A BASIC INTRODUCTION TO DEFORMATION AND CONFLUENCE OF ULTRAMETRIC DIFFERENTIAL AND q-DIFFERENCE EQUATIONS

par

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Résumé. — This is a short introduction to the phenomena of *deformation and confluence* of linear differential/difference equations, in the ultrametric context, following the papers [**ADV04**],[**Pul08b**],[**Pul08a**]. It is the transcription of a talk given at the thematic school on *Théories galoisiennes et arithmétiques des équations différentielles*, 21-25 september 2009, at the C.I.R.M. of Luminy (France). These notes are intended to be comprehensible to non specialists, and especially to the undergraduate students of that school.

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***Classification mathématique par sujets* (2000).** — Primary 12h25; Secondary 12h05; 12h10; 12h20; 12h99; 11S15; 11S20.

Introduction

The aim of this paper is to give a very simple and quick introduction to the phenomena of deformation and confluence in the ultrametric context. The aim is to explain the contents of the papers [**ADV04**], [**Pul08b**], and [**Pul08a**]. These notes are not intended to be a general survey on the topic of deformation and confluence.

The notion of stratification (cf. section 4) is essentially due to A.Grothendieck, P.Berthelot, L.Illusie, N.M.Katz, ... [(1)]. They were mainly interested in finding a substitute of the notion of linear differential equation in characteristic $p>0$, in order to obtain a good category of coefficients for a p-adic cohomological theory. Indeed, in characteristic 0, stratifications form a category which is equivalent to that of (linear) differential equations (cf. Theorem 4.1). In down to heart terms, the notion of stratification is nothing but the data of the generic Taylor solution of a differential equation (cf. section 4).

In these notes we are going to expose, in the ultrametric context, the definition of a functor called σ-deformation. Roughly speaking the functor is obtained as follows. We consider differential modules defined over a 1-dimensional affinoid X (see section 1). We consider a certain type of differential modules (M,∇) over X, whose Taylor solutions have "*large convergence*". We then prove that for all automorphism σ of X, sufficiently close to the identity, there exists a canonical semi-linear action of σ on the differential module M, making it a so called σ-module (cf. section 3). We denote by $\sigma^{\mathrm{M}}:\mathrm{M}\xrightarrow{\sim}\mathrm{M}$ this operator. The main property of this action of σ is the following: If $(\mathrm{M},\sigma^{\mathrm{M}})$ is intended as a σ-difference equation over X (see section 3), then its solutions on a disk coincide with the Taylor solutions of M intended as a differential equation. The operator σ^{M} is canonical in the sense that it commutes with the morphisms between differential modules. We hence have a functor (which is the identity on the morphisms) associating to the differential module (M,∇) the pair $(\mathrm{M},\sigma^{\mathrm{M}})$. This functor is called *σ-deformation*. The main point of this construction is the existence

(1) The notion of stratification as intended in these notes certainly date back to before, as actually Grothendieck affirms (without giving any reference, cf. [**Gro68**]). The terminology *stratification* is not standard, and our notion slightly differs from that of Grothendieck (cf.[**Ber74**]).

of σ^{M}. We deduce it by considering a certain pull back of the stratification attached to (M, ∇). In this sense we define the σ-deformation functor as the composite of the equivalence between differential equations and stratifications with a certain pull-back functor defined on the category of stratification with values on σ-modules (cf. section 6.2.1).

Structure of the paper. — In the first four sections we start by introducing differential equations, σ-modules, (elementary) stratifications, and the equivalence between differential equation and stratification. Section 5 concerns Berkovich spaces. This section is expository, and is useful in order to understand the behavior of the radius of convergence of the Taylor solutions of a differential equation, and hence the (ultrametric) convergence locus of a stratification (cf. section 5.3). Section 5 is not essential for the basic understanding of the rest of the paper. In sections 6 and 7 we introduce the σ-deformation and σ-confluence functors. We recall very roughly the method employed by Y.André and L.Di Vizio (cf. [**ADV04**]) to obtain the σ_q-confluence in the case of p-adic q-differences equations over the so called Robba ring. In section 6.2 we give an alternative construction of the σ-deformation functor as a certain pull-back of the stratification, and we compare this definition with that of Y.André and L.Di Vizio. As a main goal we obtain the σ-deformation functor for a more general class of automorphisms σ, and for more general classes of domains and of equations (cf. section 6.2.3). Moreover we obtain the analytical dependence of the operators on a parameter that can run on an ultrametric analytic variety (cf. section 6.3). In the context of q-difference equations the analytical dependence of the operator σ_q (acting on the module) with respect to q permits to reproduce the analogous of the q-confluence functor for the roots of unity (cf. section 7.4). Indeed we heuristically look to the category of differential equations as a category "over $q = 1$", and that of q-difference equations as a category "over q", where q is different to a root of unity. The classical confluence functor, as exposed in this paper, associates to a q-difference equation a differential equation having the same Taylor solutions at (one and hence) all points. This is done using the analytical dependence of σ_q (acting on the module) with respect to q (cf. section 7.3.2). We generalize this construction "over $q = \xi_{p^n}$" where ξ_{p^n} is a p^n-th root of unity instead of "over $q = 1$", by replacing the category of differential equations with a category of mixed objects formed by a q-difference module (q equal to a root of unity) together with a (compatible) differential equation. The last section 8 takes a quick look at the complex analogous. No material of this paper is new with the exception of this last section which is intended to be a very quick introduction to a forthcoming paper.

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1. Affinoids

Let K be a field, together with an ultrametric absolute value $|.|$ for which K is complete.[(2)] The basic bricks of the ultrametric geometry are the so called K-affinoids. In this paper we consider those of them that are (one dimensional, connected) affinoid sub-spaces of the affine line, defined by a family of conditions $X := \{x \mid |x-c_0| \leq R_0,\ |x-c_i| \geq R_i,\ \forall\ i=1,\ldots,n\}$, where $0 < R_1,\ldots,R_n \leq R_0$ are arbitrary real numbers, and $c_0,\ldots,c_n \in K$. We often indicate X as

$$X \ := \ \mathrm{D}^+(c_0,R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i,R_i)\ , \tag{1.1}$$

where the symbol $\mathrm{D}^+(c_0,R_0) = \{x \mid |x-c_0| \leq R_0\}$ means a *closed* disk, and $\mathrm{D}^-(c_i,R_i) = \{x \mid |x-c_i| < R_i\}$ an *open* disk. For technical reasons we assume moreover that $c_0,\ldots,c_n \in K$. If $(A,|.|)/(K,|.|)$ is a complete valued K-algebra with respect to a norm extending the absolute value of K, then we denote by $X(A)$ the set of elements in A satisfying the conditions of X. As an example if $R_0 = 1$, $R_1 = \cdots = R_n = 0$, and $c_0 = \cdots = c_n = 0$, then for all complete valued field extension Ω/K one has $X(\Omega) = \mathcal{O}_\Omega$, where $\mathcal{O}_\Omega$ is the ring of integers of Ω. In this sense, analogously to the theory of schemes, X is a functor of the category of complete normed K-algebras with values in the category of sets. By abuse of language we will write $x \in X$ to indicate "$x \in X(\Omega)$ for an unspecified complete valued field extension Ω/K".

Let now $K(T)$ be the fraction field of the ring $K[T]$ of polynomials with coefficients in K. The sub-ring $\mathcal{H}_K^{\mathrm{rat}}(X)$ of $K(T)$, formed by rational functions without poles on X, has a norm $\|.\|_X$ defined as

$$\|P(T)/Q(T)\|_X \ := \ \sup_{x\in X}|P(x)/Q(x)|\ . \tag{1.2}$$

Remark *1.1*. — *Here $x \in X$ means that x runs into the set of Ω-rational points of X, for an unspecified field Ω (large enough) equipped with an absolute value $|.|_\Omega$ extending that of K. The correct way to express the above definition would be then $\|P(T)/Q(T)\|_X := sup_{\Omega/K} sup_{x\in X(\Omega)}|f(x)|$, where $X(\Omega)$ means the "Ω-rational points of X". This formulation is possible thanks to the fact that there exists a specific $\widetilde{\Omega}/K$, together with $n+1$ points $t_{c_0,R_0},\ldots,t_{c_n,R_n} \in X(\widetilde{\Omega})$, such that for each other Ω/K one has*

$$\sup_{x\in X(\Omega)}\Big|\frac{P(x)}{Q(x)}\Big|_\Omega \ \leq \ \max_{i=0,\ldots,n}\Big|\frac{P(t_{c_i,R_i})}{Q(t_{c_i,R_i})}\Big|_{\widetilde{\Omega}} \ = \ \sup_{x\in X(\widetilde{\Omega})}\Big|\frac{P(x)}{Q(x)}\Big|_{\widetilde{\Omega}}\ . \tag{1.3}$$

The family $\{t_{c_0,R_0},\ldots,t_{c_n,R_n}\}$ is known as the (Dwork's generic points attached to the) Shilow boundary of X (cf. section 5.1). It is hence enough to consider a single field $\widetilde{\Omega}$. But we will often drop the Ω in the notations, as in the equation (1.2). Analogously, when we say that the poles of P/Q (that are algebraic over K) are not

(2) Notice that the absolute value is possibly trivial. All the statements of this paper work as well over an affinoid over a field K together with the trivial absolute value. The results of section 6.1, and also the part of the theory concerning differential/σ-difference equations over the Robba ring need the absolute value to be non-trivial, because one applies the theory of Christol-Mebkhout.

in X, we mean that there are no poles of P/Q neither in $X(K^{\mathrm{alg}})$ nor in $X(\Omega)$ for all Ω/K.

The completion $(\mathcal{H}_K(X), \|.\|_X)$ of $(\mathcal{H}_K^{\mathrm{rat}}(X), \|.\|_X)$ is called the ring of *analytic functions* over X (often called *Krasner's analytic elements* over X). If X is reduced to a closed disc $\mathrm{D}^+(c_0, R_0)$ the elements of $\mathcal{H}_K(X)$ can be expressed as power series $f = \sum_{n\geq 0} a_n(T-c_0)^n$, with $a_n \in K$, converging on $\mathrm{D}^+(c_0, R_0)$. In this case the condition of convergence becomes $\lim_{n\to\infty} |a_n|R_0^n = 0$. Indeed, in the ultrametric world, a series of elements in K converges if and only if its general term tends to 0.

More generally we define the ring of analytic functions over an *open* disk $\mathrm{D}^-(c, R)$, $c \in K$, $R > 0$ as the intersection $\mathcal{A}_K(c, R) := \cap_{R'<R}\mathcal{H}_K(\mathrm{D}^+(c, R'))$. In other words, the elements of $\mathcal{A}_K(c, R)$ are power series $\sum_{n\geq 0} a_n(T-c)^n$ verifying $\lim_n |a_n|(R')^n = 0$, for all $R' < R$.

2. Linear differential equations

The derivation d/dT is continuous with respect to $\|.\|_X$ and extends to the ring $\mathcal{H}_K(X)$. A (linear) differential equation (D.E.) over X is nothing but an expression of the type $Y' = G(T)Y$, with $G(T) \in M_n(\mathcal{H}_K(X))$. As usual we consider D.E. as objects of a category as follows. A differential module (M, ∇) is a finite free $\mathcal{H}_K$-module M, together with a map $\nabla : \mathrm{M} \to \mathrm{M}$, called *connection*, satisfying $\nabla(fm) = f'm + f\nabla(m)$ for all $f \in \mathcal{H}_K(X)$ and all $m \in \mathrm{M}$. A morphism between differential modules is an $\mathcal{H}_K(X)$-linear map commuting with the connections. The above system $Y' = GY$ corresponds to the data of a differential module (M, ∇) of dimension n, together with a fixed basis $e = \{e_1, \ldots, e_n\} \subset \mathrm{M}$, on which the action of ∇ becomes

$$\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} f_1' \\ \vdots \\ f_n' \end{pmatrix} - G(T) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \tag{2.1}$$

where ${}^t(f_1, \ldots, f_n)$ denotes $\sum f_i e_i$. A solution of $Y' = G(T)Y$ with values in $\mathcal{H}_K(X)$ is then an element of the kernel of ∇ acting on M. More generally, if (B, d) is an $\mathcal{H}_K(X)$-algebra provided with a derivation d compatible with d/dT, one is allowed to look for solutions of the expression (2.1) with $f_1, \ldots, f_n \in \mathrm{B}$. A solution of $Y' = GY$ with values in (B, d) is nothing but an element of the kernel of the map $\nabla \otimes \mathrm{Id} + \mathrm{Id} \otimes d : \mathrm{M} \otimes \mathrm{B} \to \mathrm{M} \otimes \mathrm{B}$. In another basis, the same differential module M will be associated to another differential system $Y' = \widetilde{G}Y$, then called *equivalent to* $Y' = GY$. So the differential operator $Y' = GY$ is attached to the triplet (M, ∇, e), but not unambiguously to (M, ∇). A morphism between differential modules is a $\mathcal{H}_K(X)$-linear map commuting with the ∇'s. We denote the category of differential modules over $\mathcal{H}_K(X)$ by $d - \mathrm{Mod}(\mathcal{H}_K(X))$.

3. Linear σ-difference equations

By a continuous automorphism $\sigma : X \xrightarrow{\sim} X$ we mean a continuous K-linear ring automorphism $\sigma : \mathcal{H}_K(X) \xrightarrow{\sim} \mathcal{H}_K(X)$. The continuity forces σ to be defined by the rule $\sigma(f(T)) := f(\sigma(T))$, so σ is determined by the image of T. Such an automorphism defines an automorphism of X as a functor. In other words, for all complete valued field extensions Ω/K, one has a bijection $\sigma_\Omega : X(\Omega) \xrightarrow{\sim} X(\Omega)$, compatible with the inclusions $X(\Omega) \subseteq X(\Omega')$, for all $\Omega \subseteq \Omega'$. Moreover σ_Ω is continuous with respect to the topology induced by the absolute value of Ω on $X(\Omega)$. A *σ-difference equation* over X is an expression of the type $\sigma(Y) = A(T)Y$, with $A(T) \in GL_n(\mathcal{H}_K(X))$. Analogously to the case of differential modules, a σ-difference module is the data of a finite free $\mathcal{H}_K(X)$-module S, together with an automorphism $\sigma^{\mathrm{S}} : \mathrm{S} \xrightarrow{\sim} \mathrm{S}$, satisfying $\sigma^{\mathrm{S}}(fs) = \sigma(f)\sigma^{\mathrm{S}}(s)$, for all $f \in \mathcal{H}_K(X)$, and all $s \in \mathrm{S}$. We usually drop the upper index S of σ^{S}. If a basis of S is fixed, then we can write:

$$\sigma^{\mathrm{S}} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = A(T)^{-1} \cdot \begin{pmatrix} \sigma(f_1) \\ \vdots \\ \sigma(f_n) \end{pmatrix} .$$

One sees that the solutions of $\sigma(Y) = AY$ with values in $\mathcal{H}_K(X)$ are exactly the fixed points of σ^{S}. More generally, if (B, σ') is an $\mathcal{H}_K(X)$-algebra with an automorphism $\sigma' : \mathrm{B} \xrightarrow{\sim} \mathrm{B}$ extending σ, then the solutions of $\sigma(Y) = AY$ with values in B are the fixed points in $\mathrm{S} \otimes \mathrm{B}$ under the automorphism $\sigma^{\mathrm{S}} \otimes \sigma'$. Here also we have the notion of *equivalent* equations, in complete analogy with the differential case. As in the case of differential modules, morphisms between σ-modules are $\mathcal{H}_K(X)$-linear maps commuting with the σ's. The category of σ-difference modules will be denoted by $\sigma - \mathrm{Mod}(\mathcal{H}_K(X))$.

One usually calls (q, h)-difference or simply difference equation a σ-difference equation on which $\sigma = \sigma_{q,h}$ with $\sigma_{q,h}(f(T)) := f(qT + h)$, $q, h \in K$. If $h = 0$ and $q \in K^\times$, we speak of q-difference equations, while if $q = 1$ and $h \in K - \{0\}$ we speak of finite difference equations.

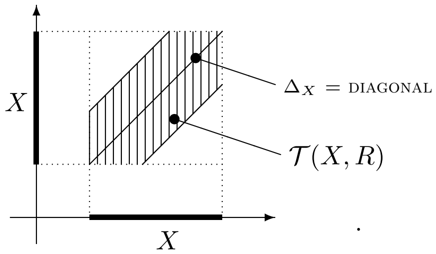
3.0.1. Ring of endomorphisms of the unit object.— The unit object $\mathbb{I}$ is the pair $(\mathrm{S}, \sigma^{\mathrm{S}})$, where $\mathrm{S} = \mathcal{H}_K(X)$, and $\sigma^{\mathrm{S}} := \sigma$. The ring of endomorphisms $\mathrm{End}(\mathbb{I})$ coincides with the ring of $f \in \mathcal{H}_K(X)$ satisfying $\sigma_{q,h}(f) = f$ (solution of the unit object). Over the field of complex numbers, in the usual settings, there exists non constant analytic functions satisfying $f(qT + h) = f(T)$ (cf. [**Duv03**], [**Duv04**]). As an example, if $h = 0$ and $|q| < 1$, the functions verifying $f(qT) = f(T)$ are the global sections of the elliptic curve $\mathbb{C}/q^{\mathbb{Z}}$ (cf. [**Sau09**]). Conversely in the p-adic context, if q is sufficiently close to 1 in order that every hole of X is stabilized by $\sigma_{q,h}$, and if q is not equal to a root of unity, then such functions do not exist. This simple fact means that the ring $\mathrm{End}(\mathbb{I})$ of endomorphisms of the unit object is reduced to K in the category $\sigma_{q,h} - \mathrm{Mod}(\mathcal{H}_K(X))$. The situation is different if q is a root of unity: we have non constant functions verifying $f(qT + h) = f(T)$, and so the category is R-linear for a ring $R = \mathrm{End}(\mathbb{I})$ strictly larger than K. This implies that if q is a root of unity we can not hope to obtain an equivalence between this category with that of differential equations, because the ring of endomorphisms of the unit objects are different.

4. Stratifications

In this section we introduce the notion of elementary stratification which is intimately related to the Taylor solution of a differential equation.

4.1. Functions on tubes around the diagonal. — A tube of radius $R > 0$ around the diagonal Δ_X is an analytic subset of $X \times X$ defined by the relation

$$\mathcal{T}(X,R) \;:=\; \{(x,y) \in X \times X \mid |x-y| < R\} \ .$$



If $R > 0$ is sufficiently small, then $\mathcal{T}(X,R)$ is isomorphic to the product of X with an open disk of radius R: $\mathcal{T}(X,R) \cong X \times \mathrm{D}^-(0,R)$ (cf. [**Pul08a**]). The isomorphism is given by $(x,y) \mapsto (y, x-y)$, with inverse $(y,\delta) \mapsto (y+\delta, y)$. This is possible because the absolute value is ultrametric. As a consequence the ring $\mathcal{A}_K(\mathcal{T}(X,R))$ of analytic functions over $\mathcal{T}(X,R)$ can be expressed as power series with coefficients in $\mathcal{H}_K(X)$ (cf. [**Pul08a**]):(3)

$$\begin{aligned}\mathcal{A}_K(\mathcal{T}(X,R)) \;=\; \{f(x,y) = \sum_{n\geq 0} f_n(y)(x-y)^n \ \text{ such that } \ f_n \in \mathcal{H}_K(X),\\ \lim_n \|f_n\|_X \cdot R_1^n = 0\ ,\ \ \forall\, R_1 < R\}\ .\end{aligned}$$

Heuristically the tube is a sort of *generic disk of radius R "around X"*, taking in account all the disks of radius R inside X. Indeed, specializing the second variable y into a point $c \in X$, the tube becomes a concrete disk $\mathrm{D}^-(c,R)$, so that the functions over $\mathcal{T}(X,R)$ specialize into functions over $\mathrm{D}^-(c,R)$. We then have a continuous ring morphism $\mathcal{A}_K(\mathcal{T}(X,R)) \to \mathcal{A}_K(c,R)$ corresponding to the map $\mathrm{D}^-(c,R) \to \mathcal{T}(X,R)$ sending x into (x,c). We will need to work with the following bigger ring

$$\mathcal{A}_K(\Delta_X^\dagger) \;:=\; \bigcup_{R>0} \mathcal{A}_K(\mathcal{T}(X,R))\ .$$

One has two maps corresponding to the two projections

$$p_1, p_2 : \mathcal{H}_K(X) \rightrightarrows \mathcal{A}_K(\Delta_X^\dagger)$$

where

$$p_1(g(T)) = \sum_{n\geq 0} (\frac{d}{dy})^n (g(y)) \frac{(x-y)^n}{n!},$$

(3) One has actually another isomorphism given by $(x,y) \mapsto (x, y-x)$. Following this isomorphism the functions around the diagonal will be written as $\sum_{n\geq 0} f_n(x)(y-x)^n$.

while $p_2(g(T))$ is the function $\sum_n f_n(y)(x-y)^n$, where $f_0 = g$, and $f_n = 0$, for all $n \geq 1$. One has moreover the map $\Delta : \mathcal{A}_K(\Delta_X^\dagger) \to \mathcal{H}_K(X)$, sending $f(x,y) = \sum_n f_n(y)(x-y)^n$ into $f(T,T) = f_0(T)$, corresponding to the diagonal embedding of X into $X \times X$. We notice that $\Delta(p_1(f)) = \Delta(p_2(f)) = f$.

4.2. Taylor solutions. — As in the complex case the study of the differential/σ-difference equations passes trough the study of the behavior of their solutions. In the complex case, for a differential equation defined over an open subset of $U \subseteq \mathbb{C}$, the radius of convergence of the Taylor solutions of a differential equation around a non singular point $c \in U$ is always the same, and it depends only on the singularities of the equation. It is the radius of the biggest open disk of $\mathbb{C}$ centered at c that does not contain any singularity of the equation. In the p-adic case the situation is different: the radius depends on the equation. As an example we can consider the equation $y' = y$, whose Taylor solution at a point $t \in K$ is the exponential $\exp(T-t) = \sum_{n\geq 0}(T-t)^n/n!$. In order to find its p-adic radius of convergence we observe that in the p-adic world an integer has usually small absolute value. Its value is given by the number of times that p divides it, it is thus as small as p divides it. Then the value $|1/n!| = 1/|n!|$ is very big, consequently the radius of convergence of the exponential is not equal to $+\infty$, but one proves that it is equal to $|p|^{\frac{1}{p-1}}$. The equation $y' = y$ does not have any singularity, so this lack of convergence is somewhat unjustified. Moreover for a more general equation, if we check the radius of convergence at another point we may have a different radius of convergence. All these numbers contain information about the equation and are actually invariants (by isomorphisms) of the differential module defined by the equation.

As already mentioned, giving a differential equation $Y' = G(T)Y$, $G(T) \in M_n(\mathcal{H}_K(X))$, is equivalent to giving a triplet (M, ∇, e), where $e \subset \mathrm{M}$ is a basis. To this basis we can attach the Taylor solution $Y(T,t) = \sum_{n\geq 0} Y^{(n)}(t,t)\frac{(T-t)^n}{n!}$ at a point $t \in X$, since Y a is solution of the equation, then $Y' = GY$, and $Y'' = (GY)' = (G' + G^2)Y$. More generally one has matrices $G_n \in M_n(\mathcal{H}_K(X))$ such that $Y^{(n)} = G_nY$. One has the recursive relations $G_0 = \mathrm{Id}$, $G_1 = G$, and $G_{n+1} = G' + G_nG$. So the Taylor expansion of the solution can be written as $Y(T,t) = [\sum_{n\geq 0} G_n(t)\frac{(T-t)^n}{n!}] \cdot Y(t,t)$. If the initial data at t is given by $Y(t,t) = \mathrm{Id}$, then $Y(T,t) = \sum_{n\geq 0} G_n(t)\frac{(T-t)^n}{n!}$. The radius of convergence of such a series around t is given by

$$\mathrm{Rad}(Y(T,t)) \quad = \quad \liminf_n \frac{1}{\sqrt[n]{\|G_n(t)/n!\|}} \ . \tag{4.1}$$

Here, if $G = (g_{i,j})$, then we set $\|(g_{i,j}(t))\| = \max_{i,j}|g_{i,j}(t)|$, in order that the above formula represents the smallest radius of convergence of the entries of $Y(T,t)$. Generally, this number depends on the chosen basis $e \subset \mathrm{M}$. In another basis $H \cdot e$, with $H \in GL_n(\mathcal{H}_K(X))$, the solution is $H(T) \cdot Y(T,t)$, hence if the radius of convergence of H at t is smaller than that of $Y(T,t)$ the radius of their product may be smaller than that of $Y(T,t)$. We notice that H converges at least on the biggest open disk

$\mathrm{D}^-(t,\rho_{t,X})$ contained in X. So we set

$$\mathrm{Rad}(\mathrm{M},t) \;\; := \;\; \min(\,\mathrm{Rad}(Y(T,t))\;,\;\rho_{t,X}\;)\;,$$

this number is now independent on the chosen basis of M. The formal power series

$$Y(x,y) \;\; := \;\; \sum_{n\geq 0} G_n(y)\frac{(x-y)^n}{n!}\;, \tag{4.2}$$

intended as a function of two variables x and y is called the *generic Taylor solution* of M in the basis $e \subset \mathrm{M}$. One proves that it lies in $GL_n(\mathcal{A}_K(\Delta^\dagger))$, indeed there exists a tube on which it converges (cf. [**Pul08a**]). This means in particular that there exists a number $R_{min} > 0$ such that, for all $t \in X$, all the entries of $Y(T,t)$ converge at least on $\mathrm{D}^-(t,R_{min})$, independently on the chosen point t. We have moreover the following properties:

- (Δ) $Y(x,y)$ is the identity on the diagonal: $Y(T,T) = \mathrm{Id}$;
- (C) $Y(x,y)$ satisfies the following cocycle relation:
$$Y(x,y)\cdot Y(y,z) = Y(x,z)$$
for all $(x,y),(y,z),(x,z) \in \mathcal{T}(X,R_{min})$.

As a direct consequence we have $Y(x,y)^{-1} = Y(y,x)$ which proves that $Y(x,y)$ is invertible.

4.3. Elementary stratifications. — An (elementary) stratification is the data of a finite free $\mathcal{H}_K(X)$-module M together with an isomorphism

$$\chi\;:\;p_2^*\mathrm{M} \xrightarrow{\sim} p_1^*\mathrm{M}\;,$$

converging over a germ of a tube around the diagonal, i.e. defined over $\mathcal{A}_K(\Delta^\dagger)$, and satisfying

- (Δ) χ is the identity on the diagonal: $\Delta^*(\chi) = \mathrm{Id}_{\mathrm{M}}$,
- (C) Cocycle relation: If $p_{i,j} : X\times X\times X \to X\times X$ denotes the projection on the (i,j)-factor, then
$$p_{1,2}^*(\chi_{\mathrm{M}})\circ p_{2,3}^*(\chi_{\mathrm{M}}) = p_{1,3}^*(\chi_{\mathrm{M}})$$
over $p_{1,2}^{-1}(\mathcal{T}(X,R))\cap p_{2,3}^{-1}(\mathcal{T}(X,R))\cap p_{1,3}^{-1}(\mathcal{T}(X,R))$,

where $\mathcal{T}(X,R)$ is the tube where χ converges. A morphism $\alpha : (\mathrm{M},\chi_{\mathrm{M}}) \to (\mathrm{N},\chi_{\mathrm{N}})$ of stratifications is a $\mathcal{H}_K(X)$-linear map $\alpha : \mathrm{M}\to\mathrm{N}$ satisfying $\chi_{\mathrm{M}} \circ (p_2^*\alpha)_{|\mathcal{T}(X,R)} = (p_1^*\alpha)_{|\mathcal{T}(X,R)} \circ \chi_{\mathrm{N}}$. We denote by $\mathrm{Hom}^\chi(\mathrm{M},\mathrm{N})$ the K-vector space of morphisms between stratifications. We denote by $\mathrm{Strat}(\mathcal{H}_K(X))$ the category of stratifications.(4)

The formal properties (Δ) and (C) coincide with the analogous properties of the generic Taylor solution of a differential equation. Indeed, we will see that the generic

(4) As already mentioned in the introduction, the notation here is not a standard one, but is general enough for our purposes. One can find a more appropriate approach in [**Kat73**], and [**Ber74**, Chap.II].

Taylor solution is always the matrix of a stratification and conversely. In other words the interest of introducing the notion of stratification lies in the following theorem

Theorem 4.1. — *The category $d-Mod(\mathcal{H}_K(X))$ is equivalent to* $\mathrm{Strat}(\mathcal{H}_K(X))$.

We skip the proof of this (well known) theorem that can be found entirely in [**Pul08a**]. We only say that if a differential equation $Y' = GY$ is given, this data corresponds to a triplet (M, ∇, e), where e is a basis of M. The attached stratification is the $\mathcal{A}_K(\Delta^\dagger)$-linear map $\chi : p_2^*\mathrm{M} \xrightarrow{\sim} p_1^*\mathrm{M}$, whose matrix in the basis $e \otimes 1$ of $p_i^*\mathrm{M}$ is given by the generic Taylor solution $Y(x,y) \in GL_n(\mathcal{A}_K(\Delta_X^\dagger))$. Conversely if $\chi : p_2^*\mathrm{M} \xrightarrow{\sim} p_1^*\mathrm{M}$ is a stratification, and if $Y(x,y)$ is its matrix in the basis $e\otimes 1$, then $Y(x,y)$ is necessarily the generic Taylor solution of an equation $Y' = G(T)Y$, with $G(T) \in M_n(\mathcal{H}_K(X))$. Indeed $Y(x,y)$ verifies the properties (Δ) and (C) of section 4.2, and so we consider the matrix G defined by $G := (\frac{d}{dx}Y(x,y)) \cdot Y(x,y)^{-1}$. One has

$$\begin{aligned} G(x) &= \Big(\frac{d}{dx}Y(x,y)\Big) \cdot Y(x,y)^{-1} = \Big(\frac{d}{dx}Y(x,y)\Big) \cdot Y(y,x) \\ &= \Big(\lim_{h\to 0} \frac{Y(x+h,y) - Y(x,y)}{h}\Big) \cdot Y(y,x) \\ &= \lim_{h\to 0} \frac{Y(x+h,x) - \mathrm{Id}}{h} \ . \end{aligned}$$

This proves that $G(x)$ does not depend on the second variable y. Now we have to prove that $G(x)$ lies in $M_n(\mathcal{H}_K(X))$. Since $Y(x,y)$ converges in a small tube, then if $h \in K$ is sufficiently small, one proves that the matrix $A(h,x) := Y(x+h,x)$ belongs to $GL_n(\mathcal{H}_K(X))$, with inverse $Y(x, x+h)$. Moreover $A(h,x)$ is also analytic with respect to h, so it is analytic as a matrix defined on $(h,x) \in \mathrm{D}^-(0,\varepsilon) \times X$, where $\varepsilon > 0$ is a small real number. So $G(x)$ coincides with the matrix $\frac{d}{dh}A(h,x)$ evaluated at $h = 0$. For this reason $G(x)$ lies in $M_n(\mathcal{H}_K(X))$.

5. The Berkovich space of an affinoid

The proofs of the assertions of this section can be founded in [**Pul08a**].

We recall that a metric space is totally disconnected if every point has a base of neighborhoods that are simultaneously closed and open. An example of a totally disconnected space is given by the rational numbers together with the euclidean usual absolute value (e.g. $]-\sqrt{2},\sqrt{2}[\cap\mathbb{Q} = [-\sqrt{2},\sqrt{2}]\cap\mathbb{Q}$). The rational numbers are totally disconnected also with respect to a p-adic absolute value, because of the ultrametric property. Moreover, in this case another phenomenon arises: the intersection of two discs is either empty or equal to one of them. When completing $\mathbb{Q}$ with respect to its euclidean usual absolute value we obtain the real numbers. In this case we are doubly lucky because firstly $\mathbb{R}$ is also "almost algebraically closed" in the sense that $\dim_{\mathbb{R}}\mathbb{C} = 2$, and secondly because $\mathbb{R}$ is also connected. Conversely, when completing $\mathbb{Q}$ with respect to a p-adic absolute value, we obtain the field of p-adic numbers $\mathbb{Q}_p$, which verifies $\dim_{\mathbb{Q}_p}\mathbb{Q}_p^{\mathrm{alg}} = +\infty$, and moreover $\mathbb{Q}_p$ remains totally disconnected. In this context the Berkovich space $\mathscr{M}(X)$ associated to an affinoid X is an archwise connected topological space that (functorially) "contains" X.

In algebraic geometry the fundamental concept of the theory of schemes is that the ring of functions over an algebraic variety is needed to recover entirely the Zarisky topology of the algebraic variety. The definition of the Berkovich spaces follows the same philosophy: the space is defined by the function ring. The elements of the Berkovich space $\mathscr{M}(X)$ attached to an affinoid X are *bounded multiplicative semi-norms* on $\mathcal{H}_K(X)$. A semi-norm $|.|$ on a normed algebra $(\mathrm{B}, \|.\|)$ is said bounded if $|.| \leq \|.\|$, it is said multiplicative if $|1| = 1$ and if $|fg| = |f||g|$, for all $f, g \in \mathrm{B}$. We notice that the kernel $\mathrm{Ker}(|.|) := \{x \in \mathcal{H}_K(X) \mid |x| = 0\}$ is a prime ideal because the semi-norm is multiplicative. But the correspondence between semi-norms and prime ideals of $\mathcal{H}_K(X)$ is not one-to-one. In this sense the Berkovich space $\mathscr{M}(X)$ is a refinement of the algebraic scheme $\mathrm{Spec}(\mathcal{H}_{\mathrm{K}}(\mathrm{X}))$ attached to $\mathcal{H}_K(X)$: it takes into account also the topology of $\mathcal{H}_K(X)$. The topology of $\mathscr{M}(X)$ is the minimal one making continuous all the maps of the form $\psi_f : \mathscr{M}(\mathcal{H}_K(X)) \to \mathbb{R}_{\geq 0}$, $\psi_f(|.|) := |f|$, for all $f \in \mathcal{H}_K(X)$.

We want now to find the points of $\mathscr{M}(X)$. If Ω/K is an extension of valued fields (it is understood that the absolute value of Ω extends that of K), then every Ω-rational point $t \in X(\Omega)$ of X defines a bounded multiplicative semi-norm $|.|_t$ of $\mathscr{M}(X)$ by the evaluation at t:

$$|f|_t \;:=\; |f(t)|_\Omega \;.$$

Lemma 5.1. — *All the points of $\mathscr{M}(X)$ are of the form $|.|_t$, for a convenient valued field extension Ω/K, and $t \in X(\Omega)$. But the point t is not uniquely determined by the semi-norm $|.|_t$, one may have $|.|_t = |.|_{t'}$ for $t \neq t' \in X(\Omega)$.*

We call *Dwork's generic point for* $|.|$ each point $t \in X(\Omega)$ satisfying $|.| = |.|_t$ as semi-norms on $\mathcal{H}_K(X)$.

5.1. Paths in $\mathscr{M}(X)$ and semi-norms of type $|.|_{t,R}$. — As usual we consider an affinoid

$$X \;=\; \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i) \;.$$

We want to describe a class of continuous paths in $\mathscr{M}(X)$. For this it is convenient to introduce the semi-norms of type $|.|_{t,R}$, $t \in \Omega$, $R \geq 0$. Of course, by the above Lemma 5.1, we will have $|.|_{t,R} = |.|_{t'}$ for some convenient point $t' \in X(\Omega)$. For $R = 0$, the semi-norm $|.|_{t,R}$ is equal to $|.|_t$ just defined. For $R > 0$ the definition is a bit more complicated: if $P \in K[T]$, then the value $|P|_{t,R}$ is the supremum on the annulus $\{x \mid |x - t| = R\}$: $|P|_{t,R} := \sup_{|x-t|=R} |P(x)|$ (see remark 1.1). For $P/Q \in K(T)$ one sets $|P/Q|_{t,R} := |P|_{t,R}/|Q|_{t,R}$ (unfortunately this is actually no longer equal to the supremum on the annulus). By restriction $|.|_{c,R}$ defines a semi-norm on $\mathcal{H}_K^{\mathrm{rat}}(X) \subset K(T)$ (cf. section 1). Finally the semi-norm $|.|_{t,R}$ extends to a bounded multiplicative semi-norm of $\mathcal{H}_K(X)$ if and only if one of the following conditions is fulfilled

– $t \in X(\Omega)$ and $R \leq R_0$, (for a convenient complete valued field Ω/K)

– t lies in a hole $\mathrm{D}^-(c_i, R_i)$ of X and $R_i \leq R \leq R_0$.

Notice that the annulus $\{x \mid |x-t| = R\}$ is not necessarily contained in X because it may encounter the holes of X. We have another important property:

$$|t-t'|_\Omega \leq R \qquad \Longrightarrow \qquad |.|_{t,R} = |.|_{t',R} \tag{5.1}$$

as semi-norms on $\mathcal{H}_K(X)$. As a consequence for all choices of $t, t' \in \mathrm{D}^+(c_0, R_0)$, one always has

$$|.|_{t,R_0} \;=\; |.|_{t',R_0}$$

as semi-norms on $\mathcal{H}_K(X)$.

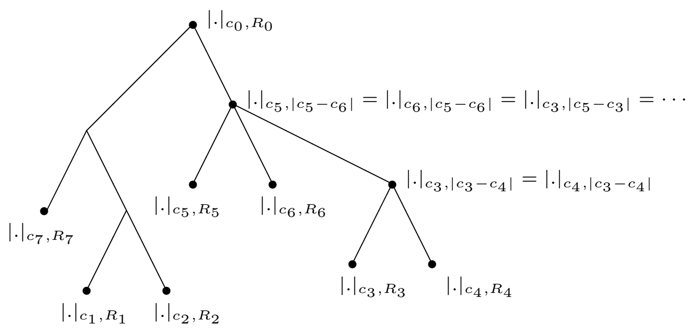
Theorem 5.2. — *The Berkovich space $\mathscr{M}(X)$ is archwise connected.*

The idea of the proof of this theorem lies in the fact that one proves that $R \mapsto |.|_{t,R}$ is a continuous path in $\mathscr{M}(X)$. The space is then connected, because every point of $\mathscr{M}(X)$ is of the type $|.|_t = |.|_{t,0}$ for a convenient Dwork's generic point $t \in X(\Omega)$. It is then connected by the above path to the point $|.|_{t,R_0}$ which is the same semi-norm for all starting points t: $|.|_{t,R_0} = |.|_{t',R_0}$, for all $t, t' \in X(\Omega)$.

As a last property we give a description of the set of maximal points of $\mathscr{M}(X)$ with respect to the natural order relation between seminorms: $|.|_1 \leq |.|_2$ if and only if $|f|_1 \leq |f|_2$ for all $f \in \mathcal{H}_K(X)$. One can prove that the maximal points of $\mathscr{M}(X)$ are those of the family

$$\mathscr{S}_X \;:=\; \cup_{i=1}^n \{\; |.|_{c_i,R} \;\}_{R\in[R_i,R_0]} \;,$$

where $c_1, \ldots, c_n$ are the centers of the holes of X. This subset of $\mathscr{M}(X)$ is called maximal Skeleton of $\mathscr{M}(X)$.



The end points of this graph $\{|.|_{c_i,R_i}\}_{i=0,\ldots,n}$ are the so called Shilov boundary of $\mathscr{M}(X)$. This finite set has the property that every function $f \in \mathcal{H}_K(X)$ assumes its maximum on it:

$$\|f\|_X \;=\; \max_{i=0,\ldots,n} |f|_{c_i,R_i} \;.$$

5.2. Radius of convergence and Berkovich space. — Let (M,∇) be a differential module, and let $Y(x,y)$ be its generic Taylor solution in a given basis. Thanks to Lemma 5.1, we can associate to every point $|.|$ of the Berkovich space the radius of $Y(x,y)$ at $|.|$ by choosing a point t satisfying $|.|=|.|_t$ and considering (non canonically) $\mathrm{Rad}(Y(x,t))$. It is clear from the definition (4.1) that $\mathrm{Rad}(Y(x,t))$ does not depend on the choice of t but only on the semi-norm $|.|_t=|.|$. Analogously one defines the number $\rho_{|.|,X}$ by $\rho_{|.|,X}:=\rho_{t,X}$, and one proves that this number is also independent from the choice of t. So we can attach to each Berkovich point $|.|$ of $\mathscr{M}(X)$ the Radius of convergence $\mathrm{Rad}(\mathrm{M},|.|)=\min(\mathrm{Rad}(Y(x,t)),\rho_{t,X})$ of M at $|.|=|.|_t$. We then obtain a function on the Berkovich space $\mathscr{M}(X)$ that is simply called *radius of convergence function*. This function satisfies many properties:

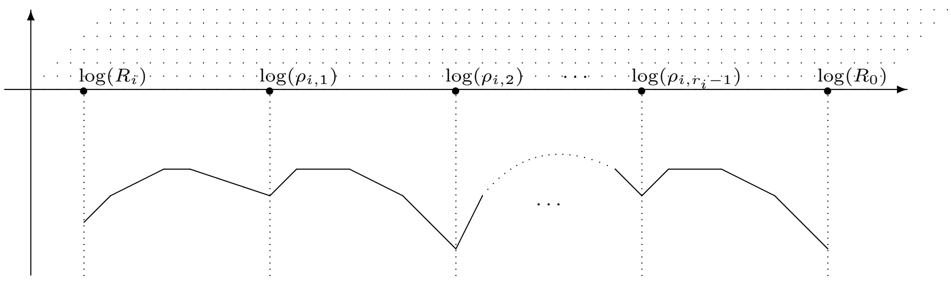
Theorem 5.3 **([CD94],[BDV08]).** — *The function* $|.|\mapsto\mathrm{Rad}(\mathrm{M},|.|)$ *is continuous on* $\mathscr{M}(X)$.

For a proof of this theorem see [**BDV08**] or [**Pul08a**]. If $I\subseteq\mathbb{R}_{\geq 0}$ and if $f: I\to\mathbb{R}_{\geq 0}$ is a function, we say that f has logarithmically a given property if the function $x\mapsto\ln(f(\exp(x))):\ln(I)\to\mathbb{R}$ has that property. As a consequence of the continuity the radius gives by restriction a continuous function on each path $R\mapsto|.|_{t,R}:I\to\mathscr{M}(X)$. This restriction is logarithmically piecewise affine (i.e. the function $\rho\mapsto\log(\mathrm{Rad}(\mathrm{M},|.|_{t,\exp(\rho)}))$ is piecewise affine). Namely it is piecewise of the form $\mathrm{Rad}(\mathrm{M},|.|_{t,R})=\alpha R^{\beta}$, $\alpha,\beta\in\mathbb{R}$. Moreover, if the annulus $\{x\in X\mid |x-t|\in I\}$ does not contain any hole of X, then the function so obtained is also log-concave (i.e. $\cap$-shaped). The slopes of the sides of this function are rational with denominator bounded by $1/n!$, where $n=\dim_{\mathcal{H}_K(X)}\mathrm{M}$.

In general the radius of convergence $\mathrm{Rad}(\mathrm{M},|.|_{t,R})$ of the function is (at the present stage of the technology) not directly computable from the knowledge of the elementary invariants of the matrix of the system $Y'=GY$ attached to M in a basis. Something can be done: if the radius is smaller than the spectral norm of d/dT with respect to the semi-norm $|.|$, and if the matrix G is in the cyclic form (i.e. attached to a differential operator in $\mathcal{H}_K(X)[d/dT]$), then the radius is explicitly related to the coefficients of the operator. But in general the radius is unknown, so usually one uses the above properties to deduce some properties of the "big" values of the radius from the knowledge of the "small" values of it.

Another important property is that the radius is an decreasing function on $\mathscr{M}(X)$. This property is often called the *transfert theorem*. Namely, if $|.|_1\leq|.|_2$ (that is if $|f|_1\leq|f|_2$ for all $f\in\mathcal{H}_K(X)$), then $\mathrm{Rad}(\mathrm{M},|.|_1)\geq\mathrm{Rad}(\mathrm{M},|.|_2)$. This follows immediately from the definition (4.1) and by the fact that if $|.|_1\leq|.|_2$, then $\rho_{|.|_1,X}=\rho_{|.|_2,X}$. It is then interesting to evaluate the radius on the maximal points of $\mathscr{M}(X)$, that is on the skeleton of $\mathscr{M}(X)$. The typical behavior of the logarithmic graph of the radius of convergence of a differential module M in a branch $R\mapsto|.|_{c_i,R}:[R_i,R_0]\to$

$\mathscr{M}(X)$ of the maximal skeleton has the following shape.



$$\ell \longmapsto \log(\mathrm{Rad}(\mathrm{M}, |.|_{c_i,\exp(\ell)})/\exp(\ell))$$

This picture represents the logarithmic graph of the function $R \mapsto \mathrm{Rad}(\mathrm{M}, |.|_{c_i,R})/R$. Notice that we normalize the function $\mathrm{Rad}(\mathrm{M}, |.|_{c_i,R})$ by dividing by R because one proves that $\rho_{|.|_{c_i,R},X} = R$, so that $\mathrm{Rad}(\mathrm{M}, |.|_{c_i,R})/R$ is smaller than 1 (because $\mathrm{Rad}(\mathrm{M}, |.|_{c_i,R})$ is by definition smaller than R). The real numbers $R_i = \rho_{i,0} < \rho_{i,1} < \cdots < \rho_{i,r} = R_0$ (over which the radius function has possibly a non concave break) are the values of R such that the annulus $\{x \mid |x - c_i| = R\}$ contains a hole of X: i.e. $\rho_{i,k} = |c_i - c_k|$ for some convenient $k \neq i$.

5.3. Convergence locus of the Taylor solution. — We have introduced the notion of Radius of convergence in order to justify the fact that the convergence locus of the Taylor solution of a differential equation (i.e. the convergence locus of the isomorphism $\chi : p_2^*\mathrm{M} \xrightarrow{\sim} p_1^*\mathrm{M}$ of a stratification) is often not reduced to a tube. Indeed, if the convergence locus is equal to a tube, the last picture would be an horizontal line. In the most part of the cases the convergence locus is a subset of $X \times X$ which is strictly larger than any tube contained in it. Moreover, the convergence locus is not *analytic* in the sense of ultrametric geometry[5], even though it satisfies, as we have seen, a lot of remarkable properties. Its study needs the introduction of the Berkovich space and some refined estimations, in order to be able to reduce its study to the case of a tube.

6. σ-Deformation

We are interested in finding couple of functors

$$\begin{tikzcd} d - \mathrm{Mod}(\mathcal{H}_K(X)) \arrow[rr, bend left, "\mathrm{Def}_\sigma"] & & \sigma - \mathrm{Mod}(\mathcal{H}_K(X)) \arrow[ll, bend left, "\mathrm{Conf}_\sigma"] \end{tikzcd}$$

(5) I.e. the definition of the convergence locus is not given by the usual conditions, like for example $\{(x, y) \text{ such that } |f(x, y)| \leq |g(x, y)|\}$.

that have to be, in the best case, quasi-inverse of each other. The idea is that the first category is equivalent to that of stratifications, and instead of defining the functor Def_σ we define a functor

$$\mathrm{Strat}(\mathcal{H}_K(X)) \longrightarrow \sigma - \mathrm{Mod}(\mathcal{H}_K(X)) .$$

We will see that although the functor Def_σ can be defined under weak assumptions on σ, it will happen that the functor Conf_σ is not always easy to be defined. This is the case, because generally the arrow $\mathrm{Hom}(\mathrm{M},\mathrm{N}) \to \mathrm{Hom}(\mathrm{Def}_\sigma(\mathrm{M}),\mathrm{Def}_\sigma(\mathrm{N}))$ is injective, but not surjective.

Notation. — If A is an abelian group, and if $\varphi : A \to A$ is an additive map, we set

$$\begin{aligned} A^{\varphi=1} &:= \{ a \in A \mid \varphi(a) = a \} , \\ A^{\varphi=0} &:= \{ a \in A \mid \varphi(a) = 0 \} . \end{aligned}$$

6.1. The Galoisian approach following André - Di Vizio. — In the framework of complex numbers the confluence of a q-difference equation into a differential equation reflects the confluence of their solutions: the q-solution tends to the solution of the limit differential equation as q tends to 1. On the other hand, in the ultrametric setting the key property of the confluence and the deformation functors is that they preserve the solutions. In the case of q-difference equations the q-solution is actually *equal* to the solution of the limit differential equation. This fact was firstly pointed out by Y. André and L. Di Vizio (cf. [**ADV04**]) in the context of "*étale*" solutions of p-adic differential/q-difference equations over a particular ring of functions $\mathcal{R}$ called the Robba ring. The elements of $\mathcal{R}$ are analytic functions over a certain germ of an annulus. The ring $\mathcal{R}$ is the p-adic analogous of the field of formal power series $\mathbb{C}(\!(T)\!)$. In the context of [**ADV04**] the residual field k of K has characteristic $p > 0$. Then $\mathcal{R}$ is a sort of "lifting in characteristic 0" of the field of formal power series $k(\!(t)\!)$. The equations considered by André-Di Vizio were supposed to have an additional structure: the presence of a Frobenius acting on the differential/q-difference module. This additional structure forces the equations to have a large radius of convergence. Their approach was Galoisian and, in a very rough and simplified way, it works as follows. Assume that we are working with a class $\mathcal{C}$ and $\mathcal{C}_q$ of differential and q-difference equations, respectively, defined over a certain ring that we indicate by $\mathcal{R}$, in order to recall that in the context of [**ADV04**] $\mathcal{R}$ is actually the Robba ring. The formalism works in general for a general integral domain $\mathcal{R}$ endowed with a derivation d and an automorphism σ_q satisfying the assumptions that we will indicate below. Intentionally we will not be precise about the nature of the real Robba ring $\mathcal{R}$, for this reason we will replace it by a general ring (still called $\mathcal{R}$) in order to avoid expository complications. The intention here is to only give a "flavour" of the formal construction of Y. André and L.Di vizio.

6.1.1. Universal covering.— The first step consists in considering a sort of universal covering $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ together with the action of its "fundamental group" $\pi_1 := \pi_1(\widetilde{\mathcal{R}}/\mathcal{R})$ that acts by $\mathcal{R}$-linear ring automorphisms of $\widetilde{\mathcal{R}}$ and such that $\widetilde{\mathcal{R}}^{\pi_1} = \mathcal{R}$. Moreover, we have to ask for the existence of a derivation $\widetilde{d}$ and an automorphism $\widetilde{\sigma}_q$, extending d and σ_q to $\widetilde{\mathcal{R}}$, satisfying the relation $\mathcal{R}^{d=0}=\mathcal{R}^{\sigma_q=1}=\widetilde{\mathcal{R}}^{\widetilde{d}=0}=\widetilde{\mathcal{R}}^{\widetilde{\sigma}_q=1}$, and such that the action of π_1 commutes with $\widetilde{d}$ and with $\widetilde{\sigma}_q$. We assume that both $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are integral domains of characteristic 0.

The real Robba ring $\mathcal{R}$ considered by [**ADV04**] is obtained as the completion with respect to a certain topology of a certain sub-ring $\mathcal{E}^\dagger$ of a Cohen ring $\mathcal{E}$ of $k(\!(t)\!)$ (cf. [**CC98**], [**Ber02**]). A finite separable extension of $k(\!(t)\!)$ is again a field of formal power series, as well as the finite extensions of $\mathbb{C}(\!(T)\!)$. Then we can consider the same construction for all finite separable extension of $k(\!(t)\!)$. In this way one proves that there is a functorial way to associate a "robba ring" to each finite separable extension of $k(\!(t)\!)$. The universal covering $\widetilde{\mathcal{R}}$ is the union of all these extensions in a fixed algebraic closure of the fraction field of $\mathcal{R}$, and we have

$$\pi_1 \;=\; \mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!))\;, \tag{6.1}$$

where $k(\!(t)\!)^{\mathrm{sep}}$ is a separable closure of $k(\!(t)\!)$. The extension $\widetilde{\mathcal{R}}/\mathcal{R}$ has to be considered as a p-adic analogue of the field of Puiseux power series $\mathbb{C}(\!(T)\!)^{\mathrm{alg}}/\mathbb{C}(\!(T)\!)$.[(6)] In [**ADV04**] one proves that σ_q can be extended to $\widetilde{\mathcal{R}}$ as mentioned above (cf. also [**Pul08b**]).

6.1.2. Equations trivialized by the universal covering.— The second step consists in proving that all the equations in $\mathcal{C}$ and in $\mathcal{C}_q$ are trivialized by $\widetilde{\mathcal{R}}$. If $Sol(\mathrm{M}_q,\widetilde{\mathcal{R}})$ denotes the $\mathcal{R}^{\sigma_q=1}$-module of solutions of M_q in $\widetilde{\mathcal{R}}$, we need also the usual condition $\dim_{\mathcal{R}^{\sigma_q=1}} Sol(\mathrm{M}_q,\widetilde{\mathcal{R}})=\dim_{\mathcal{R}}\mathrm{M}_q$, for all M_q in $\mathcal{C}_q$. This condition is often verified if the rings $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are not too bad.

6.1.3. q-Deformation and q-Confluence by Tannakian formalism.— Once these two steps have been realized, then the confluence functor can be defined using the usual Tannakian formalism that we reproduce in the following. Let $\mathrm{M}_q\in\mathcal{C}_q$. Then, by assumption, M_q is trivialized by $\widetilde{R}$ and hence $\dim_{\mathcal{R}^{\sigma_q=1}} Sol(\mathrm{M}_q,\widetilde{\mathcal{R}})=\dim_{\mathcal{R}}\mathrm{M}_q$. The group π_1 acts on the solutions $Sol(\mathrm{M}_q,\widetilde{\mathcal{R}}):=(\mathrm{M}_q\otimes_{\mathcal{R}}\widetilde{\mathcal{R}})^{\sigma_q=1}$. Namely, π_1 acts on $\mathrm{M}_q\otimes_{\mathcal{R}}\widetilde{\mathcal{R}}$ by automorphisms of the second term of the product, and it induces an action on the solutions since its action commutes with $\sigma_q\otimes\widetilde{\sigma}_q$. Moreover, the fact that M_q is trivialized by $\widetilde{\mathcal{R}}$, guarantees that the solutions generate $\mathrm{M}_q\otimes_{\mathcal{R}}\widetilde{\mathcal{R}}$ as $\widetilde{\mathcal{R}}$-module, so that in a basis of solutions the action of $\sigma_q\otimes\widetilde{\sigma}_q$ becomes the trivial one. In other words $\mathrm{M}_q\otimes_{\mathcal{R}}\widetilde{\mathcal{R}}$, together with the action of $\sigma_q\otimes\widetilde{\sigma}_q$, is isomorphic (as a $\widetilde{\sigma}_q$-module over $\widetilde{\mathcal{R}}$) to $Sol(\mathrm{M}_q,\widetilde{\mathcal{R}})\otimes_{\mathcal{R}^{\sigma_q=1}}\widetilde{\mathcal{R}}$, with the action of $1\otimes\widetilde{\sigma}_q$. This last is the trivial $\widetilde{\sigma}_q$-module over $\widetilde{\mathcal{R}}$ of dimension equal to $\dim_{\mathcal{R}^{\sigma_q=1}} Sol(\mathrm{M}_q,\widetilde{\mathcal{R}})=\dim_{\mathcal{R}}\mathrm{M}_q$.

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(6) Notice that a finite extension of $k(\!(t)\!)$ is possibly not of the type $k'(\!(t^{1/n})\!)$, for a finite separable extension k'/k of k and convenient $n\geq 1$. As an example, if $k=k'$ and if $n=p$, then $k(\!(t^{1/p})\!)/k(\!(t)\!)$ is purely *inseparable*. Separable extensions of $k(\!(t)\!)$ whose order is a power of p are described by the Artin-Schreier-Witt theory.

As a consequence of the isomorphism

$$(\mathrm{M}_q \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \ , \ \sigma_q \otimes \widetilde{\sigma}_q \ ) \ \xrightarrow{\sim} \ (\ \mathit{Sol}(\mathrm{M}_q, \widetilde{\mathcal{R}}) \otimes_{\widetilde{\mathcal{R}}^{\sigma_q=1}} \widetilde{\mathcal{R}} \ , \ 1 \otimes \widetilde{\sigma}_q \ ) \ ,$$

it follows that M_q can be entirely recovered by the representation $\mathit{Sol}(\mathrm{M}_q, \widetilde{\mathcal{R}})$ of π_1. This is done by the formula

$$\mathrm{M}_q \cong (\mathit{Sol}(\mathrm{M}_q, \widetilde{\mathcal{R}}) \otimes_{\mathcal{R}^{\sigma_q=1}} \widetilde{\mathcal{R}})^{\pi_1} \ , \tag{6.2}$$

where the action of σ_q on the tensor product is given by $1 \otimes \widetilde{\sigma}_q$. Indeed, $(\mathit{Sol}(\mathrm{M}_q, \widetilde{\mathcal{R}}) \otimes_{\mathcal{R}^{\sigma_q=1}} \widetilde{\mathcal{R}})^{\pi_1} \cong (\mathrm{M}_q \otimes_{\mathcal{R}} \widetilde{\mathcal{R}})^{\pi_1} = (\mathrm{M}_q \otimes_{\mathcal{R}} \mathcal{R}) = \mathrm{M}_q$. The isomorphism (6.2) constitutes an analogue of [**Kat82**, Prop.4.1]. It is an isomorphism of σ_q-modules over $\mathcal{R}$ that is functorial and $\otimes$-compatible. This is nothing but the usual formalism of Tannakian equivalence associating to a σ_q-module the π_1-representation given by its solutions. The identical formalism applied to differential equations associates to a differential module (M, ∇) in $\mathcal{C}$ the π_1-representation of its solution $\mathit{Sol}(\mathrm{M}, \widetilde{\mathcal{R}}) = (\mathrm{M} \otimes_{\mathcal{R}} \widetilde{\mathcal{R}})^{(\nabla \otimes 1 + 1 \otimes \widetilde{d})=0}$. As before, because M is trivialized by $\widetilde{\mathcal{R}}$, one can recover M from its π_1-representation by

$$\mathrm{M} \ \cong \ (\mathit{Sol}(\mathrm{M}, \widetilde{\mathcal{R}}) \otimes_{\mathcal{R}^{d=0}} \widetilde{\mathcal{R}})^{\pi_1} \ . \tag{6.3}$$

Now we come to the main point: we have an action of $1 \otimes \widetilde{d}$ on the tensor product in (6.2), and since it commutes with π_1, *then* (M_q, σ_q) *actually acquires a structure of differential module*. In the same way we have an action of $1 \otimes \widetilde{\sigma}_q$ on (6.3), and since it commutes with π_1, then (M, ∇) canonically has an action of σ_q and hence a σ_q-module structure. Moreover, one easily sees that a morphism commutes with σ_q if and only if it also commutes with the connection ∇. This provides an equivalence of categories between σ-modules and d-modules: the q-deformation of (M, ∇) is M itself together with the action of $1 \otimes \widetilde{\sigma}_q$ on the tensor product (6.3) just defined. On the other hand, the q-confluence of (M_q, σ_q) is M_q itself together with the action of ∇ given by $1 \otimes \widetilde{d}$ on the tensor product (6.2). The functor is the identity on the morphisms.

The idea of this equivalence has been inspired by the theory of p-adic representations (as initiated and developed by J.M.Fontaine) in which such kind of processes are largely used. These ideas are inspired by the descent theory more than the Tannakian formalism. For this reason they are a bit more flexible, and lead the field of constants to be possibly not algebraically closed.

6.1.4. Remarks.— We can now observe a certain number of facts:

1− In the context of [**ADV04**], the fact that the differential equations with Frobenius structure are trivialized by the universal covering $\widetilde{\mathcal{R}}$ is known as the Crew conjecture (cf. [**Cre98**]). It has been proved by [**And02**],[**Meb02**],[**Ked04**], and is one of the deeper result of the theory of p-adic differential equations. It constitute the p-adic analogue of the Turritin's theorem for formal differential equations over $\mathbb{C}((T))$.[7] The choice of [**ADV04**] of the Robba ring as the base

(7) Notice an essential difference between these two theorems: the classical Turritin's Theorem asserts that a differential equation over $\mathbb{C}((T))$ becomes an extension of rank one differential modules after a

ring for q-difference equations is due to the existence of this theorem. The choice of the universal covering $\widetilde{\mathcal{R}}$ (trivializing the equations in $\mathcal{C}$ and in $\mathcal{C}_q$) was then imposed in [**ADV04**] by the theory of p-adic differential equations, since the differential equations in the category $\mathcal{C}$ are trivialized by $\widetilde{\mathcal{R}}$.

The difficult part is then to prove that all the equations in $\mathcal{C}_q$ are trivialized by $\widetilde{\mathcal{R}}$. This was, in down to heart terms, the hardest part of the work of Y.André - L.Di Vizio. This fact is known as the q-analogue of the Crew conjecture. Its proof forces to employ a large part of the results of both theories of differential and difference equations. Following this approach the existence of the confluence and the deformation functors appears as a final result of a great work of description and classification of the objects of the two categories.

2– The most important fact is that the functor preserves the solutions (with values in $\widetilde{\mathcal{R}}$). In other words if M_q is the q-deformation of a differential module M, by the definition of the functor one has $\mathrm{M} = \mathrm{M}_q$ as $\mathcal{R}$-modules. Then its solutions with values in $\widetilde{\mathcal{R}}$ as a differential module *coincide* with its solutions as σ_q-module:

$$Sol(\mathrm{M},\widetilde{\mathcal{R}}) = Sol(\mathrm{M}_q,\widetilde{\mathcal{R}})\ .$$ [8]

3– So, if a fundamental matrix of solutions $\widetilde{Y} \in GL_n(\widetilde{\mathcal{R}})$ of the differential equation $\widetilde{Y}' = G \cdot \widetilde{Y}$, $G \in M_n(\mathcal{R})$ is known, then the σ_q-difference equation obtained by deformation will be $\sigma_q(\widetilde{Y}) = A \cdot \widetilde{Y}$, where $A \in GL_n(\mathcal{R})$ is given by

$$A := \sigma_q(\widetilde{Y}) \cdot \widetilde{Y}^{-1}\ .$$

It may actually be difficult to write down the elements of $\widetilde{\mathcal{R}}$, and in particular the solution $\widetilde{Y}$. In general, what we may hope to know explicitly are the Taylor solutions $Y(x,y)$ of such an equation (cf. section 4.2). But we encounter the problem that the action of σ_q on $\widetilde{\mathcal{R}}$ and on $\widetilde{Y}$ may be a priory different from the action of σ_q on $Y(x,y)$. So we possibly have $Y(\sigma_q(x),y)Y(x,y)^{-1} \neq \sigma_q(\widetilde{Y})\widetilde{Y}^{-1}$. The equality $\sigma_q(Y(x,y))Y(x,y)^{-1} = \sigma_q(\widetilde{Y})\widetilde{Y}^{-1}$ is actually verified in the context of André-Di Vizio, and in general it follows (modulo some technical verifications) from the uniqueness of the smallest differential/q-difference algebra generated by the solutions of the differential equation.

4– From this formalism it appears clearly that a differential module is automatically canonically endowed with the action of *every automorphism* σ_q on $\mathcal{R}$,

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scalar extension to a convenient (unspecified) ramified extension $\mathbb{C}((T^{1/n}))/\mathbb{C}((T))$; In the p-adic world the Crew's conjecture asserts that a differential module over $\mathcal{R}$ having an (unspecified) Frobenius Structure becomes an extension of the *trivial rank one object* after scalar extension to a convenient (unspecified) $\mathcal{R}'/\mathcal{R}$ (functorially) attached to a finite separable extension of $k((t))$. While the rank one pieces of the classical Turritin's decomposition preserve an essential information about the starting differential module, in the p-adic framework these rank one pieces are all trivial. This means that, up to the presence of logarithms, the solutions of the differential equations are all algebraic.

[8] As already mentioned there is an analogy between $\widetilde{\mathcal{R}}/\mathcal{R}$ and $\mathbb{C}((T))^{\mathrm{alg}}/\mathbb{C}((T))$. The action of π_1 on the solution in $\widetilde{\mathcal{R}}$ corresponds, in the framework of complex differential equations, to the action of the formal monodromy $\mathrm{Gal}(\mathbb{C}((T))^{\mathrm{alg}}/\mathbb{C}((T))) \cong \widehat{\mathbb{Z}}$ on the formal solutions of a differential equation over $\mathbb{C}((T))$.

for all those q such that σ_q extends to the universal covering $\widetilde{\mathcal{R}}$ and satisfies the properties of sections 6.1.1 and 6.1.2. In [**ADV04**] one proves that these properties are fulfilled by each $q \in \mathrm{D}^-(1, |p|^{\frac{1}{p-1}})$. The reason is that in the annulus $|x-1| = |p|^{1/(p-1)}$ one encouters the p-th roots of unity. Nevertheless, for all $q \in \mathrm{D}^-(1,1)$, the automorphism σ_q can be extended to $\widetilde{\mathcal{R}}$ in a way that it commutes with π_1.

5− The condition $\mathcal{R}^{d=0} = \mathcal{R}^{\sigma_q=1} = \widetilde{\mathcal{R}}^{\widetilde{d}=0} = \widetilde{\mathcal{R}}^{\widetilde{\sigma}_q=1}$ was essential to guarantee the equivalence. Indeed, the solutions are naturally modules over the ring of constants, and since the functor preserves the solutions, in order to have equivalence it is essential that the ring of q-constants is equal to the ring of d-constants. This implicitly implies that q *can not be equal to a root of unity*, indeed in that case $\mathcal{R}^{\sigma_q=1}$ is strictly bigger than $\mathcal{R}^{d=0}$. Notice that, if q is a root of unity, the formalism of section 6.1.3 works in one direction: the deformation functor is still defined, while the confluence is not.

6− The above algorithm, used define the confluence and deformation functors of q-difference equations, is very handy because it involves the Tannakian machinery, that helps in the definition of the differential/q-difference Galois group. Unfortunately, at the present state of technology, it has uniquely been proved for equations over the Robba ring $\mathcal{R}$, but not over other rings of functions. In particular it remains unproved for differential equations over an affinoid X. The main difficulty is to prove that the objects are trivialized by the chosen "universal covering". The Robba ring is the p-adic analogous of the field of formal power series $\mathbb{C}((T))$, this is the reason for which it is easier to work over $\mathcal{R}$.

7− We notice that in the usual Tannakian framework, one assumes that the rings of constants $\mathcal{R}^{d=0}$ and $\mathcal{R}^{\sigma_q=1}$ are algebraically closed fields. It is not the case here. Indeed this assumption is needed to prove that the category is equivalent to that of linear representations of the *algebraic* group of automorphisms of a fiber functor. As we have seen in section 6.1.3 we actually do not need this description in order to obtain the above equivalence.

8− For technical reasons one is obliged moreover to assume that q is very close to 1 (see point 4), and hence it can not be a root of unity. Another restriction comes from the fact that one has moreover to assume that the objects have an action of a Frobenius, which reduces considerably the class of considered equations.

9− The process of [**ADV04**] is slightly different than that of section 6.1.3. The reason is that in order to apply the traditional Tannakian formalism one needs an algebraically closed field of constants K. The Crew conjecture, as proved in [**And02**], asserts that a differential equation is trivialized by $\widetilde{\mathcal{R}}$ if K is algebraically closed. But on the other hand, to apply the theory of Christol-Mebkhout (cf. [**CM02**]) one needs also K to be discrete valuated. So what one does is considering, for all finite extension K' of K, a couple of rings $\mathcal{R}_{K'}$ and $\widetilde{\mathcal{R}}_{K'}$ functorially attached to K'. Hence one proves that a differential or a q-difference equation is trivialized by $\widetilde{\mathcal{R}}_{K'}$ for an unspecified K'. As a matter of facts one can consider $\mathcal{R} := \cup_{K'}\mathcal{R}_{K'}$ and $\widetilde{\mathcal{R}} := \cup_{K'}\widetilde{\mathcal{R}}_{K'}$, for K' running in the set of all finite extensions of K in a fixed algebraic closure, and then apply the formalism of section 6.1.3. With this setting one has $\pi_1 = \mathrm{Gal}(\overline{k}(\!(t)\!)^{\mathrm{sep}}/\overline{k}(\!(t)\!))$, where $\overline{k}$ is the residual field of K^{alg} and it is an algebraic closure of k. According to [**And02**] (and [**ADV04**]) one hence considers, for all finite extensions K'/K, the categories $\mathcal{C}(K')$ and $\mathcal{C}_q(K')$ formed by those equations that admit an (unspecified) action of a Frobenius compatible with the action of σ_q. Then one chose as $\mathcal{C} := \cup_{K'}\mathcal{C}(K')$ and $\mathcal{C}_q := \cup_{K'}\mathcal{C}_q(K')$ the inductive limits of the above categories, and one applies smoothly the formalism of section 6.1.3.

With these settings in [**And02**] (resp. [**ADV04**]) one proves that the Tannakian group of $\mathcal{C}$ (resp. $\mathcal{C}_q$) is isomorphic to the group

$$\pi_1 \times \mathbb{G}_a \;\cong\; \mathrm{Gal}(\bar{k}(\!(t)\!)^{\mathrm{sep}}/\bar{k}(\!(t)\!)) \times \mathbb{G}_a \;, \tag{6.4}$$

where $\bar{k}$ is the residual field of K^{alg}. This means that each q-difference module in $\mathcal{C}_q$ is a direct sum of sub modules of the type $\mathrm{N} \otimes U_m$, $m \geq 1$, where N is a q-difference module having a finite Tannakian group isomorphic to $\mathrm{Gal}(L/\bar{k}(\!(t)\!))$ for some finite separable extension $L/\bar{k}(\!(t)\!)$, and where U_m is the rank m module whose generic Taylor solution matrix is

$$Y_{U_m}(x,y) \;=\; \begin{pmatrix} 1 & \ell_1 & \cdots & \ell_{m-2} & \ell_{m-1} \\ 0 & 1 & \ell_1 & \cdots & \ell_{m-2} \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & 1 & \ell_1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \;, \tag{6.5}$$

where $\ell_n := [\log(x/y)]^n/n!$. A classification of the objects of $\mathcal{C}_q$ (over K^{alg})) corresponds then to a classification of the linear representations $\rho : \mathrm{Gal}(\bar{k}(\!(t)\!)^{\mathrm{sep}}/\bar{k}(\!(t)\!)) \to GL_n(K^{\mathrm{alg}})$ such that the image of ρ is finite. There exists a description of $\mathrm{Gal}(\bar{k}(\!(t)\!)^{\mathrm{sep}}/\bar{k}(\!(t)\!))$ in terms of generators and relations (cf. [**Koc65**], [**Koc67**], [**MS89**]), but this does not gives an explicit classification of the representations, and unfortunately it is not handy in order to describe the invariants of ρ.

6.2. σ-Deformation by generic Taylor solutions. — In the above section we have seen that in the Galoisian formalism the definition of the q-confluence is not harder than that of the q-deformation. If q is not equal to a root of unity the situation is perfectly symmetric. We will see now that it will actually be convenient to define the deformation first, because it always exists. Indeed the confluence does not always exist (or its definition remains unknown) when q was a root of unity (see point 5 of the final remarks of the above section). We are going to define in this section the σ-deformation functor using the Taylor solutions. We will actually obtain it as a certain pull-back of the stratification. We do not longer use the particular automorphism $\sigma_q : f(T) \mapsto f(qT)$, but a general automorphism σ of the affinoid X subject to some conditions that we will precise in the sequel. Technical details and proofs of this section can be found in [**Pul08a**].

6.2.1. Definition of the functor. — Let $\sigma : X \xrightarrow{\sim} X$ be a continuous automorphism of X as in section 3. We denote by the same symbol σ the automorphism of $\mathcal{H}_K(X)$ given by $\sigma(f(T)) := f(\sigma(T))$, for all $f \in \mathcal{H}_K(X)$. We consider the morphism

$$\Delta_\sigma \;:\; X \longrightarrow X \times X \;, \qquad \Delta_\sigma(x) \;:=\; (x, \sigma(x)) \;.$$

Assume that σ is sufficiently close to the identity of X in order to have $\Delta_\sigma(X) \subseteq \mathcal{T}(X,R)$, for some $R>0$. Clearly

$$\begin{array}{ll} p_1 \circ \Delta_\sigma = \mathrm{Id}_X \;, & p_2 \circ \Delta_\sigma = \sigma \;, \\ \Delta_\sigma^* \circ p_1^* = \mathrm{Id}_X^* \;, & \Delta_\sigma^* \circ p_2^* = \sigma^* \;. \end{array}$$

The σ-deformation functor is then defined as the composite of two functors as follows. We start from a differential module (M,∇), we consider first the functor associating to (M,∇) its stratification $\chi : p_2^*\mathrm{M} \xrightarrow{\sim} p_1^*\mathrm{M}$ that we assume to be defined over $\mathcal{T}(X,R)$, for some $R>0$. Then we consider the pull back functor by Δ_σ^*:

$$\text{(6.6)}\qquad \Big[\chi : p_2^*\mathrm{M} \xrightarrow{\sim} p_1^*\mathrm{M}\Big] \quad \overset{\Delta_\sigma^*}{\rightsquigarrow} \quad \Big[\sigma^*\mathrm{M} = \Delta_\sigma^* p_2^*\mathrm{M} \xrightarrow[\Delta_\sigma^*(\chi)]{\sim} \Delta_\sigma^* p_1^*\mathrm{M} = \mathrm{M}\Big]\;.$$

So we obtain the required isomorphism $\Delta_\sigma^*(\chi) : \sigma^*\mathrm{M} \xrightarrow{\sim} \mathrm{M}$. One proves that the σ-deformation functor so defined is a $\otimes$-compatible, additive, faithful functor (cf. section 6.2.3).

6.2.2. Conditions for the existence. — The stratification χ attached to (M,∇) is defined over a tube $\mathcal{T}(X,R)$, and in order to make sense of $\Delta_\sigma^*(\chi)$ we need the assumption $\Delta_\sigma(X) \subset \mathcal{T}(X,R)$, otherwise we can not consider the composite function $\Delta_\sigma^*(\chi)$. This means that σ has to be sufficiently close to the identity of X. Now we want to make weaken this assumption, in order to take into account a largest class of automorphisms σ. As we have seen in section 5.3, the convergence locus of the stratification is usually *not reduced* to a tube. Actually the real condition for the existence of the pull back Δ_σ^* is weaker, we only need

$$\text{(6.7)}\qquad \Delta_\sigma(X) \quad \subset \quad \{\text{Convergence locus of } \chi\}\;.$$

A technical computation in the Berkovich space $\mathscr{M}(X)$ proves that the above assumption (6.7) is reduced to the easier one

$$\text{(6.8)}\qquad |\sigma(T)-T| \quad < \mathrm{Rad}(\mathrm{M},|.|)$$

for all semi-norms $|.| \in \mathscr{M}(X)$ lying in the finite family

$$\text{(6.9)}\qquad \{|.|_{c_i,|c_i-c_j|}\}_{i\neq j; i,j=0,\ldots,n} \cup \{|.|_{c_i,R_i}\}_{i=0,\ldots,n}\;.$$

These are the end points of the maximal skeleton of $\mathscr{M}(X)$ (i.e. the Shilow Boundary), together with the "bifurcation points" of the maximal skeleton, i.e. those of the form $|.|_{c_i,|c_i-c_j|}$ (cf. section 5.2). Here again we see the interest of introducing Berkovich spaces: usually a condition involving an infinite number of points (and hence an infinite number of conditions) can be reduced to be tested in a finite number of (Berkovich) points.

As a final remark we observe that the difficult part of the proof of this last reduction is due to the fact that the convergence locus of χ (i.e. of $Y(x,y)$) is in general not an analytic subset of $X \times X$ (cf. section 5.3). It is hence difficult to prove that the composite function $\Delta_\sigma^*(\chi)$ (i.e. the matrix $A(x) = Y(\sigma(x),x)$) is actually analytic and has its coefficients in $\mathcal{H}_K(X)$. For this, under the condition (6.7), we have to prove that there exists a (admissible) covering of X, and a tube for each term of the covering whose union is contained in the convergence locus of χ, and contains the image of Δ_σ. Here again the use of the language of Berkovich spaces is of great help.

Note : — Condition (6.8) implies that σ *stabilizes globally all maximal disks of* X *(i.e. those of the form* $\mathrm{D}^-(t,\rho_{X,t})$*, cf. section 5.2)*. Indeed we recall that the radius of convergence at a point $t \in X(\Omega)$ of a differential equation is defined as the minimum between the radius of convergence of the generic Taylor solution (in a fixed basis), and the number $\rho_{X,t}$ which is the radius of the biggest open disk in X centered at t (cf. section 5.2). We did that in order to make the definition invariant under base changes (cf. section 4.2). Hence by definition the radius at t is smaller than $\rho_{X,t}$, so condition (6.8) implies

$$|\sigma(T) - T| \;<\; \rho_{X,|.|}\;, \tag{6.10}$$

for all $|.| \in \mathscr{M}(X)$. This property is implicitly necessary to define the σ-deformation, so we are lead to give the following

Definition 6.1. — *We say that* σ *is an* infinitesimal *automorphism of* X *if it stabilizes (globally) each maximal disk of* X *(i.e. each disk of the form* $\mathrm{D}^-(t,\rho_{X,t})$*, for* $t \in X(\Omega)$*, for an arbitrary complete valued field extension* Ω/K*).*

The existence of the σ-deformation functor requires σ to be infinitesimal otherwise there is no differential module satisfying the assumption (6.8). The fact that σ is infinitesimal can be checked on the family (6.9) (cf. [**Pul08a**]).

Theorem 6.2. — *Let* $\sigma : X \xrightarrow{\sim} X$ *be an infinitesimal automorphism of* X*. Let* $d-Mod(\mathcal{H}_K(X))^{\mathrm{comp}(\sigma)}$ *be the category of differential modules satisfying the condition (6.8). Then we have a deformation functor*

$$\mathrm{Def}_\sigma \;:\; d-Mod(\mathcal{H}_K(X))^{\mathrm{comp}(\sigma)} \;\longrightarrow\; \sigma - Mod(\mathcal{H}_K(X))\;.$$

The functor associates to a differential equation M *the pull-back of its stratification by the morphism* Δ_σ^* *(cf. (6.6)).*

6.2.3. Final remarks. — Finally, as in section 6.1, we observe a certain number of facts:

1– One has $\mathrm{Def}_\sigma(\mathrm{M}) = \mathrm{M}$ as $\mathcal{H}_K(X)$-modules.

2– If a basis of M is chosen, and if $Y' = GY$ is the system so defined by M, with generic Taylor solution $Y(x,y)$ (i.e. the matrix of the stratification χ), then the attached σ-difference equation is given by $\sigma(Y) = AY$, where the matrix A of the morphism $\Delta_\sigma^*(\chi) : \sigma^*\mathrm{M} \xrightarrow{\sim} \mathrm{M}$ is given by
$$A(T) \;:=\; Y(\sigma(T),T)\;.$$

3– By construction the functor preserves the generic Taylor solutions. In other words, the Taylor solution $Y(x,y)$ verifies simultaneously

$$\begin{cases} \frac{d}{dx}(Y(x,y)) &=& G(x)\cdot Y(x,y)\;, \\ Y(\sigma(x),y) &=& A(x)\cdot Y(x,y)\;. \end{cases} \tag{6.11}$$

Indeed, by the cocycle property (C) of the stratification (cf. section 4.2) we have

$$A(x) \;=\; Y(\sigma(x),x) \;=\; Y(\sigma(x),y)\cdot Y(y,x) \;=\; Y(\sigma(x),y)\cdot Y(x,y)^{-1}\;.$$

4− The functor is by construction the *identity on the morphisms*. Actually, the morphisms can be interpreted as solutions of a convenient differential equation, and this agrees with the fact that the solutions are preserved by the functor.

5− For expository reasons we have defined the σ-deformation functor only in the case of affinoids, but the functor can easily be defined over all the most important rings of functions. In particular it exists over the Robba ring $\mathcal{R}$.

These properties reflect the properties of the André-Di Vizio's "Galoisian" functor (cf. section 6.1). One actually proves that in the case of q-difference equation this functor coincides with that of Y.André and L.Di Vizio. For this we have to compare the generic Taylor solutions with the "Galoisian" solutions in the algebra $\widetilde{\mathcal{R}}$ (cf. section 6.1, 3th final remark). This is done (modulo some technical points) by using the uniqueness of the differential/σ-difference algebra generated by the solutions.

As we have seen the "Galoisian approach" needs the construction of the algebra $\widetilde{\mathcal{R}}$, trivializing all the considered equations and satisfying the "good Galoisian properties". Conversely, the σ-deformation by the Taylor solutions of section 6.2 permits to extend the result of Y.André and L.Di Vizio to more global domains like for example the affinoids, and to more general automorphisms σ. It also permits to take into account more general classes of equations, i.e. it enlarges considerably the classes $\mathcal{C}$ and $\mathcal{C}_q$ of linear differential/σ-difference equations considered in the "Galoisian approach" of [**ADV04**]. Moreover, it works over an arbitrary base field K (possibly not algebraically closed). If $\sigma = \sigma_q$ is the q-difference operator, then in the context of [**ADV04**] the deformation functor by Taylor solutions is defined for all $q \in \mathrm{D}^-(1,1)$, $q \in K$. Finally this approach permits some new considerations that we expose in the next sections.

6.3. Analytic families of operators. — Accordingly to what we have seen in the "galoisian context" (cf. section 6.1, 4th final remark) a differential module (M,∇) is canonically endowed with an action of all those operators σ that are "close enough to the identity" (i.e. satisfying (6.8)). We then obtain a family of σ-difference equations $\{\sigma(Y) = A(\sigma,x)Y\}_{\sigma\in\Sigma}$. Now the σ-deformation by Taylor solutions permits to prove the analytic dependence of the matrix of σ with respect to the variation of σ. More precisely let G be an analytic variety. Let $\{\sigma_g\}_{g\in G}$ be a family of automorphisms of X indexed by G. Assume that the family σ_g varies analytically on g, that is, the map

$$\psi_G \;:\; G\times X \longrightarrow X$$

sending (g,x) into $\sigma_g(x)$ is analytic. Then we also have the analyticity of the map $G\times X \to X\times X$, sending (g,x) into $\Delta_{\sigma_g}(x) = (x,\sigma_g(x))$. As a consequence the matrix

$$A(\sigma_g,x) \;=\; \Delta^*_{\sigma_g}(\chi) \;=\; Y(\sigma_g(x),x)$$

of the equation varies analytically on g, because it is a *composite of analytic functions*. As an example we can consider q-difference equations. In this case the analytic variety

G is an open disk $\mathrm{D}^-(1,\varepsilon)$ centered at 1, with a convenient radius $\varepsilon \leq 1$, so that, for all $q \in G = \mathrm{D}^-(1,\varepsilon)$, every automorphism σ_q satisfies (6.8) with respect to M. In this case we obtain an additional structure: since $G = \mathrm{D}^-(1,\varepsilon)$ is an analytic group, *the deformation of* M *actually is an analytic semilinear representation of* $\mathrm{D}^-(1,\varepsilon)$. The deformed object has to be considered as the data of a module M together with an *analytic action of a convenient analytic variety* G (possibly depending on M). As a matter of fact one usually fixes the analytic variety G and then one considers the deformation functor as defined on the category of all those differential equations for which the convergence locus of the stratification contains every $(x,\sigma_g(x))$. The functor then takes its values in the category of "Modules with analytic action of G".

Theorem 6.3. — *Let G be a set of operators (an analytic variety, an analytic group, respectively) acting on X by infinitesimal automorphisms, through an analytic map ψ_G as above. Denote by $d-Mod(\mathcal{H}_K(X))^{\mathrm{comp}(G)}$ the full subcategory of $d-Mod(\mathcal{H}_K(X))$ formed by $\mathcal{H}_K(X)$-differential modules satisfying* (6.8) *with respect to each σ_g, $g \in G$. Denote by $G-Mod(\mathcal{H}_K(X))$ the category of finite free $\mathcal{H}_K(X)$-modules together with an action (analytic action, semi-linear analytic action, respectively) of G. Then we have a functor*

$$\mathrm{Def}_G \;:\; d - Mod(\mathcal{H}_K(X))^{comp(\mathrm{G})} \;\longrightarrow\; G - Mod(\mathcal{H}_K(X))\ .$$

7. σ-Confluence

In this section we will give a condition for the fully faithfulness of Def_σ. The fully faithfulness is verified for a large class of automorphisms σ. While the quasi surjectivity of Def_σ is proved only in the case of automorphisms of the type $\sigma = \sigma_{q,h}$ where $\sigma_{q,h}(f(T)) := f(qT+h)$.

7.1. Fully faithfulness of the σ-deformation and non degeneracy. — As seen in the above section we fix a set of operators that can be an analytic variety G, and we consider the deformation functor as defined on the category $d-\mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{comp(G)}}$ of $\mathcal{H}_K(X)$-differential modules whose stratification has a convergence locus which contains $\{(x,\sigma_g(x))\}_{g\in G}$. Then the deformation functor takes its values in the category $G-\mathrm{Mod}(\mathcal{H}_K(X))$ of finite free $\mathcal{H}_K(X)$-modules with an action of G which is automatically analytic (resp. semi-linear and analytic) if G is an analytic variety (analytic group, respectively):

$$\mathrm{Def}_G \;:\; d - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{comp(G)}} \;\longrightarrow\; G - \mathrm{Mod}(\mathcal{H}_K(X))\ . \tag{7.1}$$

Notice that G can be reduced to a point. Under a weak assumption on the map ψ_G the G-deformation functor is *fully faithful.* These assumptions are quite technical: they essentially ask that there exists a complete valued extension Ω/K and an Ω-rational point $t \in X(\Omega)$ such that $\mathcal{A}_K(t,R)^{G'=1} = \Omega$ for all R for which the disk $\mathrm{D}^-(t,R) \subset X$ is stable under an arbitrary subset G' of G. In particular, this implies that $\mathcal{H}_K(X)^{G=1} = K$. We say that the action of G is *non degenerate* if it satisfies these assumptions. Here again we see the interest of introducing the Berkovich space, because this condition can be highly trivialized by providing the following criterion of non degeneracy:

***Lemma 7.1* (Criterion of non degeneracy).** — *If there exists a Berkovich point* $|.| \in \mathscr{M}(X)$ *such that the set* $\{|\sigma^n(T) - T|\}_{\sigma \in G', n \geq 1} - \{0\}$ *admits* $0 \in \mathbb{R}$ *as an accumulation point, then the action of* G *is non degenerate.*

Proposition 7.2. — *If the family* G *of operators is non degenerate, then the deformation functor* (7.1) *is fully faithful.*

If $G = \{\sigma_q\}$, the condition of the last lemma becomes the usual one: "q is not a root of unity".

7.2. Do we have new invariants?— In general the category of q-difference equations, with q equal to a root of unity, is intrinsically different from that of differential equations. As already mentioned in the paragraph 3.0.1, the reason is that the category of differential equations is K-linear, while, if q is a root of unity, the category of q-difference equations is R-linear for a ring R strictly larger than K. We recall that the deformation functor is the identity on the morphisms, hence the map $\mathrm{Hom}^{\nabla}(\mathrm{M}, \mathrm{N}) \to \mathrm{Hom}^{\sigma_q}(\mathrm{Def}_{\sigma_q}(\mathrm{M}), \mathrm{Def}_{\sigma_q}(\mathrm{N}))$ is an inclusion. The first Hom is a K-vector space, and the second one is an R-module. It seems difficult at the present state of technology to provide a (functorial) left inverse of this inclusion. As a consequence we see that a confluence functor is difficult to define (if it exists).

Remark *7.3.* — *In the context studied by Y.André-L.Di Vizio of differential equations with a Frobenius structure over the Robba's ring, one finds that* $G = \mathrm{D}^-(1,1)$ *(cf.* [**Pul08b**, Cor.7.14]*). As already mentioned,* σ_q *is degenerate if and only if* q *is a root of unity. Nevertheless the family* $\boldsymbol{\mu}_{p^\infty}$ *of* all roots of unity *in the disk* $\mathrm{D}^-(1,1)$ *is* non degenerate *(cf.* [**Pul08a**, Section 7.2.1]*). So the deformation functor is fully faithful, if considered as a functor with values into* $\boldsymbol{\mu}_{p^\infty}$*-semi-linear representations: if* $K = K_\infty = K(\boldsymbol{\mu}_{p^\infty})$*, then we have a fully faithful functor associating to a differential equation with an (unspecified) Frobenius structure over the Robba ring* $\mathcal{R}_{K_\infty}$ *a semi-linear* $\boldsymbol{\mu}_{p^\infty}$*-representation over the same Robba ring. For a complete statement see* [**Pul08a**, Section 7.2.1]. *The invariants (by isomorphisms) of the semi-linear representations of the p-divisible group* $\boldsymbol{\mu}_{p^\infty}$ *are hence invariants (by isomorphism) of the starting differential equation, and it would be nice to understand their meaning in term of differential equations. Notice that an invariant (by isomorphisms) of a differential equation, as for those related to its radius of convergence, are possibly not invariants of the deformed* $\boldsymbol{\mu}_{p^\infty}$*-representation, because we have more isomorphisms in the target category. As an example the radius of convergence is possibly not preserved (see the paragraph below). In this sense the invariants of the* $\boldsymbol{\mu}_{p^\infty}$*-representations should be quite somehow quite "basic".*

As in the above remark, a natural question that one may ask is the following. Assuming that the deformation is neither an equivalence nor fully faithful, it may happen that the deformed object splits into sub-objects (possibly of rank one) that do not come from a differential module by deformation. In this case one can consider these sub-objects as invariants (by isomorphisms) of the starting differential equation. They are hence useful in order to classify differential equation. This idea actually comes from the theory of p-adic representations. The p-adic representations whose corresponding (ϕ, Γ)-module admits such a splitting are called trianguline representations and were introduced by P.Colmez in the context of the p-adic Langlands correspondence. Unfortunately these invariants are all trivial in the case of q-difference equations studied by Y.André and L.Di Vizio: when deforming a differential module (over the Robba ring, with a Frobenius structure) into a σ_q-module with q equal to a root of unity, we always obtain a trivial object direct sum of the unit object. This is due to the Frobenius structure (cf. [**Pul08b**, Prop.8.6]). In order to preserve the information we need to consider the whole action of $\boldsymbol{\mu}_{p^\infty}$, as indicated in the above remark.

7.3. (q,h)**-Confluence.** — The confluence functor is (nowadays) only defined for the automorphisms of the form $\sigma_{q,h}(f(T)) = f(qT+h)$, in the case in which q is not a root of unity. In this case the situation is richer because there exists the notion of (q,h)-twisted Taylor solution (due to L.Di Vizio [**DV04**]). This notion permits to recover the stratification from the $\sigma_{q,h}$-module and obtain an analogue of theorem 4.1. In this way we get a quasi inverse of the $\sigma_{q,h}$-deformation funtor.

7.3.1. (q,h)*-twisted Taylor solution.* — If $\sigma_{q,h}(Y) = AY$ is an equation, we set

$$Y_{q,h}(x,y) \;:=\; \sum_{n\geq 0} G_{[n]}(y)\frac{(x-y)_{q,h}^{[n]}}{[n]_q^!}$$

where, if $n\in\mathbb{N}$, then $[n]_q := 1+q+q^2+\cdots+q^{n-1}$, $[n]_q^! = [n]_q\cdot[n-1]_q\cdots[1]_q$, and $(x-y)_{q,h}^{[n]} := (x-y)(x-\mathfrak{s}_{q,h}(y))(x-\mathfrak{s}_{q,h}^2(y))\cdots(x-\mathfrak{s}_{q,h}^{n-1}(y))$, where $\mathfrak{s}_{q,h}(y) := qy+h$, and $\mathfrak{s}_{q,h}^i$ denotes the composite $\mathfrak{s}_{q,h}\circ\cdots\circ\mathfrak{s}_{q,h}$ i-times. Here the matrices $G_{[n]}$ are defined, in analogy with the differential case, by the relation $d_{q,h}^n(Y) = G_{[n]}Y$, where $d_{q,h} = \frac{\sigma_{q,h}-1}{(q-1)T+h}$ is the (q,h)-derivation. Let $\mathrm{M}_{q,h}$ be the (q,h)-difference module defined by the equation $\sigma_{q,h}(Y) = AY$ in a given basis $e\subset\mathrm{M}_{q,h}$. A priori $Y_{q,h}(x,y)$ is merely a symbol, since the series may possibly not converge. However one can attach to this symbol a Radius of convergence, defied as

$$\mathrm{Rad}(Y_{q,h}(x,t)) := \liminf_n \frac{1}{\sqrt[n]{\|G_{[n]}(t)\|/|[n]_{q,h}^!|}}\;,$$

where as usual $t\in X(\Omega)$ (cf. section 5.2). As for the usual radius we generalize this definition to all points $|.|\in\mathscr{M}(X)$ by

$$\mathrm{Rad}(\mathrm{M}_{q,h},e,|.|) := \min\Big(\liminf_n \frac{1}{\sqrt[n]{\|G_{[n]}\|/|[n]_{q,h}^!|}}\;,\;\rho_{X,|.|}\Big)\,, \tag{7.2}$$

where $|.| = |.|_t$ as in section 5.2 (cf. (4.1)). This radius actually depends on the chosen basis $e\subset\mathrm{M}_{q,h}$, and is not necessarily an invariant of $\mathrm{M}_{q,h}$. One proves however that if the radius (7.2) verifies the condition (6.8), then actually the formal power series $Y_{q,h}(x,y)$ converges and defines a stratification satisfying $Y_{q,h}(qx+h,y) = A(x)\cdot Y_{q,h}(x,y)$. In this case the radius (7.2) is hence an invariant and coincides with the radius of (the differential equation defined by) the stratification as defined in section 5.2.

Note. — In the following we prefer to maintain an expository and vulgarizing description of the results. In order to skip all technical assumptions in the theorem below, and in the remainder of the paper, we are not precise about the distances of q from 1, and of h to 0. The real assumptions can be found in [**Pul08a**] and [**Pul08b**].

Theorem 7.4. — *Assume that q is not a root of unity. Let $\sigma_{q,h}-Mod(\mathcal{H}_K(X))^{Rbig}$ be the full subcategory of $\sigma_{q,h}-Mod(\mathcal{H}_K(X))$ of objects such that the radius (7.2) verifies the condition (6.8).*[(9)] *We have a functor*

$$(\Delta^*_{\sigma_{q,h}})^{-1} \;:\; \sigma_{q,h}-Mod(\mathcal{H}_K(X))^{Rbig} \longrightarrow \mathrm{Strat}(\mathcal{H}_K(X))$$

which is a section of $\Delta^*_{\sigma_{q,h}}$, and that induces an equivalence of $\sigma_{q,h}-Mod(\mathcal{H}_K(X))^{Rbig}$ with its essential image. As a consequence we have a functor

$$\mathrm{Conf}_{\sigma_{q,h}} \;:\; \sigma_{q,h}-Mod(\mathcal{H}_K(X))^{Rbig} \longrightarrow d-Mod(\mathcal{H}_K(X))$$

which is a section of $\mathrm{Def}_{\sigma_{q,h}}$ and that induces an equivalence of $\sigma_{q,h}-Mod(\mathcal{H}_K(X))^{Rbig}$ with its essential image which is $d-Mod(\mathcal{H}_K(X))^{\mathrm{comp}(\sigma_{q,h})}$ (cf. 6.2).

Proof : By section 7.1 we know that, if q is not a root of unity, the deformation functor is fully faithful. On the other hand, the respective conditions on the radii of convergences of the objects of the two categories permit to characterize the essential image of the $\sigma_{q,h}$-deformation functor (providing q not to be a root of unity). Namely, the differential equations that can be deformed are those satisfying the condition (6.8). On the other hand, if q is not a root of unity, the essential image is given by the (q,h)-difference equations satisfying the same condition (6.8) with respect to $\mathrm{Rad}(\mathrm{M}_{q,h},e,|.|)$ (cf. (7.2)). It is then immediate to prove that the σ_q-deformation is an equivalence. □

7.3.2. Comparison with the classical definition of confluence as a limit for q tending to 1. — What people do in general to obtain a confluence is to observe that the q-derivation $d_q := \frac{\sigma_q-1}{(q-1)T}$ tends to the derivation d/dT as q tends to 1 (as operators over $\mathcal{H}_K(X)$). So if we have a family of equations

$$\{\sigma_q(Y) = A(q,T)Y\ \}_q\ ,$$

we can consider the action of d_q whose matrix is given by $\frac{A(q,T)-\mathrm{Id}}{(q-1)T}$. It is intended that firstly we choose a finite free $\mathcal{H}_K(X)$-module M independent on q. Then each one of these equations defines an action of σ_q and of d_q on M. We indicate by $d_q^{\mathrm{M}}:\mathrm{M}\to\mathrm{M}$ this action. Then one consider the limit (when it exists)

$$\nabla \;:=\; \lim_{q\to 1} d_q^{\mathrm{M}}\ . \tag{7.3}$$

(9) The notation *Rbig* means "radius is big" and is placed in the notation of the category to recall that its objects satisfy condition (6.8).

This limit exists if the family of equations depends analytically on q, i.e. if $A(q,T)$ varies analytically on q (cf. section 6.3, and [**Pul08b**]). In terms of matrices this limit becomes $\lim_{q\to 1}\frac{A(q,T)-\mathrm{Id}}{(q-1)T}$, intended as a limit in $\mathrm{End}_{\mathcal{H}_K(X)}(\mathrm{M})\cong M_n(\mathcal{H}_K(X))$. One easily verifies that, when this limit converges, then ∇ is a connection on M.

Proposition 7.5. — *Let* (M,∇) *be a differential equation over* $\mathcal{H}_K(X)$. *Let* $\varepsilon>0$ *small enough. For all* q *in the disk* $G=\mathrm{D}^-(1,\varepsilon)$ *we consider the* q*-deformation* $(\mathrm{M},\sigma_q^{\mathrm{M}}):=\mathrm{Def}_{\sigma_q}(\mathrm{M})$ *of* (M,∇).(10) *Then*

$$\nabla \;=\; \lim_{q\to 1} d_q^{\mathrm{M}}\;.$$

Proof : Let $e\subset\mathrm{M}$ be a fixed basis of M, and let $A(q,T)$ be the matrix of the action of σ_q on M. By construction, if $Y(x,y)$ is the generic Taylor solution of M (i.e. the matrix of the attached stratification), then $A(q,T)=Y(qT,T)$. Then $A(q,T)$ depends analytically on q (cf. section 6.3). We have to prove that, if $G(T)$ is the matrix of ∇, then

$$G(T) \;=\; \lim_{q\to 1}\frac{A(q,T)-\mathrm{Id}}{(q-1)T}\;. \tag{7.4}$$

Indeed because $Y(x,y)$ is solution of the equation $Y'=GY$, one has $G(x)=d/dx(Y(x,y))\cdot Y(x,y)^{-1}=d/dx(Y(x,y))\cdot Y(y,x)$. On the other hand, we have $\frac{A(q,x)-\mathrm{Id}}{(q-1)x}=\frac{Y(qx,x)-\mathrm{Id}}{(q-1)x}=\frac{Y(qx,y)Y(y,x)-Y(x,y)Y(y,x)}{(q-1)x}=\frac{Y(qx,y)-Y(x,y)}{(q-1)x}Y(y,x)$. So it remains to check that $\lim_{q\to 1}\frac{Y(qx,y)-Y(x,y)}{(q-1)x}=d/dx(Y(x,y))$, but this follows from that $\lim_{q\to 1}d_q=d/dx$ as operators on the first variable of $Y(x,y)$. □

Note. — The above proposition can easily be extended to (q,h)-difference equations. Another approach of the study detailed in this section is also done in [**DR08**] in the complex situation for fuchsian systems by using the analytic description of the Galois group done by Sauloy.

7.4. q-Confluence to a root of unity. — We have seen in section 7.2 that one can consider the whole action of $\boldsymbol{\mu}_{p^\infty}$ in order to make the deformation functor fully faithful, in this case the deformed objects are semi-linear representations of $\boldsymbol{\mu}_{p^\infty}$. The approach of [**Pul08b**] to the q-difference theory with q equal to a root of unity is somehow orthogonal to that of section 7.2. One fixes a single p^n-th root of unity ξ_{p^n}, and one adds somehow artificially a data to the deformed object (namely the action of a ξ_{p^n}-tangent operator) that make the deformation $\mathrm{Def}_{\xi_{p^n}}$ an equivalence. In other words one reproduces, at each root of unity, what happens at the particular root $q=1$ (that is the q-confluence of section 7.3). In this section we expose it quickly.

(10) As above we consider $\mathrm{D}^-(1,\varepsilon)$ as an analytic variety acting on M.

7.4.1. q-tangent operators. — As we have seen if (M,∇) is a differential equation over an affinoid X, then its q-deformation $\mathrm{Def}_{\sigma_q}(\mathrm{M})$ is defined for q lying in a p-adic analytic group $G := \mathrm{D}^-(1,\varepsilon)$, for a convenient $\varepsilon > 0$. The number $0 < \varepsilon \leq 1$ depends on X and on the radius of convergence of (M,∇) (cf. (6.9)). It actually happens that ε is often quite large so that the disk $\mathrm{D}^-(1,\varepsilon)$ contains some or all p^n-th roots of unity. We recall that if $q = \xi_{p^n}$ is a p^n-th root of unity, then the deformation functor can not be full (i.e. the inclusion $\mathrm{Hom}^\nabla(\mathrm{M},\mathrm{N}) \to \mathrm{Hom}^{\sigma_q}(\mathrm{Def}_{\sigma_q}(\mathrm{M}),\mathrm{Def}_{\sigma_q}(\mathrm{N}))$ is not surjective). Indeed, even for the unit object $(\mathbb{I},\nabla) = (\mathcal{H}_K(X), d/dT)$ we have $\mathrm{Def}_\sigma(\mathbb{I}) = (\mathcal{H}_K(X),\sigma_q)$, so that $\mathrm{End}^\nabla(\mathbb{I}) \cong K$ and

$$\mathrm{End}^{\sigma_q}(\mathbb{I}) \;\cong\; \{\; f(T) \in \mathcal{H}_K(X) \mid \text{ such that } f(T) = g(T^p)\;,\; \text{with } g(T) \in \mathcal{H}_K(X)\;\}\;.$$

The deformation is the identity on the morphisms, so that the inclusion

$$\mathrm{End}^\nabla(\mathbb{I}) \cong K \quad\subset\quad \mathrm{End}^{\sigma_q}(\mathbb{I})$$

is strict. We would like to have an equivalence as in the case where q is not equal to a root of unity. The idea is to reproduce the process that we actually already have obtained for the particular root of unity $q = 1$. When q approaches 1 the object that we have is not equal to a 1-difference equation, that is simply a finite free $\mathcal{H}_K(X)$-module without any structure (because the action of σ_1 is of course the identity). But conversely, what we have "at $q = 1$" is a *differential equation*. In other words, in order to obtain an equivalence, we have to take into account the action of σ_q, for all q in a small disk around 1 as in section 7.3.2. The idea hence is the following. Fix a q_0, and let $(\mathrm{M},\sigma_{q_0}^{\mathrm{M}})$ be a q_0-difference module. Assume that q_0 is not a root of unity, and that $(\mathrm{M},\sigma_{q_0}^{\mathrm{M}})$ satisfies the conditions of theorem 7.4. Then:

1) Since by theorem 7.4 the deformation functor is an equivalence, then from the simple knowledge of $\sigma_{q_0}^{\mathrm{M}}$ we immediately have the existence of an action of a connection ∇^{M} and (by deformation of ∇^{M}) of an operator σ_q^{M} for all q in a disk $\mathrm{D}^+(1,\varepsilon)$ containing q_0, for a suitable $\varepsilon \geq |q_0 - 1| > 0$.
2) The connection and the action of σ_q so obtained are characterized by the fact that the stratification attached to $(\mathrm{M},\sigma_{q_0}^{\mathrm{M}})$ is simultaneously the Taylor solution of all these operators.

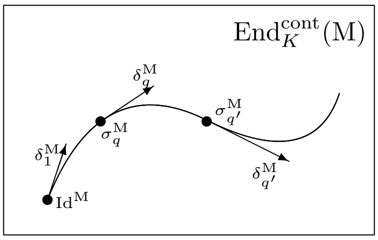
"At $q = 1$" we have a differential equation. By section 7.3.2 the connection ∇^{M} obtained by confluence, can be founded as the limit (cf. (7.3)):

$$\nabla^{\mathrm{M}} \;=\; \lim_{q\to 1} \frac{\sigma_q^{\mathrm{M}} - \mathrm{Id}^{\mathrm{M}}}{T(q-1)}\;.$$

So the operator $\delta_1^{\mathrm{M}} := T\cdot\nabla^{\mathrm{M}}$ can be seen as the value at $q = 1$ of the derivative of the map $q \mapsto \sigma_q^{\mathrm{M}}$ with values in $\mathrm{End}_K^{\mathrm{cont}}(\mathrm{M})$:

$$\delta_1^{\mathrm{M}} \;=\; \lim_{q\to 1} \frac{\sigma_q^{\mathrm{M}} - \mathrm{Id}^{\mathrm{M}}}{(q-1)} \;=\; q\frac{d}{dq}\big[(q \mapsto \sigma_q^{\mathrm{M}})\big]\Big|_{q=1}\;.$$

The main idea it to simply extend this notion to all q by considering the so called q-tangent operator acting on M:



$$\delta_q^{\mathrm{M}} := q\frac{d}{dq'}\Big(q' \mapsto \sigma_{q'}^{\mathrm{M}}\Big)\Big|_{q'=q} = q \cdot \lim_{q'\to q}\frac{\sigma_{q'}^{\mathrm{M}} - \sigma_q^{\mathrm{M}}}{q'-q} = \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}} . \tag{7.5}$$

Thus over all $q \in \mathrm{D}^-(1,\varepsilon)$, and in particular over the root of unity, we have a (pseudo differential) operator δ_q^{M} acting on M. It actually happens that $\delta_q^{\mathrm{M}} = \sigma_q^{\mathrm{M}} \circ \delta_1^{\mathrm{M}}$, so that the action of σ_q^{M} together with the action of δ_q^{M} is equivalent to the action of σ_q^{M} together with the action of ∇^{M}. What we have at each point is nothing but the same differential equations ∇^{M}. By the way, we remark that if q is not a root of unity the action of ∇^{M} (and of δ_q^{M}) is superfluous, because it can be entirely recovered from the simple knowledge of σ_q^{M} by applying the q-confluence functor.

Conversely, if q is a root of unity, the data of (δ_q^{M} or that of) ∇^{M} is necessary in order to preserve the information and make the deformation functor an equivalence. Note that if q is equal to a root of unity, then σ_q^{M} can be recovered from the knowledge of δ_q^{M} (or equivalently from ∇^{M}) by deformation. More precisely, it is then natural to introduce the following "mixed" category in order to state the analogue of theorem 7.4: For all $q \in \mathrm{D}^-(1,\varepsilon)$ let

$$(\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{H}_K(X))^{Rbig}$$

be the category of finite free modules over $\mathcal{H}_K(X)$ together with a semi-linear action of σ_q and a "compatible" action of a connection ∇^{M}. The morphisms in $(\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{H}_K(X))^{Rbig}$ commute with the actions of σ_q and ∇. The condition of compatibility between σ_q^{M} and ∇^{M} is that the Taylor solutions of ∇^{M} are also solutions of σ_q^{M} as in (6.11). Implicitly this forces the radius of convergence of the stratification to be big and to verify condition (6.8) (so the notation *Rbig* meaning "Radius is big" is placed for this reason in analogy to theorem 7.4). Note that in this definition q can be a root of unity. As already mentioned we have the following situation:

- if q is not equal to a root of unity, the data of δ_q^{M} is superfluous because it can be recovered by confluence from the knowledge of σ_q^{M}. In this case the category $(\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{H}_K(X))^{Rbig}$ is equivalent to $\sigma_q - \mathrm{Mod}(\mathcal{H}_K(X))^{Rbig}$ by the functor that "forgets δ_q".
- If q is equal to a root of unity, then the above two categories are not equivalent, but conversely the data of σ_q^{M} is superfluous because it can be recovered by deformation from the knowledge of (δ_q^{M} or equivalently of) ∇^{M}. In this case the category $(\sigma_q,\delta_q) - \mathrm{Mod}(\mathcal{H}_K(X))^{Rbig}$ is equivalent to $d - \mathrm{Mod}(\mathcal{H}_K(X))^{Rbig}$.

We summarize the above facts in the following result generalizing theorem 7.4 to roots of unity:

Theorem 7.6. — *Theorem 7.4 holds by replacing $\sigma_{q,h} - \mathit{Mod}(\mathcal{H}_K(X))^{Rbig}$ with $(\sigma_q, \delta_q) - \mathit{Mod}(\mathcal{H}_K(X))^{Rbig}$. In particular, by composition of functors, if q is not a root of unity and if ξ_{p^n} is a p^nth root of unity we have an equivalence*

$$\sigma_q - \mathit{Mod}(\mathcal{H}_K(X))^{Rbig} \xrightarrow{\sim} (\sigma_{\xi_{p^n}}, \delta_{\xi_{p^n}}) - \mathit{Mod}(\mathcal{H}_K(X))^{Rbig} \xrightarrow{\sim} d - \mathit{Mod}(\mathcal{H}_K(X))^{Rbig} .$$

The composite arrow is exactly the functor $\mathrm{Conf}_{\mathrm{q},0}$ *of Theorem 7.4.*

Note. — The notion of (σ_q, δ_q)-modules can also be interpreted as in the formalism of [**And01**, 1.4.4].

7.5. Aims, and comments. — The theory of σ-deformation as exposed in these notes has the merit of being intrinsic and to prove that in the ultrametric context the category of differential equations is equivalent to a full sub-category of that of σ-difference equations (assuming that σ is close enough to the identity, and non degenerate). This is useful to translate theorems of a theory into the other one. As an example the main result of [**DV04**] about the existence of a Frobenius antecedent (weak Frobenius structure) and transfer theorems, can actually be deduced by deformation without further computations from the same theorems in the category of differential equations. Another example for equations over the Robba ring is given by the q-analogous of the Christol-Mebkhout decomposition theorem by the slopes of the radius of convergence, and the quasi unipotence of the equations over the Robba ring with a Frobenius structure i.e. the Crew's conjecture : one of the deeper results of [**ADV04**]. All these results can be deduced *by deformation* by Taylor solution (i.e. as in section 6.2) without any computation. This is possible thanks to the fact that the deformation equivalence by the Taylor solution of section 6.2 is independent from any other result of the theory. In this sense it appears as an intrinsic.

Nevertheless, the fact that the deformation functor preserves the Taylor solutions is for some reasons highly unsatisfactory:

1.— Assume that one wants to study the solution Y of a differential equation, usually one aims to deform it into a q-difference equation whose solution Y_q approximates Y, and it approaches Y as q approaches 1. The most important examples of confluence in literature consider a deformation Y_q that is different from Y. Roughly speaking the intention is to simplify the problem by replacing Y with an easier solution matrix Y_q, but the fact that $Y_q = Y$ makes the two problems equivalent.

2.— Another problem that one encounters is the following. Assume that one works with differential and difference equations *with rational (or meromorphic) coefficients*, and that one wants to deform the differential equation into a q-difference equation in an "isomonodromic" way. The deformation functor as exposed in these notes is "*iso-galois*" so it would be the right way to deformation. But the problem is that if the starting equation has rational (or meromorphic) coefficients, then the q-difference equation having the same Taylor solutions often does not have rational (or meromorphic) coefficients. The deformation does not preserve the rationality (or the meromorphy) of the coefficients. In other words the reason is that the Taylor solution $Y(x, y)$ is analytic, but rarely rational or meromorphic. So that the matrix $A(q, T) := Y(qT, T)$ of the deformed equation is not rational or meromorphic either.

In order to satisfy the above aims we will be obliged to consider another notion of deformation that does not preserve the Taylor solutions. The fact that the Taylor solutions are preserved by the deformation functor is encoded in the fact that the functor is the identity in the morphisms (cf. section 6.2.3, point 4). Indeed the solutions can be interpreted as certain morphisms from the differential (resp. σ-difference) module with value in a certain differential (resp. σ-difference) algebra. The aim would be to have the possibility to work with *families of equations parametrized by* q in order to obtain a larger category of objects including the mentioned classical examples in a larger theory. More precisely, the idea (that is partially contained in [**Pul08b**]) is the following. As we have seen, the q-deformation functor associates to a differential module (M,∇) a σ_q-module $(\mathrm{Def}_{\sigma_q}(\mathrm{M}),\sigma_q)$ having the same solutions, since $\mathrm{Def}_{\sigma_q}(\mathrm{M}) = \mathrm{M}$, we obtain a canonical semi-linear action of σ_q on M. If we do that for all q we obtain an group action of $\mathrm{D}^+(1,\varepsilon)$ on M (cf. section 6.3), so that M is an analytic semi-linear representation of $\mathrm{D}^+(1,\varepsilon)$. This is a particular case of a family of equations indexed by q. The idea would be to drop the fact that this is a representation and instead to consider more general *sheaves of* q*-difference equations over* $\mathrm{D}^+(1,\varepsilon)$ with respect to a convenient (Grothendieck) topology of $\mathrm{D}^-(1,\varepsilon)$. Notice that if one visualizes a single q-difference equation as a "sheaf" over X, then a family of q-difference equations would be a "sheaf (over $\mathrm{D}^+(1,\varepsilon)$) of sheaves (over X)". As a matter of fact this should corresponds to a convenient notion of *sheaves of stratifications over* $\mathrm{D}^+(1,\varepsilon)$. In order to relate this to differential equations one certainly needs to obtain a generalized form of the correspondence between differential equations and stratifications (cf. theorem 4.1). The aim would be to find a global Galois groupoïd G over $\mathrm{D}^-(1,\varepsilon)$ attached to a given sheaf of stratifications, containing the information of each Galois group G_q of a single q-difference equation. In this direction it would be possible to study the eventual phenomena of iso-monodromy of equations with rational (or meromorphic) coefficients.

8. Towards a complex deformation (work in progress)

In this section we give quickly some ideas about a recent work in progress whose aim is to extend the above theory to equations over the complex numbers. The reader should consider the following as a list of obstructions to mimic the p-adic deformation process and not really as a complex deformation. One of the reasons is given at the point 2) of section 7.5 that we will also encounter in the remark 8.2. Nevertheless the ideas of section 7.2 are still valid in the complex case, and for this reason we found it interesting to notice them in this section. The reader should read the papers of Duval, Roques, Sauloy, ... (cf. [**Duv03**], [**Duv04**], [**DR08**], ...) in which one uses the analytic description of the Galois group to describe the variations of this group via confluence process. Galoisian deformation via Tannakian studies is an important part of [**And01**] which leads for instance to a computation of the Galois group of the q-hypergeometric functions.

Let $U \subseteq \mathbb{C}$ be an open subset, and let $\mathcal{H}(U)$ be the ring of analytic functions over U. If c is a point of U, it is known that the Taylor solution at c of a differential equation $Y' = G(x)Y$, with $G \in M_n(\mathcal{H}(U))$, always converges in the biggest open disk $\mathrm{D}^-(c, \rho_{c,U})$ contained in U. So the radius of convergence contains no informations about the equation. Clearly in this case $\rho_{c,U}$ is the distance of c from $\mathbb{C} - U$. Hence the generic Taylor solution $Y(x, y)$ converges in the following neighborhood of the diagonal $\mathcal{T}_U := \{(x, y) \in U \times U \mid |x-y| < \rho_{y,U}\}$. In this context the neighborhood $\mathcal{T}_U$ is independent on the equation. One easily can define the notion of stratification to this context and obtain a complete analogue of the equivalence theorem 4.1 between stratifications and differential modules. On the other hand the category of q-difference equations presents some essential difference with respect to the p-adic context. First of all we have a lack of open subsets of $\mathbb{C}$ that are stable under $x \mapsto qx$. E.g. we have $\mathbb{C} - \{0\}$, or a germ of a punctured disk at 0 or at ∞. Even assuming that U is stable under q-dilatation, the category $\sigma_q - \mathrm{Mod}(\mathcal{H}(U))$ is E_q-linear with respect to a ring E_q strictly larger than $\mathbb{C}$. This arises because of the existence of analytic functions verifying $f(qT) = f(T)$ (cf. section 3). So we can not hope to have any linear equivalence between differential and q-difference equations in this context. But we actually have a deformation functor. In order to deal with the above problems we proceed by considering the following more general situation.

8.1. — Let $\mathcal{A} \subseteq \mathbb{C}$ be an open subset containing U, and let $\sigma : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ be a bi-analytic automorphism of $\mathcal{A}$. We do not assume that U is stable under σ. As in section 6.2.1, in order to obtain a σ-difference equation we have to consider the pull-back of the stratification χ by the morphism $\Delta_\sigma : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$. The pull back exists where the composite function $\chi \circ \Delta_\sigma(T)$ exists, that is in the set $U_\sigma := \Delta_\sigma^{-1}(\mathcal{T}_U)$. In other words the matrix $A(T) := Y(\sigma(T), T)$, of the deformed equation, is defined on the open subset

$$U_\sigma \ := \ \{x \in U \mid |\sigma(x) - x| < \rho_{x,U}\} \ \subseteq \ U \ .$$

We are hence in presence of the following kind of object:

Definition 8.1. — *A* generalized *σ-difference equation is the data of a finite free $\mathcal{H}(U)$-module* M*, together with a linear isomorphism*

$$\sigma^{\mathrm{M}} \ : \ \sigma^*\mathrm{M} \xrightarrow{\sim} i^*\mathrm{M} \ ,$$

where $\sigma : U_\sigma \to U$ is the restriction of σ to U_σ, and $i : U_\sigma \to U$ is the natural inclusion. A morphism between two generalized σ-modules is an $\mathcal{H}(U)$-linear map $\alpha : \mathrm{M} \to \mathrm{N}$ commuting with σ^{M} and σ^{N}: $\sigma^{\mathrm{N}} \circ \sigma^*(\alpha) = i^*(\alpha) \circ \sigma^{\mathrm{M}}$. We still denote this category by $\sigma - Mod(\mathcal{H}(U))$.

Of course if the open U is "small", we may have $U_\sigma = \emptyset$, and in this case the object is trivial. In fact the analogous of the condition (6.8) is $U_\sigma \neq \emptyset$, so we will say that *σ acts infinitesimally on U if $U_\sigma \neq \emptyset$.* In complete analogy with section 6.2.1, by considering the pull back (by Δ_σ^*) of the stratification, one easily proves that one has a σ-deformation functor

$$\mathrm{Def}_\sigma \ : \ d - \mathrm{Mod}(\mathcal{H}(U)) \ \longrightarrow \ \sigma - \mathrm{Mod}(\mathcal{H}(U)) \ .$$

Notice that since the convergence locus $\mathcal{T}_U$ of the stratification is the same for all stratifications, then the functor is defined on all the category $d-\mathrm{Mod}(\mathcal{H}_K(X))$. In the case of q-difference equations (i.e. $\sigma = \sigma_q$), with $|q-1| < 1$, if U is a germ of punctured disk at 0, the right hand category is the usual category of q-difference equations (because if $U := \mathrm{D}^-(0, R) - \{0\}$ is a punctured disk, then $U_\sigma = \mathrm{D}^-(0, \frac{R}{|q-1|+1}) - \{0\}$ is again a punctured sub-disk of U).

8.2. — In general this deformation functor can not be an equivalence as we have seen. In order to obtain an analogue of the ideas of section 7.2 we notice that if we take into account the action of all $q \in \mathrm{D}^-(1,1)$, as in section 6.3, then on the right hand side we obtain the category of *semi-linear analytic representations of a germ of a multiplicative group*. This category is actually $\mathbb{C}$-linear, and the deformation functor is expected to be an equivalence in this case, as in the p-adic context. It is also expected to have a sort of analytic continuation permitting to extend the action of a small disk $q \in \mathrm{D}^-(1,1)$ to a larger domain. It is possibly a concrete subgroup of the multiplicative group. The aim would then be to obtain new invariants (by isomorphisms) of the differential equations by considering the invariants of this kind of representations. As an example one can restrict the representation to a germ of group of roots of unity and consider its cohomological invariants.

Remark *8.2*. —
1. *As in the p-adic case the deformation preserves, by construction, the generic Taylor solutions.*
2. *The idea of considering the definition 8.1 comes from the notion of* Frobenius structure *in the p-adic context. In that context the Frobenius map $\phi(x) = x^p$ sends an annulus $\{r < |x| < 1\}$ into another annulus $\{r^p < |x| < 1\}$. For a differential module defined over an open annulus, to have a Frobenius structure means exactly to admit the existence of an isomorphism $\phi^*\mathrm{M} \xrightarrow{\sim} i^*\mathrm{M}$, where i is the inclusion of the above annuli compatible with the connection.*
3. *As in section 7.5 the σ-deformation functor does not preserve the "meromorphy" of the coefficients: if a differential equation $Y' = G(T)Y$ has meromorphic coefficients, (i.e. if $G(T) \in M_n(\mathbb{C}(\{T\}))$), then its deformation $\sigma(Y) = A(T)Y$ does not verify $A(T) \in GL_n(\mathbb{C}(\{T\}))$. Because the matrix $A(T) = Y(\sigma(T), T)$ is obtained as the composite of the generic Taylor solutions with Δ_σ, both the maps are analytic, but not necessarily meromorphic at 0. This problem is related to the themes exposed in section 7.5: if we want to preserve the meromorphy of the coefficients, then we have to renounce to the constancy of the solutions.*

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