p-ADIC DIFFERENTIAL EQUATIONS

p-ADIC REPRESENTATIONS

AND

p-ADIC DIFFERENCE EQUATIONS

Padova,

15 March, 2007

Summary :

- **Introduction**
- p**-adic Representations**
- **(Abelian)** p**-adic differential equations**
- p**-adic difference equations**
- **Applications to** p**-adic Zeta function and to** p**-adic** L **functions**

Introduzione

Absolute values over $\mathbb{Q}$:

$$\begin{cases} |\cdot|_\infty & = & \text{Archimedean absolute value : } |\frac{m}{n}|_\infty = \max(\frac{m}{n}, -\frac{m}{n}) \ , \\ |\cdot|_p & = & p-\text{adic absolute value} \end{cases}$$

For all integer $k \in \mathbb{Z}$, $k = np^v$, $(n,p) = 1$ let

$$v_p(k) := v = \text{N}^{\text{O}} \text{ of times that } p \text{ divides } k \ .$$

Let $0 < \varepsilon_p < 1$ be an arbitrary real number, then set $|p|_p := \varepsilon_p$ and, for all $k \in \mathbb{Z}$ as above, set

$$|k|_p = |p|_p^v = \varepsilon_p^{v_p(k)} \leq 1 \ .$$

and more generally

$$\left|\frac{m}{n}\right|_p := \varepsilon_p^{v_p(m)-v_p(n)} \ .$$

Introduction

$$|\cdot|_\infty \;:\; \mathbb{Q} \;\rightsquigarrow\; \mathbb{R} = \begin{cases} \text{complete} \\ \text{connected} \\ \dim_{\mathbb{R}}\mathbb{C} = 2 \end{cases}$$

$$|\cdot|_p \;:\; \mathbb{Q} \;\rightsquigarrow\; \mathbb{Q}_p = \begin{cases} \text{complete} \\ \text{NON connected} \\ \dim_{\mathbb{Q}_p}\mathbb{C}_p = +\infty \end{cases}$$

Disks/Balls :

$$\begin{aligned} \mathrm{D}^-(a,r) &:= \{x \in \mathbb{Q}_p \mid |x-a|_p < r\} \\ \mathrm{D}^+(a,r) &:= \{x \in \mathbb{Q}_p \mid |x-a|_p \leq r\} \end{aligned}$$

Pathologies :

- If $|x-a|_p = r$, then $\mathrm{D}^-(x,r) \subset \mathrm{D}^+(a,r)$;
- We set $\mathbb{Z}_p := \mathrm{D}^+(0,1)$. One has $\mathbb{Z} \subseteq \mathbb{Z}_p$, and is dense.

p-adic representations

Let k be an algebraically closed field of characteristic p.

The object of study is

$$\mathrm{G} := \mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t))) \ .$$

• We wants to study G by classifying its *representations*, that is the groups homomorphisms

$$\rho : \mathrm{G} \longrightarrow \mathrm{GL}_n(K) \ .$$

where $K/\mathbb{Q}_p$ is a finite extension of fields.

• **Remarkable Fact :** Every finite quotient of G is solvable, more precisely

$$1 \to \mathcal{P} \to \mathrm{G} \to \mathrm{G}/\mathcal{P} \to 1 \quad , \quad \mathrm{G}/\mathcal{P} = \prod_{\ell=\mathrm{prime},\ell\neq p} \mathbb{Z}_\ell \ .$$

• $\mathcal{P}$ is a pro-p-group essentially *unknown*.

The exemple of rank one representations of rank one of $\mathcal{P}$

The theory of Artin-Schreier describes the caratters of $\mathcal{P}$:

$$0 \to \mathbb{Z}/p^n\mathbb{Z} \to \mathbf{W}_n(k(\!(t)\!)) \xrightarrow[\mathrm{F}-1]{} \mathbf{W}_n(k(\!(t)\!)) \to \mathrm{Hom}(\mathcal{P}, \mathbb{Z}/p^n\mathbb{Z}) \to 0$$

• $\mathbf{W}_n(k(\!(t)\!))$ is the *group scheme* of "Witt Vectors".
Its elements are vectors $(f_0(t), \ldots, f_n(t))$, with $f_i(t) \in k(\!(t)\!)$.

Theorem 0.1 (Pulita) *The group of characters* $\mathrm{H}^1 := \mathrm{Hom}(\mathcal{P}, \mathbb{Q}/\mathbb{Z})$ *is isomorphic to*

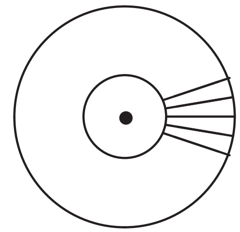
$$\mathrm{H}^1 \;\cong\; \bigoplus_{(n,p)=1} \left(\varinjlim_{m \geq 0} (\mathbf{W}_m(k) \xrightarrow{\mathrm{FV}} \mathbf{W}_{m+1}(k) \to \cdots) \right)$$

The graduated of H^1 *is isomorphic to*

$$\mathrm{Gr}_d(\mathrm{H}^1) := \mathrm{Fil}_d(\mathrm{H}^1)/\mathrm{Fil}_{d-1}(\mathrm{H}^1) \cong k \ .$$

From Representations to Equations

$$\begin{array}{ccc} k((t))^{\mathrm{sep}} & & \widetilde{\mathcal{R}}_K \\ | & \rightsquigarrow & | \\ k((t)) & & \mathcal{R}_K \end{array}$$



$\mathcal{R}_K$

- $\mathcal{R}_K$ is the ring of (germs of) analytic functions on the wedge of the unit disk $\mathrm{D}^-(0,1)$, that is functions converging on an annulus $\{1-\varepsilon < |x|_p < 1\}$ of $\mathbb{C}_p$.
- We have a functor which is *fully faithful*

$$\begin{array}{ccc} \mathrm{Rep}_K^{\mathrm{fin}}(\mathrm{G}) & \longrightarrow & \left\{ \begin{array}{c} \text{Diff.Eq.}/\mathcal{R}_K \\ \text{with a Frob.} \\ \text{structure} \end{array} \right\} \\ V & \longmapsto & (V \otimes_K \widetilde{\mathcal{R}}_K)^G \end{array}$$

Obtained results :

- Computation of this functor in the Abelian case ;
- Classification of all abelian Diff.Eq. over $\mathcal{R}_K$;
- Criteria to say when a given differential Eq. comes From a repr.

p-adic Differential Equations

- The equations considered are *linear*, homogeneous, and in normal form : $L := y^{(n)} + f_{n-1}y^{(n-1)} + \cdots + f_1 y' + f_0 y = 0$.

- **Example :** The equation $y' = y$ has solution $\exp(T) := \sum \frac{T^n}{n!}$.

- **Patology :** The function $\exp(T)$ does not converge everywhere, but converges only in a disk $\mathrm{D}^-(0,\omega_0)$, with $\omega_0 < +\infty$.

- **Invariant :** The *radius of convergence* of the Taylor solution in a point $c \in K$ is an *invariant* of the equation.

- If the equation is defined on an annulus $\{r_1 < |T| < r_2\}$, we can consider the function :

$$r \mapsto R(L,r) := \min_{|c|_p = r} \left\{ \begin{array}{l} \text{Radius of the Taylor} \\ \text{solution of } L \text{ at } c \end{array} \right\}$$

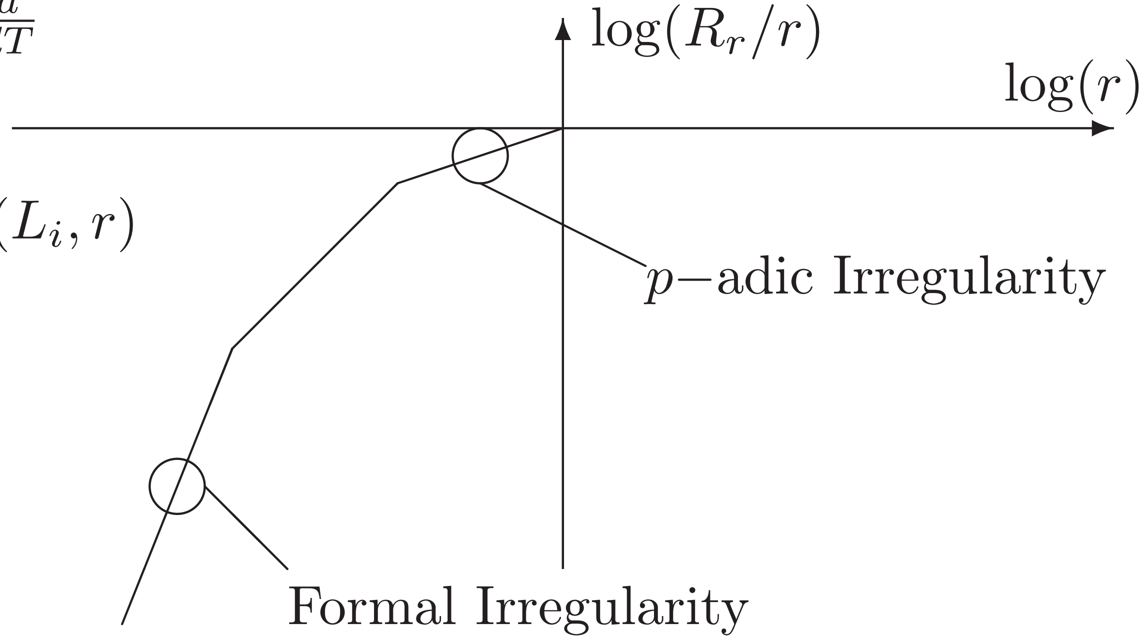
- The log-slopes of this function are *invariants* of the equation.

DWORK'S EXAMPLE

Operator	Solution at ∞	Formal Irr.	$p-$adic Irr.
$L_1 := \partial_T + \pi_0 T^{-1}$	$\exp(\pi_0 T^{-1})$	1	1
$L_2 := \partial_T + p\pi_0 T^{-p}$	$\exp(\pi_0 T^{-p})$	p	1

$\partial_T := T\frac{d}{dT}$

$R_r := R(L_i, r)$



- Since $\theta(T) := \exp(\pi_0(T^{-p} - T^{-1})) = \frac{\exp(\pi_0 T^{-p})}{\exp(\pi_0 T^{-1})}$ is *over-convergent*, then these two operators are isomorphic (over $\mathcal{R}_K$).

Theorem 0.2 (Pulita) *The rank one differential equations coming from a representation have a solution at ∞ of the type :*

$$y = T^a \cdot \exp\Big(\pi_m \phi_0(T) + \pi_{m-1}\frac{\phi_1(T)}{p} + \cdots + \pi_0 \frac{\phi_0(T)}{p^m}\Big)$$

where $\phi_j(T) = f_0(T)^{p^j} + p \cdot f_1(T)^{p^{j-1}} + \cdots + p^j \cdot f_j(T)$, with $f_1, \ldots, f_j \in T^{-1}K[T^{-1}]$, and where $\{\pi_0, \ldots, \pi_m\}$ are p^m-torsion points of a Lubin-Tate group.

- *This exponential converges for $|T| > 1$.*
- *The isomorphism class of the equation is in bijection with the pair $(\bar{a}, \rho)$ where $\bar{a} \in \mathbb{Z}_p/\mathbb{Z}$ is the residue, and $\rho \in \mathrm{H}^1$ is the character defined by the reduction of $(f_0, \ldots, f_m)$ in $\mathbf{W}_m(k((t)))$:*

$$\begin{array}{ccc} (f_0, \ldots, f_m) \in \mathbf{W}_m(T^{-1}K[T^{-1}]) & & \\ \downarrow \qquad\qquad \downarrow & & \\ (\overline{f}_0, \ldots, \overline{f}_m) \in \quad \mathbf{W}_m(k((t))) & \longrightarrow & \mathrm{H}^1 := \mathrm{Hom}(\mathrm{G}, \mathbb{Z}/p^m\mathbb{Z}) \end{array}$$

p-adic difference Equations

- Let $q, h \in \mathbb{Q}_p$, be such that $|q-1| < 1$ and $|h| < 1$. Let

$$\sigma_{q,h}(f(T)) := f(qT+h)\ , \qquad \Delta_{q,h}(f) := \frac{f(qT+h)-f(T)}{(q-1)T+h}\ .$$

- $\left\{ \begin{smallmatrix} q \to 1 \\ h \to 0 \end{smallmatrix} \right. \implies \Delta_{q,h} \longrightarrow d/dT\ .$

- Difference equations (matrix form) :

$$\sigma_{q,h}(Y) = A(T)\cdot Y\ , \quad \Longleftrightarrow \quad \Delta_{q,h}(Y) = G(T)\cdot Y(T)\ ,$$

where $G(T) = \frac{A(T)-\mathrm{I}}{(q-1)T+h}$.

Theorem 0.3 (Pulita) *A function $Y(T)$ is solution of a differential equation if and only if it is solution of a difference equation.*

In particular, for all differential equation, it exists *an* unique *difference equation having the same solutions, and reciprocally.*

Applications to p-adic Zeta and L functions

Complex Zeta function : Values of the Zeta function $\zeta : (\mathbb{C} - 1) \to \mathbb{C}$ at negative integers are known :

$$\zeta(1-n) \ = \ -\frac{B_n}{n} \ , \qquad n \geq 1 \ .$$

where $\{B_n\}_{n\geq 1}$ are the Bernulli numbers.

- *Values of ζ at positive integers are unknown.*

p-adic Zeta function : We know that $-\mathbb{N} \subseteq \mathbb{Z}_p$ is dense.

Theorem 0.4 (Kubota-Leopoldt, 1964) *It exists a unique continuous function $\zeta_p : (\mathbb{Z}_p - \{1\}) \to \mathbb{Q}_p$ such that :*

$$\zeta_p(1-n) \ = \ -(1-p^{n-1}) \cdot \frac{B_n}{n} \ , \qquad n \geq 1 \ .$$

- We define analogously $L(s, \chi)$ (complex) and $L_p(s, \chi)$ (p-adic) associate to a Dirichlet character χ.

Theorem 0.5 (Morita 1975) *It exists a unique continuous function* $\Gamma_p : \mathbb{Z}_p \to \mathbb{Q}_p$, *verifying the functional equation*

$$\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & \text{if } |x|_p=1 \\ -\Gamma_p(x) & \text{if } |x|_p<1 \end{cases} .$$

• $\Gamma_p(x+p) = A(x) \cdot \Gamma_p(x)$, with $A(x) = -(x+1)(x+2)\cdots(x+p-1)$ is a difference equation, then :

Theorem 0.6 (Pulita) *The function* Γ_p *is solution of a p-adic differential equation with coefficients converging on* $\mathrm{D}^-(0,1)$ *:*

$$\Gamma_p(T)' = g(T) \cdot \Gamma_p(T) , \qquad g(T) \text{ converges on } \mathrm{D}^-(0,1) .$$

• **Interest of this :** Let $\omega : \mathbb{Z} \to \mathbb{Z}_p$ be the Teichmüller char. Then

(Diamond 1979) : $$g(T) = \lambda_0 + \sum_{m \geq 1} L_p(1+2m, \omega^{2m}) \cdot T^{1+2m}$$

Corollary 0.1(Pulita) We obtain *congruences* involving values $L_p(1+2m, \omega^{2m})$, in *positive* integers.