsymbol{X}) \ = \ F(\boldsymbol{X}^{(0)},\ldots,\boldsymbol{X}^{(n_1)}) \ := \sum_{i,j,k} \boldsymbol{C}_{i,j,k}\cdot X_i^{(0)}\cdot X_j^{(k)} - \boldsymbol{\ell}\ . \tag{4.24}$$

This is a polynomial with coefficients in $\mathscr{O}(Y)$. Denote by $\zeta : A^n \to A^{n\cdot n_1}$ the $\mathbb{Z}$-linear map $\zeta(\boldsymbol{v}) := (\boldsymbol{v}, d(\boldsymbol{v}),\ldots,d^{n_1}(\boldsymbol{v}))$. Then we have

$$G_A(\boldsymbol{v}) \ = \ F(\zeta(\boldsymbol{v})) \ = \ F(\boldsymbol{X}^{(0)},\boldsymbol{X}^{(1)},\ldots,\boldsymbol{X}^{(n_1)})_{|X_i^{(k)}=d^k(v_i)}\ . \tag{4.25}$$

Remark 4.3.2. *The above process of linearization corresponds to working on the jet-space (tangent space if $n_1 = 1$) as recently done by B. Malgrange [Mal05].*

Since F is a polynomial expression in $\boldsymbol{X}$, Taylor formula gives

$$F(\boldsymbol{X}+\boldsymbol{Y}) \ = \ F(\boldsymbol{X}) + dF_{\boldsymbol{X}}(\boldsymbol{Y}) + N_{\boldsymbol{X}}(\boldsymbol{Y})\ . \tag{4.26}$$

where

$$dF_{\boldsymbol{X}}(\boldsymbol{Y}^{(0)},\ldots,\boldsymbol{Y}^{(n_1)}) \;=\; \sum_{\substack{i=1,\ldots,n\\ k=0,\ldots,n_1}} \frac{\partial F}{\partial X_i^{(k)}}(\boldsymbol{X})\cdot Y_i^{(k)}\;, \tag{4.27}$$

is the linear part, and $N_{\boldsymbol{X}}(\boldsymbol{Y})$ is the non linear part. So

$$G_A(\boldsymbol{v}+\boldsymbol{\xi}) \;=\; G_A(\boldsymbol{v})+dG_{A,\boldsymbol{v}}(\boldsymbol{\xi})+N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}))\;, \tag{4.28}$$

where $dG_{A,\boldsymbol{v}}(\boldsymbol{\xi})=dF_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}))$. Applied to $\boldsymbol{u}\in\mathscr{H}(x)^n$ (cf. (4.22)) this gives

$$0 \;=\; G_{\mathscr{H}(x)}(\boldsymbol{u}) \;=\; G_{\mathscr{H}(x)}(\boldsymbol{v})+dG_{\mathscr{H}(x),\boldsymbol{v}}(\boldsymbol{u}-\boldsymbol{v})+N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{u}-\boldsymbol{v}))\;. \tag{4.29}$$

In the following sections 4.3.1, 4.3.2, 4.3.3, 4.3.4, we prove that if the image of $\boldsymbol{v}\in\mathscr{O}(Y)^n$ in $\mathscr{H}(x)^n$ is close to $\boldsymbol{u}$, the map $dG_{A,\boldsymbol{v}}:A^n\to A^n$ is bijective, for $A=\mathscr{H}(x)$ and also $A=\mathscr{O}(Y)$ for some suitable Y. We then prove that, for both $A=\mathscr{H}(x)$ and $A=\mathscr{O}(Y)$, the map

$$\phi_{\boldsymbol{v},A}(\boldsymbol{\xi}) \;:=\; -dG_{A,\boldsymbol{v}}^{-1}\Big(G_A(\boldsymbol{v})+N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}))\Big) \tag{4.30}$$

is a contraction of a poly-disk $\boldsymbol{D}^+_A(0,q)=\{\boldsymbol{\xi}\in A^n \text{ such that } \|\boldsymbol{\xi}\|_A\leqslant q\}$, with $\|\boldsymbol{v}-\boldsymbol{u}\|_{\mathscr{H}(x)}<q$. Then $\phi_{\boldsymbol{v},A}$ has a unique fixed point in $\boldsymbol{D}^+_A(0,q)$. Since it is compatible with the inclusion $\boldsymbol{D}^+_{\mathscr{O}(Y)}(0,q)\subset\boldsymbol{D}^+_{\mathscr{H}(x)}(0,q)$, the fixed point of $\phi_{\boldsymbol{v},\mathscr{H}(x)}$ coincides with the fixed point of $\phi_{\boldsymbol{v},\mathscr{O}(Y)}$. If $A=\mathscr{H}(x)$ the fixed point is $\boldsymbol{u}-\boldsymbol{v}$ by (4.29). This proves that $\boldsymbol{u}-\boldsymbol{v}\in\boldsymbol{D}^+_{\mathscr{O}(Y)}(0,q)$, hence $\boldsymbol{u}\in\mathscr{O}(Y)^n$. □

4.3.1 *Computation of $dG_{A,\boldsymbol{v}}$.* We consider $(A\langle d\rangle^{\leqslant n},\|.\|_{A,d})$ as a normed K-vector space. It is isomorphic to $(A^n,\|.\|_A)$ by (4.9). The definition of differential of a function $A^n\to A^m$ then has a meaning.

If $A=\mathscr{O}(Y)$ (resp. $A=\mathscr{H}(x)$, $\mathcal{B}_\Omega(D(x,\rho))$), the product $\mathfrak{m}:A\langle d\rangle^{\leqslant n_1}\times A\langle d\rangle^{\leqslant n_2}\to A\langle d\rangle^{\leqslant n}$ is a continuous K-bilinear map with respect to $\|.\|_{op,\mathcal{P}(\mathbf{U}_Y,\rho)}$ (resp. $\|.\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$), where $\mathbf{U}_Y$ is a star around x containing the Shilov boundary of Y. By Proposition 4.2.3 the same is true with respect to the equivalent norm $\|.\|_A$ on A^n. The differential $dG_{A,\boldsymbol{v}}$ is then given by

$$dG_{A,\boldsymbol{v}} \;=\; d\mathfrak{m}_{(P_{1,\boldsymbol{v}},P_{2,\boldsymbol{v}})}\circ(\widetilde{\beta}_{n_1}\times\widetilde{\beta}_{n_2})\circ\delta\;, \tag{4.31}$$

where $(P_{1,\boldsymbol{v}},P_{2,\boldsymbol{v}}):=\gamma(\boldsymbol{v})$. Notice here that $\widetilde{\beta}_s=(d\beta_s)_{\boldsymbol{v}}$ and $d\delta_{\boldsymbol{v}}=\delta$ for all $\boldsymbol{v}\in A$, and all $s\geqslant 1$.

Now since the multiplication in $A\langle d\rangle$ is a continuous K-bilinear map, the differential of $\mathfrak{m}$ is

$$d\mathfrak{m}_{(P_{1,\boldsymbol{v}},P_{2,\boldsymbol{v}})}(Q_1,Q_2) \;=\; Q_1P_{2,\boldsymbol{v}}+P_{1,\boldsymbol{v}}Q_2\;. \tag{4.32}$$

4.3.2 *Some estimations.* Let

$$\boldsymbol{D}^+_A(0,r):=\{\boldsymbol{x}\in A^{n\cdot n_1}\;,\|\boldsymbol{x}\|_A\leqslant r\}\;, \tag{4.33}$$

and let $\|F\|_{\boldsymbol{D}^+_A(0,r)}:=\max_{\|\boldsymbol{x}\|_A\leqslant r}F(\boldsymbol{x})$. Let $\zeta:A^n\to A^{n\cdot n_1}$ be as in (4.25).

Lemma 4.3.3. *For $A=\mathscr{H}(x),\mathscr{O}(Y),\mathcal{B}_\Omega(D(x,\rho))$, the following hold:*

i) *Let $\boldsymbol{u}$ be the zero of $G_{\mathscr{H}(x)}$ (cf. (4.22)), and let $\boldsymbol{v}\in\mathcal{B}_\Omega(D(x,\rho))^n$. If $\|\boldsymbol{v}\|_{\mathcal{B}_\Omega(D(x,\rho))},\|\boldsymbol{u}\|_{\mathcal{B}_\Omega(D(x,\rho))}\leqslant r\|\zeta\|^{-1}_{op,\mathcal{B}_\Omega(D(x,\rho))}$, then*

$$\|G_{\mathcal{B}_\Omega(D(x,\rho))}(\boldsymbol{v})\|_{\mathcal{B}_\Omega(D(x,\rho))} \;\leqslant\; r^{-1}\|F\|_{\boldsymbol{D}^+_{\mathcal{B}_\Omega(D(x,\rho))}(0,r)}\|\zeta\|_{op,\mathcal{B}_\Omega(D(x,\rho))}\|\boldsymbol{v}-\boldsymbol{u}\|_{\mathcal{B}_\Omega(D(x,\rho))}\;. \tag{4.34}$$

ii) *If $\|\boldsymbol{v}\|_A\leqslant r\|\zeta\|^{-1}_{op,A}$ and $\boldsymbol{\xi}\in A^n$, then $\|dG_{A,\boldsymbol{v}}(\boldsymbol{\xi})\|_A\leqslant r^{-1}\cdot\|F\|_{\boldsymbol{D}^+_A(0,r)}\cdot\|\zeta\|_{op,A}\|\boldsymbol{\xi}\|_A$;*

iii) *If $\|\boldsymbol{v}_1\|_A,\|\boldsymbol{v}_2\|_A\leqslant r\|\zeta\|^{-1}_{op,A}$ and $\boldsymbol{\xi}\in A^n$, then*

$$\|(dG_{\boldsymbol{v}_1}-dG_{\boldsymbol{v}_2})(\boldsymbol{\xi})\|_A \;\leqslant\; r^{-2}\|F\|_{\boldsymbol{D}^+_A(0,r)}\|\zeta\|^2_{op,A}\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_A\|\boldsymbol{\xi}\|_A\;; \tag{4.35}$$

iv) *If* $\|\boldsymbol{v}\|_A, \|\boldsymbol{\xi}\|_A \leqslant r\|\zeta\|_{op,A}^{-1}$, *then*

$$\|N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}))\|_A \leqslant r^{-2} \cdot \|F\|_{\boldsymbol{D}_A^+(0,r)} \cdot \|\zeta\|_{op,A}^2 \cdot \|\boldsymbol{\xi}\|_A^2\ ; \tag{4.36}$$

v) *If* $\|\boldsymbol{v}\|_A, \|\boldsymbol{\xi}_1\|_A, \|\boldsymbol{\xi}_2\|_A \leqslant r\|\zeta\|_{op,A}^{-1}$, *then*

$$\|N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}_1)) - N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}_2))\|_A \ \leqslant\ r^{-2}\|F\|_{\boldsymbol{D}_A^+(0,r)}\|\zeta\|_{op,A}^2\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_A \max(\|\boldsymbol{\xi}_1\|_A, \|\boldsymbol{\xi}_2\|_A)\ . \tag{4.37}$$

Proof. Write Taylor's formula as $F(\boldsymbol{X}+\boldsymbol{Y}) = \sum_{|\alpha|\geqslant 0} D_\alpha(F)(\boldsymbol{X})\boldsymbol{Y}^\alpha$ with the evident meaning of the notations. The assumptions imply that $\boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{D}_A^+(0,r)$ (i.e. $\|\boldsymbol{X}\|_A, \|\boldsymbol{Y}\|_A \leqslant r$). Recall that for $\boldsymbol{X} \in \boldsymbol{D}_A^+(0,r)$ one has $\|D_\alpha(F)(\boldsymbol{X})\|_A \leqslant \|F\|_{\boldsymbol{D}_A^+(0,r)}/r^{|\alpha|}$.

iv) One has $N_{\boldsymbol{X}}(\boldsymbol{Y}) = \sum_{|\alpha|\geqslant 2} D_\alpha(F)(\boldsymbol{X})\boldsymbol{Y}^\alpha$. So for $\|\boldsymbol{X}\|_A, \|\boldsymbol{Y}\|_A \leqslant r$ one finds

$$\|N_{\boldsymbol{X}}(\boldsymbol{Y})\|_A \leqslant \sup_{|\alpha|\geqslant 2} \|D_\alpha(F)(\boldsymbol{X})\|_A\|\boldsymbol{Y}^\alpha\|_A \leqslant \sup_{|\alpha|\geqslant 2} \|F\|_{\boldsymbol{D}_A^+(0,r)}\Big(\frac{\|\boldsymbol{Y}\|_A}{r}\Big)^{|\alpha|} = \|F\|_{\boldsymbol{D}_A^+(0,r)}\Big(\frac{\|\boldsymbol{Y}\|_A}{r}\Big)^2 . \tag{4.38}$$

i) Consider (4.26) with $\boldsymbol{X} = \zeta(\boldsymbol{u})$, and $\boldsymbol{Y} = \zeta(\boldsymbol{v}-\boldsymbol{u})$. Then use the fact that $F(\zeta(\boldsymbol{u})) = G_A(\boldsymbol{u}) = 0$, together with the bound (4.38), and the following inequality (cf. (4.27))

$$\|dF_{\boldsymbol{X}}(\boldsymbol{Y})\|_{\boldsymbol{D}_A^+(0,r)} \ \leqslant\ r^{-1}\|F\|_{\boldsymbol{D}_A^+(0,r)}\|\boldsymbol{Y}\|_A\ . \tag{4.39}$$

ii) follows from (4.27), and $\|\frac{\partial F}{\partial X_i^{(k)}}\|_{\boldsymbol{D}_A^+(0,r)} \leqslant r^{-1}\|F\|_{\boldsymbol{D}_A^+(0,r)}$.

iii) Let $H : \boldsymbol{D}_A^+(0,r) \to A^n$ be any power series converging on $\boldsymbol{D}_A^+(0,r)$. The Taylor expansion of $H(\boldsymbol{X}_1)$ around $\boldsymbol{X}_2$ gives for $\boldsymbol{Z} := \boldsymbol{X}_1 - \boldsymbol{X}_2$

$$H(\boldsymbol{X}_1) - H(\boldsymbol{X}_2) = H(\boldsymbol{X}_2 + \boldsymbol{Z}) - H(\boldsymbol{X}_2) = \sum_{|\alpha|\geqslant 1} D_\alpha(H)(\boldsymbol{X}_2)\boldsymbol{Z}^\alpha\ . \tag{4.40}$$

So for $\|\boldsymbol{X}_1\|_A, \|\boldsymbol{X}_2\|_A, \|\boldsymbol{Z}\|_A \leqslant r$ one obtains

$$\|H(\boldsymbol{X}_1) - H(\boldsymbol{X}_2)\|_A \leqslant \sup_{|\alpha|\geqslant 1} \|D_\alpha(H)(\boldsymbol{X}_2)\|_A \cdot \|\boldsymbol{Z}^\alpha\|_A \leqslant \sup_{|\alpha|\geqslant 1} \|H\|_{\boldsymbol{D}_A^+(0,r)} \cdot \Big(\frac{\|\boldsymbol{Z}\|_A}{r}\Big)^{|\alpha|} = \|H\|_{\boldsymbol{D}_A^+(0,r)}\frac{\|\boldsymbol{Z}\|_A}{r} \tag{4.41}$$

We apply this to $H(\boldsymbol{X}) := dF_{\boldsymbol{X}}(\boldsymbol{Y})$, together with (4.39), to obtain $\|(dF_{\boldsymbol{X}_1} - dF_{\boldsymbol{X}_2})(\boldsymbol{Y})\|_A \leqslant r^{-2} \cdot \|F\|_{\boldsymbol{D}_A^+(0,r)}\|\boldsymbol{X}_1 - \boldsymbol{X}_2\|_A\|\boldsymbol{Y}\|_A$.

v) Taylor formula gives $N_{\boldsymbol{X}}(\boldsymbol{Y}_1) - N_{\boldsymbol{X}}(\boldsymbol{Y}_2) = \sum_{|\alpha|\geqslant 2} D_\alpha(F)(\boldsymbol{X})(\boldsymbol{Y}_1^\alpha - \boldsymbol{Y}_2^\alpha)$. Now v) follows as in the above cases using the inequality $\|\boldsymbol{Y}_1^\alpha - \boldsymbol{Y}_2^\alpha\|_A \leqslant \|\boldsymbol{Y}_1 - \boldsymbol{Y}_2\|_A \cdot \max(\|\boldsymbol{Y}_1\|, \|\boldsymbol{Y}_2\|)^{|\alpha|-1}$. Indeed

$$\boldsymbol{Y}_1^\alpha - \boldsymbol{Y}_2^\alpha \ =\ \sum_{|\beta|>1} \tbinom{\alpha}{\beta}\boldsymbol{Y}_1^{\alpha-\beta}(\boldsymbol{Y}_1 - \boldsymbol{Y}_2)^\beta \ =\ \sum_{i=1}^n (Y_{1,i} - Y_{2,i}) \cdot Q_{i,\alpha}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\ . \tag{4.42}$$

This is a sum of monomials of degree $|\alpha|$ so $Q_{i,\alpha}$ is a sum of monomials of degree $|\alpha|-1$ with integer coefficients, and hence $\|Q_{i,\alpha}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\|_A \leqslant \max(\|\boldsymbol{Y}_1\|, \|\boldsymbol{Y}_2\|)^{|\alpha|-1}$. This proves that $\|N_{\boldsymbol{X}}(\boldsymbol{Y}_1) - N_{\boldsymbol{X}}(\boldsymbol{Y}_2)\|_A \leqslant r^{-2}\|F\|_{\boldsymbol{D}_A^+(0,r)}\|\boldsymbol{Y}_1 - \boldsymbol{Y}_2\|_A \max(\|\boldsymbol{Y}_1\|_A, \|\boldsymbol{Y}_2\|_A)$. □

4.3.3 *Bijectivity of $dG_{\mathscr{O}(Y),\boldsymbol{v}}$.*

Notation 4.3.4. *Let A be one of the rings $\mathscr{O}(Y)$, $\mathscr{O}_{X,x}$, $\mathscr{H}(x)$, $\mathcal{B}_\Omega(D(x,\rho))$, and let $\boldsymbol{v} \in A^n$. The action of $dG_{\mathscr{O}(Y),\boldsymbol{v}}$ on A^n is represented by a $n \times n$ matrix $L_{\boldsymbol{v}}$ with coefficients in $A\langle d\rangle$. Note that $L_{\boldsymbol{v}}$ acts on each $A\langle d\rangle$-module of the form B^n, where B is any $A\langle d\rangle$-module.*

Proposition 4.3.5. *Let $\boldsymbol{u}$ be the zero of $G_{\mathscr{H}(x)}$ of (4.22). Then*

$$dG_{\mathscr{H}(x),\boldsymbol{u}} \;:\; B^n \xrightarrow{\sim} B^n \tag{4.43}$$

is bijective, for $B = \mathscr{H}(x)$, $\mathcal{B}_\Omega(D(x,\rho))$, $\mathscr{O}_\Omega(D(x,\rho))$.

Proof. Injectivity. Consider diagram (4.20) replacing A by the ring B. The map $d(\beta_{n_1}\times\beta_{n_2})_{\boldsymbol{u}} = \widetilde{\beta}_{n_1}\times\widetilde{\beta}_{n_2}$ is injective, and δ is bijective, so by (4.31) we have to prove that the restriction of $d\mathfrak{m}_{(P^{\geqslant\rho},P^{<\rho})}$ to the image $B\langle d\rangle^{\leqslant n_1-1}\times B\langle d\rangle^{\leqslant n_2-1}$ of $\widetilde{\beta}_{n_1}\times\widetilde{\beta}_{n_2}$ is injective. Let $(Q_1,Q_2)\in B\langle d\rangle^{\leqslant n_1-1}\times B\langle d\rangle^{\leqslant n_2-1}$ be such that $d\mathfrak{m}_{(P^{\geqslant\rho},P^{<\rho})}(Q_1,Q_2)=0$. This means that

$$P^{\geqslant\rho}Q_2 \;=\; -Q_1P^{<\rho}\;. \tag{4.44}$$

Assume, by contradiction, that $(Q_1,Q_2)\neq(0,0)$. Then Q_1 and Q_2 are non-zero and we may assume that they are monic. With the notations of section 1.2.2 one has two exact sequences $0\to \mathrm{M}_{P^{\geqslant\rho}}\to \mathrm{M}_{P^{\geqslant\rho}Q_2}\to \mathrm{M}_{Q_2}\to 0$ and $0\to\mathrm{M}_{Q_1}\to\mathrm{M}_{Q_1P^{<\rho}}\to\mathrm{M}_{P^{<\rho}}\to 0$. Now $\omega(\mathrm{M}_{P^{<\rho}},\mathscr{O}_\Omega(D(x,\rho)))=0$, so $\omega(\mathrm{M}_{Q_1},\mathscr{O}_\Omega((x,\rho)))\cong\omega(\mathrm{M}_{Q_1P^{<\rho}},\mathscr{O}_\Omega(D(x,\rho)))$. Moreover $\mathrm{M}_{P^{\geqslant\rho}Q_2}\cong\mathrm{M}_{Q_1P^{<\rho}}$ by (4.44). Applying the functor $\omega(-,\mathscr{O}(D(x,\rho)))$ one finds

$$\omega(\mathrm{M}_{P^{\geqslant\rho}},\mathscr{O}_\Omega(D(x,\rho)))\;\subseteq\;\omega(\mathrm{M}_{P^{\geqslant\rho}Q_2},\mathscr{O}_\Omega(D(x,\rho)))\;\cong\;\omega(\mathrm{M}_{Q_1P^{<\rho}},\mathscr{O}_\Omega(D(x,\rho)))\;\cong\;\omega(\mathrm{M}_{Q_1},\mathscr{O}_\Omega(D(x,\rho)))\;.$$

This is a contradiction because $\mathrm{M}_{P^{\geqslant\rho}}$ is trivialized by $\mathscr{O}_\Omega(D(x,\rho))$ and hence

$$\dim_\Omega\omega(\mathrm{M}_{P^{\geqslant\rho}},\mathscr{O}_\Omega(D(x,\rho)))\;=\;\mathrm{rank}\,\mathrm{M}_{P^{\geqslant\rho}}\;>\;\mathrm{rank}\,\mathrm{M}_{Q_1}\;\geqslant\;\dim_\Omega\omega(\mathrm{M}_{Q_1},\mathscr{O}_\Omega(D(x,\rho)))\;. \tag{4.45}$$

Surjectivity. It follows from Lemma 4.3.6 below. □

The following lemma is a generalization of Lemma 3.6.3. If $M=(m_{i,j})$ is a matrix with coefficients in $\mathcal{B}_\Omega(D(x,\rho))$, we denote by $\|M\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$ the maximum of the norms $\|m_{i,j}\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$ of the coefficients of M.

Lemma 4.3.6. *Let L be a square $n\times n$ matrix with coefficients in $\mathscr{H}(x)\langle d\rangle$. Assume that L is injective as an endomorphism of $\mathscr{O}_\Omega(D(x,\rho))^n$. Then for all $\varepsilon>0$ there exists a $n\times n$ matrix Q_ε with coefficients in $\mathscr{O}_{X,x}\langle d\rangle$ such that*

i) $\|Q_\varepsilon L-1\|_{op,\mathcal{B}_\Omega(D(x,\rho))}<\varepsilon$.

ii) Q_ε *and L are bijective as endomorphisms of $\mathscr{H}(x)$, of $\mathcal{B}_\Omega(D(x,\rho))$, and of $\mathscr{O}_\Omega(D(x,\rho))$.*

The same statements hold replacing ρ by a $\rho'<\rho$ close enough to ρ with the same matrix Q_ε.

Proof. Since $\mathscr{H}(x)$ is a field, the ring $\mathscr{H}(x)\langle d\rangle$ is (left and right) Euclidean. In particular it is a (left and right) principal ideal domain. So by the theory of elementary divisors, there exist invertible matrices $U,V\in GL_n(\mathscr{H}(x)\langle d\rangle)$ such that $\widetilde{L}=ULV$ is diagonal $\widetilde{L}=\mathrm{diag}(\widetilde{L}_1,\ldots,\widetilde{L}_n)$. By assumption L is injective on $\mathscr{O}_\Omega(D(x,\rho))^n$, then so is $\widetilde{L}$, and hence each $\widetilde{L}_i$ is injective on $\mathscr{O}_\Omega(D(x,\rho))$. By Lemma 3.6.3 $\widetilde{L}_i$ is bijective on B, with $B=\mathscr{H}(x)$, $B=\mathcal{B}_\Omega(D(x,\rho))$, and $B=\mathscr{O}_\Omega(D(x,\rho))$. So $\widetilde{L}$ is bijective on B^n, and so does L. Now by Lemma 3.6.3, for all $\varepsilon'>0$, there exists $\widetilde{Q}_i\in\mathscr{H}(x)\langle d\rangle$ such that the matrix $\widetilde{Q}:=\mathrm{diag}(\widetilde{Q}_1,\ldots,\widetilde{Q}_n)$ verifies $\|\widetilde{Q}\widetilde{L}-1\|_{op,\mathcal{B}_\Omega(D(x,\rho))}<\varepsilon'$. The matrix $Q_1:=V\widetilde{Q}U$ verifies $Q_1L-1=V(\widetilde{Q}\widetilde{L}-1)V^{-1}$. So if $\varepsilon'<\varepsilon/(\|V\|_{op,\mathcal{B}_\Omega(D(x,\rho))}\cdot\|V^{-1}\|_{op,\mathcal{B}_\Omega(D(x,\rho))})$, then Q_1 verifies i).

Let now Q_ε be a $n\times n$ matrix with coefficients in $\mathscr{O}_{X,x}\langle d\rangle$ such that $\|Q_\varepsilon-Q_1\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$ is small this is possible because $\mathscr{O}_{X,x}\langle d\rangle^{\leqslant d}\subset\mathscr{H}(x)\langle d\rangle^{\leqslant d}$ is dense for the norm $\|.\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$ as in the proof of Lemma 3.6.3. Then Q_ε verifies i), and hence $Q_\varepsilon L$ is invertible as an endomorphisms of $\mathcal{B}_\Omega(D(x,\rho))^n$. So Q_ε is surjective on it.

Let now $A,B\in GL_n(\mathscr{O}_{X,x}\langle d\rangle)$ such that $AQ_\varepsilon B=\mathrm{diag}(Q_{\varepsilon,1},\ldots,Q_{\varepsilon,n})$. Each $Q_{\varepsilon,i}$ is surjective on $\mathcal{B}_\Omega(D(x,\rho))$, so by Lemma 3.6.4 it is injective on $\mathscr{O}_\Omega(D(x,\rho))$. This proves that Q_ε is injective on $\mathscr{O}_\Omega(D(x,\rho))^n$. So we can reproduce the first part of this proof replacing L by Q_ε, to prove that Q_ε is bijective as endomorphisms of $\mathscr{H}(x)$, of $\mathcal{B}_\Omega(D(x,\rho))$, and of $\mathscr{O}_\Omega(D(x,\rho))$.

To prove the last statement we notice, in analogy with the proof of Lemma 3.6.3, that the kernel of L on $\mathscr{O}(D(x,\rho'))^n$ is a finite dimensional vector space over Ω (this is true if $n=1$, and the case $n>1$ reduces to $n=1$ by the theory of elementary divisors). The kernel has a filtration by the radii of convergence of the entries of its vectors, and we conclude as in Lemma 3.6.3. □

Recall that $L_{\boldsymbol{v}}$ is the matrix associated with the differential of G at $\boldsymbol{v}$ (cf. Notation 4.3.4).

Corollary 4.3.7. *Let $\boldsymbol{u}$ be the zero of $G_{\mathscr{H}(x)}$ of (4.22). There exists a radius $w>0$ such that*

i) *If $\boldsymbol{v}\in\mathcal{B}_\Omega(D(x,\rho))^n$ satisfies $\|\boldsymbol{v}-\boldsymbol{u}\|_{\mathcal{B}_\Omega(D(x,\rho))}<w$, then $L_{\boldsymbol{v}}$ is invertible as endomorphism of $\mathcal{B}_\Omega(D(x,\rho))^n$ and of $\mathscr{O}_\Omega(D(x,\rho))^n$.*

ii) *If moreover $\boldsymbol{v}\in\mathscr{H}(x)^n$, then $L_{\boldsymbol{v}}$ is also invertible as endomorphism of $\mathscr{H}(x)^n$.*

iii) *If $\boldsymbol{v}\in\mathscr{O}_{X,x}^n$ satisfies $\|\boldsymbol{v}-\boldsymbol{u}\|_{\mathcal{B}_\Omega(D(x,\rho))}<w$, then there exists a basis of elementary affinoid neighborhoods Y of x in X (cf. Def. 3.1.4), such that $L_{\boldsymbol{v}}$ is invertible as endomorphism of each $\mathscr{O}(Y)^n$.*

In the situation of i) and ii) there exists a square matrix Q_ε with coefficients in $\mathscr{O}_{X,x}\langle d\rangle$ such that, for all $\boldsymbol{v}$ verifying $\|\boldsymbol{v}-\boldsymbol{u}\|_{\mathcal{B}_\Omega(D(x,\rho))}<w$, one has $\|Q_\varepsilon L_{\boldsymbol{v}}-1\|_{op,\mathcal{B}_\Omega(D(x,\rho))}<\varepsilon$, and also

$$\|dG^{-1}_{\mathcal{B}_\Omega(D(x,\rho)),\boldsymbol{v}}\|_{op,\mathcal{B}_\Omega(D(x,\rho))} \;=\; \|Q_\varepsilon\|_{op,\mathcal{B}_\Omega(D(x,\rho))}\;. \tag{4.46}$$

In particular this norm is independent of $\boldsymbol{v}$.

If moreover $\boldsymbol{v}\in\mathscr{O}_{X,x}^n$, then for all $\varepsilon>0$ there exists a basis of elementary neighborhoods of x in X (cf. Def. 3.1.4) satisfying

$$\|dG^{-1}_{\mathscr{O}(Y),\boldsymbol{v}}\|_{op,\mathscr{O}(Y)} \;\leqslant\; \|Q_\varepsilon\|_{op,\mathcal{B}_\Omega(D(x,\rho))}+\varepsilon\;. \tag{4.47}$$

Proof. i) Let $\varepsilon>0$, and let Q_ε be the matrix of Lemma 4.3.6 such that $\|Q_\varepsilon L_{\boldsymbol{u}}-1\|_{op,\mathcal{B}_\Omega(D(x,\rho))^n}<\varepsilon$. Write $Q_\varepsilon L_{\boldsymbol{v}}-1=Q_\varepsilon L_{\boldsymbol{u}}-1+Q_\varepsilon(L_{\boldsymbol{v}}-L_{\boldsymbol{u}})$. Then by (4.35), there exists w such that if $\|\boldsymbol{v}-\boldsymbol{u}\|_{\mathcal{B}_\Omega(D(x,\rho))}<w$, then $\|Q_\varepsilon L_{\boldsymbol{v}}-1\|_{op,\mathcal{B}_\Omega(D(x,\rho))}<\varepsilon$. By (3.7), the map $\rho\mapsto\|Q_\varepsilon L_{\boldsymbol{v}}-1\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$ is continuous, so this proves that $Q_\varepsilon L_{\boldsymbol{v}}$ is an isomorphism on $\mathcal{B}_\Omega(D(x,\rho'))^n$ for all $\rho'\leqslant\rho$ close enough to ρ. Since Q_ε is an isomorphism too, so is $L_{\boldsymbol{v}}$. Since this holds for all $\rho'\leqslant\rho$ close enough to ρ, we deduce that $L_{\boldsymbol{v}}$ is an isomorphism on $\mathscr{O}(D(x,\rho))=\cup_{\rho'\leqslant\rho}\mathcal{B}_\Omega(D(x,\rho'))$.

ii) By Lemma 4.3.6, $L_{\boldsymbol{v}}$ is also an isomorphism on $\mathscr{H}(x)^n$.

iii) Assume now that $\boldsymbol{v}\in\mathscr{O}_{X,x}^n$. By Prop. 4.2.1 there is a basis of elementary neighborhoods Y of x in X (cf. Def. 3.1.4) such that $\boldsymbol{v}\in\mathscr{O}(Y)^n$, the coefficients of Q_ε lie in $\mathscr{O}(Y)$, and $\|Q_\varepsilon L_{\boldsymbol{v}}-1\|_{op,\mathscr{O}(Y)}<\varepsilon$. Then $Q_\varepsilon L_{\boldsymbol{v}}=1-P_{\boldsymbol{v}}$ is invertible on $\mathscr{O}(Y)^n$ with inverse $\sum_{i\geqslant 0}P_{\boldsymbol{v}}^i$. Hence Q_ε is surjective on $\mathscr{O}(Y)^n$. It is also injective because it is so as an operator on $\mathscr{H}(x)^n$. Finally Q_ε is bijective on $\mathscr{O}(Y)^n$, and so is $L_{\boldsymbol{v}}$.

Equality (4.46) follows from $L_{\boldsymbol{v}}Q_\varepsilon=1-P_{\boldsymbol{v}}$ with $\|P_{\boldsymbol{v}}\|_{op,\mathcal{B}_\Omega(D(x,\rho))}<1$. Inequality (4.47) follows from (4.6). □

4.3.4 *Contractiveness of $\phi_{\boldsymbol{v}}$.* Let $w>0$ be as in Corollary 4.3.7. Let $\boldsymbol{v}\in\mathscr{O}_{X,x}^n$ be such that $\|\boldsymbol{v}-\boldsymbol{u}\|_{\mathscr{H}(x)}<w$, so that $dG_{\mathscr{O}(Y),\boldsymbol{v}}$ is invertible on some affinoid neighborhood Y of x in X. Assume that Y is small enough in order to have (4.47). Then for $A=\mathscr{O}(Y)$, or $A=\mathscr{H}(x)$, define for all $\boldsymbol{\xi}\in A^n$

$$\phi_{\boldsymbol{v},A}(\boldsymbol{\xi}) \;:=\; -dG_{\boldsymbol{v}}^{-1}\Big(G_A(\boldsymbol{v})+N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}))\Big)\;. \tag{4.48}$$

Proposition 4.3.8. *There exist an affinoid neighborhood Y of x in X, a $\boldsymbol{v} \in \mathscr{O}(Y)^n$, and a real number $q > \|\boldsymbol{v}-\boldsymbol{u}\|_{\mathscr{H}(x)}$ such that*

$$\phi_{\boldsymbol{v},A} \;:\; A^n \longrightarrow A^n \tag{4.49}$$

is a contractive map on the disk $\boldsymbol{D}^+_A(0,q)$ for both $A = \mathscr{H}(x)$ and $A = \mathscr{O}(Y)$.

Proof. Choose r large enough to have $\zeta(\boldsymbol{u}) \in \boldsymbol{D}^-_{\mathscr{H}(x)}(0,r)$, i.e. $\|\boldsymbol{u}\|_{\mathscr{H}(x)} < r\|\zeta\|^{-1}_{op,\mathscr{H}(x)}$. Fix a real number $0 < C < 1$, and choose q such that

$$0 \;<\; q \;<\; C\cdot r\cdot \|\zeta\|^{-1}_{op,\mathcal{B}_\Omega(D(x,\rho))}\cdot \min(\kappa,1)\;, \tag{4.50}$$

where

$$\kappa \;:=\; r\cdot \|\zeta\|^{-1}_{op,\mathcal{B}_\Omega(D(x,\rho))}\cdot \|F\|^{-1}_{\boldsymbol{D}^+_{\mathcal{B}_\Omega(D(x,\rho))}(0,r)}\cdot \|Q_\varepsilon\|^{-1}_{op,\mathcal{B}_\Omega(D(x,\rho))}\;. \tag{4.51}$$

Then choose $\boldsymbol{v} \in \mathscr{O}^n_{X,x}$ such that

$$\|\boldsymbol{v}-\boldsymbol{u}\|_{\mathscr{H}(x)} \;<\; \min\Big(q,\; q\cdot\kappa,\; r\|\zeta\|^{-1}_{op,\mathscr{H}(x)}\Big)\;. \tag{4.52}$$

With these choices the following holds

i) $\phi_{\boldsymbol{v},\mathscr{H}(x)}(\boldsymbol{D}^+_{\mathscr{H}(x)}(0,q)) \subseteq \boldsymbol{D}^+_{\mathscr{H}(x)}(0,q)$;

ii) There exists Y such that $\phi_{\boldsymbol{v},\mathscr{O}(Y)}(\boldsymbol{D}^+_{\mathscr{O}(Y)}(0,q)) \subseteq \boldsymbol{D}^+_{\mathscr{O}(Y)}(0,q)$;

iii) There exists $0 < C'' < 1$ such that, for all $\boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \boldsymbol{D}^+_{\mathscr{H}(x)}(0,q)$ one has

$$\|\phi_{\boldsymbol{v},\mathscr{H}(x)}(\boldsymbol{\xi}_1)-\phi_{\boldsymbol{v},\mathscr{H}(x)}(\boldsymbol{\xi}_2)\|_{\mathscr{H}(x)} \;\leqslant\; C''\cdot\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_{\mathscr{H}(x)}\;; \tag{4.53}$$

iv) There exists Y satisfying ii), and $0 < C' < 1$, such that for all $\boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \boldsymbol{D}^+_{\mathscr{O}(Y)}(0,q)$ one has

$$\|\phi_{\boldsymbol{v},\mathscr{O}(Y)}(\boldsymbol{\xi}_1)-\phi_{\boldsymbol{v},\mathscr{O}(Y)}(\boldsymbol{\xi}_2)\|_{\mathscr{O}(Y)} \;\leqslant\; C'\cdot\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_{\mathscr{O}(Y)}\;. \tag{4.54}$$

We give the details of ii) since i) follows similarly from (4.34), (4.36) and Corollary 4.3.7. By (4.47) we can assume, restricting Y, that $\|\boldsymbol{u}\|_{\mathscr{O}(Y)}, \|\boldsymbol{v}-\boldsymbol{u}\|_{\mathscr{O}(Y)} < r\|\zeta\|^{-1}_{op,\mathscr{O}(Y)}$, and that $\|\zeta\|_{op,\mathscr{O}(Y)}$, $\|G_{\mathscr{O}(Y)}(\boldsymbol{v})\|_{\mathscr{O}(Y)}$, and $\|dG^{-1}_{\mathscr{O}(Y),\boldsymbol{v}}\|_{op,\mathscr{O}(Y)}$ are close enough to $\|\zeta\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$, $\|G_{\mathscr{O}(Y)}(\boldsymbol{v})\|_{\mathcal{B}_\Omega(D(x,\rho))}$, and $\|Q_\varepsilon\|_{op,\mathcal{B}_\Omega(D(x,\rho))}$ respectively. Then, by Corollary 4.3.7 and Prop. 4.2.1, for all $\boldsymbol{\xi} \in \boldsymbol{D}^+_{\mathscr{O}(Y)}(0,q)$ one has:

$$\begin{aligned}
\|\phi_{\boldsymbol{v},\mathscr{O}(Y)}(\boldsymbol{\xi})\|_{\mathscr{O}(Y)} \;&=\; \Big\|-dG^{-1}_{\mathscr{O}(Y),\boldsymbol{v}}\Big(G(\boldsymbol{v})+N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}))\Big)\Big\|_{\mathscr{O}(Y)} &(4.55)\\
&\leqslant\; \|-dG^{-1}_{\mathscr{O}(Y),\boldsymbol{v}}\|_{op,\mathscr{O}(Y)}\cdot\max\Big(\|G(\boldsymbol{v})\|_{\mathscr{O}(Y)}, \|N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}))\|_{\mathscr{O}(Y)}\Big) &(4.56)\\
&\leqslant\; (\|Q_\varepsilon\|_{op,\mathcal{B}_\Omega(D(x,\rho))}+\varepsilon_1)\cdot\max\Big(\|G(\boldsymbol{v})\|_{\mathcal{B}_\Omega(D(x,\rho))}+\varepsilon_2, \|N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}))\|_{\mathcal{B}_\Omega(D(x,\rho))}+\varepsilon_3\Big)\;.
\end{aligned}$$

Now (4.34) together with (4.52) gives $\|G(\boldsymbol{v})\|_{\mathcal{B}_\Omega(D(x,\rho))} < q\|Q_\varepsilon\|^{-1}_{op,\mathcal{B}_\Omega(D(x,\rho))}$. Since this inequality is strict it is possible to chose $\varepsilon_1,\varepsilon_2$ in order that $(\|Q_\varepsilon\|_{op,\mathcal{B}_\Omega(D(x,\rho))}+\varepsilon_1)\cdot(\|G(\boldsymbol{v})\|_{\mathcal{B}_\Omega(D(x,\rho))}+\varepsilon_2) \leqslant q$. Analogously, if $\|\boldsymbol{\xi}\|_{\mathscr{O}(Y)} \leqslant q$, one also has $\|N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}))\|_{\mathcal{B}_\Omega(D(x,\rho))} < q\|Q_\varepsilon\|^{-1}_{op,\mathcal{B}_\Omega(D(x,\rho))}$. This follows directly from (4.36) together with the inequality $\|\boldsymbol{\xi}\|^2_{\mathscr{O}(Y)} \leqslant q^2 < q\cdot r\cdot\|\zeta\|^{-1}_{op,\mathcal{B}_\Omega(D(x,\rho))}\cdot\kappa$ (cf. (4.50)). Now we further restrict $\varepsilon_1,\varepsilon_3$ to have $(\|Q_\varepsilon\|_{op,\mathcal{B}_\Omega(D(x,\rho))}+\varepsilon_1)\cdot(\|N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}))\|_{\mathcal{B}_\Omega(D(x,\rho))}+\varepsilon_3) \leqslant q$. So $\|\phi_{\boldsymbol{v},\mathscr{O}(Y)}(\boldsymbol{\xi})\|_{\mathscr{O}(Y)} \leqslant q$, and ii) holds.

We prove in detail iv), since iii) follows similarly from (4.37) and Corollary 4.3.7. First restrict Y further to have $q < r\|\zeta\|^{-1}_{op,\mathscr{O}(Y)}$ (cf. (4.6), and (4.50)). This guarantee that if $\|\boldsymbol{\xi}_i\|_{\mathscr{O}(Y)} \leqslant q$, then

$\zeta(\boldsymbol{\xi}_i) \in \boldsymbol{D}^{-}_{\mathscr{O}(Y)}(0,r)$. Then

$$\begin{aligned}\|\phi_{\boldsymbol{v},\mathscr{O}(Y)}(\boldsymbol{\xi}_1) - \phi_{\boldsymbol{v},\mathscr{O}(Y)}(\boldsymbol{\xi}_2)\|_{\mathscr{O}(Y)} &= \| - dG^{-1}_{\mathscr{O}(Y),\boldsymbol{v}}(N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}_1)) - N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}_2)))\|_{\mathscr{O}(Y)} \\ &\leqslant (\|Q_\varepsilon\|_{op,\mathcal{B}_\Omega(D(x,\rho))} + \varepsilon_1)\cdot \|N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}_1)) - N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}_2))\|_{\mathscr{O}(Y)} \,.\end{aligned} \tag{4.57}$$

Now by (4.37) one has

$$\begin{aligned}\|N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}_1)) - N_{\zeta(\boldsymbol{v})}(\zeta(\boldsymbol{\xi}_2))\|_{\mathscr{O}(Y)} &\leqslant r^{-2}\|F\|_{\boldsymbol{D}^{+}_{\mathscr{O}(Y)}(0,r)}\cdot\|\zeta\|^2_{op,\mathscr{O}(Y)}\cdot\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_{\mathscr{O}(Y)}\cdot\max(\|\boldsymbol{\xi}_1\|_{\mathscr{O}(Y)},\|\boldsymbol{\xi}_2\|_{\mathscr{O}(Y)}) \\ &\leqslant r^{-2}(\|F\|_{\boldsymbol{D}^{+}_{\mathcal{B}_\Omega(D(x,\rho))}(0,r)}+\varepsilon_4)\cdot(\|\zeta\|_{op,\mathcal{B}_\Omega(D(x,\rho))}+\varepsilon_5)^2\cdot\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_{\mathscr{O}(Y)}\cdot q\end{aligned}$$

Now by (4.50) inequality (4.54) holds with

$$C' = C\cdot\Big(\frac{\|Q_\varepsilon\|_{op,\mathcal{B}_\Omega(D(x,\rho))}+\varepsilon_1}{\|Q_\varepsilon\|_{op,\mathcal{B}_\Omega(D(x,\rho))}}\Big)\Big(\frac{\|F\|_{\boldsymbol{D}^{+}_{\mathcal{B}_\Omega(D(x,\rho))}(0,r)}+\varepsilon_4}{\|F\|_{\boldsymbol{D}^{+}_{\mathcal{B}_\Omega(D(x,\rho))}(0,r)}}\Big)\Big(\frac{\|\zeta\|_{op,\mathcal{B}_\Omega(D(x,\rho))}+\varepsilon_5}{\|\zeta\|_{op,\mathcal{B}_\Omega(D(x,\rho))}}\Big)^2 \tag{4.58}$$

Restricting Y the last three factors can be made arbitrarily close to 1. So there exists Y such that $0 < C' < 1$. □

4.4 Lifting to $\mathscr{O}_{X,x}$ solutions of non linear differential equations with solutions in $\mathscr{H}(x)$. With the identical proof one proves the following result. We here do not assume that K is algebraically closed.

Let Y be a fixed affinoid neighborhood of $x \in X$. Let $\boldsymbol{Z} := (Z_1,\dots,Z_{n\cdot s})$, let $m \geqslant n$, and let

$$F \;:\; \mathscr{O}(Y)^{ns} \to \mathscr{O}(Y)^m\,, \qquad F(\boldsymbol{Z}) \;:=\; \sum_{|\alpha|\geqslant 0} \boldsymbol{C}_\alpha \boldsymbol{Z}^\alpha\,, \tag{4.59}$$

where $\alpha = (\alpha_1,\dots,\alpha_{ns}) \in \mathbb{Z}^{ns}$, $|\alpha| = \sum_i \alpha_i$, $\boldsymbol{Z}^\alpha := Z_1^{\alpha_1}\cdots Z_{ns}^{\alpha_{ns}}$, $\boldsymbol{C}_\alpha \in \mathscr{O}(Y)^m$ is a power series satisfying $\|\boldsymbol{C}_\alpha\|_{\mathscr{O}(Y)}\cdot r^{|\alpha|} \to 0$ as $|\alpha| \to \infty$, i.e. F converges in a disk $\mathrm{D}^{+}_{\mathscr{O}(Y)}(0,r) \subseteq \mathscr{O}(Y)^{ns}$. For $\boldsymbol{X} = (X_1,\dots,X_n)$ consider the non-linear differential equation

$$F(\boldsymbol{X}, d(\boldsymbol{X}),\dots,d^s(\boldsymbol{X})) \;=\; 0\,. \tag{4.60}$$

Denote in analogy with the above sections $\zeta : \mathscr{O}(Y)^n \to \mathscr{O}(Y)^{ns}$, the map $\zeta(\boldsymbol{x}) = (\boldsymbol{x}, d(\boldsymbol{x}),\dots,d^s(\boldsymbol{x}))$.

Theorem 4.4.1 ([DR77, 3.1.6])**.** *Let $\boldsymbol{x} \in \mathscr{H}(x)^n$ be a solution of (4.60). If the linearized map*

$$dF_{\zeta(\boldsymbol{x})}\circ\zeta \;:\; \mathscr{H}(x)^n \to \mathscr{H}(x)^m \tag{4.61}$$

is injective, then $\boldsymbol{x} \in \mathscr{O}^n_{X,x}$.

Proof. Assume that K is algebraically closed. As in the above sections $dF_{\zeta(\boldsymbol{x})}\circ\zeta$ acts by a matrix $L_{\boldsymbol{x}} \in M_{m\times n}(\mathscr{O}(Y)\langle d\rangle)$. By the theory of elementary divisors there exists square matrices U, V with coefficients in $\mathscr{O}_{X,x}\langle d\rangle$ such that the unique non zeros terms of $UL_{\boldsymbol{x}}V$ are in the diagonal. The terms in the diagonal are either zero or differential polynomials in $\mathscr{O}_{X,x}\langle d\rangle$ that are injective as linear maps $\mathscr{H}(x)^n \to \mathscr{H}(x)^m$. If $m > n$ then we can discard $m-s$ components and reduce to the case $m = n$. The proof is then equal to that in the above sections.

Consider now the case of an general field K. To descend the theorem from K^{alg} to K, it is enough to notice that if $x_1,\dots,x_n \in X_{\widehat{K^{\mathrm{alg}}}}$ are the points with image $x \in X$, then for all x_i one has $\mathscr{O}_{X,x} = (\mathscr{H}(x)\cap\mathscr{O}_{X_{\widehat{K^{\mathrm{alg}}}},x_i}) \subset \mathscr{H}(x_i)$. □

Remark 4.4.2. *To prove the Theorem 4.1.1 of Dwork and Robba one can proceed as follows. The condition $q^2 = q$ in $\mathrm{End}_{\mathscr{O}_{X,x}}(\mathrm{M}_x)$ defines a projector. This condition is described by a non-linear differential system of the above type. Robba's theorem 3.6.9 provides a solution of such a system with coefficients in $\mathscr{H}(x)$. Dwork and Robba then relate the injectivity of (4.61) to the radii.*

5. Augmented Dwork-Robba filtration and global decomposition theorem.

From now on K is again an arbitrary complete ultrametric valued field as in our original setting 1.0.1. We begin by descending Dwork-Robba's decomposition from $\widehat{K^{\mathrm{alg}}}$ to K.

5.1 Descent of Dwork-Robba filtration

Let x be a point of type 2, 3, or 4. Let A be one of the fields $\mathscr{H}(x)$ or $\mathscr{O}_{X,x}$, and let $\overline{A}$ be one of the étale algebras $\mathscr{H}(x)\widehat{\otimes}_K\widehat{K^{\mathrm{alg}}} = \prod_{i=1}^n \mathscr{H}(x_i)$, or $\mathscr{O}_{X,x}\widehat{\otimes}_K\widehat{K^{\mathrm{alg}}} = \prod_{i=1}^n \mathscr{O}_{X_{\widehat{K^{\mathrm{alg}}}},x_i}$, where $x_1,\ldots,x_n$ are the points of $X_{\widehat{K^{\mathrm{alg}}}}$ whose image in X is x. If M is a differential module over A we denote by $\overline{\mathrm{M}} := \mathrm{M}\widehat{\otimes}_A\overline{A}$ its scalar extension to $\overline{A}$.

We recall that a semi-linear action of $\mathrm{G} := \mathrm{Gal}(K^{\mathrm{alg}}/K)$ on a $\overline{A}$-module $\overline{\mathrm{M}}$ is a map $\mu : \mathrm{G}\times\overline{\mathrm{M}} \to \overline{\mathrm{M}}$ where $g(m) := \mu(g,m)$, such that for all $g,g_1,g_2 \in \mathrm{G}$, $m,m_1,m_2 \in \overline{\mathrm{M}}$, $a \in \overline{A}$ one has

i) $g_1(g_2(m)) = (g_1g_2)(m)$, and $1_{\mathrm{G}}(m) = m$;

ii) $g(m_1+m_2) = g(m_1)+g(m_2)$;

iii) $g(am) = g(a)g(m)$.

We say that the action of G is *trivial* if $\overline{\mathrm{M}}$ is isomorphic to the representation $\overline{A}^r$ together with its natural action of G component by component.

Lemma 5.1.1. *The category of differential modules over A is equivalent to the category of differential modules over $\overline{A} = A\widehat{\otimes}_K\widehat{K^{\mathrm{alg}}}$ together with a* trivial *action of* G *commuting with the connection, and morphisms commuting with the connection and the action of* G.

Proof. The derivation $d\otimes 1$ on $\overline{A} = A\widehat{\otimes}_K\widehat{K^{\mathrm{alg}}}$ commutes with the action of G, given by $g(x\otimes y) := x\otimes g(y)$. We then have $\overline{A}^{\mathrm{G}} = A$ by Ax-Tate theorem. So for every vector space M over A we have $\overline{\mathrm{M}}^{\mathrm{G}} = (\mathrm{M}\otimes_A\overline{A})^{\mathrm{G}} = \mathrm{M}$. One sees that if M has a connection, then the corresponding connection of $\overline{\mathrm{M}}$ commutes with G, and can be descended.

Conversely if $\overline{\mathrm{M}}$ has an action of G commuting with the connection, and which is trivial, then $\overline{\mathrm{M}} = \overline{\mathrm{M}}^{\mathrm{G}}\widehat{\otimes}_A\overline{A}$, because the same holds for $\overline{A}^r$. Moreover the Leibnitz rule guarantee that the connection of $\overline{\mathrm{M}}$ is determined by its restriction to $\mathrm{M} := \overline{\mathrm{M}}^{\mathrm{G}}$.

□

Lemma 5.1.2. *Robba's and Dwork-Robba's decompositions (cf. (3.32) or (4.1)) by the spectral radii of $\overline{\mathrm{M}}$ over $\widehat{K^{\mathrm{alg}}}$ descend to decompositions of M by its spectral radii over K. Moreover the statements of Corollary 3.6.9, and of Theorem 4.1.1, also descend to K.*

Proof. It is enough to prove that for all $g\in\mathrm{G}$ one has $g((\overline{\mathrm{M}})^{\geqslant\rho}) = (\overline{\mathrm{M}})^{\geqslant\rho}$. The semi-linear bijection $g : \overline{\mathrm{M}} \xrightarrow{\sim} \overline{\mathrm{M}}$ produces a linear isomorphism $L_g : g^*\overline{\mathrm{M}} \xrightarrow{\sim} \overline{\mathrm{M}}$, where $g^*\overline{\mathrm{M}} = \overline{\mathrm{M}}\otimes_{\widehat{K^{\mathrm{alg}}},g}\widehat{K^{\mathrm{alg}}}$ is the scalar extension of $\overline{\mathrm{M}}$ by the map $g : \widehat{K^{\mathrm{alg}}} \to \widehat{K^{\mathrm{alg}}}$, and $L_g = g\otimes 1$. Since L_g is an isomorphism of differential modules, the radii of $g^*\overline{\mathrm{M}}$ at x_i coincide with those of $\overline{\mathrm{M}}$, and hence also the same as those of M at x. The same holds for $\overline{\mathrm{M}}^{\geqslant\rho}$, so the composite map $g^*(\overline{\mathrm{M}}^{\geqslant\rho}) \subseteq g^*\overline{\mathrm{M}} \xrightarrow{\sim} \overline{\mathrm{M}} \to \overline{\mathrm{M}}^{<\rho}$ is the zero map (by decomposing the modules and apply either point v) of Corollary 3.6.9, or by point ii) of Thm. 4.1.1). Hence $L_g(g^*(\overline{\mathrm{M}}^{\geqslant\rho})) \subseteq \overline{\mathrm{M}}^{\geqslant\rho}$.

The other claims of the Robba and Dwork-Robba's statements descends as follows. As we have proved, the composite morphism $c : (\mathrm{M}^*)^{\geqslant\rho} \to \mathrm{M}^* \to (\mathrm{M}^{\geqslant\rho})^*$ gives the same morphism for $\overline{\mathrm{M}}$ which is an isomorphism over $\widehat{K^{\mathrm{alg}}}$ commuting with G. So c itself is an isomorphism. Let $\alpha : \mathrm{M}^{\rho} \to \mathrm{N}^{\rho'}$

be a morphism with $\rho \neq \rho'$. This produces a morphism over $\widehat{K^{\mathrm{alg}}}$ commuting with G which is 0. So α itself is 0. □

5.2 Augmented Dwork-Robba decomposition

We now come back to the global situation.

Proposition 5.2.1 (Augmented decomposition)**.** *Let $x \in X$. Assume that i is an index separating the radii of $\mathscr{F}$ at the individual point x (cf. Def. 2.5.4). Then there exists a unique differential sub-module $(\mathscr{F}_{\geqslant i})_x \subseteq \mathscr{F}_x$ of rank $r-i+1$ over $\mathscr{O}_{X,x}$ such that*

$$\omega(x, (\mathscr{F}_{\geqslant i})_x) \;=\; \omega_{S,i}(x, \mathscr{F})\;. \tag{5.1}$$

Let $1 = i_1 < \cdots < i_h$ be the indices separating the radii of $\mathscr{F}$ at x. Then, according to (2.25), we have a corresponding decreasing sequence of differential submodules:

$$0 \;\neq\; (\mathscr{F}_{\geqslant i_h})_x \;\subset\; (\mathscr{F}_{\geqslant i_{h-1}})_x \;\subset\; \cdots \;\subset\; (\mathscr{F}_{\geqslant i_1})_x \;=\; \mathscr{F}_x\;. \tag{5.2}$$

Proof. By Dwork-Robba's decomposition (cf. Thm. 4.1.1 together with Lemma 5.1.2, and Prop. 3.5.3), the result holds for spectral radii : if $\mathrm{M}_x := \mathscr{F}_x$, and if T is a coordinate of $D(x,S)$, for which the radii of $D(x)$ and $D(x,S)$ are $r(x)$ and $R(x)$ respectively, then the result holds with $(\mathscr{F}_{\geqslant i})_x := \mathrm{M}_x^{\geqslant \rho}$ for $\rho = \mathcal{R}_{S,i}(x,\mathscr{F}) \cdot \frac{R(x)}{r(x)}$.

Let i be an over-solvable index. If K is algebraically closed then $D_{S,i}(x,\mathscr{F})$ comes by scalar extension from a disk in X containing x. Then $\omega_{S,i}(x,\mathscr{F}) = \omega(D_{S,i}(x,\mathscr{F}),\mathscr{F})$ is contained in $\omega(\mathscr{F}_x, \mathscr{O}_{X,x})$, and it generates by (1.14) a trivial differential sub-module $(\mathscr{F}_{\geqslant i})_x$ of $\mathscr{F}_x$ (this is obviously the unique sub-module of $\mathscr{F}$ whose solutions at x are identified with $\omega_{S,i}(x,\mathscr{F})$). If K is not algebraically closed this sub-module can easily be descended by Lemma 5.1.1, because $\mathrm{Gal}(\widehat{K^{\mathrm{alg}}}/K)$ preserves the modulus of the disks, hence it preserves the radii of the solutions. □

Lemma 5.2.2. *Let $x \in X$. Let i be an index separating the radii at the individual point x (cf. Definition 2.5.4). Then there exists an open neighborhood U_x of x, together with a weak triangulation S_{U_x} of U_x such that*

i) *The inclusion $(\mathscr{F}_{\geqslant i})_x \subseteq \mathscr{F}_x$ comes by scalar extension from an inclusion $(\mathscr{F}_{\geqslant i})_{|U_x} \subseteq \mathscr{F}_{|U_x}$ of differential equations over U_x;*

ii) *For all $y \in U_x$ the following conditions hold*

   (a) *one has $\omega_{S,i}(y,\mathscr{F}) \subset \omega_{S,i-1}(y,\mathscr{F})$ (i.e. i separates the global radii of $\mathscr{F}$ over U_x);*

   (b) *the restriction to U_x induces the equalities*

$$\omega_{S,i}(y,\mathscr{F}) \;=\; \omega_{S_{U_x},i}(y,\mathscr{F}_{|U_x})\;, \tag{5.3}$$

$$\omega_{S,i-1}(y,\mathscr{F}) \;=\; \omega_{S_{U_x},i-1}(y,\mathscr{F}_{|U_x})\;. \tag{5.4}$$

In particular one has:

iii) *$\mathcal{R}_{S_{U_x},i}(y,\mathscr{F}_{|U_x}) > \mathcal{R}_{S_{U_x},i-1}(y,\mathscr{F}_{|U_x})$ for all $y \in U_x$ (i.e. the index i separates the radii of $\mathscr{F}$ after localization at U_x);*

iv) *$\mathcal{R}_{S_{U_x},1}(y,(\mathscr{F}_{\geqslant i})_{|U_x}) = \mathcal{R}_{S_{U_x},i-1}(y,\mathscr{F}_{|U_x})$ for all $y \in U_x$;*

v) *$\omega_{S_{U_x},1}(y,(\mathscr{F}_{\geqslant i})_{|U_x}) = \omega_{S_{U_x},i}(y,\mathscr{F}_{|U_x}) = \omega_{S,i}(y,\mathscr{F})$, for all $y \in U_x$.*

Proof. First observe that iv) and v) are immediate consequences of i),ii),iii) by Proposition 2.9.7. Now observe that (5.4) is a consequence of (5.3) by the rule (2.33). Moreover iii) is a consequence of ii) by definitions (2.12) and (2.24). We now define U_x and S_{U_x} satisfying i) and ii).

Let U be a connected open neighborhood of x such that the coefficients of the matrix of the connection of $(\mathscr{F}_{\geqslant i})_x$ lie in $\mathscr{O}(U)$. Then i) holds for U. By continuity of the radii (cf. Thm. 2.4.1) up to restricting U one has $\mathcal{R}_{S,i}(y,\mathscr{F}) > \mathcal{R}_{S,i-1}(y,\mathscr{F})$ for all $y \in U$, hence (a) holds at each point $y \in U$. In order to find $U_x \subset U$ and a weak triangulation of U_x satisfying (b), we now proceed as follows. By Proposition 2.8.1, it is enough to define $U_x \subseteq U$ and S_{U_x} such that

$$D_{S,i-1}(y,\mathscr{F}) \text{ is } \textit{strictly} \text{ contained in } D(y,S_{U_x}), \text{ for all } y \in U_x. \tag{5.5}$$

First assume that i is over-solvable at x, $D_{S,i}(x,\mathscr{F})$ is a virtual open disk in X containing x. Then set $U_x := D_{S,i}(x,\mathscr{F})$ and $S_{U_x} = \emptyset$. Since $D_{S,i}(y,\mathscr{F}) = D_{S,i}(x,\mathscr{F}) = D(y,S_{U_x})$ for all $y \in D_{S,i}(x,\mathscr{F})$, this proves (5.5). Indeed by contrapositive the function $\mathcal{R}_{S,i-1}(-,\mathscr{F})$ is constant on $D_{S,i-1}(y,\mathscr{F})$ (cf. (2.27)), so the equality $D_{S,i-1}(y,\mathscr{F}) = D_{S,i}(x,\mathscr{F})$ implies $D_{S,i-1}(x,\mathscr{F}) = D_{S,i-1}(y,\mathscr{F}) = D_{S,i}(x,\mathscr{F})$ which is absurd because the radii of $\mathscr{F}$ are separated at x.

Assume now that i is spectral at x. This may only happen if x is of type 2, 3, or 4. By continuity and finiteness of the radii (cf. Thm. 2.4.1) we can choose $U_x \subseteq U$ such that

a) U_x is a star-shaped neighborhood of x endowed with its canonical triangulation (cf. 1.1.7) ;
b) One has $\mathcal{R}_{S,i-1}(y,\mathscr{F}) < \mathcal{R}_{S,i}(y,\mathscr{F})$, for all $y \in U_x$ (this is automatic since $U_x \subseteq U$);
c) The radius $\mathcal{R}_{S,i-1}(-,\mathscr{F})$ remains spectral and *non solvable* on the *pointed skeleton* of U_x (cf. 1.1.7);
d) either $\Gamma_S(\mathscr{F}) \cap U_x$ is empty, or $x \in \Gamma_S(\mathscr{F})$ and $\Gamma_S(\mathscr{F}) \cap U_x$ is included in the pointed skeleton of U_x.

We claim that these properties imply (5.5). By c) this is clear over the pointed skeleton of U_x by the rule (2.35) (indeed the spectral steps of the filtration of the solution are stable by localization, cf. also (2.42) and (2.43)). Now let $y \in U_x$ be outside the pointed skeleton of U_x. By contrapositive assume that $D(y,S_{U_x}) \subseteq D_{S,i-1}(y,\mathscr{F})$, then $i-1$ is solvable at y with respect to S_{U_x}. By (2.27) the radius $\mathcal{R}_{S_{U_x},i-1}(-,\mathscr{F}_{|U_x})$ is constant on $D(y,S_{U_x})$. Hence, by continuity, $\mathcal{R}_{S_{U_x},i-1}(-,\mathscr{F}_{|U_x})$ remains solvable at the topological boundary z of $D(y,S_{U_x})$. Since z lies in the pointed skeleton of U_x this is a contradiction. □

5.3 Global decomposition theorem

Recall that X is connected (cf. Setting 1.0.1).

Theorem 5.3.1. *Assume that the index i separates the radii of $\mathscr{F}$ over X (cf. Def. 2.5.4). Then $\mathscr{F}$ admits a sub-object $(\mathscr{F}_{\geqslant i},\nabla_{\geqslant i}) \subset (\mathscr{F},\nabla)$ such that for all $x \in X$ one has*

i) *rank $\mathscr{F}_{\geqslant i} = \dim_\Omega \omega_{S,i}(x,\mathscr{F}) = r-i+1$.*
ii) *For all $j = 1,\ldots,r-i+1$ the canonical inclusion $\omega(x,\mathscr{F}_{\geqslant i}) \subset \omega(x,\mathscr{F})$ identifies*

$$\omega_{S,j}(x,\mathscr{F}_{\geqslant i}) \;=\; \omega_{S,j+i-1}(x,\mathscr{F})\;. \tag{5.6}$$

Define $\mathscr{F}_{<i}$ by

$$0 \to \mathscr{F}_{\geqslant i} \to \mathscr{F} \to \mathscr{F}_{<i} \to 0\;. \tag{5.7}$$

Then, for all $x \in X$, one has

$$\mathcal{R}_{S,j}(x,\mathscr{F}) \;=\; \begin{cases} \mathcal{R}_{S,j}(x,\mathscr{F}_{<i}) & \textit{if} \quad j = 1,\ldots,i-1 \\ \mathcal{R}_{S,j-i+1}(x,\mathscr{F}_{\geqslant i}) & \textit{if} \quad j = i,\ldots,r\;. \end{cases} \tag{5.8}$$

Proof. By Lemma 5.2.2, for all point $x \in X$ there exists a neighborhood U_x of x, and a unique sub-object

$$((\mathscr{F}_{\geqslant i})_{|U_x},(\nabla_{\geqslant i})_{|U_x}) \;\subset\; (\mathscr{F}_{|U_x},\nabla_{|U_x}) \tag{5.9}$$

such that for all $y \in U_x$ one has

(A) rank $(\mathscr{F}_{\geqslant i})_{|U_x} = \dim_\Omega \omega_{S,i}(y,\mathscr{F}) = r-i+1$;

(B) $\omega(y,(\mathscr{F}_{\geqslant i})_{|U_x}) = \omega_{S,i}(y,\mathscr{F})$.

Now, by local uniqueness (cf. Prop. 5.2.1), the family $\{((\mathscr{F}_{\geqslant i})_{|U_x},(\nabla_{\geqslant i})_{|U_x})\}_x$ glues to a global sub-object $\mathscr{F}_{\geqslant i}$. By (2.21) this global sub-object satisfies point i). Point ii) follows from Proposition 2.9.7. The remaining claims follows from Proposition 2.9.5. □

Remark 5.3.2. *In Section 8 we construct an example in which $\mathscr{F}_{\geqslant i}$ is not a direct summand of $\mathscr{F}$. In Section 5.4 below we provide criteria to guarantee that $\mathscr{F}_{\geqslant i}$ is a direct summand.*

Proposition 5.3.3 (Independence of S)**.** *Let S and S' be two weak triangulations. Assume that the index i separates the radii of $\mathscr{F}$ with respect to both S and S' and denote by $\mathscr{F}_{S,\geqslant i}$ and $\mathscr{F}_{S',\geqslant i}$ the resulting sub-modules. Then $\mathscr{F}_{S,\geqslant i} = \mathscr{F}_{S',\geqslant i}$.*

Proof. Since i separates the radii in both cases one has $\omega_{S,j}(x,\mathscr{F}) = \omega_{S',j}(x,\mathscr{F})$ for all $j \leqslant i$, and all $x \in X$ (this is a consequence of (2.29)). By uniqueness of the augmented Dwork-Robba decomposition one has $(\mathscr{F}_{S,\geqslant i})_x = (\mathscr{F}_{S',\geqslant i})_x$, for all $x \in X$. Hence the composite map $\mathscr{F}_{S',i} \subset \mathscr{F} \to \mathscr{F}_{S,<i}$ is locally the zero map, so it is globally zero by Prop. 1.0.4. Then $\mathscr{F}_{S',\geqslant i} \subseteq \mathscr{F}_{S,\geqslant i}$, and by a symmetric argument $\mathscr{F}_{S,\geqslant i} = \mathscr{F}_{S',\geqslant i}$. □

Remark 5.3.4. *Let S, S' be two weak triangulations of X such that $\Gamma_S \subseteq \Gamma_{S'}$. Passing from S to S' has the effect that over-solvable radii result truncated by the rule (2.31). So if the index i separates the radii with respect to S', then it also separates the radii with respect to S. This shows that, in order to fulfill the assumptions of Thm. 5.3.1, the more convenient choice of triangulation is S.*

5.4 Conditions to have a direct sum decomposition

In this section we provide criteria to test whether $\mathscr{F}_{\geqslant i}$ is a direct summand.

Proposition 5.4.1. *The following conditions are equivalent:*

i) $\Gamma_{S,i}(\mathscr{F}) = \Gamma_{S,i}(\mathscr{F}^*)$;

ii) *For all $x \in X$ one has $\mathcal{R}_{S,i}(x,\mathscr{F}) = \mathcal{R}_{S,i}(x,\mathscr{F}^*)$.*

Proof. Clearly ii) implies i). Assume then that i) holds. Set $\Gamma := \Gamma_{S,i}(\mathscr{F}) = \Gamma_{S,i}(\mathscr{F}^*)$.

If $\Gamma = \emptyset$, then X is a virtual open disk with empty triangulation, and both $\mathcal{R}_{S,i}(-,\mathscr{F})$ and $\mathcal{R}_{S,i}(-,\mathscr{F}^*)$ are constant on X. If they are different, then one of them, say $\mathcal{R}_{S,i}(-,\mathscr{F})$, is strictly less than 1. Then, for x approaching the boundary of the disk X, the radius $\mathcal{R}_{S,i}(x,\mathscr{F})$ is spectral non-solvable. This contradicts Prop. 3.6.7. So ii) holds in this case.

Assume that $\Gamma \neq \emptyset$. Then $X - \Gamma$ is a disjoint union of virtual open disks on which $\mathcal{R}_{S,i}(x,\mathscr{F})$ and $\mathcal{R}_{S,i}(x,\mathscr{F}^*)$ are constant. So it is enough to prove that $\mathcal{R}_{S,i}(x,\mathscr{F}) = \mathcal{R}_{S,i}(x,\mathscr{F}^*)$ for all $x \in \Gamma$. By Remark 2.6.2, if $x \in \Gamma$, then $\mathcal{R}_{S,i}(x,\mathscr{F})$ and $\mathcal{R}_{S,i}(x,\mathscr{F}^*)$ are spectral, so they coincide by Proposition 2.9.8. □

Proposition 5.4.2. *If the index i separates the radii of $\mathscr{F}$ and of $\mathscr{F}^*$ (at each $x \in X$), then for all $x \in X$ one has $\mathcal{R}_{S,i}(x,\mathscr{F}) = \mathcal{R}_{S,i}(x,\mathscr{F}^*)$ and $\Gamma_{S,i}(\mathscr{F}) = \Gamma_{S,i}(\mathscr{F}^*)$.*

Proof. Let $x \in X$. It is enough to prove that $D_{S,i}(x,\mathscr{F}) = D_{S,i}(x,\mathscr{F}^*)$. If $\mathcal{R}_{S,i}(x,\mathscr{F})$ or $\mathcal{R}_{S,i}(x,\mathscr{F}^*)$ is spectral non solvable then they are equal by Prop. 2.9.8.

Assume then that both radii are solvable or over-solvable at x. By contradiction, assume that $D_{S,i}(x,\mathscr{F}) \neq D_{S,i}(x,\mathscr{F}^*)$, we may assume $D_{S,i}(x,\mathscr{F}) \subset D_{S,i}(x,\mathscr{F}^*)$ (i.e. $\mathcal{R}_{S,i}(x,\mathscr{F}) < \mathcal{R}_{S,i}(x,\mathscr{F}^*)$). We prove that this is absurd, and then exchanging the roles of $\mathscr{F}$ and $\mathscr{F}^*$ this will be enough to prove the equality of these two disks.

The radius $\mathcal{R}_{S,i}(-,\mathscr{F}^*)$ is constant over $D_{S,i}(x,\mathscr{F}^*)$, and over-solvable at each point of it. By compatibility with duals in the spectral case (cf. Prop. 3.6.7) this implies that $\mathcal{R}_{S,i}(z,\mathscr{F})$ is solvable or over-solvable for all $z\in D_{S,i}(x,\mathscr{F}^*)$. Moreover $\mathcal{R}_{S,i}(-,\mathscr{F})$ is not constant on $D_{S,i}(x,\mathscr{F}^*)$. Indeed $\mathcal{R}_{S,i}(x,\mathscr{F})<\mathcal{R}_{S,i}(x,\mathscr{F}*)$, so its constancy would imply that when z approaches the boundary of $D_{S,i}(x,\mathscr{F}^*)$ the radius $\mathcal{R}_{S,i}(z,\mathscr{F})$ becomes spectral non solvable. So $\Gamma_{S,i}(\mathscr{F})\cap D_{S,i}(x,\mathscr{F}^*)\neq\emptyset$.

Now consider an end-point z_0 of $\Gamma_{S,i}(\mathscr{F})$ in $D_{S,i}(x,\mathscr{F}^*)$. Since $\mathcal{R}_{S,i}(-,\mathscr{F})=\mathcal{R}_{S,1}(-,\mathscr{F}_{\geqslant i})$, then $\mathcal{R}_{S,i}(-,\mathscr{F})$ is a super-harmonic function on $D_{S,i}(x,\mathscr{F}^*)$ (cf. [Pul12, Thm. 4.7], see also Thm. 6.2.26). However super-harmonicity is violated at z_0 because the function is constant on each open-disk with boundary z_0, and solvable along the segment I connecting z_0 to the boundary of $D_{S,i}(x,\mathscr{F}^*)$. Indeed this implies that the slope of $\mathcal{R}_{S,1}(-,\mathscr{F}_{\geqslant i})$ along I is 1, while the slope along each other germ of segment out of x is zero. So the super-harmonicity of $\mathcal{R}_{S,1}(-,\mathscr{F}_{\geqslant i})$ is violated at z_0. □

Recall that X is connected.

Theorem 5.4.3. *Assume, as in Proposition 5.4.2, that i separates the radii of $\mathscr{F}$ and of $\mathscr{F}^*$. Assume moreover that we are in one of the following situations:*

i) *X is not a virtual disk with empty triangulation.*

ii) *X is a virtual disk with empty weak triangulation, and there exists $x\in X$ such that one of the radii $\mathcal{R}_{\emptyset,i-1}(x,\mathscr{F})$ or $\mathcal{R}_{\emptyset,i-1}(x,\mathscr{F}^*)$ is spectral non solvable.*

Then $\mathscr{F}_{\geqslant i}$ and $(\mathscr{F}^*)_{\geqslant i}$ are direct summands of $\mathscr{F}$ and $\mathscr{F}^*$ respectively.

Proof. It is enough to prove that the canonical composite morphism

$$c\;:\;(\mathscr{F}^*)_{\geqslant i}\;\to\;\mathscr{F}^*\;\to\;(\mathscr{F}_{\geqslant i})^* \tag{5.10}$$

is an isomorphism. This is a local matter. By Proposition 1.0.4 it is enough to show that c is an isomorphism at an individual point x. By Prop. 5.4.2 one has $\mathcal{R}_{S,i}(x,\mathscr{F})=\mathcal{R}_{S,i}(x,\mathscr{F}^*)$ for all $x\in X$, so the two functions have the same controlling graphs $\Gamma:=\Gamma_{S,i}(\mathscr{F})=\Gamma_{S,i}(\mathscr{F}^*)$. If we are in the case i), then $\Gamma\neq\emptyset$, and if $x\in\Gamma$, the radii are spectral at x, so c is an isomorphism at x by Prop. 3.6.7. The same holds in case ii) by the assumption. □

Remark 5.4.4. *The only pathological case which not contemplated by Thm. 5.4.3 is a virtual open disk X, with empty triangulation, on which $\mathcal{R}_{S,i}(-,\mathscr{F})$ and $\mathcal{R}_{S,i}(-,\mathscr{F}^*)$ are both the constant function with value 1, and such that $\mathcal{R}_{S,i-1}(-,\mathscr{F})$ and $\mathcal{R}_{S,i-1}(-,\mathscr{F}^*)$ are both solvable or over-solvable at all points of X. In this case if i separates the radii we do not know if $\mathscr{F}_{\geqslant i}$ is a direct factor of $\mathscr{F}$. As soon as one of $\mathcal{R}_{S,i}(-,\mathscr{F})$ and $\mathcal{R}_{S,i}(-,\mathscr{F}^*)$ is not identically equal to 1, then the index $i-1$ is spectral for $\mathscr{F}$, or for $\mathscr{F}^*$, at some point close to the boundary of the disk (because i separates the radii), and the conditions of the Thm. 5.4.3 are fulfilled.*

Remark 5.4.5. *It is interesting to notice that Thm. 5.4.3 holds with no assumptions on X if S is a (non weak) triangulation. Indeed the definition of triangulation of [Duc] always prescribes $S\neq\emptyset$.*

Corollary 5.4.6. *Assume that i separates the radii of $\mathscr{F}$, and that for $j=i,i-1$ one has $\Gamma_{S,j}(\mathscr{F})=\Gamma_{S,j}(\mathscr{F}^*)$. Then $\mathcal{R}_{S,j}(-,\mathscr{F})=\mathcal{R}_{S,j}(-,\mathscr{F}^*)$, $j=i,i-1$, so that i separates the radii of $\mathscr{F}^*$, and the sub-objects $\mathscr{F}_{\geqslant i}$ and $(\mathscr{F}^*)_{\geqslant i}$ are direct summands of $\mathscr{F}$ and $\mathscr{F}^*$ respectively.* □

5.4.1 *A criterion not involving $\mathscr{F}^*$.* We now provide a criterion to have a direct sum decomposition involving only properties of $\mathscr{F}$ and not of its dual (cf. Thm. 5.4.10). The hearth of that criterion is the following Proposition 5.4.7. Its proof is based on the Grothendieck-Ogg-Shafarevich

(G.O.S) formula in the form expressed in (8.5). We apply the G.O.S. formula to a differential module over an open disk having all the radii equal and constant as functions on the disk. So the G.O.S. formula does not need any decomposition theorem in that case, and the proof of Proposition 5.4.7 is then not circular. Note moreover that Proposition 5.4.7 follows from the more general[6] result [Ked10, Thm. 12.4.1], if D is K-rational, and by a Galois descent from $\widehat{K^{\mathrm{alg}}}$ to K otherwise.

Proposition 5.4.7. *Let $X = D$ be a virtual open disk with empty triangulation. Let $\mathscr{F}$ be a differential equation on D such that $\mathcal{R}_{\emptyset,j}(-,\mathscr{F})$ is a constant function on D for all $j = 1,\ldots,i$. Let $j \in \{1,\ldots,i\}$ be an index separating the radii. Then the index j separates the radii of $\mathscr{F}^*$ and all the assumptions of Theorem 5.4.3 and Proposition 5.4.2 are fulfilled. In particular*

i) $(\mathscr{F}_{\geqslant j},\nabla_{\geqslant j})$ *is a direct summand of* $(\mathscr{F},\nabla)$*;*

ii) $\mathcal{R}_{\emptyset,k}(x,\mathscr{F}) = \mathcal{R}_{\emptyset,k}(x,\mathscr{F}^*)$ *for all $k = 1,\ldots,i$, and all $x \in X = D$.*

Proof. We proceed by induction on i. If $i = 1$ there is nothing to prove since $\mathscr{F}_{\geqslant 1} = \mathscr{F}$. Let $i_1 > 1$ be the smallest index separating the radii. It is enough to prove that i_1 separates the radii of $\mathscr{F}^*$. Indeed in this case $\mathscr{F}_{\geqslant i_1} \subset \mathscr{F}$ is a direct summand by point ii) of Thm. 5.4.3. The conditions of Thm. 5.4.3 are fulfilled because i_1 separates the radii of $\mathscr{F}$, so the radius $\mathcal{R}_{\emptyset,i_1-1}(-,\mathscr{F})$ is a constant function with value < 1. Then if $x \in D$ is close enough to the boundary of D, the index $i_1 - 1$ is spectral non solvable at x. Finally by (5.8) the induction is evident replacing $\mathscr{F}$ by $\mathscr{F}_{\geqslant i_1}$.

It remains to prove that i_1 separates the radii of $\mathscr{F}^*$. We start by looking to the radii of $(\mathscr{F}_{<i_1})^*$. By (5.8) the radii of $\mathscr{F}_{<i_1}$ are constant functions on D all equal to $R := \mathcal{R}_{\emptyset,1}(-,\mathscr{F})$. Now the same is true for its dual:

Lemma 5.4.8. *For all $j = 1,\ldots,i_1 - 1 = \mathrm{rank}(\mathscr{F}_{<i_1})$, and all $x \in D$, one has*

$$\mathcal{R}_{\emptyset,j}(x,(\mathscr{F}_{<i_1})^*) \;=\; \mathcal{R}_{\emptyset,j}(x,\mathscr{F}_{<i_1}) \;=\; R \;<\; 1\,. \tag{5.11}$$

Proof. Let $D \xrightarrow{\sim} D^-(0,1)$ be an isomorphism. Since i_1 separates the radii of $\mathscr{F}$, we have $R = \mathcal{R}_{\emptyset,1}(-,\mathscr{F}) < 1$. So the indexes $1,\ldots,i_1 - 1$ are all spectral non solvable for $\mathscr{F}_{<i_1}$ at each point of the open segment $]x_{0,R},x_{0,1}[$. Hence compatibility with the dual holds over $]x_{0,R},x_{0,1}[$ by Prop. 3.6.7. Moreover (5.11) holds for $j = 1$ by Prop. 2.9.8, hence $\mathcal{R}_{\emptyset,j}(0,\mathscr{F}^*) \geqslant \mathcal{R}_{\emptyset,1}(0,\mathscr{F}) = R$. Since $\mathcal{R}_{\emptyset,j}(-,(\mathscr{F}_{<i_1})^*)$ is constant on $[0,x_{0,\mathcal{R}_{\emptyset,j}(0,(\mathscr{F}_{<i_1})^*)}[$, that contains $[0,x_{0,R}[$, this forces $\mathcal{R}_{\emptyset,j}(-,(\mathscr{F}_{<i_1})^*)$ to be constant on $[0,x_{0,1}[$ with value R. This is independent on the chosen isomorphism $D \xrightarrow{\sim} D^-(0,1)$, so $\mathcal{R}_{\emptyset,j}(-,(\mathscr{F}_{<i_1})^*)$ is constant on each segment $[z,x_{0,R}[$ of D, with z a rational point. If K is spherically complete this proves the claim. In general, the claim over K is deduced from that over a spherically complete extension Ω/K, since the radii are insensitive to scalar extension of K. This proves that $\mathcal{R}_{\emptyset,j}(-,(\mathscr{F}_{<i_1})^*)$ is constant over D. □

In addition to Lemma 5.4.8, for all $x \in D$, and all $j \geqslant 1$, we have

$$\mathcal{R}_{\emptyset,j}(x,\mathscr{F}_{\geqslant i_1}) \;=\; \mathcal{R}_{\emptyset,j+i_1-1}(x,\mathscr{F})\;. \tag{5.12}$$

Moreover by Prop. 2.9.8, the first radius $\mathcal{R}_{\emptyset,1}(-,(\mathscr{F}_{\geqslant i_1})^*)$ is a constant function on D and equal to $\mathcal{R}_{\emptyset,1}(-,\mathscr{F}_{\geqslant i_1})$.

Unfortunately this is not enough to guarantee that i_1 separates the radii of $\mathscr{F}^*$ because $(\mathscr{F}_{\geqslant i})^*$ is a quotient of $\mathscr{F}^*$, and the radii of the latter can be different. One has to prove that one does not have a pathology as in (8.11). For this Lemma 5.4.9 below proves that locally at each point $\mathscr{F}$ is the direct sum of $\mathscr{F}_{\geqslant i_1}$ and $\mathscr{F}_{<i}$, so that we have compatibility with duals by Prop. 2.9.6.

[6] As explained in section 5.8 the decomposition theorem [Ked10, 12.4.1] is more general because it does not assume *a priori* that the radii are separated.

Lemma 5.4.9 is based on the G.O.S. formula (8.5). Since the radii are insensitive to scalar extension of K, we can assume that K is spherically complete. This guarantee that the G.O.S. formula holds.

Lemma 5.4.9. *For all $x \in D$ there exists a spherically complete field extension Ω_x/K, and a star-shaped neighborhood Y_x of x in D such that*

i) *Y_x contains strictly $D_{\emptyset,i_1-1}(x,\mathscr{F})$. In particular one has $\mathcal{R}_{S_{Y_x},i_1-1}(x,\mathscr{F}_{|Y_x}) < \mathcal{R}_{S_{Y_x},i_1}(x,\mathscr{F}_{|Y_x})$ (the radii remains separated after localization).*[7] *Moreover if i_1-1 is solvable or over-solvable at x, then we may chose Y_x to be a virtual open disk $Y_x = D_x$ such that*
$$D_{\emptyset,i_1-1}(x,\mathscr{F}) \ \subset\ D_x \ \subseteq\ D_{\emptyset,i_1}(x,\mathscr{F})\;. \tag{5.13}$$

ii) *The restriction to $Y_x\widehat{\otimes}_K\Omega_x$ of the sequence*
$$E\ :\ 0 \to \mathscr{F}_{\geqslant i_1} \to \mathscr{F} \to \mathscr{F}_{<i_1} \to 0\;, \tag{5.14}$$
splits over $Y_x\widehat{\otimes}_K\Omega_x$.

Proof. If i_1-1 is spectral non solvable at x the statement follows from Dwork-Robba Theorem 4.1.1. Assume that i_1-1 is solvable or over-solvable at x, so that $D(x) \subseteq D_{\emptyset,i_1-1}(x,\mathscr{F})$. Note that $D_{\emptyset,i_1-1}(x,\mathscr{F})$ is not equal to D because i_1 separates the radii. Now let Ω_x/K be a field extension such that the connected components of $D_{\emptyset,i_1-1}(x,\mathscr{F})\otimes\Omega_x$ are isomorphic to $D' := D^-_{\Omega_x}(0,1)$. We consider the sequence $E\otimes\mathscr{O}^\dagger(D')$, and we use the notations of section 8.2. We now prove that the sequence splits over D'_ε for a convenient $\varepsilon > 0$. Since by assumption $D(x) \subseteq D'$, then D'_ε comes by scalar extension from a virtual disk $D_x := D'_{\varepsilon,K}$.

By (2.33), for all $\varepsilon > 0$ the radii of $(\mathscr{F}_{<i_1})_{|D'_\varepsilon}$ and $(\mathscr{F}_{\geqslant i_1})_{|D'_\varepsilon}$ are constant over D'_ε, and the index i_1 separates the radii of $\mathscr{F}_{|D'_\varepsilon}$. In particular $(\mathscr{F}_{<i_1})_{|D'_\varepsilon}$ does not verify (8.1), so the Grothendieck-Ogg-Shafarevich (8.5) formula holds. Hence there exists a virtual open disk D'_ε as above such that $h^1((\mathscr{F}_{<i_1})|_{D'_\varepsilon},\mathscr{O}(D'_\varepsilon)) = 0$ since the slope of $-\partial_b H_{\emptyset,i_1-1}(x_{0,1},(\mathscr{F}_{<i_1})|_{D'_\varepsilon})$ is zero.

The same happens for the dual $((\mathscr{F}_{<i_1})|_{D'_\varepsilon})^*$ by Lemma 5.4.8. Now, since $(\mathscr{F}_{\geqslant i_1})_{|D'_\varepsilon}$ is trivial, then by (8.5), one has
$$H^1(((\mathscr{F}_{<i})_{|D'_\varepsilon})^*\otimes(\mathscr{F}_{\geqslant i_1})_{|D'_\varepsilon}) \ =\ H^1(((\mathscr{F}_{<i})_{|D'_\varepsilon})^*)^{r-i_1+1} \ =\ 0\;. \tag{5.15}$$
Now the Yoneda group $\mathrm{Ext}^1(\mathrm{M},\mathrm{N})$ of extensions $0\to\mathrm{N}\to\mathrm{P}\to\mathrm{M}\to 0$ of differential modules can be identified with $H^1(\mathrm{M}^*\otimes\mathrm{N})$ (cf. Lemma 1.2.7). So the sequence splits over D'_ε. □

The behavior of the radii by localization to Y_x is expressed by (2.35). To show that i_1 separates the radii of $\mathscr{F}^*$, we then prove that i_1 separates the radii of $(\mathscr{F}_{|Y_x})^*$. Since the sequence (5.14) splits over $Y_x\widehat{\otimes}_K\Omega_x$, then so does the dual sequence $0 \to (\mathscr{F}_{<i_1})^* \to \mathscr{F}^* \to (\mathscr{F}_{\geqslant i_1})^* \to 0$. Now, by Lemma 5.4.8, the radii of $(\mathscr{F}_{<i_1})^*$ are all equal to $R = \mathcal{R}_{\emptyset,i_1-1}(-,\mathscr{F})$. Hence, by (5.12), they are strictly smaller than those of $(\mathscr{F}_{\geqslant i_1})^*$. This inequality is preserved by restriction to $Y_x\widehat{\otimes}_K\Omega_x$ by (5.13). We then apply Proposition 2.9.6 to prove that the radii of $\mathscr{F}^*|_{Y_x\widehat{\otimes}_K\Omega_x}$ are the union of these two families of radii, hence i_1 separates the radii of $\mathscr{F}^*|_{Y_x\widehat{\otimes}_K\Omega_x}$. So the same holds for $\mathscr{F}^*$ by (2.33). This concludes the proof of Proposition 5.4.7. □

The assumptions of the following result only involve the properties of $\mathscr{F}$. A possible criterion to guarantee condition (5.16) is discussed in section 6.3.

[7] Here S_{Y_x} denotes the canonical weak triangulation of Y_x (cf. section 1.1.7), which is the empty set if Y_x is a virtual open disk.

Theorem 5.4.10. *Assume that i separates the radii of $\mathscr{F}$, and that*

$$\Gamma_{S,1}(\mathscr{F}) \cup \cdots \cup \Gamma_{S,i-1}(\mathscr{F}) \;\subseteq\; \Gamma_{S,i}(\mathscr{F})\;. \tag{5.16}$$

Then

i) *The index i separates the radii of $\mathscr{F}^*$ and all the assumptions of Theorem 5.4.3 are fulfilled. In particular $(\mathscr{F}_{\geqslant i}, \nabla_{\geqslant i})$ is a direct summand of $(\mathscr{F}, \nabla)$.*

ii) *For all $j = 1, \ldots, i$ and all $x \in X$ one has $\mathcal{R}_{S,j}(x, \mathscr{F}) = \mathcal{R}_{S,j}(x, \mathscr{F}^*)$, hence*

$$\Gamma_{S,j}(\mathscr{F}) \;=\; \Gamma_{S,j}(\mathscr{F}^*)\;. \tag{5.17}$$

In particular $\Gamma_{S,1}(\mathscr{F}^*) \cup \cdots \cup \Gamma_{S,i-1}(\mathscr{F}^*) \;\subseteq\; \Gamma_{S,i}(\mathscr{F}^*)$.

Proof. If $\Gamma_{S,i}(\mathscr{F}) = \emptyset$, then X is a virtual open disk with empty triangulation, and the claim reduces to Proposition 5.4.7.

Assume now that $\Gamma_{S,i}(\mathscr{F}) \neq \emptyset$. Then $X - \Gamma_{S,i}(\mathscr{F})$ is a disjoint union of virtual open disks. We shall prove that i separates the radii of $\mathscr{F}^*$, and apply point i) of Theorem 5.4.3.

Firstly observe that i separates the radii of $\mathscr{F}^*$ at the points of $\Gamma_{S,i}(\mathscr{F})$. Indeed if $x \in \Gamma_{S,i}(\mathscr{F})$, then the radius $\mathcal{R}_{S,i}(x, \mathscr{F})$ is spectral at x (cf. Remark 2.6.2). Hence $\mathcal{R}_{S,1}(x, \mathscr{F}), \ldots, \mathcal{R}_{S,i-1}(x, \mathscr{F})$ are all spectral and *non solvable* at x, because i separates the radii. So by compatibility with duality in the spectral case (cf. Prop. 3.6.7) one has

$$\mathcal{R}_{S,j}(x, \mathscr{F}) \;=\; \mathcal{R}_{S,j}(x, \mathscr{F}^*)\;, \text{ for all } j = 1, \ldots, i, \text{ and all } x \in \Gamma_{S,i}(\mathscr{F})\;. \tag{5.18}$$

Let now D be a virtual open disk in X with boundary $x \in \Gamma_{S,i}(\mathscr{F})$. The assumption (5.16) implies that the radii $\mathcal{R}_{S,1}(-, \mathscr{F}), \ldots, \mathcal{R}_{S,i}(-, \mathscr{F})$ are all constant on D. Now consider the empty weak triangulation on D. The rule (2.36) proves that the radii $\mathcal{R}_{\emptyset,1}(-, \mathscr{F}_{|D}), \ldots, \mathcal{R}_{\emptyset,i}(-, \mathscr{F}_{|D})$ are all constants and that $\mathcal{R}_{\emptyset,i-1}(-, \mathscr{F}_{|D}) < \mathcal{R}_{\emptyset,i}(-, \mathscr{F}_{|D})$ (i.e. the radii remains separated after localization). Indeed if $z \in D$, we have $D = D^c_{S,i}(z, \mathscr{F})$, and hence $D_{S,i}(z, \mathscr{F}) \subseteq D$ by (2.27). So for all $j = 1, \ldots, i-1$ one also has

$$D_{S,j}(z, \mathscr{F}) \;\subset\; D_{S,i}(z, \mathscr{F}) \;\subseteq\; D\;. \tag{5.19}$$

Now by Proposition 5.4.7 the first i radii of $(\mathscr{F}_{|D})^*$ are constant functions on D equal to those of $\mathscr{F}_{|D}$. Again by (2.36), this proves that for all $j = 1, \ldots, i-1$ one has $D_{S,j}(z, \mathscr{F}^*) = D_{S,j}(z, \mathscr{F}) \subset D$. Hence the radii $\mathcal{R}_{S,1}(-, \mathscr{F}^*), \ldots, \mathcal{R}_{S,i-1}(-, \mathscr{F}^*)$ are constant functions on D, and are equal to $\mathcal{R}_{S,1}(-, \mathscr{F}), \ldots, \mathcal{R}_{S,i-1}(-, \mathscr{F})$ respectively. Together with (5.18) this gives $\mathcal{R}_{S,j}(-, \mathscr{F}^*) = \mathcal{R}_{S,j}(-, \mathscr{F})$ over the whole X.

Now (2.36) also proves that i separates the radii of $\mathscr{F}^*$ over D, and hence on the whole X by (5.18). By Proposition 5.4.2 the i-th radius of $\mathscr{F}$ and $\mathscr{F}^*$ coincide, and $\Gamma_{S,i}(\mathscr{F}) = \Gamma_{S,i}(\mathscr{F}^*)$.

By point i) of Theorem 5.4.3, we conclude. □

Remark 5.4.11. *Let S, S' be two weak triangulations of X such that $\Gamma_S \subseteq \Gamma_{S'}$. The best choice in order to fulfill condition (5.16) will be S'. By this we mean that $\Gamma_{S'} \subset \Gamma_{S',i}(\mathscr{F})$, so, as an example, we may choose S' in order to cover the pieces of $\Gamma_{S,1}(\mathscr{F}) \cup \cdots \cup \Gamma_{S,i-1}(\mathscr{F})$ that are not in $\Gamma_{S,i}(\mathscr{F})$.[8] In this way, by (2.32), we obtain $\Gamma_{S',j}(\mathscr{F}) \subseteq \Gamma_{S',i}(\mathscr{F})$ for all $j = 1, \ldots, i-1$.*

Unfortunately the increasing of S' cuts the radii, so the S'-radii can be no more separated. In fact, by Remark 5.3.4 we know that, in order to guarantee that i separates the radii, the best choice of triangulation is S. The triangulation S is also convenient for Theorem 5.4.3 which is for this reason more "natural" *than Thm. 5.4.10.*

[8]This is possible because none of the $\Gamma_{S,j}(\mathscr{F})$ contains points of type 1 or 4 by [Ked13].

5.5 Direct sum over annuli.

In section 6 we provide criteria and methods to check conditions of Theorems 5.3.1, 5.4.3, and 5.16. This section makes use of section 6, and is placed here for expository reasons.

Corollary 5.5.1. *Let X be a virtual open annulus with empty triangulation. Let I be its skeleton, and let $i \leqslant r = \mathrm{rank}(\mathscr{F})$. Assume that (cf. Corollary 6.3.7):*

i) *For all $x \in I$, and all $j \in \{1, \ldots, i-1\}$ one of the following condition holds:*[9]
   (a) *there exists an open subinterval $J \subseteq I$ containing x such that the partial height $H_{\emptyset,j}(-, \mathscr{F})$ (cf. (2.13)) is a* log-*affine map on J (cf. Section 1.1.4)*
   (b) *$\mathcal{R}_{\emptyset,j}(x, \mathscr{F})$ is solvable or over-solvable at x.*

ii) *$\mathcal{R}_{\emptyset,i-1}(x, \mathscr{F}) < \mathcal{R}_{\emptyset,i}(x, \mathscr{F})$ for all $x \in I$.*

Then the index i separates the radii of $\mathscr{F}$ globally on X, and $\Gamma_{\emptyset,j}(\mathscr{F}) = I$ for all $j = 1, \ldots, i-1$. Hence

$$\mathscr{F} \;=\; \mathscr{F}_{<i} \oplus \mathscr{F}_{\geqslant i}\;. \tag{5.20}$$

Proof. By Corollary 6.3.7, one has $\Gamma'_{\emptyset,i-1}(\mathscr{F}) = I$. Then $\mathcal{R}_{\emptyset,i-1}(-, \mathscr{F})$ is constant outside it. Since the index i separates the radii over I, it separates the radii on the whole X. Indeed if D is a disk with boundary $x \in I$, then for all $z \in D$ one has

$$\mathcal{R}_{\emptyset,i}(z, \mathscr{F}) \;\geqslant\; \mathcal{R}_{\emptyset,i}(x, \mathscr{F}) \;>\; \mathcal{R}_{\emptyset,i-1}(x, \mathscr{F}) = \mathcal{R}_{\emptyset,i-1}(z, \mathscr{F})\;. \tag{5.21}$$

This follows by concavity of $\mathcal{R}_{\emptyset,i}(-, \mathscr{F})$ outside $\Gamma'_{\emptyset,i-1}(\mathscr{F})$ (cf. point iv) of Remark 6.1.3). Then the index i separates the radii of $\mathscr{F}$, and $\Gamma'_{\emptyset,i-1}(\mathscr{F}) = I \subseteq \Gamma_{\emptyset,i}(\mathscr{F})$. Theorems 5.3.1 and 5.4.10 then apply. □

5.5.1 *Solvable equations over the Robba ring.* The so called Robba ring is the ring $\mathfrak{R} := \bigcup_{\varepsilon>0} \mathscr{O}(C_\varepsilon)$, where $C_\varepsilon := C_K^-(0; 1-\varepsilon, 1)$. Following a terminology of G. Christol and Z. Mebkhout, a differential module M over $\mathfrak{R}$ is *solvable* if $\lim_{\rho\to 1} \mathcal{R}_{\emptyset,1}(x_{0,\rho}, \mathrm{M}) = 1$. If M is solvable, then for all $i = 1, \ldots, r = \mathrm{rank}(\mathrm{M})$, there exists $\varepsilon > 0$ such that $\mathcal{R}_{S,i}(x_{0,\rho}, \mathrm{M}) = \rho^{\beta_{r-i+1}}$, for all $\rho \in]1-\varepsilon, 1[$. The numbers $\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_r$ are called the *slopes* of M.[10] We say that M is purely of slope β if $\beta_1 = \cdots = \beta_r = \beta$.

As a consequence of Corollary 5.5.1 we recover Christol-Mebkhout decomposition theorem:

Corollary 5.5.2 ([CM00])**.** *Any solvable differential module over $\mathfrak{R}$ admits a direct sum decomposition*

$$\mathrm{M} \;:=\; \bigoplus_{\beta\geqslant 0} \mathrm{M}(\beta) \tag{5.22}$$

into sub-modules $\mathrm{M}(\beta)$ that are purely of slope β.

Proof. The radii $\mathcal{R}_{S,i}(-, \mathscr{F})$ are log-affine on the skeleton of a conveniently small annulus C_ε. Then apply Corollary 5.5.1. □

5.6 Crossing points and filtration outside a locally finite set.

In this section we do not assume that the radii of $\mathscr{F}$ are separated globally on X. We investigate the existence of a filtration of $\mathscr{F}$ over some regions of X. The problem is that the radii are not

[9] In particular the condition is fulfilled if the radii $\mathcal{R}_{S,j}(-, \mathscr{F})$, $j = 1, \ldots, i-1$ are all log-affine on the skeleton $\Gamma_\emptyset$.

[10] The original terminology of Christol and Mebkhout is *p-adic slopes*. We drop the term p-adic because we allow the absolute value of K to be the extension of a trivially valued field, or to have a residual field of characteristic 0. So everything works in a more general context.

stable by localization, so we can not just localize and apply the above statements.

Definition 5.6.1. *A* separating neighborhood *of $x \in X$ is a open neighborhood U of x such that for all $i \neq j$ one has either $\mathcal{R}_{S,i}(y,\mathscr{F}) = \mathcal{R}_{S,j}(y,\mathscr{F})$ or $\mathcal{R}_{S,i}(y,\mathscr{F}) \neq \mathcal{R}_{S,j}(y,\mathscr{F})$ at every point $y \in U$. In other words, each index i separates the radii either everywhere or nowhere on U.*

Definition 5.6.2. *A point $x \in X$ is called* crossing point for $\mathscr{F}$ *if it has no separating neighborhoods. We denote by $\mathrm{Cr}_S(\mathscr{F})$ the subset of crossing points.*

Lemma 5.6.3. *The subset $\mathrm{Cr}_S(\mathscr{F})$ is a locally finite subset of X contained in $\Gamma_S(\mathscr{F})$.*

Proof. This results by finiteness (cf. Thm. 2.4.1), together with the fact that along a closed segment each radius $\mathcal{R}_{S,i}(-,\mathscr{F})$ has a finite number of breaks (cf. [Pul12, Thm. 4.7 iii)] and [PP12b]). □

Lemma 5.6.4. *Let $\mathfrak{F} \subseteq X$ be a locally finite subset containing $\mathrm{Cr}_S(\mathscr{F})$, and let Y be a connected component of $X - \mathfrak{F}$. Then Y is an separating open neighborhood of all its points. In other words if $i < j$ one has either $\mathcal{R}_{S,i}(y,\mathscr{F}) < \mathcal{R}_{S,j}(y,\mathscr{F})$ for all $y \in Y$, or $\mathcal{R}_{S,i}(y,\mathscr{F}) = \mathcal{R}_{S,j}(y,\mathscr{F})$ for all $y \in Y$.* □

Definition 5.6.5. *We denote by $X_{S,i}(\mathscr{F}) \subset X$ the open subset formed by the points $x \in X$ such that i separates the radii of $\mathscr{F}$ at x.*

Remark 5.6.6. *By definition $X_{S,1}(\mathscr{F}) = X$.*

If the open subset $X_{S,i}(\mathscr{F})$ is not the whole X, then $\Gamma_{S,i}(\mathscr{F}) \cup \Gamma_{S,i-1}(\mathscr{F}) \neq \emptyset$, and in this case $X_{S,i}(\mathscr{F})$ is the inverse image by the canonical retraction $X \to \Gamma_{S,i}(\mathscr{F}) \cup \Gamma_{S,i-1}(\mathscr{F})$ of a possibly not connected open sub-graph. In particular a connected component D of $X_{S,i}(\mathscr{F})$ such that $D \cap \Gamma_{S,i}(\mathscr{F}) = \emptyset$ is necessarily a virtual open disk with boundary in $\Gamma_{S,i}(\mathscr{F}) \cup \Gamma_{S,i-1}(\mathscr{F})$.

Moreover this proves that, if C is a connected subset of $X - \mathrm{Cr}_S(\mathscr{F})$ such that $C \cap X_{S,i}(\mathscr{F}) \neq \emptyset$, then $C \subseteq X_{S,i}(\mathscr{F})$.

5.6.1 *Existence of $\mathscr{F}_{\geqslant i}$.* We here study the existence of $\mathscr{F}_{\geqslant i}$ over some regions of X.

Proposition 5.6.7. *The restriction $\mathscr{F}_{|X_{S,i}(\mathscr{F})}$ of $\mathscr{F}$ to $X_{S,i}(\mathscr{F})$ admits a unique sub-object $(\mathscr{F}_{|X_{S,i}(\mathscr{F})})_{\geqslant i}$ of rank $r - i + 1$ such that, for all $y \in X_{S,i}(\mathscr{F})$, one has*

$$\omega(y,(\mathscr{F}_{|X_{S,i}(\mathscr{F})})_{\geqslant i}) \;=\; \omega_{S,i}(y,\mathscr{F}) \;. \tag{5.23}$$

Proof. The (global) radii $\{\mathcal{R}_{S,i}(-,\mathscr{F})\}_i$ induce a filtration $\{\omega_{S,i}(x,\mathscr{F})\}_i$ on $\omega(x,\mathscr{F})$. From that filtration we have defined in Prop.5.2.1 the augmented Dwork-Robba filtration of $\mathscr{F}_x$ for all $x \in X$. Now the connected components of $X_{S,i}(\mathscr{F})$ are the regions on which the i-th radius stays separated from the $(i-1)$-th one. So the gluing process works over such $X_{S,i}(\mathscr{F})$, and it gives a sub-object of $\mathscr{F}_{|X_{S,i}(\mathscr{F})}$. □

Corollary 5.6.8. *Let Y be a connected component of $X - \mathrm{Cr}_S(\mathscr{F})$. Let $1 = i_1 < i_2 < \ldots < i_h$ be the indexes separating the radii $\{\mathcal{R}_{S,i}(-,\mathscr{F})\}_i$ over Y. The restriction $\mathscr{F}_{|Y}$ of $\mathscr{F}$ to Y admits a filtration*

$$0 \;\neq\; (\mathscr{F}_{|Y})_{\geqslant i_h} \;\subset\; (\mathscr{F}_{|Y})_{\geqslant i_h - 1} \;\subset\; \cdots \;\subset\; (\mathscr{F}_{|Y})_{\geqslant i_1} \;=\; \mathscr{F}_{|Y} \tag{5.24}$$

such that for all $x \in Y$ one has

$$\omega(x,(\mathscr{F}_{|Y})_{\geqslant i_k}) \;=\; \omega_{S,i_k}(x,\mathscr{F}) \;. \tag{5.25}$$

Proof. By Remark 5.6.6, Y is contained in $X_{S,i_1}(\mathscr{F}) \cap X_{S,i_2}(\mathscr{F}) \cap \cdots \cap X_{S,i_h}(\mathscr{F})$. So we apply

Proposition 5.6.7. □

Remark 5.6.9. *Note that we did not endow Y with any weak triangulation, so in these propositions we do not speak about the radii of $\mathscr{F}_{|Y}$. The proposition involves only the restriction to Y of the global radii $\mathcal{R}_{S,i}(-,\mathscr{F})$. The reason is the following. It easy to see that there exists a weak triangulation S' of $X' := X - \mathrm{Cr}_S(\mathscr{F})$ such that the graph $\Gamma_{S'}$ is the minimum graph of a weak triangulation satisfying $\Gamma_S \cap X' \subseteq \Gamma_{S'}$. However, in analogy with Remark 5.4.11, the localization to X' cuts the radii, and it may happens that the radii $\mathcal{R}_{S',i}(-,\mathscr{F}_{|X'})$ are no more separated (i.e. after localization to X').*

5.6.2 *Direct sum decomposition: condition on $\mathscr{F}$ and $\mathscr{F}^*$.* Here we gives conditions to have a direct sum decomposition.

We firstly consider the open subset $X_{S,i}(\mathscr{F},\mathscr{F}^*) := X_{S,i}(\mathscr{F}) \cap X_{S,i}(\mathscr{F}^*)$, over which i separates the radii of both $\mathscr{F}$ and $\mathscr{F}^*$. Set $\Gamma := \Gamma_{S,i}(\mathscr{F}) \cup \Gamma_{S,i}(\mathscr{F}^*) \cup \Gamma_{S,i-1}(\mathscr{F}) \cup \Gamma_{S,i-1}(\mathscr{F}^*)$. If $X_{S,i}(\mathscr{F},\mathscr{F}^*)$ is not equal to X, then $\Gamma \neq \emptyset$, hence $X_{S,i}(\mathscr{F},\mathscr{F}^*)$ is the inverse image by the retraction $X \to \Gamma$ of a possibly not connected open sub-graph of Γ.

Proposition 5.6.10. *Let $\mathscr{F}$ be a differential equation over X, and let C be a connected component of $X_{S,i}(\mathscr{F},\mathscr{F}^*)$. If C satisfies $C \cap \big(\Gamma_{S,i}(\mathscr{F}) \cup \Gamma_{S,i}(\mathscr{F}^*)\big) = \emptyset$, then assume moreover that there exists $x \in C$ such that $i-1$ is spectral for $\mathscr{F}$, or for $\mathscr{F}^*$. Then $(\mathscr{F}_{|Y})_{\geqslant i}$ is a direct summand of $\mathscr{F}_{|Y}$.*

Proof. It is enough to prove that the canonical composite morphism $c : (\mathscr{F}^*_{|Y})_{\geqslant i} \subseteq \mathscr{F}^*_{|Y} \to ((\mathscr{F}_{|Y})_{\geqslant i})^*$ is an isomorphism. For this, by Prop. 1.0.4, we only need a point $x \in Y$ at which $i-1$ is spectral non solvable for $\mathscr{F}$ or $\mathscr{F}^*$ (as in the proof of Thm. 5.4.3). If $Y \cap \big(\Gamma_{S,i}(\mathscr{F}) \cup \Gamma_{S,i}(\mathscr{F}^*)\big) \neq \emptyset$, then we are done since, by Remark 2.6.2, i is spectral, for $\mathscr{F}$ and $\mathscr{F}^*$, at each point of that intersection (and hence $i-1$ is spectral non solvable). □

Corollary 5.6.11. *Let Y be a connected component of $X - (\mathrm{Cr}_S(\mathscr{F}) \cup \mathrm{Cr}_S(\mathscr{F}^*))$. Let $1 = i'_1 < i'_2 < \cdots < i'_{h'}$ be the indexes separating simultaneously the global radii of $\mathscr{F}$ and of $\mathscr{F}^*$ over Y. Let C be the connected component of $X_{S,i'_k}(\mathscr{F},\mathscr{F}^*)$ containing Y. Assume that for all i'_k one of the two situation is realized:*

i) $C \cap (\Gamma_{S,i'_k}(\mathscr{F}) \cup \Gamma_{S,i'_k}(\mathscr{F}^*)) \neq \emptyset$

ii) $C \cap (\Gamma_{S,i'_k}(\mathscr{F}) \cup \Gamma_{S,i'_k}(\mathscr{F}^*)) = \emptyset$, *and the index $i'_k - 1$ is spectral for $\mathscr{F}$ or for $\mathscr{F}^*$ at some point of C.*

Then the terms of the filtration (5.24) of $\mathscr{F}$ over Y corresponding to an index of the family $\{i'_k\}_k$ are direct summands of $\mathscr{F}_{|Y}$. □

5.6.3 *Direct sum decomposition: conditions on $\mathscr{F}$.* In analogy with Thm. 5.4.10, we now provide conditions on the controlling graphs.

Denote by $X_{|\Gamma_{S,i}(\mathscr{F})}$ the affinoid domain of X obtained removing the connected components of $X - \Gamma_{S,i}(\mathscr{F})$ intersecting $\Gamma_{S,j}(\mathscr{F})$ for some $j = 1,\ldots,i-1$. Such connected components form a locally finite family of virtual open disks.

Proposition 5.6.12. *Let $Y := X_{S,i}(\mathscr{F}) \cap X_{|\Gamma_{S,i}(\mathscr{F})}$. Then $(\mathscr{F}_{|Y})_{\geqslant i}$ is a direct summand of $\mathscr{F}_{|Y}$.*

Proof. By definition $\Gamma_S \subseteq \Gamma_{S,i}(\mathscr{F}) \subseteq X_{|\Gamma_{S,i}(\mathscr{F})}$. One sees that there exists a smallest weak triangulation S' of $X_{|\Gamma_{S,i}(\mathscr{F})}$ containing S and the boundaries of the disks removed. and the bifurcation points of $\Gamma_{S,i}(\mathscr{F})$. The set S' is a locally finite set in $\Gamma_{S,i}(\mathscr{F})$. Then S' verifies $\Gamma_S \subseteq \Gamma_{S'} \subseteq \Gamma_{S,i}(\mathscr{F})$.

By Prop. 2.8.2, for all $j = 1, \dots, i$ one has

$$\Gamma_{S',j}(\mathscr{F}_{|X'}) \;=\; (\Gamma_{S,j}(\mathscr{F}) \cap X') \cup \Gamma_{S'} \;\subseteq\; \Gamma_{S',i}(\mathscr{F}_{|X'}) \;=\; \Gamma_{S,i}(\mathscr{F})\;, \tag{5.26}$$

where $X' := X_{|\Gamma_{S,i}(\mathscr{F})}$. Moreover the first i S-radii are all spectral along $\Gamma_{S,i}(\mathscr{F})$ by Remark 2.6.2. So by (2.37), the index i separates the radii at $y \in X_{|\Gamma_{S,i}(\mathscr{F})}$ with respect to S if and only if it separates the radii with respect to S'. This proves that $X_{S,i}(\mathscr{F}) \cap X_{|\Gamma_{S,i}(\mathscr{F})}$ is the inverse image by the retraction $X_{|\Gamma_{S,i}(\mathscr{F})} \to \Gamma_{S',i}(\mathscr{F}_{|X_{|\Gamma_{S,i}(\mathscr{F})}})$ of the subset of points on which i separates the radii. In other words $X_{S,i}(\mathscr{F}) \cap X' = X'_{S',i}(\mathscr{F}_{|X'})$, where $X' := X_{|\Gamma_{S,i}(\mathscr{F})}$. Hence, replacing X by $X_{|\Gamma_{S,i}(\mathscr{F})}$, we can assume $\Gamma_{S,j}(\mathscr{F}) \subseteq \Gamma_{S,i}(\mathscr{F})$, for all $j = 1, \dots, i$.

Now with the same proof as Thm. 5.4.10 one shows that i separates the radii of $\mathscr{F}^*$ at each point of $X_{S,i}(\mathscr{F})$ (i.e. $X_{S,i}(\mathscr{F}) = X_{S,i}(\mathscr{F}, \mathscr{F}^*)$), and the assumptions of Prop. 5.6.10 are fulfilled. □

Corollary 5.6.13. *Let Y be a connected analytic domain of $X - \mathrm{Cr}_S(\mathscr{F})$. Let $1 = i_1 < \dots < i_h$ be the indexes separating the radii $\{\mathcal{R}_{S,i}(-, \mathscr{F})\}_i$ over Y. If*

$$Y \;\subseteq\; \Big(X_{|\Gamma_{S,i_1}(\mathscr{F})} \cap X_{|\Gamma_{S,i_2}(\mathscr{F})} \cap \dots \cap X_{|\Gamma_{S,i_h}(\mathscr{F})}\Big) \tag{5.27}$$

then $(\mathscr{F}_{|Y})_{\geqslant i_k}$ is a direct summand of $\mathscr{F}_{|Y}$ for all k.

Proof. By Remark 5.6.6, $Y \subseteq X_{S,i_k}(\mathscr{F})$ for all k, so $(\mathscr{F}_{|Y})_{\geqslant i_k}$ exists by Corollary 5.6.8. Now (5.27) means $Y \subseteq X_{|\Gamma_{S,i_k}(\mathscr{F})}$ for all k. We then apply Proposition 5.6.12. □

5.6.4 *Clean decomposition.* Until now, we have restricted the module $\mathscr{F}$ to a subset Y but have always computed its radii with respect to the original weak triangulation S, which is a weak triangulation on X. Now, we will allow us to change the weak triangulation as well and compute the radii with respect to a weak triangulation of Y. In the process, the radii may be truncated, which would lead to a decomposition that is less precise. On the other hand, we will show that such a decomposition always exists.

Recall that, by [Ked13, Thm. 4.5.15], every radius of convergence is constant in the neighborhood of a point of type 4. In particular, every graph $\Gamma_{S,j}(\mathscr{F})$ is the skeleton of a weak triangulation. It is the only time in this paper where we shall use this difficult result.

Theorem 5.6.14 (Clean decomposition)**.** *Set $r = \mathrm{rank}(\mathscr{F}) = \mathrm{rank}(\mathscr{F}^*)$. There exists a weak triangulation S_d of X containing S such that the following holds:*

i) *$\Gamma_{S_d} = \Gamma_S(\mathscr{F}) \cup \Gamma_S(\mathscr{F}^*)$ (cf. (2.18));*
ii) *for every $x \in X$ and every $j \in \{1, \dots, r\}$, we have $\mathcal{R}_{S_d,j}(x, \mathscr{F}) = \mathcal{R}_{S_d,j}(x, \mathscr{F}^*)$;*
iii) *every connected component of $X \setminus S_d$ is a separating neighborhood of all its points for both $\mathscr{F}$ and $\mathscr{F}^*$, with respect to the weak triangulation S_d.*

Moreover, let C be a connected component of $X \setminus S_d$ (necessarily a virtual open disk or annulus) and endow it with the empty weak triangulation. Let $1 = i_1 < i_2 < \dots < i_h$ be the indexes separating the radii of $\mathscr{F}_{|C}$. Then, we have a direct sum decomposition

$$\mathscr{F}_{|C} \;=\; \bigoplus_{1 \leqslant m \leqslant h} (\mathscr{F}_{|C})_{i_m} \tag{5.28}$$

such that, for every $m \in \{1, \dots, h\}$, every $j \in \{1, \mathrm{rank}((\mathscr{F}_{|C})_{i_m})\}$ and every $x \in C$, we have

$$\mathcal{R}_{\emptyset,j}(x, (\mathscr{F}_{|C})_{i_m}) \;=\; \mathcal{R}_{\emptyset,i_m}(x, \mathscr{F}_{|C}) \;=\; \mathcal{R}_{S_d,i_m}(x, \mathscr{F})\;. \tag{5.29}$$

The same result hold for $\mathscr{F}^*$ and, for every $m \in \{1, \dots, h\}$, we have $(\mathscr{F}_{|C})^*_{i_m} = (\mathscr{F}^*_{|C})_{i_m}$.

Proof. Let us first choose a weak triangulation S' such that $\Gamma_{S'} = \Gamma_S(\mathscr{F}) \cup \Gamma_S(\mathscr{F}^*)$. Such an S' exists by [Ked13, Thm. 4.5.15]. By Proposition 2.7.1, one has $\Gamma_{S'}(\mathscr{F}) = \Gamma_{S'}(\mathscr{F}^*) = \Gamma_{S'}$, and $\mathcal{R}_{S',i}(-,\mathscr{F}) = \mathcal{R}_{S',i}(-,\mathscr{F}^*)$ for all i. Hence $\mathrm{Cr}_{S'}(\mathscr{F}^*) = \mathrm{Cr}_{S'}(\mathscr{F}) \subset \Gamma_{S'}$. Define $S_d := S' \cup \mathrm{Cr}_{S'}(\mathscr{F})$. Proposition 2.7.1 then ensures that the S'-radii coincides with the S_d-radii, hence on an edge of Γ_{S_d} two S_d-radii of $\mathscr{F}$ (resp. $\mathscr{F}^*$) are either always equal or always different. One sees that $X_{|\Gamma_{S_d,i}(\mathscr{F})} = X$, for all i. The statement is now a consequence of Proposition 5.6.12, and Corollary 5.6.11. □

Remark 5.6.15. *Without using Kedlaya's result, it is still possible to prove a slightly weaker statement by replacing X by $X \setminus T$, where T is the locally finite subset of X formed by the type 4 points in $\Gamma_S(\mathscr{F}) \cup \Gamma_S(\mathscr{F}^*)$. Then S_d is a weak triangulation of $X \setminus T$ whose skeleton is $\Gamma_{S_d} = \big(\Gamma_S(\mathscr{F}) \cup \Gamma_S(\mathscr{F}^*)\big) \setminus T$. The other properties hold unchanged.*

Remark 5.6.16. *An alternative proof of the global decomposition Theorem 5.3.1 is possible by using the principle of Thm. 5.6.14. Namely if S'_d is a weak triangulation of $\Gamma_S(\mathscr{F})$, containing the triangulation S_d of Thm. 5.6.14, and satisfying*

i) *Each connected component of $X - S'_d$ is either an open virtual disk or annulus;*
ii) *The radii are all $\log$-linear on each edge of $\Gamma_{S'_d}$ (i.e. on the interior of each connected component of $\Gamma_{S'_d} - S'_d$).*
iii) *Let C be an annulus which is a connected component of $\Gamma_{S'_d}$. Then each index $i = 1, \dots, r$ is either solvable at each point of Γ_C, or never solvable at the points of Γ_C.*

One sees that in the situation of iii), if i is the smallest index which is over-solvable over Γ_C, then C is contained in the disk $D_{S,i}(x,\mathscr{F})$. So there exists $(\mathscr{F}_{|C})_{\geqslant i} \subseteq \mathscr{F}_{|C}$ generated by $\omega_{S,i}(x,S)$, for any point $x \in \Gamma_C$. Then uses [Ked10, 12.4.2] to decompose $\mathscr{F}_{|C}$ by the spectral radii. This together with the augmented Dwork-Robba decomposition at the points of S'_d this glue, and one have the result of Theorem 5.3.1.

5.7 Formal differential equations

Assume that K is trivially valued. This is a somehow degenerate situation since the punctured disk $D^-_K(0,1) - \{0\}$ coincides with the open segment $]0, x_{0,1}[$.

Remark 5.7.1. *Let $K((T))$ be the field of formal power series with coefficients in K. If $f = \sum_{i\geqslant n} a_i T^i \in K((T))$, then $|f|_{0,\rho} = \sup_{i\geqslant n} |a_i|\rho^i$, and $|a_i|$ is either equal to 0 or 1. Hence for all $0 < \rho < 1$ one has $|f|_{0,\rho} = \rho^{v_T(f)}$, where $v_T(f) = \min_{a_i \neq 0}\{i\}$ is the T-adic valuation of f.*

The Remark shows that we have equalities of rings

$$K((T)) \;=\; \left\{\begin{array}{ll} \mathscr{H}(x_{0,\rho}) & \text{for all } \rho \in]0,1[; \\ \mathscr{O}(C) & \text{for all annulus } C \subseteq D^-_K(0,1) \text{ centered at } 0; \\ \mathfrak{R} := \bigcup_{\varepsilon>0} \mathscr{O}(C_\varepsilon) & \text{where } C_\varepsilon := C^-_K(0; 1-\varepsilon, 1) \text{ (this is the Robba ring).} \end{array}\right. \tag{5.30}$$

A differential module M over $K((T))$ then have 3 kind of decompositions:

i) If M is viewed as a module over $\mathscr{H}(x_{0,\rho})$, one has the Robba's decomposition (3.32);
ii) If M is viewed as a module over $K((T))$, one has the decomposition [DMR07, p. 97-107] below by the slopes of its formal Newton polygon of B. Malgrange and J. P. Ramis (cf. [Ram78]);
iii) If M is viewed as a module over the Robba's ring $\mathfrak{R}$, one has the decomposition by the slopes of Christol-Mebkhout of section 5.5.1 (in fact we prove that such a module is always solvable);

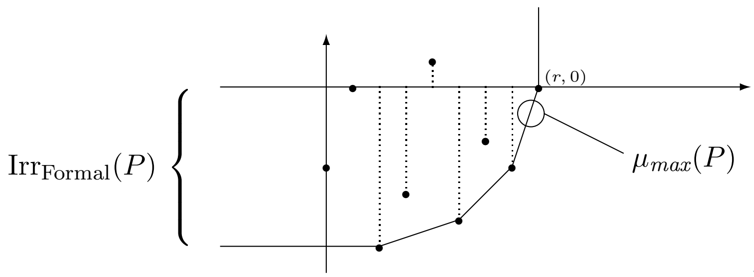
Definition 5.7.2 (Formal Newton polygon)**.** *Let $P := \sum_{k=0}^n g_k(T)(\frac{d}{dT})^k \in K((T))\langle \frac{d}{dT}\rangle$, with $g_n =$*

1, *be a differential operator corresponding to a cyclic basis of* M. *For* $(u,v) \in \mathbb{R}^2$ *let* $\mathcal{Q}(u,v) = \{(x,y) \in \mathrm{R}^2 \mid x \leqslant u, y \geqslant v\}$. *Then the* formal Newton polygon *is the convex hull of the family of quadrants* $\{\mathcal{Q}(k, v_T(g_k)+n-k)\}_{k=0,\dots,n}$. *The numbers*

$$\mathrm{Irr}_{\mathrm{Formal}}(P) := \max_{0\leqslant k\leqslant n}\{k - v_T(g_k)\} - (n - v_T(g_n)) \,, \tag{5.31}$$

$$\mu_{\max}(P) := \max\Big(0 \,,\, \max_{k=1,\dots,n} \frac{v_T(g_k)}{k-n} - 1 \Big) \,. \tag{5.32}$$

is called the Formal irregularity and the Poincaré-Katz rank of M *respectively. These are height and the largest slope of the formal Newton polygon respectively:*



. (5.33)

The formal Newton polygon is independent on the chosen cyclic basis of M.

In order to be coherent with the rest of the paper, by convention we say that the formal Newton polygon has r slopes $\mu_1 \leqslant \cdots \leqslant \mu_r$ defined as $\mu_i := h(i) - h(i-1)$, where $h : [-\infty, r] \to \mathbb{R}$ is the function whose epigraph is the formal Newton polygon.

The module M is said pure of formal slope μ if $\mu_1 = \cdots = \mu_r = \mu$.

Theorem 5.7.3 ([DMR07, p. 97-107])**.** *Any solvable differential module over* $K((T))$ *admits a direct sum decomposition*

$$\mathrm{M} := \bigoplus_{\mu\geqslant 0} \mathrm{M}(\mu) \tag{5.34}$$

into sub-modules $\mathrm{M}(\mu)$ *that are purely of slope* μ. □

In this section we prove the following:

Proposition 5.7.4. *The above* 3 *decompositions of* M *coincide. More precisely let* S *be a weak triangulation of* $D^-_K(0,1) - \{0\}$.[11] *Let* $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_r$ *be the slopes of the formal Newton polygon of* M. *Then*

i) *For all* $i = 1,\dots,r = \mathrm{rank}(\mathrm{M})$ *and all* $\rho \in]0,1[$ *one has*

$$\mathcal{R}_{S,i}(x_{0,\rho}, \mathrm{M}) = \rho^{\mu_{r-i+1}} \,. \tag{5.35}$$

In particular M *viewed as a differential module over* $\mathfrak{R}$ *is solvable following the terminology of Christol-Mebkhout (cf. Section 5.5.1).*

ii) *One has* $\Gamma_{S,i}(\mathrm{M}) = \Gamma_S =]0, x_{0,1}[= D^-_K(0,1) - \{0\}$. *The radii of* M *are separated over* $D^-_K(0,1) - \{0\}$, *and there exists a global decomposition by the radii over* $K((T)) = \mathscr{O}(D^-_K(0,1) - \{0\})$.

iii) *The formal slopes* $\mu_1 \leqslant \cdots \leqslant \mu_r$ *coincide with the slopes* $\beta_1 \leqslant \cdots \leqslant \beta_r$ *of Christol-Mebkhout.*

[11] In this degenerate situation we automatically have $\Gamma_S =]0, x_{0,1}[$, and S is a sequence of points in $]0, x_{0,1}[$ whose limit is 0.

In particular one has

$$\partial_b \mathcal{R}_{S,1}(x_{0,1}, M) \;=\; \mu_r = \beta_r = \mu_{max}(P) \;, \tag{5.36}$$
$$\partial_b H_{S,r}(x_{0,1}, M) \;=\; \mathrm{Irr}_{Formal}(P) \;, \tag{5.37}$$

where $H_{S,r} = \prod_{i=1}^{n} \mathcal{R}_{S,i}(x, \mathrm{M})$ is the highest partial height of $\mathscr{F}$ (cf. (2.13)), and $b =]x_{0,1-\varepsilon}, x_{0,1}[$ is a germ of segment oriented as out of $x_{0,1}$.

We sum up these facts by saying that the formal Newton polygon equals the Christol-Mebkhout Newton polygon, and it coincides with the derivative *of the convergence Newton polygon.*

Proof. Decomposing the module with respect to all decomposition results Cor. 3.6.9, Cor. 5.5.2, and Thm. 5.7.3 we can assume that the three newton polygons of M all have an individual slope with multiplicity rank(M). The statement then follows from (5.35) for $i = 1$. Now with the notations of (2.50) one has

$$\mathcal{R}_{S,1}(x_{0,\rho}, \mathrm{M}) \;=\; \min(1, \mathcal{R}^Y(x_{0,\rho})/\rho) \;, \tag{5.38}$$

where $\mathcal{R}^Y(x_{0,\rho}) = \liminf_n(|G_n|_{0,\rho}/|n!|)^{-1/n}$. By Remark 5.7.1, the functions $\log(\rho) \mapsto \log(|G_n|_{0,\rho})$ are all lines passing through the origin. Hence the same happens for the functions $\log(\rho) \mapsto \log \mathcal{R}^Y(x_{0,\rho})$ and $\log(\rho) \mapsto \log \mathcal{R}_{S,1}(x_{0,\rho}, \mathrm{M})$. In particular one has

$$\lim_{\rho\to 1^-} \mathcal{R}_{S,1}(x_{0,\rho}, \mathrm{M}) \;=\; 1 \;. \tag{5.39}$$

Now by [CM02, Thm.6.2] we obtain for all $\rho \in]0, 1[$

$$\mathcal{R}_{S,1}(x_{0,\rho}, \mathrm{M}) \;=\; \rho^{\mu_{\max}(P)} \;. \tag{5.40}$$

This proves the claim. □

Remark 5.7.5. A posteriori*, one sees that the decomposition result of [DMR07, p. 97-107], coincides with that of the prior paper [Rob75b] exposed in section 3.*

5.8 Notes.

The decomposition theorem 5.3.1 is not a simple consequence of Robba's and Dwork-Robba's decompositions by the spectral radii (cf. Cor. 3.6.9, and Thm. 4.1.1). Indeed the proof of Theorem 5.3.1 uses the continuity of all the radii (cf. Prop. 2.9.7), which is a consequence of the local finiteness of $\Gamma_S(\mathscr{F})$. The proof of the finiteness involves again Robba's decomposition, the results of [Ked10], and in particular another decomposition result due to Kedlaya [Ked10, 12.4.1].

The decomposition theorem [Ked10, 12.4.1] is a crucial point for both proofs of the finiteness of the controlling graphs of [Pul12] and [Ked13]. It is a kind of analogous of Prop. 5.4.7, and a posteriori the assumptions of [Ked10, 12.4.1] implies those of Prop. 5.4.7. The power of [Ked10, 12.4.1] is really that it *does not assume* that the radii are separated, but *it implies* the separateness condition. In fact theorem [Ked10, 12.4.1] is actually used in [Pul12] and [Ked13] to prove that the radii are separated and constant on certain regions (see point iii) of Remark 6.1.3, and [Ked13, Lemma 4.3.11]), this is the heart of both proofs of the finiteness result.

6. An operative description of the controlling graphs

In this section we provide a description of the controlling graphs which is useful to control the conditions of Theorems 5.3.1, 5.4.3, and 5.16.

6.1 Partial heights and their behavior.

In this section we recall some results and methods of [Pul12].

Set $\Gamma'_{S,0}(\mathscr{F}) := \Gamma_S$, and for $i = 1, \ldots, r$, set

$$\Gamma'_{S,i}(\mathscr{F}) \ := \ \Gamma_{S,1}(\mathscr{F}) \cup \cdots \cup \Gamma_{S,i}(\mathscr{F}) \ . \tag{6.1}$$

The graph $\Gamma'_{S,i}(\mathscr{F})$ is a substitute of $\Gamma_{S,i}(\mathscr{F})$. Remark that

$$\Gamma_S \ = \ \Gamma'_{S,0}(\mathscr{F}) \ \subseteq \ \Gamma'_{S,1}(\mathscr{F}) \ \subseteq \ \Gamma'_{S,2}(\mathscr{F}) \ \subseteq \ \cdots \ \subseteq \ \Gamma'_{S,r}(\mathscr{F}) \ . \tag{6.2}$$

For this reason, it is more convenient to use $\Gamma'_{S,i}(\mathscr{F})$ in proofs by induction on i.

Lemma 6.1.1 ([Pul12, Thm.4.7])**.** *Let $i \leqslant r$. Let $I \subseteq \Gamma_S$ be the skeleton of an annulus in X. Then the index i is spectral on I, and $H_{S,i}(-,\mathscr{F})$ is* log*-concave on I.* □

Lemma 6.1.2 ([Ked10, 11.3.2])**.** *Let $i \leqslant r$. Let $D \subseteq X$ be a disk such that $D \cap S = \emptyset$. Let $]x, y[$ be an open segment in D oriented towards the exterior of D. If the index i is spectral non solvable at each point of $]x, y[$, then $H_{S,i}(-,\mathscr{F})$ is decreasing on it.* □

Remark 6.1.3 (From [Pul12])**.** *The graphs $\Gamma'_{S,i}(\mathscr{F})$ satisfy the following properties.*

i) *For every $i = 0, \ldots, r$, the topological space $X - \Gamma'_{S,i}(\mathscr{F})$ is a disjoint union of virtual open disks of the form $D(y, \Gamma'_{S,i}(\mathscr{F}))$ (cf. Def. 2.2.1), where $y \notin \Gamma'_{S,i}(\mathscr{F})$ is a rigid point.*

ii) *The radii $\mathcal{R}_{S,1}(-,\mathscr{F}), \ldots, \mathcal{R}_{S,i-1}(-,\mathscr{F})$ are constant functions on the disk $D(x, \Gamma'_{S,i-1}(\mathscr{F}))$, for all $x \in X$. In particular the ratio $\mathcal{R}_{S,i}(-,\mathscr{F})/H_{S,i}(-,\mathscr{F})$ is constant on $D(x, \Gamma'_{S,i-1}(\mathscr{F}))$. This implies that the controlling graphs and the* log*-slopes of $\mathcal{R}_{S,i}(-,\mathscr{F})$ and $H_{S,i}(-,\mathscr{F})$ coincide on $D(x, \Gamma'_{S,i-1}(\mathscr{F}))$ (cf. Def. 2.4.2):*

$$\Gamma_{S,i}(\mathscr{F}) \cap (X - \Gamma'_{S,i-1}(\mathscr{F})) \ = \ \Gamma_S(H_{S,i}(-,\mathscr{F})) \cap (X - \Gamma'_{S,i-1}(\mathscr{F})) \ . \tag{6.3}$$

Hence

$$\Gamma'_{S,i}(\mathscr{F}) \ = \ \bigcup_{j=1}^{i} \Gamma_S(H_{S,j}(-,\mathscr{F})) \ . \tag{6.4}$$

iii) *Either $\mathcal{R}_{S,i-1}(-,\mathscr{F}) = \mathcal{R}_{S,i}(-,\mathscr{F})$ as functions over $D(x, \Gamma'_{S,i-1}(\mathscr{F}))$, or they are separated at each point of it by [Pul12, Prop. 7.5]. This follows from the fact that the restriction of $\mathscr{F}$ to $D := D(x, \Gamma'_{S,i-1}(\mathscr{F}))$ decomposes by [Ked10, 12.4.1] as a direct sum $\mathscr{F}_{|D} = (\mathscr{F}_{|D})_{\geqslant i} \oplus (\mathscr{F}_{|D})_{<i}$. Hence $\mathcal{R}_{S,i}(-,\mathscr{F})$ and $H_{S,i}(-,\mathscr{F})$ behave as first radii of convergence outside $\Gamma_{S,i-1}(\mathscr{F})$. So they have the concavity property of point iv) below.*

Point iii) is used in [Pul12, Section 7.4] to prove that the radii are separated over $D(x, \Gamma'_{S,i-1}(\mathscr{F}))$. As a consequence they have the following property:

iv) Let $]z, y[$ be a segment in X such that $\Gamma'_{S,i-1}(\mathscr{F}) \cap]z, y[= \emptyset$. We consider $]z, y[$ as oriented towards the exterior of the disk $D(x, \Gamma'_{S,i-1}(\mathscr{F}))$ containing it. Then the functions $\mathcal{R}_{S,i}(-,\mathscr{F})$ and $H_{S,i}(-,\mathscr{F})$ are log-concave and decreasing on $]z, y[$ (cf. [Pul12, sections 3.1,3.2]). We refer to this property by saying that $\mathcal{R}_{S,i}(-,\mathscr{F})$ and $H_{S,i}(-,\mathscr{F})$ have the concavity property *outside $\Gamma'_{S,i-1}(\mathscr{F})$.*

In particular let $D \subset X$ be a virtual disk with boundary $x \in X$, such that $D \cap \Gamma'_{S,i-1}(\mathscr{F}) = \emptyset$, and let b be the germ of segment out of x inside D. Then (cf. (1.8)):

$$b \in \Gamma_{S,i}(\mathscr{F}) \ \text{ if and only if } \ \partial_b \mathcal{R}_{S,i}(x,\mathscr{F}) \ \neq \ 0 \ . \tag{6.5}$$

Namely by iv) one has

$$\partial_b \mathcal{R}_{S,i}(x,\mathscr{F}) = 0 \ \text{ if and only if } \ \mathcal{R}_{S,i}(-,\mathscr{F}) \text{ is constant on } D \ . \tag{6.6}$$

The same is true replacing $\mathcal{R}_{S,i}$ by $H_{S,i}$ in (6.5) and (6.6).

The following lemma studies the structure of the controlling graphs in the solvable case. It is somehow a key lemma in what follows.

Lemma 6.1.4 ([Pul12, Lemma 7.7])**.** *Assume that the index $i \in \{1, \dots, r\}$ is solvable at $x \in X$, and that $x \in \Gamma_{S,i}(\mathscr{F})$. Then the following holds:*

i) *If $x \in \Gamma_{S,i}(\mathscr{F}) - \Gamma'_{S,i-1}(\mathscr{F})$, then x is an end point of $\Gamma_{S,i}(\mathscr{F})$;*

ii) *If $x \in \Gamma'_{S,i-1}(\mathscr{F})$, then $\Gamma_{S,i}(\mathscr{F}) \subseteq \Gamma'_{S,i-1}(\mathscr{F})$ around x (i.e. if $[x, y[\subseteq \Gamma_{S,i}(\mathscr{F})$, then $[x, y[\subseteq \Gamma'_{S,i-1}(\mathscr{F})$ if y is close enough to x).*

Proof. Let D be a virtual disk in $X - \Gamma'_{S,i-1}(\mathscr{F})$ with boundary $x \in \Gamma_{S,i}(\mathscr{F})$. With the notations of (6.5), solvability at x implies $\partial_b \mathcal{R}_{S,i}(x, \mathscr{F}) \leqslant 0$, while the concavity property iv) of Remark 6.1.3 implies $\partial_b \mathcal{R}_{S,i}(x, \mathscr{F}) \geqslant 0$. So $\partial_b \mathcal{R}_{S,i}(x, \mathscr{F}) = 0$ and $D \cap \Gamma_{S,i}(\mathscr{F}) = \emptyset$. □

Reasoning as in [Pul12, Remark 7.1] one proves the following characterization of $\Gamma'_{S,i}(\mathscr{F})$. It does not follow directly from from Definition 2.4.2 as explained in Remark 2.4.3.

Proposition 6.1.5 ([Pul12, Thm.4.7 iii), iv)])**.** *Let Γ_i be the union of the closed segments $[x, y] \subseteq X$ on which at least one of the partial heights $H_{S,1}(-, \mathscr{F}), \dots, H_{S,i}(-, \mathscr{F})$, or equivalently one of the radii $\mathcal{R}_{S,1}(-, \mathscr{F}), \dots, \mathcal{R}_{S,i}(-, \mathscr{F})$, is never constant on $[x, y]$. Then*

$$\Gamma'_{S,i}(\mathscr{F}) \;=\; \Gamma_S \cup \Gamma_i \;. \tag{6.7}$$

Proof. The proof is an induction on i. Namely $\mathcal{R}_{S,1}(-, \mathscr{F}) = H_{S,1}(-, \mathscr{F})$ has the concavity property iv) of Remark 6.1.3 outside Γ_S, so (6.7) holds for $i = 1$. Assume inductively that (6.7) holds for $i - 1$. Again $H_{S,i}$ has the concavity property outside $\Gamma'_{S,i-1}(\mathscr{F})$, so (6.7) holds for i. □

Remark 6.1.6. *In section 6.3 we give another, more operative, description of $\Gamma'_{S,i}(\mathscr{F})$.*

6.2 Weak super-harmonicity of partial heights

In this section, we are interested in super-harmonicity properties of the partial heights $H_{S,i}(-, \mathscr{F})$. Using Dwork-Robba's Theorem 4.1.1, we give a proof of a formula that was first stated by K. Kedlaya (see [Ked13, Thm. 5.3.6]) for spectral radii. We generalize it somewhat by taking into account solvable and over-solvable radii too.

We begin with a few definitions.

Definition 6.2.1. *Let x be a point of X and b be a branch out of x. The preimage of the branch b on $X_{\widehat{K^{\mathrm{alg}}}}$ is a finite number of branches and we define the degree of the branch b as*

$$\deg(b) \;=\; \mathrm{Card}\big(\pi^{-1}_{\widehat{K^{\mathrm{alg}}}/K}(b)\big) \;. \tag{6.8}$$

The degree of a branch may also be computed another way. Let us first recall a definition.

Definition 6.2.2 ([Duc, 3.1.1.4])**.** *For any non-empty connected open subset U of X, we denote $\mathfrak{s}(U)$ the algebraic closure of K in $\mathscr{O}(U)$. It is a finite extension of K.*

Proposition 6.2.3 ([Duc, 4.4.26])**.** *Let x be a point of X of type 2 or 3 and b be a branch out of x. For every open virtual annulus C that is a section of b and whose closure contains x, we have*

$$\deg(b) \;=\; [\mathfrak{s}(C) \colon k] \;. \tag{6.9}$$

Notation 6.2.4. *For every point x in X, we denote by $N_S(x)$ the number of branches out of x, counted with their degrees, that belong to Γ_S.*

Definition 6.2.5 (Laplacian)**.** *Let $x \in X$. Let $f\colon X \to \mathbb{R}$ be a map such that $\partial_b f(x) = 0$, for almost every branch b out of x. Set*

$$dd^c f(x) \;=\; \sum_b \deg(b) \cdot \partial_b f(x)\;, \tag{6.10}$$

where b runs through the family of branches out of x.

Let $\Gamma \subseteq X$ be a graph containing x, or, more generally, a set of branches out of x. We set

$$dd^c_{\not\subseteq\Gamma} f(x) \;:=\; \sum_{b\notin\Gamma} \deg(b) \cdot \partial_b f(x)\;, \qquad dd^c_{\subseteq\Gamma} f(x) \;:=\; \sum_{b\in\Gamma} \deg(b) \cdot \partial_b f(x)\;. \tag{6.11}$$

Remark 6.2.6. *It is actually possible to define a Laplacian operator for a much larger class of maps. See [Thu05] (briefly summarized in [PP12a, Section 3.2]) for a detailed treatment of those questions.*

In [PP12b, Section 2.2], we carefully investigated the fibers of the base- change maps $\pi_{L/K}\colon X_L \to X$, where L is a complete valued extension of K. Thanks to those results, it is possible to compute the behavior of the previous quantities after extension of scalars.

Lemma 6.2.7. *Let L be a complete valued extension of K. Let L_0 be the completion of the algebraic closure of K in L. Let $\pi^{-1}_{L_0/K}(x) := \{x_1, \dots, x_n\}$. Then for all i we have $N_{S_{L_0}}(x_i) = N_S(x)/n$, and for every point $y \in \pi^{-1}_{L/L_0}(x_i)$, we have*

$$N_{S_L}(y) \;=\; \begin{cases} 0 & \text{if} \quad y \neq \sigma_L(x_i) \\ N_{S_{L_0}}(x_i) & \text{if} \quad y = \sigma_L(x_i) \end{cases}\;. \tag{6.12}$$

Lemma 6.2.8. *Let $x \in X$. Let $f\colon X \to \mathbb{R}$ be a map such that, for almost every branch b out of x, we have $\partial_b f(x) = 0$. Let L be a complete valued extension of K. Set $f_L = f \circ \pi_{L/K}$. Then,*

i) *for every point y in $\pi^{-1}_{L/K}(x)$ and almost every branch c out of y, we have $\partial_c f_L(y) = 0$;*

ii) *With the notations of Lemma 6.2.7, we have $dd^c f_{L_0}(x_i) = dd^c f(x)/n$, and for all $y \in \pi^{-1}_{L/L_0}(x_i)$ we have*

$$dd^c f_L(y) \;=\; \begin{cases} 0 & \text{if} \quad y \neq \sigma_L(x_i) \\ dd^c f_{L_0}(x_i) & \text{if} \quad y = \sigma_L(x_i) \end{cases}\;. \qquad \square \tag{6.13}$$

We will now study the Laplacian of the partial heights. Let us begin with the points that lie on the skeleton of the curve.

Lemma 6.2.9. *Let S and S' be two weak triangulations of X. Let $x \in \Gamma_S \cap \Gamma_{S'}$. Then, for all spectral non-solvable index i,*[12] *we have*

$$dd^c \mathcal{R}_{S',i}(x, \mathscr{F}) - N_{S'}(x) \;=\; dd^c \mathcal{R}_{S,i}(x, \mathscr{F}) - N_S(x)\;. \tag{6.14}$$

Proof. We may assume that K is algebraically closed. Let b be a branch out of x. If b belongs to the complement of $\Gamma_S \cup \Gamma_{S'}$, or to $\Gamma_S \cap \Gamma_{S'}$, then we have $\partial_b \mathcal{R}_{S',i}(x, \mathscr{F}) = \partial_b \mathcal{R}_{S,i}(x, \mathscr{F})$.

Assume that b belongs to $\Gamma_{S'}$, but not to Γ_S. Remark that we are reduced to computing radii on an annulus. By [PP12b, Lemma 3.3.3, (c)], we have $\partial_b \mathcal{R}_{S',i}(x, \mathscr{F}) = \partial_b \mathcal{R}_{S,i}(x, \mathscr{F}) + 1$.

Analogously if b belongs to Γ_S, but not to $\Gamma_{S'}$, then $\partial_b \mathcal{R}_{S',i}(x, \mathscr{F}) = \partial_b \mathcal{R}_{S,i}(x, \mathscr{F}) - 1$.

[12] Note that i is spectral non-solvable with respect to S if and only if it is so with respect to S'.

The result follows by summing up all the contributions. □

Definition 6.2.10 (Vertex free of solvability)**.** *We say that $i = 1, \dots, r$ is a* vertex at x, *of the convergence Newton polygon, if $i = r$, or if $i + 1$ separates the radii at x. We say that i is a* vertex free of solvability *at x if i is a vertex at x, and if none of the indexes $j \in \{1, \dots, i\}$ is solvable at x.*

Proposition 6.2.11. *Assume that X is an analytic domain of $\mathbb{A}^{1,\mathrm{an}}_K$, and let S be a weak triangulation of X. Let $i \in \{1, \dots, r\}$. For every $x \in \Gamma_S \cap \mathrm{Int}(X)$, we have*

$$dd^c H_{S,i}(x, \mathscr{F}) \;\leqslant\; (N_S(x) - 2) \cdot \min(i, i^{\mathrm{sp}}_x)\;. \tag{6.15}$$

Moreover equality holds if i is a vertex free of solvability at x.

Proof. We may assume that K is algebraically closed. First assume that $i \leqslant i^{\mathrm{sp}}_x$. Then, for all $j \leqslant i$, $\mathcal{R}_{S,j}(x, \mathscr{F})$ is spectral non-solvable. Let $Y \subseteq X$ be an affinoid domain of A^1_K which is a neighborhood of x in X. Assume moreover that $\Gamma_S \cap Y$ is the skeleton of Y corresponding to its minimal triangulation. Then Y contains the maximal disks $D(y, S) \subseteq X$ of all its points $y \in Y$. Then the radii and their Laplacians are stable by localization to Y by Proposition 2.8.2. Hence we may assume that X is an affinoid domain of the affine line.

By Lemma 6.2.9, we may endow X with any weak triangulation, as soon as it contains x. Up to reducing X again, we may assume that x belong to its minimal triangulation S_0. With the notations of [PP12b, Section 2.4.2], and obvious generalizations, by [Pul12, Thm. 4.7], we have $dd^c H^{\mathrm{emb}}_i(x, \mathscr{F}) \leqslant 0$, and equality holds if x is a vertex free of solvability.

By [PP12b, Formula (2.4.2)], for every $j \in \{1, \dots, i\}$, we have

$$\mathcal{R}_{S_0,j}(x, \mathscr{F}) \;=\; \frac{\mathcal{R}^{\mathrm{emb}}_j(x, \mathscr{F})}{\rho_{S_0}(x)}\;. \tag{6.16}$$

The map ρ_{S_0} is constant outside Γ_{S_0}, has slope 1 on the branch of Γ_{S_0} out of x towards infinity (in $\mathbb{A}^{n,\mathrm{an}}_K$) and slope -1 on every other branch of Γ_{S_0} out of x. The result follows.

Now assume that $i > i^{\mathrm{sp}}_x$. Since $x \in \Gamma_S$, then i is solvable at x by Remark 2.6.2. In this case, we have $\mathcal{R}_{S,i}(x, \mathscr{F}) = 1$, hence $\partial_b \mathcal{R}_{S,i}(x, \mathscr{F}) \leqslant 0$ for every branch b out of x, and $dd^c \mathcal{R}_{S,i}(x, \mathscr{F}) \leqslant 0$. We deduce that the result for every index $i > i^{\mathrm{sp}}_x$ is a consequence of the result for $i = i^{\mathrm{sp}}_x$. □

6.2.1 *Generalities.* We will now extend this result to arbitrary curves. To do so, we will map the curve to the affine line by a finite étale morphism and use the previous proposition. We will need to understand how the radii of convergence change in the process. For this we recall some notions of [Duc].

Let $f : X \to Y$ be a finite flat morphism between quasi-smooth K-analytic curves. Recall that if $y \in Y$ the inverse image of a connected affinoid domain V containing y is an affinoid domain $U = f^{-1}(V)$ of X such that the algebra $\mathscr{O}(U)$ is a locally free $\mathscr{O}(V)$-module of finite type (cf. [Duc, 3.1.13, 3.1.14]). The local rank of $\mathscr{O}(U)$ over $\mathscr{O}(V)$ is independent on the choice of U and V, and is called the *degree* $\deg_y(f)$ *of f at* y. It coincides with the rank of the free $\mathscr{O}_{Y,y}$-module $\prod_{f(x)=y} \mathscr{O}_{X,x}$. The degree $y \mapsto \deg_y(f)$ is a locally constant function on Y, and if Y is connected it is called the *degree* $\deg(f)$ *of* f. The rank of the free $\mathscr{O}_{Y,y}$-module $\mathscr{O}_{X,x}$ is called *the degree* $\deg^x(f)$ *of f at x*. By [PP12b, Lemma 3.2.1], if $\mathscr{O}_{X,x}$ is a field, then $\deg^x(f) = [\mathscr{H}(x) : \mathscr{H}(y)]$.

Notation 6.2.12. *Let x be a point of X of type 2. If K is algebraically closed, then the residue field $\widetilde{\mathscr{H}(x)}$ of $\mathscr{H}(x)$ is the function field of a unique projective smooth connected curve $\mathcal{C}_x$ over $\widetilde{K}$. Moreover if $x \in \mathrm{Int}(X)$, there is a canonical bijection between the set of branches out of x and the set of rational points of $\mathcal{C}_x$ (cf. [Duc, 4.2.11.1]). In the sequel we identify these two sets.*

Let now $f : X \to Y$ be an étale morphism between quasi-smooth K-analytic curves. Let $x \in \mathrm{Int}(X)$ be a point of type 2, and let $y := f(x)$. If K is algebraically closed, the following properties hold:

i) $[\mathscr{H}(x) : \mathscr{H}(y)] = [\widetilde{\mathscr{H}(x)} : \widetilde{\mathscr{H}(y)}]$ (cf. [Duc, 4.3.15]);

ii) The map f induces a morphism $\mathcal{C}_x \to \mathcal{C}_y$ of algebraic curves over $\widetilde{K}$, associating to each branch b out of x the branch $f(b)$ out of y;

iii) Reciprocally if $\mathcal{C}_x \to \mathcal{C}_y$ is a morphism, then we may lift it into a morphism $\varphi\colon Z \to W$, where Z and W are affinoid neighborhoods of x in X and of y in Y respectively;

iv) There exist open annuli C_b and $C_{f(b)}$, that are sections of b and $f(b)$ respectively, such that
   (a) f induces an étale morphism $C_b \to C_{f(b)}$. We denote by d_b or $\deg(f_{|b})$ its degree;
   (b) the inverse image in C_b of the skeleton $I_{f(b)}$ of $C_{f(b)}$ is the skeleton I_b of C_b, and for all point $z \in I_b$ one has $[\mathscr{H}(z) : \mathscr{H}(f(z))] = d_b$.

v) If $Q \in \mathcal{C}_x$ is the point corresponding to b, then the degree d_b coincides with the ramification index e_Q of $\mathcal{C}_x \to \mathcal{C}_y$ at Q (cf. [Duc, 4.3.15]).

vi) For all branch c out of $y = f(x)$, we have

$$\deg^x(f) \;=\; [\mathscr{H}(x) : \mathscr{H}(y)] \;=\; \sum_{f(b)=c} d_b \,. \tag{6.17}$$

Lemma 6.2.13 ([PP12b, Lemma 3.4.2])**.** *Assume that K is algebraically closed. Let Y and Z be quasi-smooth K-analytic curves with weak triangulations S and T respectively. Let $f\colon Y \to Z$ be a finite étale morphism. Let $y \in \Gamma_S \cap f^{-1}(\Gamma_T)$. If $d_y = [\mathscr{H}(y) \colon \mathscr{H}(f(y))]$ is prime to p, then for all $i \in \{1,\dots,r\}$ and all $j \in \{1,\dots,d_y\}$, we have $\mathcal{R}_{T,d_y(i-1)+j}(f(y), f_*\mathscr{F}) = \mathcal{R}_{S,i}(y,\mathscr{F})$:*

$$\boldsymbol{\mathcal{R}}_T(f(y), f_*\mathscr{F}) \;=\; \big(\underbrace{\mathcal{R}_{S,1}(y,\mathscr{F}),\dots,\mathcal{R}_{S,1}(y,\mathscr{F})}_{d_y\ \textit{times}}\,,\ \dots\ ,\ \underbrace{\mathcal{R}_{S,r}(y,\mathscr{F}),\dots,\mathcal{R}_{S,r}(y,\mathscr{F})}_{d_y\ \textit{times}}\big)\,. \quad \square \tag{6.18}$$

Corollary 6.2.14. *Assume that K is algebraically closed. Let Y and Z be quasi- smooth K-analytic curves with weak triangulations S and T respectively. Let $f\colon Y \to Z$ be a finite étale morphism. Let $y \in \Gamma_S \cap f^{-1}(\Gamma_T)$. Let b be a branch out of y such that the branches b and $f(b)$ belong to Γ_S and Γ_T respectively. There exists sections C_b and $C_{f(b)}$ of b and $f(b)$ that are open annuli such that f induces a finite étale morphism $f_b\colon C_b \to C_{f(b)}$ of degree $d_b = \deg(f_{|b})$. Endow $C_{f(b)}$ with the empty weak triangulation. For $z \in C_{f(b)}$, denote by $\partial_{[z,f(y)[}\mathcal{R}_{\emptyset,i}(-,(f_{|b})_*(\mathscr{F}_{|C_b}))$ the slope of $\mathcal{R}_{\emptyset,i}(-,(f_{|b})_*(\mathscr{F}_{|C_b}))$ on $[z,f(y)[$. If d_b is prime to p, then the limit when z tends to $f(y)$ of the tuple of slopes $\{\partial_{[z,f(y)[}\mathcal{R}_{\emptyset,i}(z,(f_{|b})_*(\mathscr{F}_{|C_b}))\}_{i=1,\cdots,r\cdot d_b}$ is*

$$\Big(\underbrace{\frac{1}{d_b}\,\partial_b\mathcal{R}_{S,1}(y,\mathscr{F}),\dots,\frac{1}{d_b}\,\partial_b\mathcal{R}_{S,1}(y,\mathscr{F})}_{d_b\ \textit{times}}\,,\ \dots\ ,\ \underbrace{\frac{1}{d_b}\,\partial_b\mathcal{R}_{S,r}(y,\mathscr{F}),\dots,\frac{1}{d_b}\,\partial_b\mathcal{R}_{S,r}(y,\mathscr{F})}_{d_b\ \textit{times}}\Big)\,. \tag{6.19}$$

Proof. Endow C_b with the empty weak triangulation. With obvious notations, the last tuple is the limit when $z' \in C_b$ tends to y of the same tuple with $\partial_b\mathcal{R}_{S,i}(y,\mathscr{F})$ replaced by $\sigma_{[z',y[}\mathcal{R}_{\emptyset,i}(-,\mathscr{F}_{|C_b})$. The result now follows from Lemma 6.2.13 and [Duc, Thm 4.4.33] in order to take into account the dilatation of distances induced by the finite map f. $\square$

Corollary 6.2.15. *Assume that K is algebraically closed. Let Y and Z be quasi-smooth K-analytic curves with weak triangulations S and T respectively. Let $f\colon Y \to Z$ be a finite étale morphism. Let $y \in \Gamma_S \cap f^{-1}(\Gamma_T)$ such that $f^{-1}(f(y)) = \{y\}$. Let c be a branch out of $f(y)$ that belongs to Γ_T. Assume that every branch over b belongs to Γ_S and that all the degrees d_b are prime to p. Then we* *have*

$$\partial_b H_{S,r}(y,\mathscr{F}) \;=\; \partial_{f(b)} H_{S,r\cdot\deg(f)}(f(y),f_*\mathscr{F}) \;. \tag{6.20}$$

Proof. Let $b_1,\dots,b_t$ be the branches out of y over c. There exists a section C_c of c and sections C_i's of the b_i's such that f induces finite étale morphisms $f_i\colon C_i \to C_c$ of degree d_{b_i}. We may assume that all those sections are open annuli. Let U denote the union of the C_i's. We have $f_*(\mathscr{F}_{|U}) = \bigoplus_{1\leqslant i\leqslant t}(f_i)_*(\mathscr{F}_{|C_i})$ and the result now follows from the previous corollary. □

We need to find conditions ensuring that we can apply the previous results.

Notation 6.2.16. *Let x be a point of X of type 2. If K is algebraically closed, we denote by $g(x)$ the genus of the curve $\mathcal{C}_x$. In general, we set*

$$g(x) \;:=\; \sum_{y\in\pi^{-1}_{\widehat{K^{\mathrm{alg}}}/K}(x)} g(y) \;. \tag{6.21}$$

Let x be a point of X of type 3. We set $g(x)=0$.

Definition 6.2.17. *Let x be a point of X of type 2. If K is algebraically closed, we say that the point x satisfies the condition (TR) if there exists a finite morphism*

$$f \;:\; \mathcal{C}_x \longrightarrow \mathbb{P}^1_{\widetilde{K}} \tag{6.22}$$

that is tamely ramified everywhere and unramified almost everywhere, i.e. the degree of f is prime to the characteristic of $\widetilde{K}$ at a finite number of closed point of $\mathcal{C}_x$ and equal to 1 at every other.

In general, we say that the point x satisfies the condition (TR) if one of its inverse image of x on $X_{\widehat{K^{\mathrm{alg}}}}$ satisfies the condition (TR) (by Galois action all other inverse image of x satisfies (TR)).

We say that X satisfies the condition (TR) if, for every point x of type 2 of X, the curve $\mathcal{C}_x$ satisfies the condition (TR).

Proposition 6.2.18. *Let x be a point of X of type 2. If $g(x)=0$ or if $\mathrm{char}(\widetilde{K})\neq 2$, then x satisfies the condition (TR).*

In particular, if the curve $X_{\widehat{K^{\mathrm{alg}}}}$ contains no type 2 points of positive genus (for instance, if X may be embedded in $\mathbb{P}^{1,\mathrm{an}}_K$, a Tate curve or, more generally, a Mumford curve) or if $\mathrm{char}(\widetilde{K})\neq 2$, then X satisfies the condition (TR).

Proof. We may assume that K is algebraically closed. If $g(x)=0$, then $\mathcal{C}_x$ is isomorphic to $\mathbb{P}^1_{\widetilde{K}}$ and the result is obvious.

If $\mathrm{char}(\widetilde{K})\neq 2$, then the result follows from [Ful69, Prop. 8.1]. □

Proposition 6.2.19. *Let x be a point in $\Gamma_S\cap\mathrm{Int}(X)$ of type 2 that satisfies the condition (TR). Assume that all the radii are spectral non-solvable at x. Then, we have*

$$dd^cH_{S,r}(x,\mathscr{F}) \;=\; (2g(x)-2+N_S(x))\cdot r \;. \tag{6.23}$$

Proof. We may assume that K is algebraically closed. By assumption, there exists a finite morphism $f\colon \mathcal{C}_x \to \mathbb{P}^1_{\widetilde{K}}$ that is tamely ramified everywhere, and unramified almost everywhere. We may lift it to a morphism $\varphi\colon Y\to W$, where Y is an affinoid neighborhood of x in X and W an affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_K$. By restricting Y, we may assume that $\varphi^{-1}(\varphi(x))=\{x\}$. By [Duc, Thm. 4.3.15], the degree of the restriction of φ to any branch of Y out of x is prime to p and it is equal to 1 for almost all of them.

By Lemma 6.2.9, the result does not depend on the chosen triangulation on X. We may endow Y and W with triangulations S and T respectively such that

i) $\varphi^{-1}(\Gamma_T) = \Gamma_S$,

ii) Γ_S contains all the branches b out of x at which the degree of the restriction of φ is not 1,

iii) the radii are locally constant outside Γ_S and Γ_T.

By applying Corollary 6.2.15 to every branch of Γ_S out of x, and by (6.17), we show that

$$dd^c H_{S,r}(x, \mathscr{F}) \;=\; dd^c H_{T,dr}(\varphi(x), \varphi_*\mathscr{F}), \tag{6.24}$$

where $d = [\mathscr{H}(x) : \mathscr{H}(\varphi(x))]$ is the degree of φ at x.

Moreover, by Proposition 6.2.11, we have

$$dd^c H_{T,dr}(\varphi(x), \varphi_*\mathscr{F}) \;=\; (N_T(\varphi(x)) - 2) \cdot d \cdot r. \tag{6.25}$$

Denote by $\Gamma_S(x)$ the set of branches of Γ_S out of x. We have

$$N_T(\varphi(x)) \cdot d \;=\; \sum_{b \in \Gamma_S(x)} \deg(\varphi_{|b}) = N_S(x) + \sum_{b \in \Gamma_S(x)} (\deg(\varphi_{|b}) - 1). \tag{6.26}$$

Recall that we chose Γ_S in order that it contains every branch of $\Gamma_S(x)$ where the degree of φ is not 1. Hence, by Riemann-Hurwitz formula (cf. [Har77, Cor. 2.4]), we have

$$\sum_{b \in \Gamma_S(x)} (\deg(\varphi_{|b}) - 1) \;=\; \sum_{c \in \mathcal{C}_x(\widetilde{K})} (e_c(\varphi) - 1) = 2d + 2g(x) - 2, \tag{6.27}$$

where $e_c(\varphi)$ denotes the ramification index of φ at c. The result follows. □

Proposition 6.2.20. *Let x be a point in $\Gamma_S \cap \mathrm{Int}(X)$. If it is of type 2, assume that it satisfies the condition (TR). Then, for every $i \in \{1, \dots, r\}$, we have*

$$dd^c H_{S,i}(x, \mathscr{F}) \;\leqslant\; (2g(x) - 2 + N_S(x)) \cdot \min(i, i_x^{\mathrm{sp}}). \tag{6.28}$$

Moreover equality holds if i is a vertex free of solvability at x.

Proof. If x has type 3, then it has a neighborhood that is isomorphic to an annulus, and the result follows from Proposition 6.2.11.

Assume that x has type 2. As in the proof of Proposition 6.2.11, the result for $i > i_x^{\mathrm{sp}}$ follows from that for $i = i_x^{\mathrm{sp}}$.

Assume that $i \leqslant i_x^{\mathrm{sp}}$, *i.e.* the radius $\mathcal{R}_{S,i}(x, \mathscr{F})$ is spectral non-solvable. As in the proof of Proposition 6.2.11, this allows us to localize in the neighborhood of x. Recall that, by Dwork-Robba's Theorem 4.1.1, the differential module $\mathscr{F}_x$ may be written as a direct sum $\mathscr{F}_{1,x} \oplus \cdots \oplus \mathscr{F}_{s,x} \oplus \mathscr{F}_{s+1,x}$ where, for every $k \in \{1, \dots, s\}$, the radii of $\mathscr{F}_{k,x}$ at x are spectral non-solvable and equal, and the radii of $\mathscr{F}_{s+1,x}$ at x are all equal to 1. It is easy to see that the result for $\mathscr{F}_x$ follows from the result for the $\mathscr{F}_{k,x}$'s with $k \leqslant s$. Replacing $\mathscr{F}_x$ by one of those $\mathscr{F}_{k,x}$'s, we may assume that all the $\mathcal{R}_{S,i}(x, \mathscr{F})$'s are equal.

Let $i \leqslant j \in \{1, \dots, r\}$. Since $\mathcal{R}_{S,i}(x, \mathscr{F}) = \mathcal{R}_{S,j}(x, \mathscr{F})$, for every branch b out of x, we have $\partial_b \mathcal{R}_{S,i}(x, \mathscr{F}) \leqslant \partial_b \mathcal{R}_{S,j}(x, \mathscr{F})$, hence $dd^c \mathcal{R}_{S,i}(x, \mathscr{F}) \leqslant dd^c \mathcal{R}_{S,j}(x, \mathscr{F})$. Now, consider the polygon whose vertices are the points $(i, dd^c H_{S,i}(x, \mathscr{F}))$ with $i \in \{1, \dots, r\}$. The previous inequalities show that it is convex, so we have $dd^c H_{S,i}(x, \mathscr{F})/i \leqslant dd^c H_{S,r}(x, \mathscr{F})/r$. By Proposition 6.2.19, we have $dd^c H_{S,r}(x, \mathscr{F})/r \leqslant 2g(x) - 2 + N_S(x)$ and the result follows. □

We will now consider points outside the skeleton of the curve. We will be interested in super-harmonicity properties of the partial heights. We will use a notation for the exceptional set where this property does not hold.

Definition 6.2.21. *For every $i \in \{1, \dots, r\}$, set*

$$\mathscr{E}_{S,i}(\mathscr{F}) \ := \ \{x \in X \mid dd^c H_{S,i}(x, \mathscr{F}) > 0\} \ . \tag{6.29}$$

In the sequel, if no confusion is possible, we write $\mathscr{E}_{S,i} := \mathscr{E}_{S,i}(\mathscr{F})$ for short.

This set has been precisely investigated by the second author in the case of the affine line (see [Pul12, Thm. 4.7, (v)]).

Definition 6.2.22. *We define inductively a sequence of locally finite sets*

$$\mathscr{C}_{S,1}(\mathscr{F}) \ \subseteq \ \dots \ \subseteq \ \mathscr{C}_{S,r}(\mathscr{F}) \ \subseteq \ X \tag{6.30}$$

as follows. Let $\aleph_1 := \emptyset$, and for $2 \leqslant i \leqslant r$ let $\aleph_i$ be the locally finite set of points $x \in X - \Gamma_S$ satisfying

i) *$\mathcal{R}_{S,i}(-, \mathscr{F})$ is solvable at x;*
ii) *x is an end point of $\Gamma_{S,i}(\mathscr{F})$;*
iii) *$x \in \Gamma'_{S,i-1}(\mathscr{F}) \cap \Gamma_{S,i}(\mathscr{F}) \cap \Gamma_S(H_{S,i}(-, \mathscr{F}))$.*[13]

Define

$$\mathscr{C}_{S,i}(\mathscr{F}) \ := \ \bigcup_{j=1,\dots,i} \aleph_j \ . \tag{6.31}$$

In the sequel if no confusion is possible we write $\mathscr{C}_{S,i} := \mathscr{C}_{S,i}(\mathscr{F})$ for short.

Remark 6.2.23. *Let $i \in \{1, \dots, r\}$. By definition one has*

$$\mathscr{C}_{S,i} \cap \Gamma_S \ = \ \emptyset \ , \qquad \mathscr{C}_{S,i} \ \subseteq \ \Gamma'_{S,i-1}(\mathscr{F}) \ . \tag{6.32}$$

The graph $\Gamma_{S,i}(\mathscr{F}) \cap \Gamma'_{S,i-1}(\mathscr{F})$ contains Γ_S and the points of $\mathscr{C}_{S,i}$ are some of its end-points. We deduce that the cardinal of $\mathscr{C}_{S,i}$ is at most the number of end-points of $\Gamma'_{S,i-1}(\mathscr{F})$ that do not belong to Γ_S.

Lemma 6.2.24. *Let $i \in \{1, \dots, r\}$. For every $x \in \mathscr{C}_{S,i}$, we have*

$$dd^c H_{S,i}(x, \mathscr{F}) \leqslant i - 1 \ . \tag{6.33}$$

Proof. Let $x \in \mathscr{C}_{S,i}$. Since $x \notin \Gamma_S$, we may assume that X is an open disk with empty weak triangulation. By definition $\mathscr{C}_{S,1} = \emptyset$, and the first radius is super-harmonic on each maximal disk of X. So the claim holds for $i = 1$.

Let $i \geqslant 2$. By definition $\mathcal{R}_{\emptyset,i}(-, \mathscr{F})$ is solvable at x. For all $j \leqslant i_x^{\mathrm{sp}}$ one has $dd^c H_{\emptyset,j}(x, \mathscr{F}) \leqslant 0$, so it is enough to prove that for all $\max(1, i_x^{\mathrm{sp}}) < j \leqslant i$ one has $dd^c \mathcal{R}_{\emptyset,j}(x, \mathscr{F}) \leqslant 1$.

If b is a branch out of x going towards the interior of the disk, then $\partial_b \mathcal{R}_{\emptyset,j}(x, \mathscr{F}) \leqslant 0$. Indeed otherwise $D_{S,j}(x, \mathscr{F})$ is a virtual disk in X containing x, so $x \notin \Gamma_{S,j}(\mathscr{F})$, and the indexes $j, j{+}1, \dots, i$ are over-solvable at x, which is absurd.

If b is the branch going towards the exterior of the disk, then $\partial_b \mathcal{R}_{\emptyset,j}(x, \mathscr{F}) \leqslant 1$. Indeed otherwise we would have as above over-solvability at x, hence a contradiction. The result follows. □

Proposition 6.2.25. *Let $i \in \{1, \dots, r\}$. For every $x \notin (S \cup \mathscr{C}_{S,i})$, we have*

$$dd^c H_{S,i}(x, \mathscr{F}) \ \leqslant \ 0. \tag{6.34}$$

In other words, we have $\mathscr{E}_{S,i} \subseteq S \cup \mathscr{C}_{S,i}$. In particular, the set $\mathscr{E}_{S,i}$ is locally finite.

Moreover equality holds in (6.34) if i is a vertex free of solvability at x (cf. Def. 6.2.10).

[13] Here $\Gamma_S(H_{S,i}(-, \mathscr{F}))$ have been defined in Def. 2.4.2.

Proof. Let $x \in X \setminus (S \cup \mathscr{C}_{S,i})$. Let us first assume that $x \in \Gamma_S$. Since $x \notin S$, we have $g(x) = 0$, which ensures that condition (TR) is satisfied, and $N_S(x) = 2$, hence the result follows from Proposition 6.2.20.

Let us now assume that $x \notin \Gamma_S$. We may also assume that K is algebraically closed. Let D be the connected component of $X \setminus \Gamma_S$ containing x. It is an open disk. Let us identify it with $D^-(0,R)$ for some $R > 0$ and endow it with the empty triangulation. Remark that, for every $j \in \{1,\dots,r\}$, the maps $\mathcal{R}_{S,j}(-,\mathscr{F})_{|D}$ and $\mathcal{R}_{\emptyset,j}(-,\mathscr{F}_{|D})$ coincide, hence $\mathscr{C}_{S,j}(\mathscr{F}) \cap X = \mathscr{C}_{\emptyset,j}(\mathscr{F}_{|D})$ (cf. definition 6.2.22). We will now consider the embedded radii in the sense of [PP12b, Section 2.4.2]. By [PP12b, Formula (2.4.2)], for every $j \in \{1,\dots,r\}$, we have

$$\mathcal{R}_{\emptyset,j}(x,\mathscr{F}_{|D}) \;=\; \frac{\mathcal{R}_j^{\mathrm{emb}}(x,\mathscr{F}_{|D})}{R}, \tag{6.35}$$

hence it is enough to prove the results for the embedded partial heights.

The $\mathscr{C}_{\emptyset,j}(\mathscr{F})$ now coincide with the $\mathscr{C}_j$'s of [Pul12, Thm. 4.7]. Namely, assume first that $\mathcal{R}_i^{\mathrm{emb}}(x,\mathscr{F}_{|D}) \in]0,R[$. Let $R' \in]\mathcal{R}_i^{\mathrm{emb}}(x,\mathscr{F}),R[$ and set $D' = D^+(0,R')$. By continuity, there exists a neighborhood V of x in D' such that, for every $j \in \{1,\dots,i\}$ and every $z \in V$, we have $\mathcal{R}_j^{\mathrm{emb}}(z,\mathscr{F}_{|D}) < R'$, and

$$\mathcal{R}_j^{\mathrm{emb}}(z,\mathscr{F}_{|D}) \;=\; \mathcal{R}_j^{\mathrm{emb}}(z,\mathscr{F}_{|D'}). \tag{6.36}$$

When restricted to D', the $\mathscr{C}_j$'s increase by the individual point $x_{0,R'} \neq x$, and we may replace D with D'. Since D' is an affinoid domain of the affine line, the result now follows from [Pul12, Thm. 4.7].

Now, assume that $\mathcal{R}_i^{\mathrm{emb}}(x,\mathscr{F}_{|D}) = R$. Then the map $\mathcal{R}_i^{\mathrm{emb}}(x,\mathscr{F}_{|D})$ is constant on D, hence $dd^c\mathcal{R}_i^{\mathrm{emb}}(x,\mathscr{F}_{|D}) = 0$. If $i = 1$, then we are done. Otherwise, the result follows from the result for $i-1$ and we may proceed by descending induction until we reach a case where we have either $i = 0$ or $\mathcal{R}_i^{\mathrm{emb}}(x,\mathscr{F}_{|D}) < R$. □

Let us now sum up the results.

Theorem 6.2.26. *Let $x \in X$. If it is of type 2, assume that it satisfies the condition (TR). Let $i \in \{1,\dots,r\}$.*

i) *If $x \in \Gamma_S \cap \mathrm{Int}(X)$, then*

$$dd^c H_{S,i}(x,\mathscr{F}) \;\leqslant\; (2g(x) - 2 + N_S(x)) \cdot \min(i, i_x^{\mathrm{sp}}). \tag{6.37}$$

ii) *If $x \notin (S \cup \mathscr{C}_{S,i})$, then*

$$dd^c H_{S,i}(x,\mathscr{F}) \;\leqslant\; 0. \tag{6.38}$$

Moreover equalities hold in (6.37) and (6.38), if i is a vertex free of solvability at x (cf. Def. 6.2.10).

6.3 An operative description of $\Gamma'_{S,i}(\mathscr{F})$

Notation 6.3.1. *Let $\Gamma \subseteq X$ be a locally finite graph.*[14] *Let $x \in \Gamma$, and let $b \notin \Gamma$ be a germ of segment out of x. We denote by $D_b \subset X$ the virtual open disk with boundary x containing b.*

Proposition 6.3.2. *Let $i \leqslant r$, let $\Gamma \subseteq X$ be a locally finite graph containing Γ_S. Let $x \in \Gamma$. The following conditions are equivalent:*

i) *One has $\Gamma'_{S,i}(\mathscr{F}) \cap D_b = \emptyset$, for all direction b out of x such that $b \notin \Gamma$ (i.e. $\Gamma'_{S,i}(\mathscr{F}) \subseteq \Gamma$ around x).*

[14] Recall that $X - \Gamma$ is a disjoint union of virtual disks, cf. 1.1.5.

ii) *For all $j = 1, \ldots, i$, and for all germs of segment b out of x such that $b \notin \Gamma$, one has*

$$\partial_b \mathcal{R}_{S,j}(x, \mathscr{F}) \;=\; 0 \qquad (\textit{resp. } \partial_b H_{S,j}(x,\mathscr{F}) \;=\; 0)\;. \tag{6.39}$$

iii) *For all $j = 1, \ldots, i$ the function $\mathcal{R}_{S,j}(-, \mathscr{F})$ (resp. $H_{S,j}(-,\mathscr{F})$) verifies*

$$dd^{c}_{\not\subseteq\Gamma} \mathcal{R}_{S,j}(x, \mathscr{F}) \;=\; 0 \qquad (\textit{resp. } dd^{c}_{\not\subseteq\Gamma} H_{S,j}(x,\mathscr{F}) \;=\; 0)\;. \tag{6.40}$$

iv) *Same as iii), replacing the equalities by $\leqslant$.*

v) *Same as ii), replacing the equalities by $\leqslant$.*

In particular these equivalent conditions holds at each $x \in \Gamma$ if, and only if, one has

$$\Gamma'_{S,i}(\mathscr{F}) \;\subseteq\; \Gamma\;. \tag{6.41}$$

Proof. We can assume K algebraically closed. Clearly i) $\Rightarrow$ ii) $\Rightarrow$ iii) $\Rightarrow$ iv), and ii) $\Rightarrow$ v) $\Rightarrow$ iv). We now prove that iv) imply i). We proceed by induction on i.

If $i = 1$, the first radius has the concavity property on each disk D' such that $D' \cap \Gamma_S = \emptyset$. So both iv) and v) imply ii) for $\mathcal{R}_{S,1}(x, \mathscr{F})$. By (6.5) one has i).

Let now $i > 0$. Assume inductively that all conditions hold for $i-1$. Firstly notice that, for all $b \notin \Gamma$, one has $\partial_b \mathcal{R}_{S,i}(x,\mathscr{F}) = \partial_b H_{S,i}(x,\mathscr{F})$ by point ii) of Remark 6.1.3. In fact by i) one has $\Gamma_{S,i-1}(\mathscr{F}) \cap D_b = \emptyset$. Now, since $\mathcal{R}_{S,i}(-,\mathscr{F})$ has the concavity property on D_b (cf. point iv) of Remark 6.1.3), then $\partial_b \mathcal{R}_{S,i}(x,\mathscr{F}) \geqslant 0$ for all $b \notin \Gamma$. Hence both v) and iv) imply ii), and by (6.5) one has $\Gamma_{S,i}(\mathscr{F}) \cap D_b = \emptyset$, for all $b \notin \Gamma$. This implies i) since $\Gamma'_{S,i}(\mathscr{F}) = \Gamma'_{S,i-1}(\mathscr{F}) \cup \Gamma_{S,i}(\mathscr{F})$. □

Lemma 6.3.3. *If $i^{\mathrm{sp}}_x = 0$ (i.e. if all the radii are solvable or over-solvable at x), the conditions of Proposition 6.3.2 are automatically fulfilled at $x \in \Gamma$.*

Proof. Let $b \notin \Gamma$ be a germ of segment out of x. Since the first radius satisfies the concavity property outside Γ_S (cf. point iv) of Remark 6.1.3). Then $\mathcal{R}_{S,1}(-,\mathscr{F})$ is constant and over-solvable at all point of D_b. This will imply the same property for $\mathcal{R}_{S,i}(-,\mathscr{F})$, for all $i \geqslant 1$. □

Combining Theorem 6.2.26, Lemma 6.1.4, and Prop. 6.3.2, one finds the following result.

Definition 6.3.4. *Let $\Gamma \subseteq X$ be a locally finite graph containing Γ_S. We denote by $B(\Gamma)$ the set of points $x \in \Gamma$ such that either $x \in \partial X$, or x does not satisfy the (TR) condition.*

Corollary 6.3.5. *Let $i \leqslant r$. Let $\Gamma \subseteq X$ be a locally finite graph containing Γ_S. Assume that the following conditions hold:*

i) *For all $x \in B(\Gamma)$, one of the equivalent conditions of Proposition 6.3.2 holds at x;*

ii) *For all $x \in \Gamma - B(\Gamma)$, either the conditions of Proposition 6.3.2 holds at x, or $\Gamma \neq \{x\}$ and for all $j = 1, \ldots, \min(i, i^{\mathrm{sp}}_x)$, one has*

$$dd^{c}_{\subseteq\Gamma} H_{S,j}(x,\mathscr{F}) \;\geqslant\; \begin{cases} 0 & \textit{if} \quad x \notin \Gamma_S \\ (2g(x) - 2 + c(x)) \cdot j & \textit{if} \quad x \in \Gamma_S\;. \end{cases} \tag{6.42}$$

Then

$$\Gamma'_{S,i}(\mathscr{F}) \;\subseteq\; \Gamma\;. \tag{6.43}$$

Proof. We can assume K algebraically closed. We fix $x \in \Gamma - B(\Gamma)$, and we prove that (6.42) implies that the equivalent conditions of Proposition 6.3.2 at x.

Assume that $i \leqslant i^{\mathrm{sp}}_x$. Then the indexes $j = 1, \ldots, i$ are all spectral-non solvable, hence $x \notin \mathscr{C}_{S,i}$. By Theorem 6.2.26, condition (6.42) implies $\sum_{b \notin \Gamma} \partial_b H_{S,j}(x,\mathscr{F}) \leqslant 0$, for all $j = 1, \ldots, i$. So we are done by iv) of Proposition 6.3.2.

Assume now that $i > i_x^{\mathrm{sp}}$. If $j \leqslant \min(i_x^{\mathrm{sp}}, i)$, we proceed as above to prove that $\Gamma'_{S,j}(\mathscr{F}) \subseteq \Gamma$ around x. So we consider $i_x^{\mathrm{sp}} < j \leqslant i$, and, inductively, we assume that for all $k = 1, \ldots, j-1$, and all $b \notin \Gamma$ one has $\partial_b \mathcal{R}_{S,k}(x, \mathscr{F}) = 0$ (i.e. $\Gamma'_{S,j-1}(\mathscr{F}) \subseteq \Gamma$ around x). If $x \notin \Gamma_{S,j}(\mathscr{F})$, then $\partial_b \mathcal{R}_{S,j}(x, \mathscr{F}) = 0$ for all b, so we are done. Assume then that $x \in \Gamma_{S,j}(\mathscr{F})$. In this case, since $j > i_x^{\mathrm{sp}}$ is solvable or over-solvable at x, Lemma 6.1.4 shows that $\partial_b \mathcal{R}_{S,j}(x, \mathscr{F}) = 0$, for all $b \notin \Gamma'_{S,j-1}(\mathscr{F}) \subseteq \Gamma$. In particular this holds if $b \notin \Gamma$, and Proposition 6.3.2 implies $\Gamma'_{S,j}(\mathscr{F}) \subseteq \Gamma$ around x. □

Remark 6.3.6 (Annuli in Γ)**.** *Let Γ be a locally finite graph containing Γ_S. Assume that $]y, z[\subset \Gamma$ is the skeleton of a virtual open annulus in X, such that no bifurcation point of Γ lie in $]y, z[$. Condition (6.42) means, in this case, that the radii that are spectral non solvable at x are all* log*-affine over an open interval in $]y, z[$ containing x, while radii that are solvable or over-solvable at x, are allowed to have a break at x. Indeed (6.42) implies convexity of $H_{S,j}(-, \mathscr{F})$ along $]y, z[$, and it is known that $H_{S,j}(-, \mathscr{F})$ is concave if j is spectral non solvable at x [Pul12, Thm. 4.7, i)].*

Corollary 6.3.7 (Annuli)**.** *Let X be an open annulus with empty triangulation. Let I be the skeleton of the annulus. Let $i \leqslant r = \mathrm{rank}(\mathscr{F})$. Assume that at each point x of I, and for all $j \in \{1, \ldots, i\}$ one of the following conditions holds:*

i) *there exists an open subinterval $J \subseteq I$ containing x such that the partial height $H_{S,j}(-, \mathscr{F})$ (cf. (2.13)) is a* log*-affine map on J (cf. section 1.1.4 or [PP12a, Def. 3.1.1])*

ii) *$\mathcal{R}_{S,j}(x, \mathscr{F})$ is solvable or over-solvable at x.*

Then

$$\Gamma_{S,j}(\mathscr{F}) \;=\; \Gamma'_{S,j}(\mathscr{F}) \;=\; I\,, \quad \textit{for all } j = 1, \ldots, i\,. \tag{6.44}$$

Proof. Apply Corollary 6.3.5 to $\Gamma = I$. □

Remark 6.3.8. *Let $1 \leqslant i_1 \leqslant i_2 \leqslant r$. Let Γ be a locally finite graph containing Γ_S. Assume that*

i) *For all germ of segment b in $\Gamma - \Gamma_S$ at least one of the radii $\mathcal{R}_{S,j}(-, \mathscr{F})$, $j \in \{1, \ldots, i_2\}$, has a non zero slope on b.*

ii) *The conditions of Corollary 6.3.5, are fulfilled for $i = i_1$.*

Then by Prop. 6.1.5 we have

$$\Gamma'_{S,i_1}(\mathscr{F}) \;\subseteq\; \Gamma \;\subseteq\; \Gamma'_{S,i_2}(\mathscr{F})\,. \tag{6.45}$$

If we replace condition i) by

i′) *For each end point x of Γ such that $x \notin \Gamma_S$, one has $D^{c}_{S,i_2}(x, \mathscr{F}) = D(x)$ (cf. section 2.6.1).*

Then by Prop. 2.6.3 one obtains

$$\Gamma'_{S,i_1}(\mathscr{F}) \;\subseteq\; \Gamma \;\subseteq\; \Gamma_{S,i_2}(\mathscr{F})\,. \tag{6.46}$$

Indeed the $X - \Gamma_{S,i_2}(\mathscr{F})$ is a disjoint union of virtual open disks. So if an end point x of Γ belongs to $\Gamma_{S,i_2}(\mathscr{F})$, the whole segment joining x to Γ_S is also included in $\Gamma_{S,i_2}(\mathscr{F})$.

7. Explicit bounds on the size of the controlling graphs

In this section, using techniques based on section 6, we provide explicit bounds on the size of the controlling graphs $\Gamma'_{S,j}(\mathscr{F})$.

In this section, we assume that the curve X satisfies the condition (TR) (see Definition 6.2.17). Recall that, by Proposition 6.2.18, this is always the case if char $\widetilde{K} \neq 2$ or if X is locally embeddable into the line.

Recall that the curve X is assumed to be connected and that r denotes the rank of the locally free sheaf $\mathscr{F}$. Let us remark that Theorem 6.2.26 gives a bound on $dd^c\mathcal{R}_{S,1}(-,\mathscr{F})$ at every point of X outside the topological boundary. In particular, if this boundary is empty, we find a bound that holds everywhere. In order for a global finiteness result to be possible, we will also need the curve to be compact. Putting this conditions together, we are led to consider smooth projective curves (cf. [Duc, Thm. 3.2.82]). In this case, we show that the controlling graph $\Gamma_{S,1}(\mathscr{F})$ may only contain at most $\max(4r(g-1),0)$ more edges than the skeleton of the curve, where g is the genus of the curve. This gives an explicit version of the finiteness result for the first radius of convergence from [PP12b] and [PP12a].

As for the higher radii, the situation is more intricate since it is not clear whether the bounds of Theorem 6.2.26 always hold. However, there are many interesting case where they do hold, and then the controlling graph $\Gamma_{S,j}(\mathscr{F})$ may only contain at most $\max(4jr(g-1),0)$ more edges than the skeleton of the curve. Let us mention that our results apply unconditionally for elliptic curves (still under condition (TR)). We deduce that every radius of convergence on such a curve is constant and that differential modules split as direct sums of modules with all radii equal.

When the result of Theorem 6.2.26 fails to hold, we also manage to compute bounds, though more complicated, on the size of the $\Gamma'_{S,j}(\mathscr{F})$'s by relying on the study of the locus where super-harmonicity fails (see Definition 6.2.21 and the results that follow).

7.1 Local and global effective estimations

Recall that we started with a triangulation S and associated a graph Γ_S to it. We did it in such a way that the set of vertices of Γ_S is S. In particular, every point of x of positive genus is a vertex of Γ_S.

Next, in Section 6.1, for every $j \in \{1,\dots,r\}$, we extended this graph to a graph $\Gamma'_{S,j}(\mathscr{F})$, which is the smallest graph containing Γ_S outside which the maps $\mathcal{R}_{S,1}(-,\mathscr{F})$, ..., $\mathcal{R}_{S,j}(-,\mathscr{F})$ are locally constant. In this section, we will work with a further refinement.

Definition 7.1.1. *Let $j \in \{1,\dots,r\}$. Let $\Gamma''_{S,j}(\mathscr{F})$ be the graph built from $\Gamma'_{S,j}(\mathscr{F})$ by adding a vertex at every break-point of one of the maps $\mathcal{R}_{S,i}(-,\mathscr{F})$, with $i \in \{1,\dots,j\}$.*

Remark 7.1.2. *As $\Gamma''_{S,j}(\mathscr{F})$ contains Γ_S as a* subgraph*, it is understood that all points of S are vertexes of $\Gamma''_{S,j}(\mathscr{F})$.*

Remark 7.1.3. *The graph $\Gamma''_{S,j}(\mathscr{F})$ may be characterized as the smallest locally finite subgraph of X, containing Γ_S, such that the maps $\mathcal{R}_{S,1}(-,\mathscr{F}),\dots,\mathcal{R}_{S,j}(-,\mathscr{F})$ are all*

i) *locally constant outside $\Gamma''_{S,j}(\mathscr{F})$;*
ii) log-*affine on every edge of $\Gamma''_{S,j}(\mathscr{F})$.*

We would like to be able to count the number of vertices and edges of graphs with some degrees taken into account.

Definition 7.1.4. *Let x be a point of X. We set*

$$\deg(x) \;=\; \mathrm{Card}(\pi_0(\pi^{-1}_{\widehat{K^{\mathrm{alg}}}}(x)))\ . \tag{7.1}$$

It is also the degree over K of the algebraic closure of K inside $\mathscr{H}(x)$.

Definition 7.1.5. *Let V be a finite subset of X. We define the weighted cardinal of V as the sum of the degrees of its points.*

Let us now adapt these definitions in the case of segments.

Definition 7.1.6. *Let J be a segment whose interior lie inside $X \setminus S$. Its interior is the skeleton of a unique open virtual annulus C. We set*

$$\deg(J) \ := \ [\mathfrak{s}(C) : K] \ = \ \mathrm{Card}(\pi_0(\pi^{-1}_{\widehat{K^{\mathrm{alg}}}}(C))) \ = \ \mathrm{Card}(\pi_0(\pi^{-1}_{\widehat{K^{\mathrm{alg}}}}(J))) \ . \tag{7.2}$$

Definition 7.1.7. *Let Γ be a finite graph whose open edges contain no points of S. We define the weighted number of edges of Γ as the sum of the degrees of its edges.*

These notions behave well with respect to extensions of scalars.

Lemma 7.1.8 ([PP12b]). *Let Ω/K be a complete valued field extension of K, and let L be the completion of the algebraic closure of K in Ω. Let $V \subseteq X$ be a finite set and let $\Gamma \subseteq X$ be a finite graph, the interiors of all of whose edges lie in $X \setminus S$. Then $V_L := \pi^{-1}_{L/K}(V)$ is a finite set and its weighted cardinal is that of V. Similarly, $\Gamma_L := \pi^{-1}_{L/K}(\Gamma)$ is a finite graph, the interiors of all its edges lie in $X_L \setminus S_L$, and its weighted number of edges equal to that of Γ.*

Moreover the projection $\pi_{\Omega/L}$ identifies $V_\Omega := \sigma_{\Omega/L}(V_L)$ with V_L, $\Gamma_\Omega := \sigma_{\Omega/L}(\Gamma_L)$ with Γ_L and does not change the weighted number of points or edges. □

Let us begin with a few easy computations.

Proposition 7.1.9. *Let $j \in \{1, \dots, r\}$. Let D be a virtual open disc inside X. Assume that $D \cap \Gamma'_{S,j-1}(\mathscr{F}) = \emptyset$. Assume that there exists a segment I approaching the boundary of D such that the map $H_{S,j}(-, \mathscr{F})$ is linear on I. Let σ be its slope, computed towards the interior of D. Then $\sigma \geqslant 0$, and it is zero if and only if $\Gamma''_{S,j}(\mathscr{F}) \cap D = \emptyset$. If $\sigma > 0$, the weighted number of edges of $\Gamma''_{S,j}(\mathscr{F}) \cap D$ is at most $\deg(I)(2r\sigma - 1)$ and its weighted number of end-points is at most $r\sigma$.*

Proof. By point iii) of Remark 6.1.3, outside $\Gamma'_{S,j-1}(\mathscr{F})$, the map $H_{S,j}(-, \mathscr{F})$ behaves like a first radius of convergence, hence we may assume that $j = 1$.

By Lemmas 6.2.8 and 7.1.8, we may assume that K is algebraically closed and spherically complete (hence $\deg(I) = 1$). Indeed if K is algebraically closed, then $\Gamma_{S_\Omega}(\mathscr{F} \widehat{\otimes}_K \Omega) \subseteq X_\Omega$ coincides with $\sigma_\Omega(\Gamma_S(\mathscr{F}))$.

Let α be a rational point of D. The map $\mathcal{R}_{S,1}(-, \mathscr{F})$ is constant in the neighborhood of α and non-increasing along the segment I_α joining α to the boundary of D. Since I_α and I coincide in the neighbourhood of the boundary of D, we have $\sigma \geqslant 0$. The same argument proves that any slope on D is non-negative (when computed towards the interior).

Moreover, the map $\mathcal{R}_{S,1}(-, \mathscr{F})$ is super-harmonic on D and its slopes are of the form m/i with $m \in \mathbb{N}$ and $1 \leqslant i \leqslant r$. In particular, σ takes its values in a well-ordered set and we may argue by induction. If $\sigma = 0$, then, by the same argument as above, the map $\mathcal{R}_{S,1}(-, \mathscr{F})$ is constant on every segment inside D, hence on D and the number of edges of $\Gamma''_{S,1}(\mathscr{F}) \cap D$ is 0, as well as its number of end-points.

Let us now assume that $\sigma > 0$ (hence $\sigma \geqslant 1/r$) and that we proved the result for every smaller value. Let us start from the boundary of D and consider the first edge J of $\Gamma''_{S,1}(\mathscr{F}) \cap D$. Let y be its endpoint. Denote by $\mathcal{C}$ the set of branches out of y. For every $c \in \mathcal{C}$, denote by σ_c the slope of $\mathcal{R}_{S,1}(-, \mathscr{F})$ out of y in the direction of c. Denote by b the direction associated to J. We have $\sigma_b = -\sigma$.

By super-harmonicity, we have $\sum_{c \neq b} \sigma_c \leqslant \sigma$. Recall that every σ_c is non-negative, hence, for every $c \neq b$, we have $\sigma_c \leqslant \sigma$. Moreover, if there exists $c \neq b$ such that $\sigma_c = \sigma$, then every other $\sigma_{c'}$

is 0 and $\mathcal{R}_{S,1}(-,\mathscr{F})$ is constant in the corresponding directions. This implies that J is not an edge of $\Gamma''_{S,1}(\mathscr{F})\cap D$ and we get a contradiction. We deduce that, for every $c\neq b$, we have $\sigma_c<\sigma$, hence $\sigma_c\leqslant\sigma-1/r$.

Let $c_1,\dots,c_s$ be the branches out of y different from b such that $\sigma_{c_i}>0$ (since a positive slope is always at least $1/r$, we actually have $s\leqslant r\sigma$). For every $i\in\{1,\dots,s\}$, let D_i be the connected component of $X\setminus\{y\}$ that lies in the direction of c_i. By induction, the number of edges of $\Gamma''_{S,1}(\mathscr{F})\cap D_i$ is at most $2r\sigma_{c_i}-1$.

If $s=0$, then $\Gamma''_{S,1}(\mathscr{F})\cap D=J$ and the result holds, since $2r\sigma-1\geqslant 1$ and $r\sigma\geqslant 1$.

If $s=1$, the total number of edges of $\Gamma''_{S,1}(\mathscr{F})\cap D$ is at most

$$1+2r\sigma_{c_1}-1\ \leqslant\ 2r(\sigma-1/r)\ \leqslant\ 2r\sigma-1 \tag{7.3}$$

and its number of end-points is $1\leqslant r\sigma$.

Finally, assume that $s\geqslant 2$. Then the total number of edges of $\Gamma''_{S,1}(\mathscr{F})\cap D$ is at most

$$1+\sum_{i=1}^{s}(2r\sigma_{c_i}-1)\ \leqslant\ 1+2r\sigma-s\ \leqslant\ 2r\sigma-1\ , \tag{7.4}$$

since we have $\sum_{i=1}^{s} r\sigma_{c_i}\leqslant r\sigma$ by super-harmonicity. Similarly, the total number of end-points of $\Gamma''_{S,1}(\mathscr{F})\cap D$ is at most

$$\sum_{i=1}^{s} r\sigma_{c_i}\ \leqslant\ r\sigma\ . \tag{7.5}$$

□

Corollary 7.1.10. *Let $x\in X$. Let $j\in\{1,\dots,r\}$. Let $\mathscr{D}_x^j$ be a set of connected components of $X\setminus\{x\}$ that are virtual open discs and do not meet $\Gamma'_{S,j-1}(\mathscr{F})$. Set $D_x^j=\Big(\bigcup_{D\in\mathscr{D}_x^j}D\Big)\cup\{x\}$. Let*

$$\sigma_x^d\ :=\ dd^c_{\subseteq D_x^j}H_{S,j}(x,\mathscr{F})\ . \tag{7.6}$$

Then $\sigma_x^d\geqslant 0$, and it is zero if and only if $\Gamma''_{S,j}(\mathscr{F})\cap D_x^j=\emptyset$. If $\sigma_x^d>0$, the weighted number of edges of $\Gamma''_{S,j}(\mathscr{F})\cap D_x^j$ is at most $2r\sigma_x^d-1$, and its weighted number of end-points is at most $r\sigma_x^d$.□

Recall that, in Definition 6.2.21, we introduced a subset $\mathscr{E}_{S,j}$ of X outside which the map $H_{S,j}(-,\mathscr{F})$ is super-harmonic. By Proposition 6.2.25, it is locally finite.

Lemma 7.1.11. *Let $j\in\{1,\dots,r\}$. Let C be a virtual open annulus of skeleton J inside X. Assume that $C\cap\Gamma'_{S,j-1}\subseteq J$ and $J\cap\mathscr{E}_{S,j}=\emptyset$. Then the map $H_{S,j}(-,\mathscr{F})$ is* log*-concave on J.*

Proof. Let $x\in J$. Let σ_- and σ_+ be the slopes of the map $H_{S,j}(-,\mathscr{F})$ at x in the two directions that belong to J. Let $\mathscr{D}_x^j$ be the set of the other directions. Every one of them corresponds to a virtual open disk with boundary x that does not meet $\Gamma'_{S,j-1}$. By Corollary 7.1.10, the sum σ_x^d of the slopes of $H_{S,j}(-,\mathscr{F})$ at x in those direction is non-negative. Since $x\notin\mathscr{E}_{S,j}$, we have $dd^cH_{S,j}(x,\mathscr{F})\leqslant 0$ and we finally deduce that

$$\deg(J)\cdot(\sigma_-+\sigma_+)\ =\ dd^cH_{S,j}(x,\mathscr{F})-\sigma_x^d\ \leqslant\ 0\ . \tag{7.7}$$

□

Proposition 7.1.12. *Let $j\in\{1,\dots,r\}$. Let C be a virtual open annulus of skeleton J inside X. Assume that $C\cap\Gamma'_{S,j-1}\subseteq J$ and $J\cap\mathscr{E}_{S,j}=\emptyset$. Assume that there exists two disjoint segments J_- and J_+ inside J approaching the two boundaries of C on which the map $H_{S,j}(-,\mathscr{F})$ is linear. Let σ_-*

and σ_+ be the respective associated slopes, computed towards the interior of C. Then $\sigma_- + \sigma_+ \geqslant 0$, and it is zero if and only if $\Gamma''_{S,j}(\mathscr{F}) \cap C = J$. If $\sigma_- + \sigma_+ > 0$, the weighted number of edges of $\Gamma''_{S,j}(\mathscr{F}) \cap C$ is at most $\deg(J)(1 + 2r(\sigma_- + \sigma_+))$ and its weighted number of end-points is at most $r(\sigma_- + \sigma_+)$.

Proof. By Lemmas 6.2.8 and 7.1.8, we may assume that K is algebraically closed. Consider the map $H_{S,j}(-,\mathscr{F})$ along J in the direction from the boundary associated to J_- to the one associated to J_+. By Lemma 7.1.11, it is concave. By assumption, its first slope is σ_- and its last slope is $-\sigma_+$. We deduce that $\sigma_- + \sigma_+ \geqslant 0$. As in the proof of Proposition 7.1.9, all the slopes of $H_{S,j}(-,\mathscr{F})$ are of the form $\pm m/i$, with $m \in \mathbb{Z}$, and $1 \leqslant i \leqslant r$, and we deduce that there are at most $r(\sigma_- + \sigma_+)$ break points on J. Let us call them $x_1, \dots, x_s$. For each i, let $\tau_i < 0$ be the difference between the slope going out and the slope going in at the point x_i. We have $\sum_{i=1}^{s} \tau_i = -(\sigma_- + \sigma_+)$.

Let $z \in J$. Denote by $dd^c_{\not\subseteq J} H_{S,j}(z,\mathscr{F})$ the sum of the slopes out of z, in the directions that do not belong to J. Since $J \cap \mathscr{E}_{S,j} = \emptyset$, we have $dd^c H_{S,j}(z,\mathscr{F}) \leqslant 0$.

If z is not a break-point, then the sum of the two slopes in the directions that belong to J is zero, hence $dd^c_{\not\subseteq J} H_{S,j}(z,\mathscr{F}) \leqslant 0$. By Corollary 7.1.10, we deduce that the number of edges of $\Gamma''_{S,j}(\mathscr{F})$ out of z outside J is 0, as well as its number of end-points.

If $z = x_i$ for some i, we find $dd^c_{\not\subseteq J} H_{S,j}(x_i,\mathscr{F}) \leqslant -\tau_i$. By Corollary 7.1.10, the number of edges of $\Gamma''_{S,j}(\mathscr{F})$ out of z outside J is at most $-2r\tau_i - 1$, and its number of end-points is at most $-2\tau_i$.

Summing up, we find that, inside J, the number of edges of $\Gamma''_{S,j}(\mathscr{F})$ is $s+1$ and, out of J, at most

$$\sum_{i=1}^{s} (-2r\tau_i - 1) \;=\; 2r(\sigma_- + \sigma_+) - s\;. \tag{7.8}$$

As for the number of end-points, it is at most

$$\sum_{i=1}^{s} -r\tau_i \;=\; r(\sigma_- + \sigma_+)\;. \tag{7.9}$$

□

Remark 7.1.13. *From the above Lemmas it is possible to derive the following simple criterion.*

Assume that the following conditions are fulfilled :

i) $\mathcal{R}_{S,1}(s,\mathscr{F}) = 1$ *for all* $s \in S$,

ii) *Let C be an open virtual annulus such that*

 (a) *C is a connected component of $X - S$.*

 (b) *The topological closure I of Γ_C in X is an open or semi-open interval (not a loop).*

 Identify Γ_C with $]0,1[$. If $I = \Gamma_C$ is open, then assume that

$$\lim_{x\to 0^+} \mathcal{R}_{S,1}(x,\mathscr{F}) \;=\; \lim_{x\to 1^-} \mathcal{R}_{S,1}(x,\mathscr{F}) \;=\; 1\;. \tag{7.10}$$

 If $I - \Gamma_C$ is a point, say $\{0\}$, then assume that

$$\lim_{x\in\Gamma_C,\, x\to 1^-} \mathcal{R}_{S,1}(x,\mathscr{F}) \;=\; 1\;. \tag{7.11}$$

Then

$$\mathcal{R}_{S,i}(x,\mathscr{F}) = 1, \quad \textit{for all } i = 1,\dots,r, \textit{ and all } x \in X\;. \tag{7.12}$$

In alternative, consider a triangulation S', i.e. a weak triangulation such that each connected component of $X - S'$ is relatively compact in X. Then (7.12) holds if $\mathcal{R}_{S',1}(x,\mathscr{F}) = 1$ for all $x \in S'$. This is a slight generalization of [Bal10, Theorem 0.1.8].

Let us put the results together.

Proposition 7.1.14. *Assume that X is compact. Let $j \in \{1, \ldots, r\}$. Let E''_{j-1} be the weighted number of edges of $\Gamma''_{S,j-1}(\mathscr{F})$. Let V''_{j-1} be the set of vertices of $\Gamma''_{S,j-1}(\mathscr{F})$. Set $\mathscr{E}''_j = \mathscr{E}_{S,j} \setminus V''_{j-1}$ and, for every $e \in \mathscr{E}''_j$, let d_e be the degree of the edge of $\Gamma''_{S,j-1}(\mathscr{F})$ containing e. Then, if $\mathscr{Q}$ is any set such that $\mathscr{E}_{S,j} \subseteq \mathscr{Q} \subseteq V''_{j-1} \cup \mathscr{E}_{S,j}$, the weighted number of edges of $\Gamma''_{S,j}(\mathscr{F})$ is at most*

$$E''_{j-1} + \sum_{e \in \mathscr{E}''_j} d_e + 2r \cdot \sum_{x \in \mathscr{Q}} dd^c H_{S,j}(x, \mathscr{F}) \;, \tag{7.13}$$

and its weighted number of end-points that are not end-points of $\Gamma''_{S,j-1}(\mathscr{F})$ is at most

$$r \cdot \sum_{x \in \mathscr{Q}} dd^c H_{S,j}(x, \mathscr{F}) \;. \tag{7.14}$$

Proof. We can assume K algebraically closed. By Proposition 6.2.25 and Definition 6.2.22, the set $\mathscr{E}''_j$ is contained in $\Gamma''_{S,j-1}(\mathscr{F})$. Let us first modify $\Gamma''_{S,j-1}(\mathscr{F})$ by adding a vertex at each point of $\mathscr{E}''_j$. The number of edges E'''_{j-1} of the resulting graph $\Gamma'''_{S,j-1}(\mathscr{F})$ is at most $E''_{j-1} + \sum_{e \in \mathscr{E}''_j} d_e$.

Let $x \in V''_{j-1} \cup \mathscr{E}''_j$. Let σ^d_x be the sum of the slopes of $H_{S,j}(-, \mathscr{F})$ at x in the directions that do not belong to $\Gamma'''_{S,j-1}(\mathscr{F})$. Let $x' \in (V''_{j-1} \cup \mathscr{E}''_j) \setminus \{x\}$ such that $[x, x']$ is an edge of $\Gamma'''_{S,j-1}(\mathscr{F})$. Denote by $\sigma_{x,x'}$ the slope of $H_{S,j}(-, \mathscr{F})$ at x in the direction of x'.

By Corollary 7.1.10, for every $x \in V''_{j-1} \cup \mathscr{E}''_j$, the number of edges of $\Gamma''_{S,j}(\mathscr{F})$ out of x outside $\Gamma'''_{S,j-1}(\mathscr{F})$ is at most $2r\sigma^d_x$. By Proposition 7.1.12, for every $x \neq x'$ in $V''_{j-1} \cup \mathscr{E}''_j$ such that $[x, x']$ is an edge of $\Gamma'''_{S,j-1}(\mathscr{F})$, the number of edges of $\Gamma''_{S,j}(\mathscr{F})$ inside the virtual annulus whose skeleton is $]x, x'[$ is at most $1 + 2r(\sigma_{x,x'} + \sigma_{x',x})$. This proves (7.13) with $\mathscr{Q} = V''_{j-1} \cup \mathscr{E}_{S,j}$. Now for all $x \in V''_{j-1} - \mathscr{E}_{S,j}$ one has $dd^c H_{S,j}(x, \mathscr{F}) \leqslant 0$. This proves (7.13) in the general case.

As regards, the number of end-points of $\Gamma''_{S,j}(\mathscr{F})$ that are not end-points of $\Gamma''_{S,j-1}(\mathscr{F})$, it is the sum of the number of end-points of the trees that grow out of each open edge and each vertex of $\Gamma''_{S,j-1}(\mathscr{F})$. The result is now proved as in the previous paragraph. □

Remark 7.1.15. *By Proposition 6.2.25 one has $\mathscr{E}_{S,j} \subseteq S \cup \mathscr{C}_{S,j}$, so*

$$\mathscr{E}''_j \;\subseteq\; \mathscr{C}_{S,j} \;, \tag{7.15}$$

because $S \subseteq V''_{j-1}$ by definition. In particular, by Remark 6.2.23, the cardinality $\mathrm{Card}(\mathscr{E}''_j) \leqslant \mathrm{Card}(\mathscr{C}_{S,j})$ is less than the number of end-points of $\Gamma''_{S,j-1}(\mathscr{F})$ (and the same results holds with degrees taken into account). This number will be bounded in Corollary 7.2.8.

7.2 The case of smooth, geometrically connected, projective curves.

We will now assume that X is a smooth geometrically connected projective curve in order to get a more concrete result. We will also assume that X satisfies the condition (TR). By Theorem 6.2.26, for every $s \in S$ and every $j \in \{1, \ldots, r\}$, we have

$$dd^c H_{S,j}(s, \mathscr{F}) \;\leqslant\; \max(2g(s) - 2 + N_S(s), 0) \cdot j \;. \tag{7.16}$$

The following lemma computes the sum of the terms at the right-hand side of the inequality.

Lemma 7.2.1. *Assume that K is algebraically closed. Assume that X is a smooth connected projective curve of genus g. We have*

$$\sum_{s \in S} (2g(s) - 2 + N_S(s)) \;=\; 2g - 2 \;. \tag{7.17}$$

Proof. Extending the base field K does not change any of the quantities above, hence we may assume that K is non-trivially valued.

Let E_S be the number of edges of Γ_S. We have

$$\sum_{s\in S} 2g(s)-2+N_S(s) \;=\; 2\Big(\sum_{s\in S} g(s)\Big)-2\mathrm{Card}(S)+2E_S\;. \tag{7.18}$$

Moreover, by [Ber90, p.82, before Theorem 4.3.1], we have

$$\sum_{s\in S} g(s)+1-\mathrm{Card}(S)+E_S \;=\; g \tag{7.19}$$

and the result follows. □

Remark 7.2.2. *The number $2g(s)-2+N_S(s)$ may only be negative if $g(s)=0$ and $N_S(s)\in\{0,1\}$.*

Assume that K is algebraically closed and that $g(s)=0$. If $N_S(s)=0$, then X is isomorphic to $\mathbb{P}^{1,\mathrm{an}}_K$. If $N_S(s)=1$, then the skeleton Γ_S contains a branch that ends at an interior point of genus 0, hence is not minimal (cf. [Duc, 5.2.2.3]).

By GAGA, differential modules on the projective line are algebraic, hence trivial. We may then concentrate on curves of positive genus.

Corollary 7.2.3. *Assume that X is a smooth geometrically connected projective curve of genus $g\geqslant 1$ that satisfies the condition (TR). Let E_S be the weighted number of edges of Γ_S. Then the weighted number of edges of $\Gamma''_{S,1}(\mathscr{F})$ is at most*

$$E_S+4r(g-1) \tag{7.20}$$

and its weighted number of end-points that are not end-points of Γ_S is at most $2r(g-1)$.

Proof. By Lemma 7.1.8, we may assume that K is algebraically closed. We may also assume that the triangulation is minimal. Recall that $\mathscr{E}_{S,1}\subseteq S\subseteq V''_0\cup\mathscr{E}_{S,1}$. By Proposition 7.1.14 applied to $\mathscr{Q}=S$, the number of edges of $\Gamma''_{S,1}(\mathscr{F})$ is at most $E_S+2r\cdot\sum_{s\in S} dd^c\mathcal{R}_{S,1}(x,\mathscr{F})$ and its number of end-points that do not belong to Γ_S is at most $r\cdot\sum_{s\in S} dd^c\mathcal{R}_{S,1}(x,\mathscr{F})$.

In order to prove the result, it is now enough to bound from above the number $\sum_{s\in S} dd^c\mathcal{R}_{S,1}(s,\mathscr{F})$. By Theorem 6.2.26 and Remark 7.2.2, for every $s\in S$, we have $dd^c\mathcal{R}_{S,1}(s,\mathscr{F})\leqslant 2g(s)-2+N_S(s)$, hence the result follows from Lemma 7.2.1. □

For higher radii, we may use the same proof in order to get a similar result under additional hypotheses. We will first handle the cases where super-harmonicity holds everywhere outside Γ_S.

Corollary 7.2.4. *Assume that X is a smooth geometrically connected projective curve of genus $g\geqslant 1$ that satisfies the condition (TR). Let $j\in\{2,\dots,r\}$. Assume that $\mathscr{E}_{S,j}\subseteq S$. Let E''_{j-1} be the weighted number of edges of $\Gamma''_{S,j-1}(\mathscr{F})$. Then the weighted number of edges of $\Gamma''_{S,j}(\mathscr{F})$ is at most $E''_{j-1}+4r(g-1)j$.*

Proof. Using (7.16), the proof is the same as that of Corollary 7.2.3. □

Corollary 7.2.5. *Assume that X is a smooth geometrically connected projective curve of genus $g\geqslant 1$ that satisfies the condition (TR). Let $j\in\{1,\dots,r\}$. Assume that $\bigcup_{1\leqslant i\leqslant j}\mathscr{E}_{S,i}\subseteq S$. Let E_S be the weighted number of edges of Γ_S. Then the weighted number of edges of $\Gamma''_{S,j}(\mathscr{F})$ is at most $E_S+2r(g-1)j(j+1)$.* □

Remark 7.2.6. *In Prop. 8.2.2, we show that the condition $\mathscr{E}_{S,i}\subseteq S$ is related to the presence of*

Liouville numbers in the equation at the points of $\mathscr{C}_{S,i}$.

We now give more general results by taking arbitrary exceptional sets $\mathscr{E}_{S,j}$ into account.

Corollary 7.2.7. *Assume that X is a smooth geometrically connected projective curve of genus $g \geqslant 1$ that satisfies the condition (TR). Let $j \in \{2, \ldots, r\}$. Let E''_{j-1} be the weighted number of edges of $\Gamma''_{S,j-1}(\mathscr{F})$, and let L''_{j-1} be its number of end-points that are not end-points of Γ_S. Then the weighted number of edges of $\Gamma''_{S,j}(\mathscr{F})$ is at most*

$$E''_{j-1} + L''_{j-1} + 4r(g-1)j + 2r(j-1)L''_{j-1} \tag{7.21}$$

and its weighted number of end-points that do not belong to $\Gamma''_{S,j-1}$ is at most

$$2r(g-1)j + r(j-1)L''_{j-1} \, . \tag{7.22}$$

In particular, if $j \leqslant r-1$, the weighted cardinal of $\mathscr{C}_{S,j+1}$ is at most

$$2r(g-1)j + (r(j-1)+1)L''_{j-1} \, . \tag{7.23}$$

Proof. We may assume that K is algebraically closed. Remark 6.2.23 implies that the cardinal of $\mathscr{C}_{S,j}$ is at most L''_{j-1}. Set $\mathscr{Q} = S \cup \mathscr{E}_{S,j}$, hence $\mathscr{E}_{S,j} \subseteq \mathscr{Q} \subseteq V''_{j-1} \cup \mathscr{E}_{S,j}$, and $\mathscr{E}''_j \subseteq \mathscr{C}_{S,j}$ by Remark 7.1.15. By Lemma 6.2.24, for every $x \in \mathscr{C}_{S,j}$, we have $dd^c H_{S,j}(x,\mathscr{F}) \leqslant j-1$, hence

$$\sum_{x \in \mathscr{Q}} dd^c H_{S,j}(x,\mathscr{F}) \leqslant \sum_{x \in S} dd^c H_{S,j}(x,\mathscr{F}) + \sum_{x \in \mathscr{E}_{S,j} \cap \mathscr{C}_{S,j}} dd^c H_{S,j}(x,\mathscr{F}) \tag{7.24}$$

$$\leqslant (2g-2)j + (j-1)L''_{j-1}. \tag{7.25}$$

The result now follows from Proposition 7.1.14. □

Corollary 7.2.8. *Assume that X is a smooth geometrically connected projective curve of genus $g \geqslant 1$ that satisfies the condition (TR). Let E_S be the weighted number of edges of Γ_S and let L_S be its number of end-points. Define sequences $(\ell_n)_{n\geqslant 0}$ and $(e_n)_{n \geqslant 0}$ by $\ell_0 = 0$, $e_0 = E_S$ and, for every $n \geqslant 0$,*

$$\ell_{n+1} = 2r(g-1)(n+1) + (rn+1)\,\ell_n \, ; \tag{7.26}$$

$$e_{n+1} = e_n + 4r(g-1)(n+1) + (2rn+1)\,\ell_n \, . \tag{7.27}$$

Let $j \in \{1, \ldots, r\}$. Then the weighted number of edges of $\Gamma''_{S,j}(\mathscr{F})$ is at most e_j, its weighted number of end-points is at most $L_S + \ell_j$ and the weighted cardinal of $\mathscr{C}_{S,j}$ is at most ℓ_{j-1}. □

Remark 7.2.9. *One has*

$$\ell_n \; = \; O(r^{n-1}(n-1)!) \, , \qquad e_n \; = \; O(r^n n!) \, . \tag{7.28}$$

7.3 Elliptic curves.

Let X be an elliptic curve over an algebraically closed field. By [Ber90, p.82, before Thm. 4.3.1], we have

$$\sum_{s \in S} g(s) + \chi(\Gamma_S) = 1, \tag{7.29}$$

where $\chi(\Gamma_S)$ denotes the Euler-Poincaré characteristic of the skeleton.

There are two cases.

i) If $\chi(\Gamma_S) = 1$, then Γ_S is homotopy equivalent to a circle and X contains no points with positive genus. In this case, X is a Tate curve and it has bad reduction.

ii) If $\chi(\Gamma_S) = 0$, then there exists point $x_0 \in X$ such that $g(x_0) = 1$ and every other point has genus 0. It is easy to check that the singleton $\{x_0\}$ is a triangulation of X. In this case, X has good reduction.

Corollary 7.3.1. *Assume that X is an elliptic curve that satisfies the condition (TR).* [15] *Let $j \in \{1, \dots, r\}$. Then the map $\mathcal{R}_{S,j}(-, \mathscr{F})$ is constant on X and $\Gamma_{S,j}(\mathscr{F}) = \Gamma_S$.*

Moreover, the module with connection $(\mathscr{F}, \nabla)$ admits a direct sum decomposition

$$\mathscr{F} = \bigoplus_{\rho \in]0,1]} \mathscr{F}^{\rho} \tag{7.30}$$

with the property that, for every $\rho \in]0,1]$ such that $\mathscr{F}^{\rho} \neq 0$ and every $i \in \{1, \dots, \mathrm{rank}(\mathscr{F}^{\rho})\}$, one has $\mathcal{R}_{S,i}(-, \mathscr{F}^{\rho}) = \rho$.

Proof. As before, we may assume that K is algebraically closed and that X is endowed with a minimal triangulation S. From Corollary 7.2.3 it follows that $\Gamma''_{S,1}(\mathscr{F}) = \Gamma_S$.

Moreover, by Proposition 6.2.25 and Remark 6.2.23 we have $\mathscr{E}_{S,2} \subseteq \Gamma'_{S,1}(\mathscr{F})$ and $\mathscr{E}_{S,2} \cap \Gamma_S \subseteq S$. Hence $\mathscr{E}_{S,2} \subseteq S$. We may now prove that $\Gamma''_{S,j}(\mathscr{F}) = \Gamma_S$ for all $j \in \{1, \dots, r\}$ by induction on j by using the same arguments and Corollary 7.2.4 instead of Corollary 7.2.3.

Let $j \in \{1, \dots, r\}$. Over a non-archimedean algebraically closed field, there are two types of elliptic curves.

a) Assume first that X has good reduction. Then its skeleton Γ_S is a point. As a consequence, the map $\mathcal{R}_{S,j}(-, \mathscr{F})$ is constant on Γ_S, hence on X.
b) Assume now that X has bad reduction, *i.e.* X is a Tate curve. Then its triangulation is a singleton $S = \{\gamma\}$ and its skeleton is a circle. The graph Γ_S has one vertex: γ and one edge: Γ_S. By the definition of $\Gamma''_{S,j}(\mathscr{F})$, the map $\mathcal{R}_{S,j}(-, \mathscr{F})$ is log-affine and continuous on Γ_S and we deduce that it is constant.

The second part of the theorem now follows from Theorem 5.4.10. □

Remark 7.3.2. *The same proof applied to $\mathbb{P}^{1,\mathrm{an}}_K$, with a minimal triangulation consisting on a point, gives that any differential equation $\mathscr{F}$ on it satisfy $\mathcal{R}_{S,1}(x, \mathscr{F}) = 1$, for all $x \in X$. This confirms the fact that all differential equation over $\mathbb{P}^{1,\mathrm{an}}_K$ is trivial, because it is algebraic by GAGA.*

8. Some counterexamples.

In this section we provide the following counterexamples:

i) Non compatibility of solvable and over-solvable radii with duals (cf. section 8.4);
ii) Uncontrolled behavior of solvable and over-solvable radii by exact sequences (cf. section 8.4);
iii) An explicit example of differential module over a disk for which $\mathscr{F}_{\geqslant i}$ is not a direct summand (cf. section 8.3);
iv) Some basic relations between the Grothendieck-Ogg-Shafarevich formula and super-harmonicity of partial heights (cf. Remark 8.2.3).

All the examples involve a differential module over an open disk with empty triangulation. Indeed any possible counterexample to the above situations is reduced to this case by localizing to a maximal disk $D(x, S)$. This is because on Γ_S all the radii are spectral and are compatible with duality and

[15] If it is a Tate curve, this is automatic. Otherwise, there is one condition to check at the only point of X that has positive genus. Recall that it is always satisfied if char $\widetilde{K} \neq 2$.

spectral sequences. Concerning iv) the potential failure of super-harmonicity at some points was one of the crucial difficulties of [Pul12], we here relate this to the presence of Liouville numbers.

8.1 Setting.

Let $D := D^-(0,1)$ be the unit open disk, and let T be its coordinate. In this section all differential module will be defined over the ring $\mathscr{O}^\dagger(D) := \cup_{\varepsilon>0}\mathscr{O}(D_\varepsilon)$, where $D_\varepsilon := D^-(0,1+\varepsilon)$. Namely $\mathscr{O}^\dagger(D)$ is formed by power series $f(T) = \sum_{i\geqslant 0} a_i T^i$ satisfying $\lim_i |a_i|\rho^i = 0$, for some unspecified $\rho > 1$. The data of a differential module M over $\mathscr{O}^\dagger(D)$ is equivalent to that of a differential module M_ε over $\mathscr{O}(D_\varepsilon)$ for some unspecified $\varepsilon > 0$. The triangulation on D_ε will always be the empty one. Moreover the radii are assumed to be *all solvable or over-solvable* at the boundary $x_{0,1}$ of D. By Lemma 6.1.4, this implies M is trivial on all sub-disks of D_ε with boundary $x_{0,1}$.

All differential modules will have the property **NL** of non Liouville exponents. By definition this means that the exponents, and their differences, are not Liouville numbers (cf. [CM93], [CM97], [CM00], [CM01]). Under this condition $H^1(\mathrm{M},\mathscr{O}^\dagger(D)) := \mathrm{Coker}(\nabla : \mathrm{M} \to \mathrm{M})$ has finite dimension.

The **NL** condition is quite implicit. The effective way to ensure it is to assume either that the restriction of M to the annulus $C_\varepsilon = \{|T(x)| \in]1, 1+\varepsilon[\}$ has a Frobenius structure, or that none of the radii $\mathcal{R}_{\emptyset,i}(-,\mathrm{M})$ of M verifies the Robba condition along $b = [x_{0,1}, x_{0,1+\varepsilon}[$:

$$\mathcal{R}_{\emptyset,i}(-,\mathrm{M}) \text{ is solvable at } x_{0,1}, \text{ and } \partial_b \mathcal{R}_{\emptyset,i}(x_{0,1},\mathrm{M}) = 1 \ . \tag{8.1}$$

Remark 8.1.1. *Condition (8.1) differs from the same condition for the radii of $\mathrm{M}_{|C_\varepsilon}$. Indeed over-solvable radii over D_ε are truncated by localization to C_ε and become solvable. This corresponds to the existence of trivial submodules of M_ε that of course satisfy the condition* **NL***. So the* **NL** *condition really arises from the presence of a break of some $\mathcal{R}_{\emptyset,i}(-,\mathrm{M})$ at $x_{0,1}$, as in (8.1), before localization.*

8.2 Grothendieck-Ogg-Shafarevich formula and super-harmonicity

In this section M is a differential module over $\mathscr{O}^\dagger(D)$ of rank $r = \mathrm{rank}(\mathrm{M})$, and $b =]x_{0,1}, x_{0,1+\varepsilon}[$ is a germ of segment oriented as out of $x_{0,1}$.

Definition 8.2.1 ([CM00],[CM01])**.** *One defines the p-adic irregularity of* M *at* ∞ *as*

$$\mathrm{Irr}_\infty \mathrm{M} \ := \ -\partial_b H_{\emptyset,r}(x_{0,1},\mathrm{M}_{|C_\varepsilon}) \ = \ -\sum_{i=0}^{r} \partial_b \mathcal{R}_{\emptyset,i}(x_{0,1},\mathrm{M}_{|C_\varepsilon}) \ . \tag{8.2}$$

Over-solvable radii of M corresponds to solutions of M on some D_ε. These radii are truncated by localization to C_ε. As a result their slope remains zero, but the localization to C_ε adds -1 to the slope of each other radius (cf. Prop. 2.8.2). One finds $\partial_b H_{\emptyset,r}(x_{0,1},\mathrm{M}_{|C_\varepsilon}) = \partial_b H_{\emptyset,r}(x_{0,1},\mathrm{M}) - r + h^0(\mathrm{M})$:

$$\mathrm{Irr}_\infty(\mathrm{M}) \ = \ \mathrm{rank}(\mathrm{M}) - \partial_b H_{\emptyset,r}(x_{0,1},\mathrm{M}_\varepsilon) - h^0(\mathrm{M}) \ . \tag{8.3}$$

Assume now that M has the **NL** property, and that K is spherically complete. Then one has the Grothendieck-Ogg-Shafarevich formula (often called Euler-Poincaré formula):

$$h^0(\mathrm{M}) - h^1(\mathrm{M}) \ = \ \mathrm{rank}(\mathrm{M}) - \mathrm{Irr}_\infty(\mathrm{M}) \ . \tag{8.4}$$

By (8.3) this formula can be written as :

$$h^1(\mathrm{M}) \ = \ -\partial_b H_{\emptyset,r}(x_{0,1},\mathrm{M}_\varepsilon) \ . \tag{8.5}$$

Proposition 8.2.2. *Let M_ε be a differential module over a disk D_ε, such that $\mathcal{R}_{\emptyset,1}(-,\mathrm{M}_\varepsilon)$ is solvable or over-solvable at $x_{0,1}$, and such that M_ε has the* **NL** *property. Then for all $i = 1,\dots,r = \mathrm{rank}(\mathrm{M}_\varepsilon)$ the i-th partial height $H_{\emptyset,i}(-,\mathrm{M}_\varepsilon)$ is super-harmonic at $x_{0,1}$: $dd^c H_{\emptyset,i}(-,\mathrm{M}_\varepsilon) \leqslant 0$.*

Proof. Super-harmonicity is insensitive to scalar extensions of K. So we can assume that K is spherically complete. If $\mathcal{R}_{\emptyset,1}(-,\mathrm{M}_\varepsilon)$ is over-solvable at $x_{0,1}$, then M_ε is trivial over D_ε for some ε, and the statement is trivial. Over-solvable radii do not contribute to super-harmonicity, so by Prop. 2.9.5, we can assume that M_ε has no trivial submodules over D_ε, i.e. $\mathcal{R}_{\emptyset,i}(x_{0,1},\mathrm{M}_\varepsilon) = 1$ for all i. For small values of ε, the radii are all log-affine on $[x_{0,1}, x_{0,1+\varepsilon}[$. So Lemma 6.1.4, together with Cor. 6.3.5, and Remark 6.3.6 prove that $\Gamma_{S,i}(\mathrm{M}_\varepsilon)$ is either equal to $[x_{0,1}, x_{0,1+\varepsilon}[$, or empty if $\mathcal{R}_{\emptyset,i}(-,\mathrm{M}_\varepsilon)$ is constant. By (8.5), $H_{\emptyset,r}(-,\mathrm{M}_\varepsilon)$ is concave at $x_{0,1}$, and constant outside $[x_{0,1}, x_{0,1+\varepsilon}[$, hence $H_{\emptyset,r}(-,\mathrm{M}_\varepsilon)$ is super-harmonic on D_ε. The assertion for $H_{\emptyset,i}(-,\mathrm{M}_\varepsilon)$ is deduced from that of $H_{\emptyset,r}(-,\mathrm{M}_\varepsilon)$ by interpolation. Namely by convexity of the convergence Newton polygon one has $H_{\emptyset,i}(-,\mathrm{M}_\varepsilon) \leqslant \frac{i}{r} H_{\emptyset,r}(-,\mathrm{M}_\varepsilon)$, moreover these two functions coincide at $x_{0,1}$. This proves that $H_{\emptyset,i}(-,\mathrm{M}_\varepsilon)$ is super-harmonic. □

Remark 8.2.3. *In [Pul12, Thm.4.7] one proves that $H_{\emptyset,i}(-,\mathrm{M}_\varepsilon)$ are super-harmonic over D_ε outside the* finite *set $\mathscr{C}_i$ of Def. 6.2.22. However $H_{\emptyset,1}(-,\mathrm{M}_\varepsilon)$ is super-harmonic on the whole disk. Proposition 8.2.2 shows that if $H_{\emptyset,i}(-,\mathrm{M}_\varepsilon)$ is not super-harmonic then M_ε has some Liouville exponent, or possibly some non solvable radii in order to be outside the range of validity of G.O.S. formula.*

Remark 8.2.4. *Since $\mathrm{Irr}_\infty(\mathrm{M})$ involves spectral radii, these are stable by duality and one has $\mathrm{Irr}_\infty(\mathrm{M}) = \mathrm{Irr}_\infty(\mathrm{M}^*)$. Hence $h^0(\mathrm{M}) - h^1(\mathrm{M}) = h^0(\mathrm{M}^*) - h^1(\mathrm{M}^*)$ as soon as Grothendieck-Ogg-Shafarevich formula holds.*

8.3 An example where $\mathscr{F}_{\geqslant i}$ is not a direct summand.

The Yoneda group $\mathrm{Ext}^1(\mathrm{M},\mathrm{N})$ of extensions $0 \to \mathrm{N} \to \mathrm{P} \to \mathrm{M} \to 0$ of differential modules can be identified with $H^1(\mathrm{M}^* \otimes \mathrm{N})$ (cf. Lemma 1.2.7).

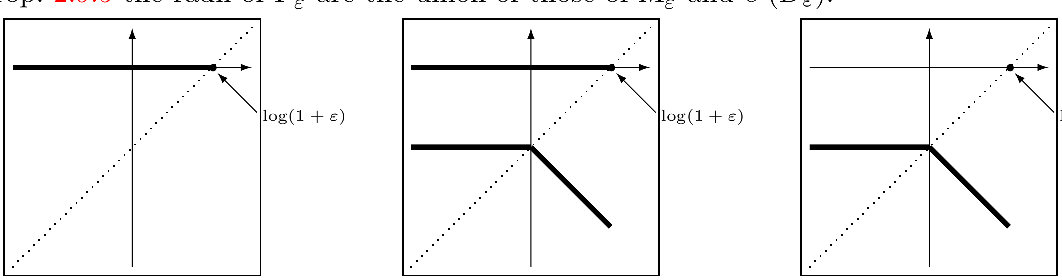
Lemma 8.3.1. *Assume that* M *has the property* **NL**. *If $\partial_b H_{\emptyset,r}(x_{0,1},\mathrm{M}_\varepsilon) \leqslant -1$ there exists a non splitting exact sequence $0 \to \mathrm{M}_\varepsilon \to \mathrm{P}_\varepsilon \to \mathscr{O}(D_\varepsilon) \to 0$.*

Proof. $h^1(\mathrm{M}) = -\partial_b H_{\emptyset,r}(x_{0,1},\mathrm{M}) \geqslant 1$. So $\mathrm{Ext}^1(\mathscr{O}^\dagger(D),\mathrm{M}) = H^1(\mathrm{M}\otimes\mathscr{O}^\dagger(D)) = H^1(\mathrm{M}) \neq 0$. □

Let M be a rank one differential module over $\mathscr{O}^\dagger(D)$ such that $\mathcal{R}_{\emptyset,1}(x_{0,1},\mathrm{M}_\varepsilon)$ is solvable, and $\mathrm{Irr}_\infty(\mathrm{M}) \geqslant 2$ (i.e. $\partial_b H_{\emptyset,r}(x_{0,1},\mathrm{M}_\varepsilon) \leqslant -1$). Such differential modules have been classified in [Pul07]. Consider the dual of the non splitting sequence of Lemma 8.3.1

$$0 \to \mathscr{O}(D_\varepsilon) \to \mathrm{P}_\varepsilon^* \to \mathrm{M}_\varepsilon^* \to 0\ . \tag{8.6}$$

By Prop. 2.9.5 the radii of P_ε^* are the union of those of M_ε^* and $\mathscr{O}(D_\varepsilon)$:



Radius of $\mathscr{O}(D_\varepsilon)$ Radii of P_ε^* Radius of M_ε^* (8.7)

The functions of the pictures are precisely

$$\widetilde{\rho} \ \mapsto\ \log(\mathcal{R}_{\emptyset,i}(x_{0,\exp(\widetilde{\rho})},\bullet))\ , \tag{8.8}$$

where ε is *unspecified*, in order to avoid complicate behavior of the radius of M_ε^*. The dotted line denotes the region of solvability. At the left hand side of this line one has over-solvable radii (always constant functions), and on its right hand side one has spectral non solvable radii. By section 6 (cf. Corollary 6.3.5) one sees that

$$\Gamma_{\emptyset,1}(\mathrm{M}_\varepsilon^*) = \Gamma_{\emptyset,1}(\mathrm{P}_\varepsilon^*) = [x_{0,1}, x_{0,1+\varepsilon}[\ , \qquad \Gamma_{\emptyset,2}(\mathrm{P}_\varepsilon^*) = \Gamma_{\emptyset,1}(\mathscr{O}(D_\varepsilon)) = \emptyset\ . \tag{8.9}$$

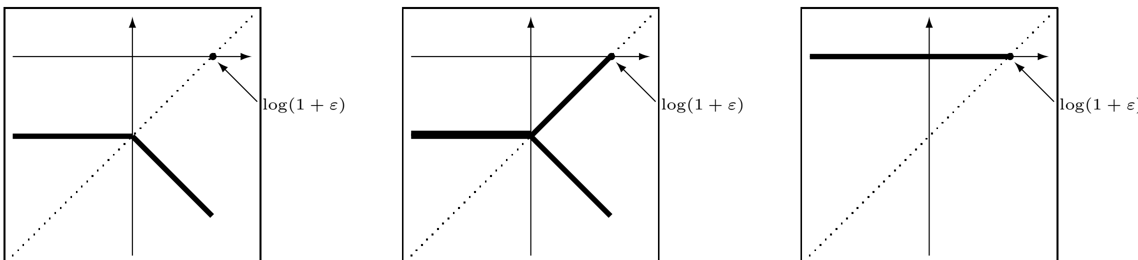
The radii of P_ε^* are then separated on the whole disk D_ε, if $\varepsilon > 0$ is small enough. In this case $\mathscr{O}(D_\varepsilon) = (\mathrm{P}_\varepsilon^*)_{\geqslant 2}$. Moreover, by construction, P_ε^* in not isomorphic to the direct sum of $\mathrm{M}_\varepsilon^* = (\mathrm{P}_\varepsilon^*)_{<2}$ and $(\mathrm{P}_\varepsilon^*)_{\geqslant 2} = \mathscr{O}(D_\varepsilon)$.

8.4 Non compatibility of solvable or over-solvable radii with duality and with exactness.

Consider now the sequence of Lemma 8.3.1

$$0 \to \mathrm{M}_\varepsilon \to \mathrm{P}_\varepsilon \to \mathscr{O}(D_\varepsilon) \to 0\ . \tag{8.10}$$

We now show that over-solvable radii of this sequence do not behave as spectral one (cf. Thm. 2.9.1). Indeed a solution on D_ε of P_ε generates a trivial submodule whose intersection with M_ε is zero because M_ε is not trivial. This is absurd since the sequence does not split. On the other hand spectral non solvable radii are stable by duality (cf. Prop. 2.9.8) so the pictures of the radii of (8.10) coincides with (8.11) under the doted line. By continuity one finds:



Radius of M_ε Radii of P_ε Radius of $\mathscr{O}(D_\varepsilon)$ (8.11)

This proves that over-solvable radii of P_ε (nor their controlling graphs) are not preserved by duality (cf. Remark 2.9.9 and Prop. 2.9.8). Indeed by section 6 (cf. Corollary 6.3.5) one sees that

$$\Gamma_{\emptyset,1}(\mathrm{M}_\varepsilon) = \Gamma_{\emptyset,1}(\mathrm{P}_\varepsilon) = \Gamma_{\emptyset,2}(\mathrm{P}_\varepsilon) = [x_{0,1}, x_{0,1+\varepsilon}[\ . \tag{8.12}$$

Remark 8.4.1. *Among all the extensions of $\mathscr{O}(D_\varepsilon)$ by M_ε^*, the unique non trivial extension P_ε^* is also the unique one for which the radii are not separated. And its controlling graphs contradicts the assumption of Theorem 5.4.10. Notice that that information is written in the radii of P_ε and P_ε^*, while the radii and the controlling graphs of M_ε, $\mathscr{O}(D_\varepsilon)$, and their duals, are stable by duality.*

Remark 8.4.2. *The differential module P_ε satisfies $\mathscr{C}_1 = \mathscr{C}_2 = \emptyset$ (cf. (6.30)). Indeed $H_{\emptyset,2}(-, P_\varepsilon)$ is a constant function on D_ε, and $\Gamma(H_{\emptyset,2}(-,\mathrm{P}_\varepsilon)) = \emptyset$ (cf. point iii) of Def. 6.2.22).*

Also P_ε^* verifies $\mathscr{C}_1 = \mathscr{C}_2 = \emptyset$, but the reason is that $\mathcal{R}_{\emptyset,2}(-,\mathrm{P}_\varepsilon^*)$ is constant and $\Gamma_{\emptyset,2}(\mathrm{P}_\varepsilon^*) = \emptyset$.

Notes.

- The global decomposition results of this paper, together with those of [Pul12] and [PP12b], are of a *pre-cohomological nature*, in the sense that they do not involve any cohomological consideration.
- The graphs $\Gamma_{S,i}(\mathscr{F})$ are essentially unknown at the present state of technology. No general algorithms are known. In rank one case there exists an algorithm due to Christol [Chr11], based on [Pul07], that actually computes explicitly the radius of a rank one equation of the form $y' = g(T)y$, with $g(T) \in K[T]$. In a work in progress we extended such an algorithm to equation over an affinoid domain Y of the affine line (i.e. $g(T) \in \mathscr{O}(X)$) to find the graphs on

a maximal disk, but not to the whole X.

Remark 8.4.3. *A result of [Ked13] proves that the controlling graphs of $\mathscr{F}$ do not contain any point of type 4. The proofs of this paper work on points of type 4, so that result of Kedlaya is not used in the present paper.*

Appendix A. A note about the definition of the radius

As already observed the radii only depends on the skeleton Γ_S, in the sense that if $\Gamma_S = \Gamma_{S'}$ then $\mathcal{R}_{S,i}(-,-) = \mathcal{R}_{S',i}(-,-)$ (cf. Remark 2.3.3).

In this section we consider a more general definition of the radii based on the idea that the datum defining the radii is a graph Γ_S instead of a triangulation. So we define the radii starting from a graph which is not necessarily the skeleton of a weak triangulation. We here show that such a point of view is not a real generalization, in the sense that the main theorems (finiteness, continuity, and decomposition) in this new context can be deduced from the same results in the framework of weak triangulations. For this reason the framework of weak triangulations seems to us the optimal one.

A.1 Definition of the radii

As a mater of fact all one needs to define the radii is the notion of maximal disks. For this we proceed as follow. Let $\mathscr{F}$ be a differential equation over X, and let $x \in X$.

A graph $\mathfrak{G} \subseteq X$ is called a *weakly admissible graph* if $X - \mathfrak{G}$ is a disjoint union of virtual open disks.[16]

For all $x \in X$ we call maximal disk the Ω-rational disk $D(x, \mathfrak{G})$ (cf. Def. 2.2.1). The disk $D(x, \mathfrak{G})$ is empty if and only if $x \in \mathfrak{G}$ and if x is a point of type 1. In this case set $\mathcal{R}_{\mathfrak{G},i}(x, \mathscr{F}) := 1$, for all $i = 1, \ldots, r$.

Otherwise, imitating section 2.3, choose an isomorphism $D(x, \mathfrak{G}) \xrightarrow{\sim} D^-_\Omega(0, R)$ sending t_x into 0. Consider the restriction $\widetilde{\mathscr{F}}$ of $\mathscr{F}$ to $D^-_\Omega(0, R)$. And then define $\mathcal{R}^{\widetilde{\mathscr{F}}}_{\mathfrak{G},i}(x)$ as the radius of the largest open disk D centered at 0, contained in $D^-_\Omega(0, R)$, such that $\widetilde{\mathscr{F}}$ has at least $r - i + 1$ linearly independent solutions on D, where $r = \mathrm{rank}(\mathscr{F})$. Now set $\mathcal{R}_{\mathfrak{G},i}(x, \mathscr{F}) := \mathcal{R}^{\widetilde{\mathscr{F}}}_{\mathfrak{G},i}(x)/R$, for all $i = 1, \ldots, r$.

We call $\mathfrak{G}$-multiradius of $\mathscr{F}$ the tuple $(\mathcal{R}_{\mathfrak{G},1}(x, \mathscr{F}), \ldots, \mathcal{R}_{\mathfrak{G},r}(x, \mathscr{F}))$.

The following definition coincides with Def. 2.4.2 replacing Γ_S with $\mathfrak{G}$.

Definition A.1.1 (cf. Def. 2.4.2)**.** *We call $\mathfrak{G}$-controlling graph, or $\mathfrak{G}$-skeletons, of $\mathcal{R}_{\mathfrak{G},i}(-, \mathscr{F})$ the set of points $x \in X$ that do not admit as a neighborhood in X a virtual open disk D, such that $D \cap \mathfrak{G} = \emptyset$, on which $\mathcal{R}_{\mathfrak{G},i}(-, \mathscr{F})$ is constant on D. We denote it by $\Gamma_{\mathfrak{G},i}(\mathscr{F})$.*

It follows by the definition that $\mathfrak{G} \subseteq \Gamma_{\mathfrak{G},i}(\mathscr{F})$.

A.2 Properties

In this section we prove that the radii $\mathcal{R}_{\mathfrak{G},i}(x, \mathscr{F})$ are subjected to analogous properties of the radii $\mathcal{R}_{S,i}(x, \mathscr{F})$ attached to a weak triangulation S. Indeed their restriction to a maximal disk $D(x, \mathfrak{G})$

[16] This generalizes a terminology of Ducros [Duc, (5.1.3)], where one defines an *analytically admissible graph* as a weakly admissible graph such that the disks that are connected components of $X - \mathfrak{G}$ are relatively compact in X. In analogy with the definition of weak triangulations, we allow the empty set of a virtual open disk to be a weakly admissible graph.

is determined by the functions $\mathcal{R}^{\widetilde{\mathscr{F}}}_{\mathfrak{G},i}(x)$ in the spirit of [Pul12]. Below, one shows how to reduce the study of the $\mathfrak{G}$-radii to the case of the radii defined by a weak triangulation.

Proposition A.2.1 ([Duc, (1.3.15.2)]). *A weakly admissible graph is a closed connected subset of X, hence it is a graph (cf. Section 1.1.5).*

Proposition A.2.2. *Let $\mathfrak{G}$ be a weakly admissible graph. Then there exists a weak triangulation S of X such that $\Gamma_S \subseteq \mathfrak{G}$.*

Proof. Let S' be an arbitrary triangulation. The intersection $\Gamma := \mathfrak{G} \cap \Gamma_{S'}$ is a locally finite graph whose complement $X - \Gamma$ is a disjoint union of virtual disks. Since Γ is contained in $\Gamma_{S'}$, then Γ is the skeleton of a weak triangulation S by [Duc, 5.2.2.3]. □

Proposition A.2.3. *Let $\Gamma_S \subseteq \mathfrak{G}$ be the skeleton of a triangulation contained in the weakly admissible graph $\mathfrak{G}$. Then for all $i = 1, \ldots, r$ one has*

$$\mathcal{R}_{\mathfrak{G},i}(x,\mathscr{F}) \;=\; \min\Big(\,1\;,\; f_{S,\mathfrak{G}}(x)\cdot\mathcal{R}_{S,i}(x,\mathscr{F})\,\Big) \tag{A.1}$$

where $f_{S,\mathfrak{G}} : X \to [1,+\infty[$ is the modulus of the inclusion of disks $D(x,\mathfrak{G}) \subseteq D(x,S)$ (cf. (1.1.2)).
□

The following Proposition A.2.4 together with Theorem 2.4.1 shows that the $\mathfrak{G}$-controlling graph $\Gamma_{\mathfrak{G},i}(\mathscr{F})$ is locally finite if and only if so $\mathfrak{G}$ is.

Proposition A.2.4. *One has $\Gamma_{\mathfrak{G},i}(\mathscr{F}) = \mathfrak{G} \cup \Gamma_{S,i}(\mathscr{F})$.*

Proof. The proof coincides with that of Proposition 2.7.2 (replacing S' with $\mathfrak{G}$). □

Lemma A.2.5. *The function $x \mapsto f_{S,\mathfrak{G}}(x)$ is continuous if and only if $\mathfrak{G}$ is a locally finite graph.* □

Theorem A.2.6. *If $\mathfrak{G}$ is a locally finite graph, the functions $\mathcal{R}_{\mathfrak{G},i}(-,\mathscr{F})$ are continuous.*

Proof. This follows directly from (A.1) and the continuity of both $\mathcal{R}_{S,i}(-,\mathscr{F})$ and $f_{S,\mathfrak{G}}$. □

Remark A.2.7. *The trivial equation $(\mathscr{O}_X, d)$ satisfies $\mathcal{R}_{\mathfrak{G},1}(x,\mathscr{F}) = 1$ for all $x \in X$ and all weakly admissible graph $\mathfrak{G}$. So it is always continuous.*

Remark A.2.8. *If $\mathfrak{G}$ is not locally finite, then there are differential equation with non continuous radii. Here we provide a basic explicit example.*

Let X be the the disk $\mathrm{D}^-_K(0,1)$ with empty weak triangulation. Denote by $\mathscr{F}_a$ the rank one differential equation over X, given by $y' = a \cdot y$, with $a \in K$. A direct computation of the Taylor expansion of its solution shows that for all $x \in X$ the function $\mathcal{R}_{\emptyset,1}(x,\mathscr{F}_a)$ is constant on X with value $R_a := \min(1, |p|^{\frac{1}{p-1}}/|a|)$. If the valuation of K is non trivial, it is then possible to construct differential equations having arbitrarily small and constant radii on a disk.

Let now $\mathfrak{G}$ be a weakly admissible graph in X. By (A.1), this shows that, for all bifurcation point $x \in \mathfrak{G}$, there exists $a \in K$, such that $\mathcal{R}_{\mathfrak{G},1}(y,\mathscr{F}_a) = f_{S,\mathfrak{G}}(y) \cdot R_a$ for all y close enough to x. This proves that $\Gamma_{\mathfrak{G},1}(\mathscr{F}_a) = \mathfrak{G}$ around x.

If $\mathfrak{G}$ is not locally finite, then, by Lemma A.2.5, the function $\mathcal{R}_{\mathfrak{G},1}(y,\mathscr{F}_a)$ is not continuous. So locally finiteness of $\mathfrak{G}$ is a necessary condition to have continuity.

Theorem A.2.9. *If $\mathcal{R}_{\mathfrak{G},i-1}(x,\mathscr{F}) < \mathcal{R}_{\mathfrak{G},i}(x,\mathscr{F})$ for all $x \in X$, then there exists a sub-object $\mathscr{F}_{\geqslant i}$ of $\mathscr{F}$ satisfying analogous properties to Thm. 5.3.1, and of section 5.4, replacing everywhere the* *index S by $\mathfrak{G}$. Moreover $\mathscr{F}_{\geqslant i}$ is independent on the choice of $\mathfrak{G}$.*

Proof. Let S be a weak triangulation such that $\Gamma_S \subseteq \mathfrak{G}$. For all $x \in X$ one has $D(x, \mathfrak{G}) \subseteq D(x, S)$. So $\omega_{S,j}(x, \mathscr{F}) = \omega_{\mathfrak{G},j}(x, \mathscr{F})$ for all $j \leqslant i$. the index i also separates the Γ_S-radii. By Theorem 5.3.1 we have the existence of a submodule $\mathscr{F}_{S,\geqslant i}$ separating the S-radii. As in Remark 5.3.4, and Prop. 5.3.3 one shows that $\mathscr{F}_{S,\geqslant i}$ also separates the $\mathfrak{G}$-radii : $\mathscr{F}_{S,\geqslant i} = \mathscr{F}_{\mathfrak{G},\geqslant i}$. □

Remark A.2.10. *According to Remark 5.4.11, to prove that $\mathscr{F}_{\geqslant i}$ is a direct factor, it is convenient to chose $\mathfrak{G}$ as small as possible. This can be done by replacing $\mathfrak{G}$ by a convenient weak triangulation S such that $\Gamma_S \subseteq \mathfrak{G}$.*

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