

Banachoid spaces

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Contents

1	Definitions	2
1.1	Normoid and Banachoid spaces	2
1.2	Bounded morphisms	5
1.3	Strict morphisms	7
2	Properties	8
2.1	Products and coproducts in Ban_K^b	8
2.2	Čech cohomology	9
2.3	Products and coproducts in $\text{Ban}_{M,K}$	9
2.4	Direct limits in $\text{Ban}_{M,K}$	10
2.5	Decomposable spaces	11
3	Tensor products	13
3.1	Definition and first properties	13
3.2	Exactness properties	14
3.3	Normoid Fréchet spaces	18
4	Cohomology of curves	22
4.1	Stein curves	22
4.2	de Rham cohomology	25
5	The Christol-Mebkhout limit formula for analytic cohomology	28

When computing de Rham cohomology, it is often better to work with spaces with no boundary rather than affinoid spaces. A typical example is that of the trivial equation on the unit disk: the first de Rham cohomology group is 0 on the open disk whereas it is infinite-dimensional on the closed one.

As a consequence, we are led to replace Banach algebras by Fréchet algebras. For our purpose, it is very important to be able to carry out extensions of scalars. For instance, if we want to apply Christol and Mebkhout's results from [CM00] and [CM01], as in Section ??, we need the base field to be algebraically closed and maximally complete. More generally, we would like to define tensor products and show that they behave as expected. For Banach spaces, this has been carried out by Gruson in [Gru66] and we will follow his arguments closely.

In what follows, we will not deal with arbitrary Fréchet spaces but with a more restrictive class of spaces where norms are part of the data. We believe that this is in accordance with the overall philosophy of Berkovich spaces where norms plays a prominent role and not only the induced topology.

Conventions

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Keywords:

We denote by \mathbb{N} the set of non-negative integers. We denote by \mathbb{R} the field of real numbers and by \mathbb{R}_+ the subset of real numbers that are greater than or equal to zero.

In all the text, K will be denote a fixed field endowed with a non-archimedean (possibly trivial) absolute value $|\cdot|: K \rightarrow \mathbb{R}_+$, with respect to which it is complete.

We set $K^\circ := \{x \in K, |x| \leq 1\}$. It is a local ring with maximal ideal $K^{\circ\circ} := \{x \in K, |x| < 1\}$.

In all the text, we will consider K -analytic space in the sense of Berkovich (see [Ber90] and [Ber93]), although choosing another another theory would lead to the same results and proofs.

1. Definitions

1.1. Normoid and Banachoid spaces

Definition 1.1. *A non-archimedean seminorm on a K -vector space U is a map $\|\cdot\|: U \rightarrow \mathbb{R}_+$ such that*

- i) $\forall x, y \in U, \|x + y\| \leq \max(\|x\|, \|y\|)$;
- ii) $\forall \lambda \in K, \forall x \in U, \|\lambda x\| = |\lambda| \|x\|$.

In the sequel of this paper, every seminorm will be non-archimedean and we simply speak about seminorms, without further mention of the terminology ultrametric.

Here, we will use the setting of uniform spaces from [Bou71, Chapitre II]. Recall that a family of pseudometrics on a space induces a uniform structure on it (see [Bou74, Chapitre IX, § 1]).

Definition 1.2. *Let M be a non-empty set. An M -normoid space (over K) is a K -vector space U together with a family of seminorms $u = (u_m)_{m \in M}$. We endow it with the uniform structure (and the topology) induced by u .*

A normoid space is a K -vector space that is M -normoid for some non-empty set M .

Definition 1.3. *Let U be a K -vector space. Let M and N be non-empty sets. Let $u = (u_m)_{m \in M}$ and $u' = (u'_n)_{n \in N}$ be two families of seminorms on U . We say that u' is finer than u if, for every $m \in M$, there exists a finite subset I_m of N and $C_m \in \mathbb{R}_+$ such that*

$$\forall x \in U, \quad u_m(x) \leq C_m \cdot \max_{n' \in I_m} (u'_{n'}(x)). \quad (1.1)$$

We say that u and u' are equivalent if u is finer than u' and u' is finer than u . This is an equivalence relation.

Notation 1.4. *When $(u_m)_{m \in M}$ is a family of seminorms and I a finite subset of M , we set*

$$u_I := \max_{m \in I} (u_m). \quad (1.2)$$

Remark that equivalent families of seminorms define the same uniform structure and topology.

Example 1.5. *Let $(u_m)_{m \in M}$ be a family of seminorms on U .*

- i) *Let $N \subseteq M$ be a subset such that for all $m \in M - N$ there exists a finite subset $I \subset N$ such that the norm u_I is finer than u_m . Then the family $(u_n)_{n \in N}$ is equivalent to the family $(u_m)_{m \in M}$ and also to the family $(u'_m)_{m \in M}$ given by*

$$u'_m = \begin{cases} u_m & \text{if } m \in N \\ 0 & \text{if } m \in M - N. \end{cases} \quad (1.3)$$

- ii) In particular, if $I \subset M$ is a finite subset and if $m \in I$, we may replace the seminorm u_m by u_I without changing the equivalence class of the family $(u_m)_{m \in M}$.
- iii) More generally, if $A \subseteq M$ is a subset, and B is a collection of finite subsets of M such that, for all $m \in A$, there is $I_m \in B$ such that $m \in I_m$, then $(u_m)_{m \in M}$ is equivalent to $(u_m)_{m \in M-A} \cup (u_I)_{I \in B}$. In particular, the family $(u_m)_{m \in M}$ is always equivalent to the family $(u_I)_I$, where I runs through the set of all finite subsets of M .
- iv) If for some $m \in M$ there exists a finite subset $I \subset M$ and a constant $C \in \mathbb{R}_+$ such that $m \notin I$ and $u_m \leq C \cdot u_I$, then we can remove u_m from the family without changing the equivalence class.
- v) More generally, if $A \subseteq M$ is a subset and if, for all $m \in A$, there exists a finite subset $I_m \subseteq M$ and a constant $C_m \in \mathbb{R}_+$ such that $I_m \cap A = \emptyset$ and $u_m \leq C_m \cdot u_{I_m}$, then the family $(u_m)_{m \in M}$ is equivalent to the family $(u_m)_{m \in M-A}$.

Definition 1.6. Let M and N be non-empty sets. We say that an M -normoid space (U, u) is equivalent to an N -normoid space (V, v) if there exists a K -linear isomorphism $f: U \rightarrow V$ such that the family of seminorms $f^*(v)$ that is induced by v on U by transport of structure is equivalent to u . This is an equivalence relation.

Remark that equivalent normoid spaces define isomorphic uniform and topological spaces.

Lemma 1.7. Let M and N be non-empty sets. Let $u = (u_m)_{m \in M}$ and $u' = (u'_n)_{n \in N}$ be two families of seminorms on U . Consider the following assertions:

- (1) the identity map $\text{Id}: (U, u') \rightarrow (U, u)$ is continuous;
- (2) the uniform structure defined by u' is finer than that defined by u ;
- (3) the topology defined by u' is finer than that defined by u ;
- (4) u' is finer than u .

Then (1), (2), (3) are equivalent and (4) implies them. If K is not trivially valued, then all the conditions are equivalent.

Let (V, v) be an N -normoid space. Consider the following assertions:

- (i) there exists a K -linear homeomorphism $(U, u) \xrightarrow{\sim} (V, v)$;
- (ii) (U, u) and (V, v) are equivalent normoid spaces.

Then (ii) always implies (i) and, if K is not trivially valued, then (i) implies (ii).

In particular, if (U, u) and (V, v) are equivalent normoid spaces, then (U, u) is Hausdorff (resp. complete) if, and only if, (V, v) is. \square

Definition 1.8. Let M be a non-empty set. An M -Banachoid space is a Hausdorff complete M -normoid space.

A Banachoid space is a K -vector space that is M -Banachoid for some non-empty set M .

Remark 1.9. Let $(U, u = (u_m)_{m \in M})$ be an M -normoid space. It is Hausdorff if, and only if, for each $x \in U - \{0\}$, there exists $m \in M$ such that $u_m(x) \neq 0$.

A net $(x_\alpha)_{\alpha \in D}$ (for some directed set D) is Cauchy if, and only if, it is Cauchy with respect to each seminorm u_m . It converges to an element x of U if, and only if, for each $m \in M$, the net $(u_m(x_\alpha - x))_{\alpha \in D}$ converges to 0.

The space (U, u) is complete if, and only if, every Cauchy net converges.

Lemma 1.10. *Let U be a finite-dimensional K -vector space. Let u be a family of seminorms on U such that (U, u) is a Hausdorff normoid space. Let $\|\cdot\|$ be a norm on U . Then, u is equivalent to $\|\cdot\|$, (U, u) is a Banachoid space and $(U, \|\cdot\|)$ is a Banach space.*

Proof. For every seminorm v_0 on U , we define the kernel of v_0 as

$$\text{Ker}(v_0) = \{x \in U \mid v_0(x) = 0\}. \quad (1.4)$$

It is a subspace of U . For every family of seminorms $v = (v_m)_{m \in M}$ on U , we set

$$d_v = \min(\{\dim_K(\text{Ker}(v_m)), m \in M\}). \quad (1.5)$$

Among the family of seminorms on U that are equivalent to u , let us now choose a family $u' = (u'_n)_{n \in N}$ such that $d_{u'}$ is minimal. There exists $n_0 \in N$ such that $\dim_K(\text{Ker}(u'_{n_0})) = d_{u'}$. Let us assume by contradiction that $d_{u'} > 0$. Pick $x \in U \setminus \{0\}$ such that $u'_{n_0}(x) = 0$. Since (U, u') is Hausdorff, there exists $n_1 \in N$ such that $u'_{n_1}(x) \neq 0$. Replace in u' the seminorm u'_{n_0} by the seminorm $\max(u'_{n_0}, u'_{n_1})$, we find an equivalent family u'' with $d_{u''} < d_{u'}$, which contradicts the definition of u' .

We have just proven that $d_{u'} = 0$. In other terms, u'_{n_0} is a norm. We now construct a new family of seminorms $u'' = (u''_n)_{n \in N}$ on U by setting, for every $n \in N$, $u''_n = \max(u'_n, u'_{n_0})$. It is equivalent to u' and it is a family of norms. Since all the norms on a finite-dimensional space over a complete valued field are equivalent¹, the family u'' is equivalent to the family with one element $(\|\cdot\|)$.

Using the equivalence of norms again, it is easy to show that $(U, \|\cdot\|)$ is a Banach space and to deduce that (U, u) is a Banachoid space. \square

Definition 1.11. *Let X be a K -analytic space. Let \mathcal{V} be an affinoid covering of X for the G -topology.*

- i) *For every $V \in \mathcal{V}$, denote by u_V the norm on the K -affinoid algebra $\mathcal{O}(V)$ and by u'_V the composition of u_V with the restriction map $\mathcal{O}(X) \rightarrow \mathcal{O}(V)$. We call $(u'_V)_{V \in \mathcal{V}}$ the normoid structure on $\mathcal{O}(X)$ associated to \mathcal{V} .*
- ii) *Let \mathcal{F} be a coherent sheaf on X . For every $V \in \mathcal{V}$, there exists a surjection $\mathcal{O}(V)^{n_V} \rightarrow \mathcal{F}(V)$. Denote by v_V the residue seminorm on $\mathcal{F}(V)$ (which is actually a norm, see [Ber90, Proposition 2.1.9 and its proof]) and by v'_V the composition of v_V with the restriction map $\mathcal{F}(X) \rightarrow \mathcal{F}(V)$. We call such a normoid structure a basic admissible structure on $\mathcal{F}(X)$ associated to \mathcal{V} . A normoid structure on $\mathcal{F}(X)$ is called admissible if it is equivalent to a basic admissible normoid structure (for some choices of covering and surjections).*
- iii) *Let U be an analytic domain of X such that $\mathcal{V}_U := \{V \in \mathcal{V} \mid V \subseteq U\}$ is a covering of U for the G -topology.
The previous construction endows $\mathcal{F}(U)$ with a \mathcal{V}_U -normoid structure. We turn it into a \mathcal{V} -normoid structure by adding, for all $V \in \mathcal{V}$ not contained in U , the zero seminorm on $\mathcal{F}(U)$. We say that this normoid structure on $\mathcal{F}(U)$ is induced by that on $\mathcal{F}(X)$.*
- iv) *Let L be a complete non-trivially valued extension of K . Then $\mathcal{V}_L := \{V_L \mid V \in \mathcal{V}\}$ is a covering of X_L for the G -topology. Moreover, for every $V \in \mathcal{V}$, v_V induces a norm on $\mathcal{F}_L(V_L) \simeq \mathcal{F}(V) \hat{\otimes}_K L$.² The previous construction endows $\mathcal{F}_L(X_L)$ with a \mathcal{V} -normoid structure. We say that this normoid structure on $\mathcal{F}_L(X_L)$ is induced by that on $\mathcal{F}(X)$.*

¹This classical result remains true when K is trivially valued, see for instance [Ked10, Theorem 1.3.6].

²The isomorphism $\mathcal{F}_L(V_L) \simeq \mathcal{F}(V) \hat{\otimes}_K L$ is classical in the affinoid case, it may be proved by choosing a presentation and using the right exactness of the completed tensor product by L (see [Gru66]). We will prove a generalization of this fact in Corollary 3.22.

Lemma 1.12. *Retain the notations of Definition 1.11. The space $\mathcal{O}(X)$ endowed with the normoid structure associated to some covering is a Banachoid space. The normoid structures on $\mathcal{O}(X)$ associated to two affinoid coverings of X are equivalent. If X is reduced, they induce the topology of compact convergence.*

The space $\mathcal{F}(X)$ endowed with an admissible normoid structure is a Banachoid space. All the admissible normoid structures on $\mathcal{F}(X)$ are equivalent.

The induced normoid structures on the spaces $\mathcal{F}(U)$ and $\mathcal{F}(X)_L$ are admissible. In particular, endowed with the latter, the spaces $\mathcal{F}(U)$ and $\mathcal{F}(X)_L$ are Banachoid spaces.

Proof. The proofs of those results are similar to the proofs of analogous results in Berkovich theory (in a Banach setting) and use classical arguments. The first one essentially follows from Tate's acyclicity theorem (see [Ber93, Lemma 1.2.12 and the discussion following it]). If X is reduced, then each affinoid domain V of X is reduced, hence the given norm on $\mathcal{O}(V)$ is equivalent to the sup-norm on V by [Ber90, Proposition 2.1.4]. It follows that the induced topology is that of compact convergence.

For the second result, we refer to the proof of [Ber90, Proposition 2.1.9] for some details. The last result is a consequence of the definitions. □

1.2. Bounded morphisms

We now define *bounded* morphisms. Recall Notation 1.4.

Definition 1.13. *Let $(U_1, u^1), \dots, (U_r, u^r), (V, v)$ be normoid spaces whose families of seminorms are indexed by sets M_1, \dots, M_r, N respectively. An r -linear map $f: (U_1, u^1) \times \dots \times (U_r, u^r) \rightarrow (V, v)$ is said to be bounded if, for every finite subset J of N , there exists $C_J \in \mathbb{R}_+$ and finite subsets J_1, \dots, J_r of M_1, \dots, M_r respectively such that*

$$\forall (x_1, \dots, x_r) \in U_1 \times \dots \times U_r, \quad v_J(f(x_1, \dots, x_r)) \leq C_J \prod_{i=1}^r u_{J_i}^i(x_i). \quad (1.6)$$

Let $f: (U_1, u^1) \times \dots \times (U_r, u^r) \rightarrow (V, v)$ be a bounded r -linear map. For every finite subsets J, I_1, \dots, I_r of N, M_1, \dots, M_r , we set

$$N'_{J, I_1, \dots, I_r}(f) = \inf(\{D \geq 0 \mid \forall (x_1, \dots, x_r) \in U_1 \times \dots \times U_r, v_J(f(x_1, \dots, x_r)) \leq D \prod_{i=1}^r u_{I_i}^i(x_i)\}) \quad (1.7)$$

and

$$N_{J, I_1, \dots, I_r}(f) = \begin{cases} N'_{J, I_1, \dots, I_r}(f) & \text{if } N'_{J, I_1, \dots, I_r}(f) < +\infty; \\ 0 & \text{otherwise.} \end{cases} \quad (1.8)$$

Remark 1.14. *We get an equivalent definition if we require that the property holds for all singletons of N instead of all finite subsets. Similarly, the family of seminorms $(N_{\{n\}, I_1, \dots, I_r})_{n, I_1, \dots, I_r}$ and $(N_{J, I_1, \dots, I_r})_{J, I_1, \dots, I_r}$ are equivalent since $N_{J, I_1, \dots, I_r} = \max_{n \in J} (N_{\{n\}, I_1, \dots, I_r})$.*

We also introduce a more restrictive notion of *contraction*.

Definition 1.15. *Let $(U_1, u^1), \dots, (U_r, u^r), (V, v)$ be normoid spaces whose families of seminorms are all indexed by the same set M . An r -linear map $f: (U_1, u^1) \times \dots \times (U_r, u^r) \rightarrow (V, v)$ is said to*

be a contraction if, for every $m \in M$, we have

$$\forall(x_1, \dots, x_r) \in U_1 \times \dots \times U_r, \quad v_m(f(x_1, \dots, x_r)) \leq \prod_{i=1}^n u_m^i(x_i). \quad (1.9)$$

For every $m \in M$, we set

$$N_m(f) = \inf(\{D \geq 0 \mid \forall(x_1, \dots, x_r) \in U_1 \times \dots \times U_r, v_m(f(x_1, \dots, x_r)) \leq D \prod_{i=1}^n u_m^i(x_i)\}). \quad (1.10)$$

Given normoid spaces $(U_1, u^1), \dots, (U_r, u^r), (V, v)$, we denote by $\text{Mult}_b((U_1, u^1) \times \dots \times (U_r, u^r), (V, v))$ (resp. $\text{Mult}_{b,1}((U_1, u^1) \times \dots \times (U_r, u^r), (V, v))$) the set of bounded r -linear maps (resp. r -linear contractions) between them. When $r = 1$, we simply write $\mathcal{L}_b((U_1, u^1), (V, v))$ (resp. $\mathcal{L}_{b,1}((U_1, u^1), (V, v))$). Each N_{J, I_1, \dots, I_r} (resp. N_m) is a seminorm on this space.

Lemma 1.16. *If (V, v) is Hausdorff (resp. complete), then the space $\text{Mult}_b((U_1, u^1) \times \dots \times (U_r, u^r), (V, v))$ (resp. $\text{Mult}_{b,1}((U_1, u^1) \times \dots \times (U_r, u^r), (V, v))$) together with the family $(N_{J, I_1, \dots, I_r})_{J, I_1, \dots, I_r}$ (resp. (N_m)) is a Hausdorff (resp. complete) normoid space. \square*

Lemma 1.17. *Let $(U_1, u^1), \dots, (U_r, u^r), (V, v)$ be normoid spaces. The families of seminorms that we have defined on $\text{Mult}_b(U_1 \times \dots \times U_r, V)$ and $\text{Mult}_b(U_1 \times \dots \times U_{r-1}, \mathcal{L}_b(U_r, V))$ are compatible with the natural isomorphism*

$$\text{Mult}_b(U_1 \times \dots \times U_r, V) \xrightarrow{\sim} \text{Mult}_b(U_1 \times \dots \times U_{r-1}, \mathcal{L}_b(U_r, V)). \quad (1.11)$$

The analogous result holds for

$$\text{Mult}_{b,1}(U_1 \times \dots \times U_r, V) \xrightarrow{\sim} \text{Mult}_{b,1}(U_1 \times \dots \times U_{r-1}, \mathcal{L}_{b,1}(U_r, V)). \quad (1.12)$$

Proof. It follows from a direct computation with the help of Remark 1.14. \square

Lemma 1.18. *Let U be a K -vector space. Let u and u' be two normoid structures on U (possibly indexed by different sets). Then the map $\text{Id}: (U, u') \rightarrow (U, u)$ is bounded (resp. bi-bounded) if, and only if, u' is finer than u (resp. u' is equivalent to u). \square*

Lemma 1.19. *Let $(U_1, u^1), \dots, (U_r, u^r), (V, v)$ be normoid spaces and let $f: U_1 \times \dots \times U_r \rightarrow V$ be an r -linear map. If f is bounded, then it is a continuous map. If K is not trivially valued, then the converse also holds. \square*

Lemma 1.20. *Let M_1, \dots, M_r, N be non-empty sets. Let $(U_1, u^1 = (u_m^1)_{m \in M_1}), \dots, (U_r, u^r = (u_m^r)_{m \in M_r})$ and $(V, v = (v_n)_{n \in N})$ be normoid spaces. Let $f: (U_1, u^1) \times \dots \times (U_r, u^r) \rightarrow (V, v)$ be a bounded r -linear map. Then, there exist a non-empty set M and families of seminorms a^1, \dots, a^r, b on U_1, \dots, U_r, V , all indexed by M , that are equivalent to u^1, \dots, u^r, v respectively and such that the morphism $(U_1, a^1) \times \dots \times (U_r, a^r) \rightarrow (V, b)$ induced by f is a contraction.*

Proof. Let $n \in N$. Consider subsets J_1, \dots, J_r of M_1, \dots, M_r and a constant C_n as in (1.6).

Denote by M the disjoint union of the M_i 's and N . For all $m \in M$, we set

$$b_m := \begin{cases} v_m & \text{if } m \in N \\ 0 & \text{if } m \notin N \end{cases} \quad a_m^i := \begin{cases} u_m^i & \text{if } m \in M_i \\ 0 & \text{if } m \in M_j \text{ and } j \neq i \\ C_m^{1/r} u_m^i & \text{if } m \in N \end{cases} \quad (1.13)$$

By construction, f is a contraction with respect to the M -normoid structures given by a^1, \dots, a^r, b . Moreover, proceeding as in Example 1.5, it is not hard to show that the families of seminorms a^1, \dots, a^r, b are equivalent to u^1, \dots, u^r, v .

□

Let us now give the basic examples of bounded maps.

Lemma 1.21. *Let X be a K -analytic space, let U be an analytic domain of X and let \mathcal{F} and \mathcal{G} be coherent sheaves on X . Endow $\mathcal{F}(X)$, $\mathcal{F}(U)$ and $\mathcal{G}(X)$ with admissible normoid structures. Then, the following results hold.*

- i) *The restriction map $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is bounded.*
- ii) *For each morphism of coherent sheaves $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ that is \mathcal{O}_X -linear, the associated map $\varphi(X): \mathcal{F}(X) \rightarrow \mathcal{G}(X)$ is bounded.*

Proof. Point i) follows from the definitions. For point ii), we may assume that $\mathcal{F}(X)$ and $\mathcal{G}(X)$ are endowed with basic admissible normoid structures associated to an affinoid covering \mathcal{V} of X for the G -topology. We then argue as in the proof of [Ber90, Proposition 2.1.9]. □

In Proposition 4.13, we will show that connections on locally free sheaves over curves give rise to bounded maps too.

Definition 1.22. *The category Norm_K^b is the category whose objects are normoid spaces and whose morphisms are bounded K -linear maps. The category Ban_K^b is the full subcategory of Norm_K^b whose objects are Banachoid.*

Let M be a non-empty set. The category $\text{Norm}_{M,K}$ is the category whose objects are M -normoid spaces and whose morphisms are K -linear contractions. The category $\text{Ban}_{M,K}$ is the full subcategory of $\text{Norm}_{M,K}$ whose objects are M -Banachoid.

1.3. Strict morphisms

Let us now describe explicitly strict morphisms in the category $\text{Norm}_{M,K}$.

Definition 1.23. *Let $(U, u = (u_m)_{m \in M})$ and $(V, v = (v_m)_{m \in M})$ be normoid spaces.*

A linear contraction $f: (U, u) \rightarrow (V, v)$ is said to be strict if the canonical map from its coimage to its image is an isomorphism in $\text{Norm}_{M,K}$, i.e.

$$\forall m \in M, \quad \forall y \in \text{Im}(f), \quad v_m(y) = \inf_{x \in f^{-1}(y)} (u_m(x)). \quad (1.14)$$

A strict injection will be called an isometry.

In this paper, we will not consider strict morphisms in Norm_K^b , hence the phrase “strict morphism” will always refer to the previous definition (and similarly for “isometry”). On the other hand, the topological version of the notion will be useful.

Definition 1.24. *A continuous map $f: (U, u) \rightarrow (V, v)$ between normoid spaces is said to be topologically strict if the canonical map*

$$U / \text{Ker}(f) \xrightarrow{\sim} \text{Im}(f) \quad (1.15)$$

is a homeomorphism.

Under some conditions, topologically strict maps give rise to strict maps.

Lemma 1.25. *Let (U, u) and (V, v) be normoid spaces and let $f: U \rightarrow V$ be a topologically strict K -linear map. Assume, moreover, that $\text{Im}(f)$ is topologically complemented in V . Then, there exist*

a non-empty set M and families of seminorms u' and v' indexed by M on U and V that induce the same topologies as u and v respectively such that the map $f: (U, u') \rightarrow (V, v')$ is strict.

Proof. Denote $u = (u_m)_{m \in M}$ and $v = (v_n)_{n \in N}$. By assumption, the induced map $U/\text{Ker}(f) \rightarrow \text{Im}(f)$ is a homeomorphism. For every $m \in M$, denote by w_m the seminorm on $\text{Im}(f)$ induced by u_m via the previous homeomorphism. The family $w = (w_m)_{m \in M}$ induces the topology of $\text{Im}(f)$.

Let W be a topological complement of $\text{Im}(f)$ in V . For every $(m, n) \in M \times N$, define a seminorm $v'_{m,n}$ on V by

$$v'_{m,n}: \begin{array}{ccc} V = \text{Im}(f) \oplus W & \rightarrow & \mathbb{R} \\ y \oplus z & \mapsto & \max(w_m(y), v_n(z)) \end{array} . \quad (1.16)$$

For every $(m, n) \in M \times N$, set $u'_{m,n} = u_m$. The families u' and v' satisfy the required properties. \square

Lemma 1.26. *Let $(U, (u_m)_{m \in M})$ be a normoid space. Then, there exists a Hausdorff normoid space $(U_H, (u_{H,m})_{m \in M})$ and a contraction $h: U \rightarrow U_H$ such that, for every contraction $f: U \rightarrow V$ from U to a Hausdorff normoid space V , there exists a unique contraction g such that $f = g \circ h$.*

Moreover, the morphism h is surjective and isometric in the sense that

$$\forall x \in U, \quad \forall m \in M, \quad u_{H,m}(h(x)) = u_m(x) . \quad (1.17)$$

Proof. Set

$$U_0 := \{x \in U \mid \forall m \in M, u_m(x) = 0\} . \quad (1.18)$$

It is a subspace of U and the quotient $U_H := U/U_0$ satisfies the required properties. \square

Definition 1.27. *The space U_H of the previous lemma is called the biggest Hausdorff quotient of U . Its equivalence class only depends on the equivalence class of U . We usually denote the seminorms on U_H by u_m instead of $u_{H,m}$.*

Lemma 1.28. *Let $(U, (u_m)_{m \in M})$ be a normoid space. Then, there exists a Banachoid space $(\hat{U}, (\hat{u}_m)_{m \in M})$ and a contraction $c: U \rightarrow \hat{U}$ such that, for every contraction $f: U \rightarrow V$ from U to a Banachoid space V , there exists a unique contraction g such that $f = g \circ c$.*

Moreover, the morphism c is isometric in the sense that

$$\forall x \in U, \quad \forall m \in M, \quad \hat{u}_m(c(x)) = u_m(x) \quad (1.19)$$

and, if U is Hausdorff, then c is injective.

Proof. Using the notation of Lemma 1.26, we define \hat{U} as the completion of U_H with respect to the family of seminorms $(u_{H,m})_{m \in M}$ (see [Bou74, Chapitre IX, § 1, N° 3, Proposition 1]). Since every $u_{H,m}$ is uniformly continuous, it extends to \hat{U} and it is easy to check that it is still a seminorm. \square

Definition 1.29. *The space \hat{U} of the previous lemma is called the Hausdorff completion of U . Its equivalence class only depends on the equivalence class of U . We usually denote the seminorms on \hat{U} by u_m instead of \hat{u}_m .*

2. Properties

2.1. Products and coproducts in Ban_K^b

The category Ban_K^b admits small products and coproducts. The construction is described in the following lemma whose proof is left to the reader.

Lemma 2.1. Let $\mathcal{E} = (E_i, (u_{m_i}^i)_{m_i \in M_i})_{i \in I}$ be a family of Banachoid spaces. Let $F := \prod_{i \in I} E_i$ be the product of the E_i 's in the category of K -vector spaces and, for each $i \in I$, denote by $p_i : F \rightarrow E_i$ the projection map. Set $M := \bigsqcup_{i \in I} M_i$. For each $m \in M$, set $v_m = u_m^i \circ p_i$, where i is the unique element of I such that $m \in M_i$.

Then $(F, (v_m)_{m \in M})$ is the product and the coproduct of the family \mathcal{E} in Ban_K^b . \square

Remark 2.2. If, in the setting of the previous lemmas, the family of Banachoid spaces is finite (i.e. I is finite), there is another natural construction that is sometimes convenient to work with. Set $F := \prod_{i \in I} E_i = \bigoplus_{i \in I} E_i$ and $N := \prod_{i \in I} M_i$. For every $n = (m_i)_{i \in I} \in N$ and every $x = (x_i)_{i \in I} \in F$, set $v_n(x) := \max_{i \in I} (u_{m_i}^i(x_i))$.

Then $(F, (v_n)_{n \in N})$ is the product and the coproduct of the family \mathcal{E} in Ban_K^b .

Remark 2.3. The normoid structure on the product or the coproduct of a family of Banachoid spaces is well defined only up to equivalence. Changing each element of the family by an equivalent element turns the product and the coproduct into equivalent spaces.

2.2. Čech cohomology

We now show that the product in Ban_K^b is well-suited for Čech cohomology.

Lemma 2.4. Let X be a K -analytic space and let \mathcal{V} be an affinoid covering of X for the G -topology. Let \mathcal{F} be a coherent sheaf on X . For each $V \in \mathcal{V}$, endow $\mathcal{F}(V)$ with a norm v_V as in item ii) of Definition 1.11. Consider the product $\prod_{V \in \mathcal{V}} \mathcal{F}(V)$ in Ban_K^b . Then, the normoid structure on $\mathcal{F}(X)$ induced by the injection

$$\mathcal{F}(X) \hookrightarrow \prod_{V \in \mathcal{V}} \mathcal{F}(V) \quad (2.1)$$

coincides with the basic admissible normoid structure from Definition 1.11.

For each affinoid domain U of X that is a finite intersection of elements of \mathcal{V} , endow $\mathcal{F}(U)$ with a norm v_U as in item ii) of Definition 1.11.

If X is separated, then the cohomology of \mathcal{F} on X is the cohomology of the complex of Banachoid spaces with bounded maps

$$\mathcal{C} : 0 \rightarrow \prod_{V \in \mathcal{V}} \mathcal{F}(V) \rightarrow \prod_{V \neq W \in \mathcal{V}} \mathcal{F}(V \cap W) \rightarrow \dots, \quad (2.2)$$

where the products are taken in Ban_K^b .

Proof. The first part of the result follows directly from the definitions. The second part follows from the fact that the complex of K -vector spaces underlying \mathcal{C} is nothing but the Čech complex of \mathcal{F} associated to \mathcal{V} and that, in the case where X is separated, finite intersections of affinoid domains of X are either empty or affinoid domains, hence \mathcal{F} -acyclic, by Tate's acyclicity theorem. \square

2.3. Products and coproducts in $\text{Ban}_{M,K}$

We now turn to the category $\text{Ban}_{M,K}$. It admits small products and coproducts too.

Lemma 2.5. Let $\mathcal{E} = (E_i, (u_m^i)_{m \in M})_{i \in I}$ be a family of Banachoid spaces.

Let F be the K -vector space of families $(x_i)_{i \in I} \in \prod_{i \in I} E_i$ such that, for each $m \in M$, $(u_m^i(x_i))_{i \in I}$ is bounded. For each $m \in M$ and each $x = (x_i)_{i \in I} \in F$, we set $v_m(x) := \sup_{i \in I} (u_m^i(x_i))$.

Then $(F, (v_m)_{m \in M})$ is the product of the family \mathcal{E} in $\text{Ban}_{M,K}$. \square

Lemma 2.6. Let $\mathcal{E} = (E_i, (u_m^i)_{m \in M})_{i \in I}$ be a family of Banachoid spaces.

Let G be the K -vector space of families $(x_i)_{i \in I} \in \prod_{i \in I} E_i$ such that, for each $m \in M$, $(u_m^i(x_i))_{i \in I}$ tends to 0 along the filter of complements of finite subsets of I . For each $m \in M$ and each $x = (x_i)_{i \in I} \in G$, we set $v_m(x) := \sup_{i \in I} (u_m^i(x_i))$.

Then $(G, (v_m)_{m \in M})$ is the coproduct of the family \mathcal{E} in $\text{Ban}_{M,K}$. \square

2.4. Direct limits in $\text{Ban}_{M,K}$

Since the category of Banachoid spaces $\text{Ban}_{M,K}$ is additive and has kernels and cokernels³ (hence equalizers and coequalizers), we deduce that it is complete and cocomplete (see [ML98, Theorem V.2.1]). We describe direct limits (*i.e.* colimits of directed families of objects) explicitly.

Let (I, \leq) be a directed set. Let $(U_i, (u_{i,m})_{m \in M})_{i \in I}$ be a family of Banachoid spaces and $(f_{j,i}: U_i \rightarrow U_j)_{i \leq j}$ be a family of linear contractions such that the family $(U_i, f_{i,j})$ is a direct system in $\text{Ban}_{M,K}$.

Let V_0 be the coproduct of the U_i 's in the category of sets (*i.e.* their disjoint union). Let \sim be the equivalence relation on V_0 defined by

$$(x_i \in U_i) \sim (x_j \in U_j) \quad \text{if} \quad \exists k \in I, \quad k \geq i, \quad k \geq j, \quad f_{k,i}(x_i) = f_{k,j}(x_j). \quad (2.3)$$

The quotient V_0 / \sim is the set-theoretic direct limit of $(U_i, f_{i,j})$.

Let $m \in M$ and $x \in V_0$. Choose $i \in I$ and $x_i \in U_i$ such that x_i belongs to the class defined by x . We then set

$$v_m(x) = \inf_j (u_{j,m}(f_{j,i}(x_i))). \quad (2.4)$$

Since I is a directed set, v_m is well defined and it is a seminorm on V_0 .

The Hausdorff completion V of V_0 / \sim with respect to the family of the v_m 's is the direct limit of $(U_i, f_{i,j})$ in the category of Banachoid spaces $\text{Ban}_{M,K}$.

The explicit construction of direct limits allows to prove basic properties easily.

Proposition 2.7. *Direct limits in $\text{Ban}_{M,K}$ preserve strict morphisms, isometries, and exact sequences of strict morphisms.*

Proof. The fact that direct limits preserve strict morphisms and isometries follows from a simple computation (see the proof of [Gru66, Proposition 1]).

Let us now consider a directed family of exact sequences $(\mathcal{C}_i: A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i)_{i \in I}$ with strict morphisms. We want to prove that the sequence

$$\mathcal{C}: \varinjlim_{i \in I} A_i \xrightarrow{f} \varinjlim_{i \in I} B_i \xrightarrow{g} \varinjlim_{i \in I} C_i \quad (2.5)$$

is exact. It is enough to prove that $\text{Ker}(g) \subseteq \text{Im}(f)$.

Let $x \in \text{Ker}(g)$. By the first point, the morphism $\varinjlim_{i \in I} \text{Im}(g_i) \rightarrow \varinjlim_{i \in I} C_i$ is an isometry, hence an injection. We deduce that g sends x to the element 0 in $\varinjlim_{i \in I} \text{Im}(g_i)$.

For every $i \in I$, we have a short exact sequence in $\text{Ban}_{M,K}$

$$\mathcal{D}_i: 0 \rightarrow A_i / \text{Ker}(f_i) \rightarrow B_i \rightarrow \text{Im}(g_i) \rightarrow 0. \quad (2.6)$$

We deduce that, for every Banachoid space E , we have an exact sequence

$$0 \rightarrow \varinjlim_{i \in I} \mathcal{L}_{b,1}(\text{Im}(g_i), E) \rightarrow \varinjlim_{i \in I} \mathcal{L}_{b,1}(B_i, E) \rightarrow \varinjlim_{i \in I} \mathcal{L}_{b,1}(A_i / \text{Ker}(f_i), E), \quad (2.7)$$

³By Lemma 1.28, the Hausdorff completion of the set-theoretical cokernel of a contraction in $\text{Ban}_{M,K}$ is the cokernel in $\text{Ban}_{M,K}$.

which is to say an exact sequence

$$0 \rightarrow \mathcal{L}_{b,1}(\varinjlim_{i \in I} \text{Im}(g_i), E) \rightarrow \mathcal{L}_{b,1}(\varinjlim_{i \in I} B_i, E) \rightarrow \mathcal{L}_{b,1}(\varinjlim_{i \in I} A_i / \text{Ker}(f_i), E). \quad (2.8)$$

We deduce by Yoneda's Lemma that the sequence $\varinjlim_{i \in I} \mathcal{D}_i$ is right-exact and the result follows. \square

2.5. Decomposable spaces.

Definition 2.8. A finite-dimensional normed K -vector space $(U, \|\cdot\|)$ is said to be decomposable if there exists a basis (e_1, \dots, e_d) of U such that

$$\forall \lambda_1, \dots, \lambda_d \in K, \quad \left\| \sum_{i=1}^d \lambda_i e_i \right\| = \max_{1 \leq i \leq d} (|\lambda_i| \|e_i\|). \quad (2.9)$$

A finite-dimensional Banachoid space $(U, (u_m)_{m \in M})$ is said to be decomposable if, for every $m \in M$, u_m is a norm on U and the space (U, u_m) is decomposable.

Under some assumptions on the base field, we will prove that Banachoid spaces are direct limits of decomposable spaces. We first need a few preparatory lemmas.

If $(U, \|\cdot\|)$ is a seminormed vector space over K , we set

$$U^\circ = \{x \in U \mid \|x\| \leq 1\}. \quad (2.10)$$

Thanks to the non-archimedean triangle inequality, it is a K° -module.

Recall also that K° is either a field (if K is trivially valued) or a valuation ring (otherwise). In any case, every finitely generated K° -module with no torsion is free. In particular, the property holds for K° -submodules of K -vector spaces.

Lemma 2.9. Let $(U, \|\cdot\|)$ be a finite-dimensional seminormed vector space over K . Let N be a finitely generated K° -submodule of U° such that the natural map $N \otimes_{K^\circ} K \xrightarrow{\sim} U$ is an isomorphism of K -vector spaces.

Set $\|0\|_N = 0$. For every $x \in U - \{0\}$, we consider the Minkowski functional

$$\|x\|_N = \inf\{|\lambda| \mid \lambda \in K^*, x \in \lambda N\}. \quad (2.11)$$

The map $\|\cdot\|_N$ is a norm on U .

Moreover, let (e_1, \dots, e_d) be a basis of N over K° . Then, for every $\lambda_1, \dots, \lambda_d \in K$, we have

$$\left\| \sum_{i=1}^d \lambda_i e_i \right\|_N = \max_{1 \leq i \leq d} (|\lambda_i|) \geq \left\| \sum_{i=1}^d \lambda_i e_i \right\|. \quad (2.12)$$

In particular, the identity map $(U, \|\cdot\|_N) \rightarrow (U, \|\cdot\|)$ is a contraction.

Proof. The fact that $\|\cdot\|_N$ is a norm on U is a simple verification.

Let $\lambda_1, \dots, \lambda_d \in K$. The inequality

$$\left\| \sum_{i=1}^d \lambda_i e_i \right\|_N \leq \max_{1 \leq i \leq d} (|\lambda_i|) \quad (2.13)$$

comes readily from the non-archimedean triangle inequality. Since e_1, \dots, e_d belong to U° , the inequality

$$\left\| \sum_{i=1}^d \lambda_i e_i \right\| \leq \max_{1 \leq i \leq d} (|\lambda_i|) \quad (2.14)$$

is a consequence of the non-archimedean triangle inequality too.

It remains to prove the converse of inequality (2.13). If every λ_i is zero, it is obvious, so we assume otherwise. Up to multiplying the λ_i 's by some scalar, we may assume that

$$\max_{1 \leq i \leq d} (|\lambda_i|) = 1. \quad (2.15)$$

Set $x := \sum_{i=1}^d \lambda_i e_i$. Let $\lambda \in K$ such that $|\lambda| < 1$. Since (e_1, \dots, e_d) is a basis of N , x cannot belong to λN . We deduce that $\|x\|_N \geq 1$, which finishes the proof. \square

Lemma 2.10. *Maintain the assumptions of Proposition 2.9. Let N' be another K° -module with the same properties as N . The following properties are equivalent :*

i) $N \subseteq N'$;

ii) $\|x\|_N \geq \|x\|_{N'}$, for all $x \in U$. \square

Proof. i) \Rightarrow ii). Assume $N \subseteq N'$. Let $x \in U$. We have $\{\lambda \in K^*, x \in \lambda N\} \subseteq \{\lambda \in K^*, x \in \lambda N'\}$. This implies that $\inf\{|\lambda| \mid \lambda \in K^*, x \in \lambda N\} \geq \inf\{|\lambda| \mid \lambda \in K^*, x \in \lambda N'\}$.

ii) \Rightarrow i). Assume $\|\cdot\|_N \geq \|\cdot\|_{N'}$. Then we have an inclusion of unit balls $N = \{x \in U, \|x\|_N \leq 1\} \subseteq \{x \in U, \|x\|_{N'} \leq 1\} = N'$. \square

Proposition 2.11. *Assume that K is densely valued, i.e. $|K|$ is dense in \mathbb{R}_+ . Then, every Banachoid space is a direct limit in $\text{Ban}_{M,K}$ of finite-dimensional decomposable Banachoid spaces.*

Proof. Let $(U, (u_m)_{m \in M})$ be a Banachoid space. Let \mathcal{S} be the family of finite subsets of U .

Let $m \in M$. By induction on $n \in \mathbb{N}^*$, to each couple (x, S) with $x \in S$ and $S \in \mathcal{S}$ of cardinal n , we may associate $\lambda_{S,x,m} \in K$ such that

$$\begin{cases} 1 - (n+1)^{-1} \leq |\lambda_{S,x,m}| \cdot u_m(x) \leq 1 & \text{if } u_m(x) \neq 0; \\ n+1 \leq |\lambda_{S,x,m}| & \text{if } u_m(x) = 0 \end{cases} \quad (2.16)$$

and, for every finite subset T of S containing x , $|\lambda_{T,x,m}| \leq |\lambda_{S,x,m}|$.

Let $S \in \mathcal{S}$. Denote by V_S the subvector-space of U generated by S . Consider the K° -submodule $N_{S,m}$ of $(U, u_m)^\circ$ generated by the family $(\lambda_{S,x,m} x)_{x \in S}$ and denote by $v_{S,m}$ the norm associated to it on V_S by the construction of Lemma 2.9.

For every $S' \in \mathcal{F}$ that contains S , by construction, we have a natural injection $j_{S',S}: V_S \rightarrow V_{S'}$ that sends $N_{S,m}$ into $N_{S',m}$. We deduce that the injection $j_{S',S}: (V_S, v_{S,m}) \rightarrow (V_{S'}, v_{S',m})$ is a contraction.

Finally, remark that, for every $x \in S$, we have

$$u_m(x) \leq v_{S,m}(x) \leq |\lambda_{S,x,m}|^{-1} \leq \frac{\text{Card}(S) + 1}{\text{Card}(S)} \cdot u_m(x) \quad \text{if } u_m(x) \neq 0 \quad (2.17)$$

and

$$v_{S,m}(x) \leq |\lambda_{S,x,m}|^{-1} \leq \frac{1}{\text{Card}(S) + 1} \quad \text{if } u_m(x) = 0. \quad (2.18)$$

In particular, for all $S \in \mathcal{S}$, the natural inclusion $(V_S, (v_{S,m})_{m \in M}) \rightarrow (U, (u_m)_{m \in M})$ is a contraction. Moreover, the characterization of direct limits from Section 2.4 shows that the canonical map $\lim_{\rightarrow S \in \mathcal{S}} ((V_S, (v_{S,m})_{m \in M}), (j_{S',S})) \rightarrow (U, (u_m)_{m \in M})$ is isometric and surjective. The result follows. \square

3. Tensor products

3.1. Definition and first properties

Let $(U, (u_m)_{m \in M})$ and $(V, (v_n)_{n \in N})$ be Banachoid spaces. For every $m \in M$, $n \in N$ and $z \in U \otimes_K V$, set

$$u_m \otimes v_n(z) := \inf \left\{ \max_{1 \leq i \leq r} (u_m(x_i) \cdot v_n(y_i)) \text{ such that } z = \sum_{i=1}^r x_i \otimes y_i \right\}. \quad (3.1)$$

The map $u_m \otimes v_n$ is a seminorm on $U \otimes_K V$. We denote by $U \hat{\otimes}_K V$ the Hausdorff completion of $U \otimes_K V$ with respect to the uniform structure induced by the $u_m \otimes v_n$'s. The seminorms $u_m \otimes v_n$ extend naturally to it and endow it with a structure of Banachoid space.

The following result is now easily proven from the definitions.

Proposition 3.1. *The natural map $\pi: U \times V \rightarrow U \hat{\otimes}_K V$ is a bilinear contraction between $M \times N$ -Banachoid spaces.*

Moreover, for every Banachoid space (W, w) , the composition with π induces natural isomorphisms of Banachoid spaces

$$\mathcal{L}_b(U \hat{\otimes}_K V, W) \xrightarrow{\sim} \text{Mult}_b(U \times V, W) \quad (3.2)$$

and, in the case where w is indexed by $M \times N$,

$$\mathcal{L}_{b,1}(U \hat{\otimes}_K V, W) \xrightarrow{\sim} \text{Mult}_{b,1}(U \times V, W). \quad (3.3)$$

□

It follows that $-\hat{\otimes}_K V$ is a functor from $\text{Ban}_{M,K}^b$ to $\text{Ban}_{M \times N, K}^b$, Ban_K^b to Ban_K^b and $\text{Ban}_{M,K}$ to $\text{Ban}_{M \times N, K}$. Thanks to Lemma 1.18, we deduce that the equivalence class of $U \hat{\otimes}_K V$ only depends on the equivalence classes of U and V .

Moreover, using Lemma 1.17 to identify $\text{Mult}_{b,1}(U \times V, W)$ and $\mathcal{L}_{b,1}(U, \mathcal{L}_{b,1}(V, W))$, we may write this functor $-\hat{\otimes}_K V$ as a left-adjoint.

Corollary 3.2. *The functor $-\hat{\otimes}_K V$ from the category $\text{Ban}_{M,K}^b$ (resp. Ban_K^b , resp. $\text{Ban}_{M,K}$) to the category $\text{Ban}_{M \times N, K}^b$ (resp. Ban_K^b , resp. $\text{Ban}_{M \times N, K}$) commutes with colimits. In particular, it is right exact.* □

Remark 3.3. *By Lemma 2.1, products in Ban_K^b coincides with coproducts, hence the functor $-\hat{\otimes}_K V$ commutes with products in Ban_K^b , by right-exactness.*

An explicit computation proves that the functor $-\hat{\otimes}_K V$ from $\text{Ban}_{M,K}$ to $\text{Ban}_{M \times N, K}$ also commutes with products.

Remark that the tensor product with a finite-dimensional decomposable Banachoid space may be easily described.

Lemma 3.4. *Let $(U, (u_m)_{m \in M})$ and $(V, (v_n)_{n \in N})$ be Banachoid spaces. Assume that V is finite-dimensional. Then, we have a natural isometric isomorphism*

$$U \otimes_K V \xrightarrow{\sim} U \hat{\otimes}_K V.$$

Moreover, assume that there exists $n \in N$ and a basis (e_1, \dots, e_d) of V such that

$$\forall \lambda_1, \dots, \lambda_d \in K, \quad v_n \left(\sum_{i=1}^d \lambda_i e_i \right) = \max_{1 \leq i \leq d} (|\lambda_i| \|e_i\|). \quad (3.4)$$

Then, for every $m \in M$, we have

$$\forall x_1, \dots, x_d \in U, \quad (u_m \otimes v_n) \left(\sum_{i=1}^d x_i \otimes e_i \right) = \max_{1 \leq i \leq d} (u_m(x_i) \|e_i\|). \quad (3.5)$$

Proof. The part of the statement concerning the explicit form of $u_m \otimes v_n$ comes from a straightforward computation.

To prove the first part, it is enough to prove that $U \otimes_K V$ is complete. For this purpose, we may replace every norm v_m by an equivalent one. As a consequence, we may assume that the norms v_m are all given as in (3.4) with the same basis for all. The result follows easily. \square

We now compute a particularly simple kind of tensor product. Let $r \in \mathbb{R}_+^* - \sqrt{|K^*|}$. Following [Ber90, after Definition 2.1.1], we set

$$K_r = \left\{ f = \sum_{i \in \mathbb{Z}} a_i T^i \mid a_i \in K, \lim_{i \rightarrow \pm\infty} |a_i| r^i = 0 \right\} \quad (3.6)$$

and endow it with the norm

$$|f|_r = \max_{i \in \mathbb{Z}} (|a_i| \cdot r^i). \quad (3.7)$$

The K -algebra $(K_r, |\cdot|_r)$ is actually a valued field.

Lemma 3.5. *Let $(U, (u_m)_{m \in M})$ be a Banachoid space over K . Let $r \in \mathbb{R}_+^* - \sqrt{|K^*|}$.*

Set

$$U_r = \left\{ f = \sum_{i \in \mathbb{Z}} x_i T^i \mid x_i \in U, \forall m \in M, \lim_{i \rightarrow \pm\infty} u_m(x_i) r^i = 0 \right\} \quad (3.8)$$

and endow it with the family of seminorms

$$u_{m,r}(f) = \max_{i \in \mathbb{Z}} (u_m(a_i) \cdot r^i). \quad (3.9)$$

Then $(U_r, (u_{m,r})_{m \in M})$ is a Banachoid space over K_r and it is isomorphic to the tensor product $U \hat{\otimes}_K K_r$ in $\text{Ban}_{M,K}$.

Proof. For every $i \in \mathbb{Z}$, consider the Banach space (L_i, v_i) , where L_i is a one-dimensional vector space over K with basis (e_i) and

$$\forall \lambda \in K, v_i(\lambda \cdot e_i) = |\lambda| \cdot r^i. \quad (3.10)$$

Then, the space K_r is the direct sum of the L_i 's (cf. Lemma 2.6).

The result nows follows from Corollary 3.2 since direct sums are direct limits. \square

3.2. Exactness properties

We first adapt [Ber90, Proposition 2.1.2] to the setting of Banachoid spaces.

Proposition 3.6. *Let $r \in \mathbb{R}_+^* - \sqrt{|K^*|}$. Let U, V, W be M -Banachoid spaces over K .*

- i) The natural map $x \in U \mapsto x \otimes 1 \in U \hat{\otimes}_K K_r$ is an isometry.*
- ii) A linear bounded map $f: U \rightarrow V$ is a contraction (resp. strict) if, and only if, the map $f_r := f \otimes 1: U \hat{\otimes}_K K_r \rightarrow V \hat{\otimes}_K K_r$ is.*
- iii) Let $S: U \xrightarrow{f} V \xrightarrow{g} W$ be a sequence of bounded maps. Set $S_r := S \hat{\otimes}_K K_r$. If S_r is exact, then S is exact. If f is strict and S is exact, then S_r is exact. \square*

Remark 3.7. Assume that K is discretely but not trivially valued. Let $r \in \mathbb{R}_+^* - \sqrt{|K^*|}$. Then the field K_r is densely valued.

Assume that K is trivially valued. Let $r \in \mathbb{R}_+^* - \{1\}$. Then K_r is discretely but not trivially valued. We may now apply the previous argument to construct a field $K_{r,s}$ that is densely valued.

Remark also that, if K is not discretely valued, then it is densely valued.

Proposition 3.8. Let U, V, W, E be M -Banachoid spaces over K .

i) If a linear map $f: U \rightarrow V$ is strict, then so is the map $f_E: U \hat{\otimes}_K E \rightarrow V \hat{\otimes}_K E$.

In particular, for every complete valued extension L of K , the map $x \in U \mapsto x \otimes 1 \in U \hat{\otimes}_K L$ is an isometry.

ii) If $S: U \xrightarrow{f} V \xrightarrow{g} W$ is an exact sequence of strict linear maps, then so is the sequence $S_E: U \hat{\otimes}_K E \xrightarrow{f_E} V \hat{\otimes}_K E \xrightarrow{g_E} W \hat{\otimes}_K E$.

iii) If a linear map $f: U \rightarrow V$ is strict, then we have natural strict isomorphisms

$$\text{Ker}(f) \hat{\otimes}_K E \xrightarrow{\sim} \text{Ker}(f_E) \quad \text{and} \quad \text{Coker}(f) \hat{\otimes}_K E \xrightarrow{\sim} \text{Coker}(f_E). \quad (3.11)$$

Proof. By Remark 3.7 and Proposition 3.6, we may assume that K is densely valued. Then, by Proposition 2.11, E is a direct limit in $\text{Ban}_{M,K}$ of finite-dimensional decomposable Banachoid spaces.

On the other hand, using Lemma 3.4, it is easy to check that tensoring by a finite-dimensional decomposable Banachoid space preserves exactness and strictness in $\text{Ban}_{M,K}$. By Proposition 2.7, those properties are also preserved by direct limits. Finally, by Corollary 3.2, the tensor product commutes with colimits, and we conclude that assertions i) and ii) hold. Assertion iii) follows from ii). □

For future reference, we note that the strictness property descends.

Lemma 3.9. Let $f: (U, u) \rightarrow (V, v)$ be a bounded K -linear map between Banachoid spaces over K . Let E be a Banachoid space over K . Denote by u_E and v_E the families of seminorms induced by u and v on $U \hat{\otimes}_K E$ and $V \hat{\otimes}_K E$ respectively as in Section 3.1. Assume that there exist equivalent families u'_E and v'_E such that the map $f_E: (U \hat{\otimes}_K E, u'_E) \rightarrow (V \hat{\otimes}_K E, v'_E)$ is strict.

Denote by u' and v' the families of seminorms induced on U and V by u'_E and v'_E respectively. Then u' and v' are equivalent to u and v and the map $f: (U, u') \rightarrow (V, v')$ is strict. □

We now show that kernels of bounded linear maps also commute with extension of scalars when they are not too big.

Proposition 3.10. Let $f: U \rightarrow V$ be a bounded linear map between Banachoid spaces over K . Let E be a Banach space over K of countable type (i.e. that has a dense subspace of dimension at most countable over K). Then, we have a natural isomorphism in Ban_K^b

$$\text{Ker}(f) \hat{\otimes}_K E \xrightarrow{\sim} \text{Ker}(f_E). \quad (3.12)$$

Proof. By Proposition 3.8, the map $\text{Ker}(f) \hat{\otimes}_K E \rightarrow U \hat{\otimes}_K E$ is injective. Moreover, its image clearly sits inside $\text{Ker}(f_E)$. Hence, it is enough to prove that the natural map $\text{Ker}(f) \hat{\otimes}_K E \rightarrow \text{Ker}(f_E)$ is surjective.

Let $x \in U \hat{\otimes}_K E$ such that $f_E(x) = 0$. If E is finite-dimensional over K , then, by Lemma 3.4, we have $U \hat{\otimes}_K E = U \otimes_K E$ and $V \hat{\otimes}_K E = V \otimes_K E$ and the result holds thanks to the usual properties of the tensor product.

Let us now assume that E has infinite dimension over K . We may replace the norm on E by an equivalent one $\|\cdot\|$. Hence, by [Ked10, Lemma 1.3.8], we may assume that there exists a sequence $(e_i)_{i \geq 0}$ of elements of E such that

- i) for every $a \in E$, there exists a unique sequence $(a_i)_{i \geq 0}$ of elements of K such that the series $\sum_{i \geq 0} a_i e_i$ converges to y ;
- ii) with the same notations, we have

$$\|a\| = \max_{i \geq 0} (|a_i| \|e_i\|). \quad (3.13)$$

One checks that similar properties hold for $U \hat{\otimes}_K E$:

- i) for every $y \in U \hat{\otimes}_K E$, there exists a unique sequence $(y_i)_{i \geq 0}$ of elements of U such that the series $\sum_{i \geq 0} y_i \otimes e_i$ converges to y ;
- ii) with the same notations, for every $m \in M$, we have

$$(u_m \otimes \|\cdot\|)(y) = \max_{i \geq 0} (u_m(y_i) \|e_i\|), \quad (3.14)$$

where $u = (u_m)_{m \in M}$.

Of course, we have an analogous statement for $V \hat{\otimes}_K E$.

Let us now write $x = \sum_{i \geq 0} x_i \otimes e_i$, with $x_i \in U$ for all i . We have

$$0 = f_E(x) = \sum_{i \geq 0} f(x_i) \otimes e_i, \quad (3.15)$$

hence $f(x_i) = 0$ for all i , by uniqueness. The result follows. \square

Definition 3.11. *Let E be a K -vector space endowed with a norm w . Let $\alpha \geq 1$.*

A family $(f_i)_{i \in I}$ of elements of E is said to be α -cartesian with respect to w if, for each family $(a_i)_{i \in I}$ of elements of K with finite support, we have

$$w\left(\sum_{i \in I} a_i f_i\right) \geq \alpha^{-1} \max_{i \in I} (|a_i| w(f_i)). \quad (3.16)$$

Lemma 3.12. *Let E be a K -vector space endowed with a norm w . Let $(f_i)_{i \in I}$ be a family of elements of E that is α -cartesian with respect to w for some $\alpha \geq 1$. Let $(U, u = (u_m)_{m \in M})$ be a normoid K -vector space.*

Then, for each $m \in M$ and each family $(c_i)_{i \in I}$ of elements of U with finite support, we have

$$(u_m \otimes w)\left(\sum_{i \in I} c_i \otimes f_i\right) \geq \alpha^{-1} \max_{i \in I} (u_m(c_i) w(f_i)). \quad (3.17)$$

In particular, if (U, u) is Hausdorff, then an element of $U \otimes_K E$ may be written in the form $\sum_{i \in I} c_i \otimes f_i$, for some family $(c_i)_{i \in I}$ of elements of U with finite support, in at most one way. \square

Proposition 3.13. *Let (E, e) be a Banachoid space over K . Assume that there exists a norm w on E that is coarser than the family of seminorms e and a family $(f_n)_{n \in \mathbb{N}}$ of elements of E that generates a dense subspace of E and is α -cartesian with respect to w for some $\alpha \geq 1$. Let (U, u) be a Banachoid space. Then, for each $x \in U \hat{\otimes}_K E$, there exists a unique family $(c_n)_{n \in \mathbb{N}}$ such that the series $\sum_{n \in \mathbb{N}} c_n \otimes f_n$ converges to x in $U \hat{\otimes}_K E$.*

Proof. Let $x \in U \hat{\otimes}_K E$. It follows from the assumptions that there exists a net $(x_\lambda)_{\lambda \in D}$ (for some directed set D) of elements of $U \otimes_K E$ that converges to x in $U \hat{\otimes}_K E$ and such that, for each $\lambda \in D$,

there exists a family $(c_{\lambda,n})_{n \in \mathbb{N}}$ of elements of U with finite support such that

$$x_\lambda = \sum_{n \in \mathbb{N}} c_{\lambda,n} \otimes f_n. \quad (3.18)$$

It follows from Lemma 3.12 that, for each $n \in \mathbb{N}$, the net $(c_{\lambda,n})_{\lambda \in D}$ is Cauchy. The existence part of the result follows. Uniqueness follows from Lemma 3.12 too. \square

Definition 3.14. *A Banachoid space (E, e) over K is said to be of countable type if*

- i) *E contains a dense subspace of dimension at most countable over K ;*
- ii) *there exists a norm w on E that is coarser than the family of seminorms e .*

Proposition 3.15. *Let $f: U \rightarrow V$ be a bounded linear map between Banachoid spaces over K . Let (E, e) be a Banachoid space over K of countable type. Then, we have a natural isomorphism in Ban_K^b*

$$\text{Ker}(f) \hat{\otimes}_K E \xrightarrow{\sim} \text{Ker}(f_E). \quad (3.19)$$

Proof. By Proposition 3.8, the map $\text{Ker}(f) \hat{\otimes}_K E \rightarrow U \hat{\otimes}_K E$ is injective. Moreover, its image clearly sits inside $\text{Ker}(f_E)$. Hence, it is enough to prove that the natural map $\text{Ker}(f) \hat{\otimes}_K E \rightarrow \text{Ker}(f_E)$ is surjective.

Let $x \in U \hat{\otimes}_K E$ such that $f_E(x) = 0$. If E is finite-dimensional over K , then, by Lemma 3.4, we have $U \hat{\otimes}_K E = U \otimes_K E$ and $V \hat{\otimes}_K E = V \otimes_K E$ and the result holds thanks to the usual properties of the tensor product.

Let us now assume that E has infinite dimension over K . Let E_0 be a dense subspace of countable dimension. By [BGR84, Proposition 2.6.2/3]⁴, E_0 admits a basis $(f_n)_{n \in \mathbb{N}}$ that is α -cartesian with respect to w for some $\alpha \geq 1$.

By Proposition 3.13, there exists a unique family $(c_n)_{n \in \mathbb{N}}$ such that the series $\sum_{n \in \mathbb{N}} c_n \otimes f_n$ converges to x in $U \hat{\otimes}_K E$. Since f is bounded, the series $\sum_{n \in \mathbb{N}} f(c_n) \otimes f_n$ converges to $f_E(x) = 0$ in $U \hat{\otimes}_K E$. The uniqueness statement of Proposition 3.13 ensures that we have $f(c_n) = 0$ for all $n \in \mathbb{N}$. The result follows. \square

Corollary 3.16. *Let X be a K -analytic space, let \mathcal{F} be a coherent sheaf on X and let L be a complete valued extension of K of countable type. Endow $\mathcal{F}(X)$ with an admissible normoid structure and $\mathcal{F}_L(X_L)$ with the induced structure. Then, we have a canonical isomorphism in Ban_K^b*

$$\mathcal{F}(X) \hat{\otimes}_K L \xrightarrow{\sim} \mathcal{F}_L(X_L). \quad (3.20)$$

Proof. Let \mathcal{V} be an affinoid covering of X for the G-topology. For each affinoid domain U of X that is a finite intersection of elements of \mathcal{V} , endow $\mathcal{F}(U)$ with a norm v_U as in item ii) of Definition 1.11.

Endow $\mathcal{F}(X)$ with the basic admissible normoid structure induced by the family $(v_V)_{V \in \mathcal{V}}$. Since, by Lemma 1.12, all admissible normoid structures on $\mathcal{F}(X)$ are equivalent, it is enough to prove the result for this one.

Let us consider the morphism of Banachoid spaces

$$f: \prod_{V \in \mathcal{U}} \mathcal{F}(V) \rightarrow \prod_{V \neq W \in \mathcal{U}} \mathcal{F}(V \cap W), \quad (3.21)$$

⁴This reference only covers the non-trivially valued case. If K is trivially valued, one can apply [Ked10, Lemma 1.3.8] to the completion of E_0 with respect to w and check that the elements given by the construction actually belong to E_0 .

where the products are taken in Ban_K^b . By Lemma 2.1, these products coincide with the set theoretical products together with a certain family of seminorms.

By Remark 3.3, we have a natural isomorphism

$$\left(\prod_{V \in \mathcal{V}} \mathcal{F}(V) \right) \hat{\otimes}_K L \xrightarrow{\sim} \prod_{V \in \mathcal{V}} \mathcal{F}(V) \hat{\otimes}_K L, \quad (3.22)$$

and similarly for the other product in the target of f .

It follows that the morphism $f \hat{\otimes}_K L$ may be identified with

$$f_L: \prod_{V \in \mathcal{V}} \mathcal{F}_L(V_L) \longrightarrow \prod_{V \neq W \in \mathcal{V}} \mathcal{F}_L(V_L \cap W_L). \quad (3.23)$$

In particular, its kernel is $\mathcal{F}_L(X_L)$ endowed with the right Banachoid structure by Lemma 2.4.

The result now follows from Proposition 3.15. □

Corollary 3.17. *Let X be a K -analytic space and Y be a K -affinoid space. Denote by pr_X and pr_Y the canonical projections from $X \times_K Y$ to X and Y respectively. Let \mathcal{F} and \mathcal{G} be coherent sheaves on X and Y respectively. Endow $\mathcal{F}(X)$ and $\mathcal{G}(Y)$ with admissible normoid structures. Then, we have a canonical isomorphism in Ban_K^b*

$$\mathcal{F}(X) \hat{\otimes}_K \mathcal{G}(Y) \xrightarrow{\sim} (\mathcal{F} \boxtimes_k \mathcal{G})(X \times_K Y), \quad (3.24)$$

where $\mathcal{F} \boxtimes_k \mathcal{G} := \text{pr}_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_K Y}} \text{pr}_Y^* \mathcal{G}$.

Proof. First note that $\mathcal{O}(Y)$ is a quotient of a Tate algebra with a finite number of variables, hence a Banach space of countable type. The same property holds for $\mathcal{G}(Y)$ since it is a quotient of some power of $\mathcal{O}(Y)$ (see item ii) of Definition 1.11).

We may then use the same strategy as in the proof of Corollary 3.16, with L replaced by $\mathcal{G}(Y)$, to reduce to the case where X is affinoid, which is well-known. □

3.3. Normoid Fréchet spaces

We now adapt the definition of Fréchet space to our setting. The difference with the usual ones is that the seminorms defining the topology are part of the data.

Definition 3.18. *An M -Banachoid space (U, u) is said to be an M -normoid Fréchet space if it is equivalent to an N -Banachoid space (V, v) where N is (at most) countable.*

Note that a completed tensor product of two normoid Fréchet spaces is still a normoid Fréchet space.

Lemma 3.19. *Every M -normoid Fréchet space is metrizable.*

If K is not trivially valued, then every metrizable M -Banachoid space is normoid Fréchet.

Proof. Let N be a countable set and $(U, (u_n)_{n \in N})$ be an N -Banachoid space. We may assume that N is infinite by adding zero semi-norms if need be and identify N with \mathbb{N} . The map

$$d: (x, y) \mapsto \sum_{n \in \mathbb{N}} 2^{-n} \frac{v_n(y - x)}{1 + v_n(y - x)} \quad (3.25)$$

defines a distance on U that induces the same uniform structure as the family $(v_n)_{n \in \mathbb{N}}$. It follows

that U is normoid Fréchet.

Assume that K is not trivially valued. Let $(U, (u_m)_{m \in M})$ be an M -Banachoid space that is metrizable. In this case, 0 has a countable basis of neighborhoods, and we may assume that each of them is defined using only finitely many u_m 's. Denote by N the set of elements $m \in M$ that appear in the definition of one of the neighborhoods. It is countable and the families $(u_m)_{m \in M}$ and $(u_n)_{n \in N}$ are equivalent by Lemma 1.7. \square

Remark 3.20. *Let X be a K -analytic space and let \mathcal{F} be a coherent sheaf on X . If the space X is countable at infinity (i.e. a countable union of compact subsets), then every admissible normoid structure on $\mathcal{F}(X)$ is actually a normoid Fréchet structure. Indeed, by Lemma 1.12, all such structures are equivalent and, by choosing a countable affinoid cover and carrying out the construction of item ii) of Definition 1.11, one clearly gets a normoid Fréchet structure. The difference with the usual Fréchet spaces is that the seminorms defining the topology are part of the data.*

The case we have just described will actually be the most interesting for us. Indeed, since K -analytic curves are paracompact (see [Duc, Théorème 4.5.10]), every K -analytic curve with countably many connected components is countable at infinity.

In the normoid Fréchet setting, we can remove the countable type assumption from Proposition 3.15.

Proposition 3.21. *Let U be a normoid Fréchet space over K , let V be a Banachoid space over K and let $f: U \rightarrow V$ be a bounded linear map. Let E be a Banach space over K . Then, we have a natural isomorphism in Ban_K^b*

$$\text{Ker}(f) \hat{\otimes}_K E \xrightarrow{\sim} \text{Ker}(f_E). \quad (3.26)$$

Proof. Starting as in the proof of Proposition 3.15, we are reduced to proving that the natural map $\text{Ker}(f) \hat{\otimes}_K E \rightarrow \text{Ker}(f_E)$ is surjective.

Let $x \in U \hat{\otimes}_K E$ such that $f_E(x) = 0$. Since $U \hat{\otimes}_K E$ is normoid Fréchet, x may be obtained as a limit of countably many elements in $U \otimes_K E$. We deduce that there exists a sub- K -vector-space E_0 of E with countable dimension over K such that x is in the image of $U \hat{\otimes}_K E_0$ in $U \hat{\otimes}_K E$. By Proposition 3.8, we may assume that $E = E_0$ and we may then apply Proposition 3.15 to conclude. \square

Using Proposition 3.21 instead of Proposition 3.15, we immediately derive an analogue of Corollary 3.16.

Corollary 3.22. *Let X be a K -analytic space countable at infinity, let \mathcal{F} be a coherent sheaf on X and let L be a complete valued extension of K . Endow $\mathcal{F}(X)$ with an admissible normoid structure and $\mathcal{F}_L(X_L)$ with the induced structure. Then, we have a canonical isomorphism in Ban_K^b*

$$\mathcal{F}(X) \hat{\otimes}_K L \xrightarrow{\sim} \mathcal{F}_L(X_L). \quad (3.27)$$

\square

We recall the following criterion result to ensure that a continuous map is topologically strict. For a reference, see for instance [Sch02, Lemma 22.2] (and note that the assumption that the kernel is finite-dimensional is not used in the proof).

Proposition 3.23. *Assume that K is not trivially valued. Let $f: E \rightarrow F$ be a continuous map between Fréchet spaces over K that has finite-dimensional cokernel. Then, the image of f is closed and topologically complemented and f is topologically strict.* \square

Corollary 3.24. *Let $f: (U, u) \rightarrow (V, v)$ be a bounded K -linear map between normoid Fréchet spaces. Let L be a complete non-trivially valued extension of K such that f_L has finite-dimensional cokernel. Then, there exist families of seminorms u' and v' equivalent to u and v respectively such that the map $f: (U, u') \rightarrow (V, v')$ is strict. In particular, f is topologically strict.*

Proof. By Proposition 3.23, f_L is topologically strict and its image is topologically complemented. By Lemmas 1.25 and 1.7, we may replace the families of seminorms on $U \hat{\otimes}_K L$ and $V \hat{\otimes}_K L$ by equivalent ones so that f_L becomes strict. The result now follows from Lemma 3.9. \square

Combining this corollary with Proposition 3.8, we obtain the following descent result.

Corollary 3.25 (Descent). *Let $f: (U, u) \rightarrow (V, v)$ be a bounded K -linear map between normoid Fréchet spaces. Let E be Banachoid space over K . Assume that there exists a complete non-trivially valued extension L of K such that f_L has finite-dimensional cokernel. Then, f has finite-dimensional cokernel and we have canonical isomorphisms in Ban_K^b*

$$\text{Ker}(f) \hat{\otimes}_K E \xrightarrow{\sim} \text{Ker}(f_E) \quad (3.28)$$

and

$$\text{Coker}(f) \otimes_K E \xrightarrow{\sim} \text{Coker}(f_E). \quad (3.29)$$

\square

Corollary 3.26 (Descent). *Let*

$$\mathcal{C}: \cdots \rightarrow (U^{n-1}, u^{n-1}) \xrightarrow{f_n} (U^n, u^n) \xrightarrow{f_{n+1}} (U^{n+1}, u^{n+1}) \rightarrow \cdots \quad (3.30)$$

be a complex of normoid Fréchet spaces with bounded K -linear maps. Let E be a Banach space over K and consider the complex

$$\mathcal{C}_E: \cdots \rightarrow (U^{n-1}, u^{n-1}) \hat{\otimes}_K E \xrightarrow{f_{n,E}} (U^n, u^n) \hat{\otimes}_K E \xrightarrow{f_{n+1,E}} (U^{n+1}, u^{n+1}) \hat{\otimes}_K E \rightarrow \cdots \quad (3.31)$$

Let L be a complete non-trivially valued extension of K and consider the complex

$$\mathcal{C}_L: \cdots \rightarrow (U^{n-1}, u^{n-1}) \hat{\otimes}_K L \xrightarrow{f_{n,L}} (U^n, u^n) \hat{\otimes}_K L \xrightarrow{f_{n+1,L}} (U^{n+1}, u^{n+1}) \hat{\otimes}_K L \rightarrow \cdots \quad (3.32)$$

Let $n \in \mathbb{Z}$ and assume that $H^n(\mathcal{C}_L)$ is finite-dimensional. Then, $H^n(\mathcal{C})$ is finite-dimensional and we have a canonical isomorphism

$$H^n(\mathcal{C}) \otimes_K E \xrightarrow{\sim} H^n(\mathcal{C}_E). \quad (3.33)$$

Proof. Let $n \in \mathbb{Z}$. Let $F \in \{K, L\}$. By Proposition 3.21, we have a canonical isomorphism

$$\text{Ker}(f_{n+1}) \hat{\otimes}_K F \xrightarrow{\sim} \text{Ker}(f_{n+1,F}). \quad (3.34)$$

Consider the map $g_n: U^{n-1} \rightarrow \text{Ker}(f_{n+1})$ induced by f_n . It is a bounded map of normoid Fréchet spaces and we have $\text{Coker}(g_n) = H^n(\mathcal{C})$. Applying the functor $-\hat{\otimes}_K F$, we find a map $g_{n,F}: U^{n-1} \hat{\otimes}_K F \rightarrow \text{Ker}(f_{n+1}) \hat{\otimes}_K F$. Thanks to (3.34), we have $\text{Coker}(g_{n,F}) = H^n(\mathcal{C}_F)$ and the result now follows from Corollary 3.25. \square

Arguing as in the proof of Corollary 3.16, we deduce the following result.

Corollary 3.27. *Let X be a K -analytic space that is countable at infinity, let \mathcal{F} be a coherent sheaf on X and let L be a complete valued extension of K . Endow $\mathcal{F}(X)$ with an admissible normoid structure and $\mathcal{F}_L(X_L)$ with the induced structure. Then, we have a canonical isomorphism in Ban_K^b*

$$\mathcal{F}(X) \hat{\otimes}_K L \xrightarrow{\sim} \mathcal{F}_L(X_L). \quad (3.35)$$

Let $n \geq 1$. Assume that X is separated and that there exists $M \in \{K, L\}$ such that M is not trivially valued and $H^n(X_M, \mathcal{F}_M)$ is finite-dimensional. Then, $H^n(X, \mathcal{F})$ and $H^n(X_L, \mathcal{F}_L)$ are both finite-dimensional and we have a canonical isomorphism

$$H^n(X, \mathcal{F}) \otimes_K L \xrightarrow{\sim} H^n(X_L, \mathcal{F}_L). \quad (3.36)$$

□

We may also generalize the result of Corollary 3.17 to higher cohomology groups.

Corollary 3.28. *Let X be K -analytic space that is separated and countable at infinity and Y be a K -affinoid space. Denote by pr_X and pr_Y the canonical projections from $X \times_K Y$ to X and Y respectively. Let \mathcal{F} and \mathcal{G} be coherent sheaves on X and Y respectively. Endow $\mathcal{F}(X)$ and $\mathcal{G}(Y)$ with admissible normoid structures. Let $n \geq 1$ and assume that there exists a complete non-trivially valued extension L of K such that $H^n(X_L, \mathcal{F}_L)$ is finite-dimensional. Then, $H^n(X, \mathcal{F})$ is finite-dimensional and we have a canonical isomorphism*

$$H^n(X, \mathcal{F}) \otimes_K \mathcal{G}(Y) \xrightarrow{\sim} H^n(X \times_K Y, \mathcal{F} \boxtimes_K \mathcal{G}). \quad (3.37)$$

□

Under more restrictive conditions, we can also consider tensor products of complexes by normoid Fréchet spaces.

Corollary 3.29. *Let*

$$\mathcal{C}: 0 \xrightarrow{f_0} (U^0, u^0) \xrightarrow{f_1} (U^1, u^1) \xrightarrow{f_2} \dots \xrightarrow{f_n} (U^n, u^n) \rightarrow 0 \quad (3.38)$$

be a finite complex of normoid Fréchet spaces with bounded K -linear maps. Let E be a normoid Fréchet space over K , let L be a complete non-trivially valued extension of K and consider the complexes

$$\mathcal{C}_E: 0 \xrightarrow{f_{0,E}} (U^0, u^0) \hat{\otimes}_K E \xrightarrow{f_{1,E}} \dots \xrightarrow{f_{n,E}} (U^n, u^n) \hat{\otimes}_K E \rightarrow 0 \quad (3.39)$$

and

$$\mathcal{C}_L: 0 \xrightarrow{f_{0,L}} (U^0, u^0) \hat{\otimes}_K L \xrightarrow{f_{1,L}} \dots \xrightarrow{f_{n,L}} (U^n, u^n) \hat{\otimes}_K L \rightarrow 0. \quad (3.40)$$

Assume that, for each $m \in \{1, \dots, n\}$, $H^m(\mathcal{C}_L)$ is finite-dimensional over L . Then, for each $m \in \{0, \dots, n\}$, we have a canonical isomorphism in Ban_K^b

$$H^m(\mathcal{C}) \hat{\otimes}_K E \xrightarrow{\sim} H^m(\mathcal{C}_E). \quad (3.41)$$

Proof. By a decreasing induction on $m \in \{0, \dots, n\}$, we will prove that, for $F \in \{E, L\}$, we have canonical isomorphisms

$$H^m(\mathcal{C}) \hat{\otimes}_K F \xrightarrow{\sim} H^m(\mathcal{C}_F) \quad (3.42)$$

and

$$\text{Ker}(f_m) \hat{\otimes}_K F \xrightarrow{\sim} \text{Ker}(f_{m,F}). \quad (3.43)$$

Let us start with $m = n$. For $G \in \{K, E, L\}$, the map f_n is a bounded map of normoid Fréchet spaces and we have $\text{Coker}(f_{n,G}) = H^n(\mathcal{C}_G)$. The results then follow from Corollary 3.25.

Let us now assume that the result holds for some $m \in \{1, \dots, n\}$. Consider the map $g_{m-1}: U^{m-1} \rightarrow \text{Ker}(f_m)$ induced by f_{m-1} . It is a bounded map of normoid Fréchet spaces and we have $\text{Coker}(g_{m-1}) = H^{m-1}(\mathcal{C})$. Let $F \in \{E, L\}$. Applying the functor $-\hat{\otimes}_K F$, we find a map $g_{m-1,F}: U^{m-1} \hat{\otimes}_K F \rightarrow \text{Ker}(f_m) \hat{\otimes}_K F$. By induction, we have $\text{Ker}(f_m) \hat{\otimes}_K F \xrightarrow{\sim} \text{Ker}(f_{m,F})$ and the map $g_{m-1,F}$ identifies to the map induced by $f_{m-1,F}$. We then conclude by Corollary 3.25 again. □

We may now also remove the assumption that Y is affinoid in Corollaries 3.17 and 3.28.

Corollary 3.30. *Let X and Y be K -analytic spaces that are separated and countable at infinity. Denote by pr_X and pr_Y the canonical projections from $X \times_K Y$ to X and Y respectively. Let \mathcal{F} and \mathcal{G} be coherent sheaves on X and Y respectively. Endow $\mathcal{F}(X)$ and $\mathcal{G}(Y)$ with admissible normoid structures. Assume that X is of finite dimension and that there exists a complete non-trivially valued extension L of K such that, for each $m \geq 1$, $H^m(X_L, \mathcal{F}_L)$ is finite-dimensional and $H^m(Y_L, \mathcal{G}_L) = 0$. Then, we have a canonical isomorphism in Ban_K^b*

$$\mathcal{F}(X) \hat{\otimes}_K \mathcal{G}(Y) \xrightarrow{\sim} (\mathcal{F} \boxtimes_K \mathcal{G})(X \times_K Y) \quad (3.44)$$

and, for each $n \geq 1$, $H^n(X, \mathcal{F})$ is finite-dimensional and we have

$$H^n(X, \mathcal{F}) \otimes_K \mathcal{G}(Y) \xrightarrow{\sim} H^n(X \times_K Y, \mathcal{F} \boxtimes_K \mathcal{G}). \quad (3.45)$$

Proof. Let \mathcal{V} be an affinoid covering of X for the G -topology and consider the Čech complex

$$\mathcal{C}: 0 \rightarrow \prod_{V \in \mathcal{V}} \mathcal{F}(V) \rightarrow \prod_{V \neq W \in \mathcal{V}} \mathcal{F}(V \cap W) \rightarrow \dots \quad (3.46)$$

By Corollary 3.27, the higher cohomology groups of the complex \mathcal{C}_L are finite-dimensional over L . It then follows from Corollary 3.29 that the groups $H^n(X, \mathcal{F}) \hat{\otimes}_K \mathcal{G}(Y)$ are the cohomology groups of the complex $\mathcal{C}_{\mathcal{G}(Y)}$. By Corollary 3.17, for each affinoid domain U of X , we have $\mathcal{F}(U) \hat{\otimes}_K \mathcal{G}(Y) \simeq (\mathcal{F} \boxtimes_K \mathcal{G})(U \times_K Y)$, hence $\mathcal{C}_{\mathcal{G}(Y)}$ is a Čech complex for $\mathcal{F} \boxtimes_K \mathcal{G}$ on $X \times_K Y$. By Corollary 3.28, for each affinoid domain U of X , the sheaf $\mathcal{F} \boxtimes_K \mathcal{G}$ is acyclic on $U \times_K Y$, hence the previous Čech complex does compute the cohomology of $\mathcal{F} \boxtimes_K \mathcal{G}$ on $X \times_K Y$. \square

4. Cohomology of curves

4.1. Stein curves

Let us recall the definition of a quasi-Stein space (see [Kie67, Definition 2.3]).

Definition 4.1 (Quasi-Stein). *We say that a K -analytic space X is quasi-Stein if there exists a covering $(X_n)_{n \geq 0}$ of X for the G -topology such that*

- i) $X_n \subseteq X_{n+1}$;
- ii) X_n is an affinoid domain of X_{n+1} ;
- iii) the map $\mathcal{O}(X_{n+1}) \rightarrow \mathcal{O}(X_n)$ has dense image.

Let us now recall a result of Kiehl. Recall that a point x in a K -analytic space is said to be rigid if the extension $\mathcal{H}(x)/K$ is finite.

Theorem 4.2 ([Kie67, Satz 2.4]). *Let X be a K -analytic quasi-Stein space. Let \mathcal{F} be a coherent sheaf on X . Then the following results hold:*

- i) For every $q \geq 1$, we have $H^q(X, \mathcal{F}) = 0$.
- ii) For every rigid point $x \in X$, the stalk \mathcal{F}_x is generated by $\mathcal{F}(X)$ as an $\mathcal{O}_{X,x}$ -module.

\square

Remark 4.3. *Kiehl actually gave the definition in the setting of rigid geometry, i.e. for strictly K -analytic spaces with strictly K -affinoid domains and for a non-trivially valued field K . However, Theorem 4.2 is easily seen to hold in the more general case with the same proof.*

As in the complex setting, the obstruction for a K -analytic curve to be Stein lies in the presence of proper (or, equivalently, projective) connected components.

Theorem 4.4 ([LvdP95, Theorem 3.4]). *Assume that K is not trivially valued. Let X be a quasi-smooth strictly K -analytic curve. If no connected component of X is proper, then X is quasi-Stein.* \square

Definition 4.5 (Cohomologically Stein). *We say that a K -analytic space X is cohomologically Stein if, for every coherent sheaf \mathcal{F} on X and every $q \geq 1$, we have*

$$H^q(X, \mathcal{F}) = 0. \quad (4.1)$$

Thanks to the techniques developed in the previous sections, we can prove the following result.

Corollary 4.6. *Let X be a quasi-smooth K -analytic curve. If no connected component of X is proper, then X is cohomologically Stein.*

Proof. We may assume that X is connected. By [Duc, Théorème 4.5.10]), X is paracompact, hence countable at infinity.

Let \mathcal{F} be a coherent sheaf on X . Let L be a complete non-trivially valued extension of K such that X_L is strictly L -analytic. By Theorem 4.4, X_L is quasi-Stein, hence by Kiehl's Theorem 4.2, we have $H^q(X_L, \mathcal{F}_L) = 0$. The result now follows from Corollary 3.27. \square

Remark 4.7. *The converse of Corollary 4.6 also holds.*

As usual, the cohomological vanishing has consequences in terms of global generation of coherent sheaves.

Corollary 4.8. *Let X be a quasi-smooth K -analytic curve with no proper connected components. Then, every coherent sheaf \mathcal{F} on X is generated by its global sections: for each $x \in X$, the stalk \mathcal{F}_x is generated by $\mathcal{F}(X)$ as an $\mathcal{O}_{X,x}$ -module.*

Proof. We may assume that X is connected. Let us first assume that X contains only type 3 points. If k is not trivially valued, then X is reduced to a point and the result is obvious.

If k is trivially valued, then, there exists an irreducible polynomial $P \in k[T]$ and an interval $I \subseteq]0, 1[$ such that X is isomorphic to the analytic domain of the line defined by $\{|P| \in I\}$. Let us write the interval I as an increasing union of closed intervals I_n , with $n \in \mathbb{N}$. We may also write X as the increasing union of the affinoid domains $X_n = \{|P| \in I_n\}$. Note that, for every $n \in \mathbb{N}$, the ring $\mathcal{O}(X_n)$ is a field isomorphic to the P -adic completion of $k[T]$ and the restriction map $\mathcal{O}(X_{n+1}) \rightarrow \mathcal{O}(X_n)$ is an isomorphism. It follows that $\mathcal{O}(X)$ is also isomorphic to the P -adic completion of $k[T]$ and that the global section functor induces an equivalence of categories between coherent sheaves on X and finite-dimensional vector spaces over $\mathcal{O}(X)$.

We now assume that X does not contain only points of type 3. In this case, every non-empty closed analytic subset of X contains a rigid point.

Let \mathcal{F} be a coherent sheaf on X . Let z be a rigid point in X . Let \mathcal{I}_z be the sheaf of ideals that defines $\{z\}$ with its reduced structure. Using the exact sequence

$$0 \rightarrow \mathcal{I}_z \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{F} / \mathcal{I}_z \mathcal{F} \rightarrow 0 \quad (4.2)$$

and the fact that $H^1(X, \mathcal{I}_z \mathcal{F}) = 0$, one shows that the morphism

$$\mathcal{F}(X) \rightarrow (\mathcal{F} / \mathcal{I}_z \mathcal{F})(X) \simeq \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x, \quad (4.3)$$

where \mathfrak{m}_x denotes the maximal ideal of the local ring \mathcal{O}_x , is surjective. It now follows from Nakayama's lemma that the stalk \mathcal{F}_x is generated by $\mathcal{F}(X)$ as an \mathcal{O}_x -module.

Let \mathcal{G} be the sheaf of \mathcal{O}_X -modules generated by $\mathcal{F}(X)$. The sheaf \mathcal{F}/\mathcal{G} is of finite type. It follows that its support is a closed analytic subset Z of X . The previous argument shows that Z contains no rigid point, hence Z is empty. We deduce that $\mathcal{F} = \mathcal{G}$ and the result follows. \square

We say that a coherent sheaf \mathcal{F} on a K -analytic space X is of bounded rank if the family $(\text{rank}_{\mathcal{H}(x)}(\mathcal{F}(x)))_{x \in X}$ is bounded.

Corollary 4.9. *Let X be a quasi-smooth K -analytic curve with no proper connected components. Let \mathcal{F} be a coherent sheaf of bounded rank on X . Then the module of global sections $\mathcal{F}(X)$ is of finite type over $\mathcal{O}(X)$.*

In particular, there exist an integer q and a surjective morphism $\mathcal{O}^q \rightarrow \mathcal{F}$.

Proof. As in the proof of Corollary 4.8, we may assume that X is connected and contains a rigid point z . Let \mathcal{F} be a coherent sheaf of bounded rank on X . By Corollary 4.8, there exists an integer q_0 and a morphism $\varphi: \mathcal{O}^{q_0} \rightarrow \mathcal{F}$ on X such that the induced morphism $\varphi_z: \mathcal{O}_z^{q_0} \rightarrow \mathcal{F}_z$ is surjective.

Let us now consider the coherent sheaf \mathcal{G} given by the cokernel of φ . Its support Z is a closed analytic subset of X that does not contain z . If it is empty, then we are done. Otherwise, it is a locally finite subset of rigid points of X . Remark that \mathcal{G} is also of bounded rank, hence, by Corollary 4.8, there exists an integer q_1 and a surjective morphism $\psi: \mathcal{O}^{q_1} \rightarrow \mathcal{G}$ on X .

Since X is cohomologically Stein, the morphism ψ induce a surjective morphism $\mathcal{O}(X)^{q_1} \rightarrow \mathcal{G}(X)$, hence $\mathcal{G}(X)$ is of finite type. Similarly, we have an exact sequence $\mathcal{O}(X)^{q_0} \rightarrow \mathcal{F}(X) \rightarrow \mathcal{G}(X) \rightarrow 0$, which shows that $\mathcal{F}(X)$ is of finite type.

The last statement follows from the first and from Corollary 4.8. \square

Corollary 4.10. *Let X be a quasi-smooth K -analytic curve with no proper connected components and let Y be an analytic domain of X such that the restriction map $\mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ has dense image (with respect to the topology of compact convergence). Let \mathcal{F} be a coherent sheaf of bounded rank on X and endow $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ with admissible normoid structures. Then the restriction map $\mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ has dense image.*

Proof. By Corollary 4.9, there exist an integer q and a surjective morphism $\varphi: \mathcal{O}^q \rightarrow \mathcal{F}$. Denote by \mathcal{K} the kernel of φ . It is a coherent sheaf. By Corollary 4.6, for each analytic domain U of X , we have $H^1(U, \mathcal{K}) = 0$, hence the map $\mathcal{O}(U)^q \rightarrow \mathcal{F}(U)$ induced by φ is surjective.

Let \mathcal{V} be an affinoid covering of X for the G -topology such that $\{V \in \mathcal{V} \mid V \subseteq Y\}$ is a covering of Y for the G -topology. Let us endow $\mathcal{F}(X)$ with the basic admissible structure defined the covering \mathcal{V} and the surjections above and let us endow $\mathcal{F}(Y)$ with the induced normoid structure. It is now easy to check that the restriction map $\mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ has dense image. \square

Corollary 4.11. *Let X be a quasi-smooth K -analytic curve with no proper connected components. The functor $\mathcal{F} \mapsto \mathcal{F}(X)$ induces an equivalence between the category of coherent sheaves of bounded rank (resp. locally free sheaves of bounded rank) and the category of $\mathcal{O}(X)$ -modules of finite type (resp. projective $\mathcal{O}(X)$ -modules of finite type).*

Proof. The equivalence between the category of coherent sheaves of bounded rank and the category of $\mathcal{O}(X)$ -modules of finite type follows from Corollaries 4.8 and 4.9.

Let \mathcal{F} be a coherent sheaf of bounded rank. We want to prove that it is locally free if, and only if, $\mathcal{F}(X)$ is projective. If $\mathcal{F}(X)$ is projective, then it is a direct factor of a free $\mathcal{O}(X)$ -module, hence,

by Corollary 4.8, for every $x \in X$, \mathcal{F}_x is a direct factor of a free $\mathcal{O}_{X,x}$ -module, hence projective, hence free.

Let us now assume that \mathcal{F} is locally free. By Corollary 4.9, there exist an integer q and a surjective morphism $\mathcal{O}^q \rightarrow \mathcal{F}$. It gives rise to a morphism of coherent sheaves $\mathcal{H}om(\mathcal{F}, \mathcal{O}^q) \rightarrow \mathcal{H}om(\mathcal{F}, \mathcal{F})$. This morphism is surjective because it is surjective on each stalk since \mathcal{F} is locally free. Since X is cohomologically Stein, passing to global sections, we find a surjective morphism

$$\mathrm{Hom}(\mathcal{F}(X), \mathcal{O}(X)^q) = H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{O}^q)) \rightarrow H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{F})) = \mathrm{Hom}(\mathcal{F}(X), \mathcal{F}(X)). \quad (4.4)$$

It follows that the identity endomorphism on \mathcal{F} has a preimage, hence the surjective map $\mathcal{O}(X)^q \rightarrow \mathcal{F}(X)$ has a section, hence $\mathcal{F}(X)$ is a direct factor of a free module and $\mathcal{F}(X)$ is projective. \square

4.2. de Rham cohomology

In this section, we will derive some consequences of our results for de Rham cohomology on curves.

To begin with, let us show that connections on locally free sheaves on curves give rise to bounded maps.

Lemma 4.12. *Let V be a K -affinoid quasi-smooth curve such that there exists an affinoid domain W of the affine line and a finite \tilde{A} - \otimes -tale morphism $\varphi: V \rightarrow W$. Let \mathcal{F} be a free \mathcal{O}_V -module of finite rank endowed with a connexion $\nabla: \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_V^1$. Then, there exist a norm $\|\cdot\|$ on $\mathcal{F}(V)$ and a norm $\|\cdot\|'$ on $\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \Omega^1(V)$ such that*

- i) *the norm $\|\cdot\|$ (resp. $\|\cdot\|'$) is the sup-norm (as in (2.9)) with respect to some $\mathcal{O}(V)$ -basis of $\mathcal{F}(V)$ (resp. $\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \Omega^1(V)$);*
- ii) *($\mathcal{F}(V), \|\cdot\|$) and ($\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \Omega^1(V), \|\cdot\|'$) are Banach spaces;*
- iii) *the connection $\nabla: (\mathcal{F}(V), \|\cdot\|) \rightarrow (\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \Omega^1(V), \|\cdot\|')$ is a contraction.*

Proof. Let us first work on W . Let T be a coordinate on $\mathbb{A}_K^{1,\mathrm{an}}$. Then, we may identify Ω_W^1 with \mathcal{O}_W by choosing the basis dT of Ω_W^1 . Let us endow $\mathcal{O}(W)$ with the sup-norm $\|\cdot\|_W$ on W . It is a Banach ring. Denote by $\|\cdot\|'_W$ the norm induced by $\|\cdot\|_W$ on $\Omega^1(W)$. It makes it a Banach space too. Explicit computations show that the derivation d/dT is a bounded map. This is easy to check if W is a closed disk and, in the general case, one may use the Mittag-Leffler decomposition (see [FvdP04, Proposition 2.2.6]). It follows that the natural map

$$d_W: (\mathcal{O}(W), \|\cdot\|_W) \rightarrow (\Omega^1(W), \|\cdot\|'_W) \quad (4.5)$$

is bounded: there exists $C \in \mathbb{R}_+$ such that, for each $f \in \mathcal{O}(W)$, $\|d_W(f)\|'_W \leq C \|f\|_W$.

The finite morphism φ induces a finite morphism $\mathcal{O}(W) \rightarrow \mathcal{O}(V)$. Let us choose a finite generating family e_1, \dots, e_n of $\mathcal{O}(V)$ over $\mathcal{O}(W)$. Consider the corresponding surjection $\mathcal{O}(W)^n \rightarrow \mathcal{O}(V)$ and endow $\mathcal{O}(V)$ with the quotient norm $\|\cdot\|_{V,q}$ induced by $\|\cdot\|_W$. It gives $\mathcal{O}(V)$ a structure of Banach ring.

Since φ is étale, we have an isomorphism $\varphi^* \Omega_W^1 \xrightarrow{\sim} \Omega_V^1$, hence an isomorphism of global sections $\Omega^1(W) \otimes_{\mathcal{O}(W)} \mathcal{O}(V) \xrightarrow{\sim} \Omega^1(V)$. Let us endow $\Omega^1(V)$ with the tensor norm $\|\cdot\|'_{V,q}$ induced by $\|\cdot\|'_W$ and $\|\cdot\|_{V,q}$. It gives $\Omega^1(V)$ a structure of Banach space.

Let us now prove that the natural map

$$d_V: (\mathcal{O}(V), \|\cdot\|_{V,q}) \rightarrow (\Omega^1(V), \|\cdot\|'_{V,q}) \quad (4.6)$$

is bounded. Let $f \in \mathcal{O}(V)$ and let $\varepsilon > 0$. There exist $a_1, \dots, a_n \in \mathcal{O}(W)$ such that $f = \sum_{i=1}^n a_i e_i$

and $\|f\|_{V,q} \geq \max_{1 \leq i \leq n} (\|a_i\|_W) - \varepsilon$. We have

$$\|d_V(f)\|'_{V,q} \leq \max \left(\left\| \sum_{i=1}^n d_W(a_i) e_i \right\|'_{V,q}, \left\| \sum_{i=1}^n a_i d_W(e_i) \right\|'_{V,q} \right) \quad (4.7)$$

$$\leq \max_{1 \leq i \leq n} (\|d_W(a_i)\|_W, \|a_i\|_W \|d_V(e_i)\|'_{V,q}) \quad (4.8)$$

$$\leq \max \left(C, \max_{1 \leq j \leq n} \|d_V(e_j)\|'_{V,q} \right) \cdot (\|f\|_{V,q} + \varepsilon). \quad (4.9)$$

It follows that d_V is bounded.

Since V is reduced, the sup-norm on $\mathcal{O}(V)$ is equivalent to the norm $\|\cdot\|_{V,q}$. To prove this, by Proposition 3.6, one may tensor with K_r finitely many times, hence reduce to the case where K is non-trivially valued and V is strictly K -affinoid. The result then follows from [BGR84, Theorem 6.2.4/1] (see also [Ber90, Proposition 2.1.4 (ii)]).

From now on, we endow $\mathcal{O}(V)$ with the sup-norm $\|\cdot\|_V$ on V and $\Omega^1(V) \simeq \Omega^1(W) \otimes_{\mathcal{O}(W)} \mathcal{O}(V)$ with the tensor norm $\|\cdot\|'_V$ induced by $\|\cdot\|'_W$ and $\|\cdot\|_V$. The map $d_V : (\mathcal{O}(V), \|\cdot\|_V) \rightarrow (\Omega^1(V), \|\cdot\|'_V)$ is still bounded.

Remark that $\|\cdot\|'_V$ is a norm of $\mathcal{O}(V)$ -modules in the sense that

$$\forall a \in \mathcal{O}(V), \forall s \in \Omega^1(V), \|as\|'_V \leq \|a\|_V \|s\|'_V. \quad (4.10)$$

By assumption, $\mathcal{F}(V)$ is a free $\mathcal{O}(V)$ -module. Let us choose a basis (e'_1, \dots, e'_m) of it and endow $\mathcal{F}(V)$ with the sup-norm $\|\sum f_i e'_i\| := \max_i (\|f_i\|_V)$ associated to it. Endow $\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \Omega^1(V)$ with the tensor norm $\|\cdot\|'$ induced by $\|\cdot\|$ and $\|\cdot\|'_V$. The norms $\|\cdot\|$ and $\|\cdot\|'$ are norms of $\mathcal{O}(V)$ -modules.

Now remark that, for every $a_1, \dots, a_m \in \mathcal{O}(V)$, we have

$$\nabla \left(\sum_{i=1}^m a_i e'_i \right) = \sum_{i=1}^m e_i \otimes d_V(a_i) + \sum_{i=1}^m a_i \nabla(e'_i) \in \mathcal{F}(V) \otimes_{\mathcal{O}(V)} \Omega^1(V). \quad (4.11)$$

It follows that the map ∇ is bounded. Multiplying the norm $\|\cdot\|'_V$ by a constant, we can ensure that ∇ is a contraction. \square

Proposition 4.13. *Let X be a quasi-smooth K -analytic curve. Let \mathcal{F} be a locally free sheaf of finite rank endowed with a connection ∇ . Endow $\mathcal{F}(X)$ and $\mathcal{F}(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$ with admissible normoid structures. Then the map*

$$\nabla(X) : \mathcal{F}(X) \rightarrow \mathcal{F}(X) \otimes_{\mathcal{O}(X)} \Omega^1(X) \quad (4.12)$$

induced by ∇ is bounded.

Proof. Since X is quasi-smooth, it admits an affinoid covering \mathcal{V} for the G -topology such that, for each $V \in \mathcal{V}$, there exists a finite étale morphism from V to an affinoid domain of the affine line. Up to refining the covering, we may assume moreover that, for each $V \in \mathcal{V}$, \mathcal{F}_V is free.

By Lemma 4.12, for every $V \in \mathcal{V}$, there exist a norm $\|\cdot\|_V$ on $\mathcal{F}(V)$ and a norm $\|\cdot\|'_V$ on $\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \Omega^1(V)$ such that the map

$$(\mathcal{F}(V), \|\cdot\|_V) \rightarrow (\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \Omega^1(V), \|\cdot\|'_V) \quad (4.13)$$

induced by ∇ is bounded. By composing those norms with the restriction maps, we define Banachoid spaces $(\mathcal{F}(X), u)$ and $(\mathcal{F}(X) \otimes_{\mathcal{O}(X)} \Omega^1(X), u')$. By construction, the map induced by ∇ is bounded. Moreover, the first property Lemma 4.12 ensures that u and u' can be chosen basic admissible. \square

Corollary 4.14. *Let X be a quasi-smooth K -analytic curve. Let (\mathcal{F}, ∇) be a module with connection on X . Let L be a complete valued extension of K . Assume that there exists $M \in \{K, L\}$ such that M*

is not trivially valued and $H_{\text{dR}}^1(X_M, \mathcal{F}_M)$ is finite-dimensional. Then, $H_{\text{dR}}^1(X, \mathcal{F})$ and $H_{\text{dR}}^1(X_L, \mathcal{F}_L)$ are both finite-dimensional and we have natural isomorphisms

$$H_{\text{dR}}^0(X, \mathcal{F}) \otimes_K L \xrightarrow{\sim} H_{\text{dR}}^0(X_L, \mathcal{F}_L) \quad \text{and} \quad H_{\text{dR}}^1(X, \mathcal{F}) \otimes_K L \xrightarrow{\sim} H_{\text{dR}}^1(X_L, \mathcal{F}_L). \quad (4.14)$$

Proof. We may assume that X is connected. If X is proper, then it is projective and we get back to a purely algebraic situation where the result is known to hold.

Let us now assume that X is not proper. By Proposition 4.13, there exist admissible Banachoid structures on $\mathcal{F}(X)$ and $\mathcal{F}(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$ such that the map $\nabla(X): \mathcal{F}(X) \rightarrow \mathcal{F}(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$ is bounded. Moreover, by [Duc, Théorème 4.5.10], X is paracompact, hence by Remark 3.20, $\mathcal{F}(X)$ and $\mathcal{F}(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$ are normoid Fréchet spaces.

By Corollary 4.6, we have $H^1(X, \mathcal{F}) = H^1(X, \mathcal{F} \otimes_{\mathcal{O}(X)} \Omega^1(X)) = 0$, hence $H_{\text{dR}}^0(X, \mathcal{F})$ and $H_{\text{dR}}^1(X, \mathcal{F})$ are respectively the kernel and cokernel of the map $\nabla(X)$.

The same results hold for X_L . Moreover, by Corollary 3.22, the map $\mathcal{F}_L(X_L) \rightarrow \mathcal{F}_L(X_L) \otimes_{\mathcal{O}(X_L)} \Omega^1(X_L)$ is obtained from $\nabla(X)$ by applying $-\hat{\otimes}_K L$. The result now follows from Corollary 3.25. \square

For later use, we record here some surjectivity results in de Rham cohomology.

Lemma 4.15. *Assume that X has no proper connected component. Let W be an analytic domain of X such that the restriction map $\mathcal{O}(X) \rightarrow \mathcal{O}(W)$ has dense image. Assume that there exists a complete non-trivially valued extension L of K such that $H_{\text{dR}}^1(W_L, (\mathcal{F}_L)_{|W_L})$ is finite-dimensional. Then, the map*

$$H_{\text{dR}}^1(X, \mathcal{F}) \longrightarrow H_{\text{dR}}^1(W, \mathcal{F}_{|W}) \quad (4.15)$$

is surjective.

Proof. We may assume that X is connected. It follows from the density hypothesis that W is connected too. Set $\mathcal{F}' := \mathcal{F} \otimes_{\mathcal{O}} \Omega^1$. We have a commutative diagram

$$\begin{array}{ccccccc} \mathcal{F}(X) & \xrightarrow{\nabla} & \mathcal{F}'(X) & \longrightarrow & H_{\text{dR}}^1(X, \mathcal{F}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{F}(W) & \xrightarrow{\nabla} & \mathcal{F}'(W) & \longrightarrow & H_{\text{dR}}^1(W, \mathcal{F}_{|W}) & \longrightarrow & 0 \end{array} \quad (4.16)$$

By Proposition 4.13, we can endow $\mathcal{F}(W)$ and $\mathcal{F}'(W)$ with structures of admissible normoid Fréchet spaces such that the map $\nabla: \mathcal{F}(W) \rightarrow \mathcal{F}'(W)$ is bounded. Moreover, by Corollary 4.10, the map $\mathcal{F}'(X) \rightarrow \mathcal{F}'(W)$ has dense image.

By Corollary 4.6, W_L is cohomologically Stein, hence the cokernel of the map $\nabla_L: \mathcal{F}_L(W_L) \rightarrow \mathcal{F}'_L(W_L)$ coincides with $H_{\text{dR}}^1(W_L, (\mathcal{F}_L)_{|W_L})$, which is finite-dimensional. By Corollary 3.24, ∇ is topologically strict. In particular, its image is closed, hence its cokernel $H_{\text{dR}}^1(W, \mathcal{F}_{|W})$ is naturally endowed with a structure of normoid Fréchet space.

By Corollary 4.6, W is cohomologically Stein, hence the bottom line is exact. It follows that the image of the map $H_{\text{dR}}^1(X, \mathcal{F}) \rightarrow H_{\text{dR}}^1(W, \mathcal{F}_{|W})$ is dense. By Lemma 1.10, it is a closed subspace of $H_{\text{dR}}^1(W, \mathcal{F}_{|W})$, hence it coincides with it. \square

A similar result holds for meromorphic de Rham cohomology. Before stating it, let us recall a few definitions. We still denote by X a quasi-smooth K -analytic curve. Let Z be a locally finite subset of rigid points of X . We denote by $\mathcal{O}_X[*Z]$ the sheaf of meromorphic functions on P that are holomorphic on $X - Z$ (hence have poles at worst on Z).

Let \mathcal{F} be a locally free $\mathcal{O}_X[*Z]$ -module of finite rank on X . Following [Del70], we define a

meromorphic connection on \mathcal{F} with poles on Z to be a K -linear map

$$\nabla: \mathcal{F} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{F} \quad (4.17)$$

that satisfies the Leibniz rule: for every open subset U of X and every $f \in \mathcal{O}_X[*Z](U)$ and $s \in \mathcal{F}(U)$, we have

$$\nabla(fs) = df \otimes s + f\nabla s. \quad (4.18)$$

We then define, as usual, the de Rham cohomology groups of $H_{\text{dR}}^i(X(*Z), (\mathcal{F}, \nabla))$ to be the hypercohomology groups of the complex

$$\cdots \rightarrow 0 \rightarrow \mathcal{F} \xrightarrow{\nabla} \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{F} \rightarrow 0 \rightarrow \cdots, \quad (4.19)$$

where \mathcal{F} is placed in degree 0 and $\Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}$ in degree 1.

If Z is empty, we recover the usual de Rham cohomology groups of X .

Lemma 4.16. *There exists a locally free \mathcal{O}_X -module G of finite rank such that we have an isomorphism of $\mathcal{O}_X[*Z]$ -modules (without connections) $G \otimes_{\mathcal{O}_X} \mathcal{O}_X[*Z] \simeq \mathcal{F}$.*

Proof. Every point z of Z admits a neighborhood U_z of z on which the restriction of \mathcal{F} is isomorphic to $\mathcal{O}_X[*\{z\}]^d$ for some d . In particular, it is isomorphic to \mathcal{O}_X^d over $U_z - \{z\}$, hence extends to \mathcal{O}_X^d over U_z .

Since Z is locally finite, the U_z 's can be chosen disjoint, and the different extensions can then be glued together. \square

Lemma 4.17. *Let W be an analytic domain of $X - Z$ such that the restriction map $(\mathcal{O}_X[*Z])(X) \rightarrow \mathcal{O}(W)$ has dense image. Assume that there exists a complete non-trivially valued extension L of K such that $H_{\text{dR}}^1(W_L, \mathcal{F}_L)$ is finite-dimensional. Then the map*

$$H_{\text{dR}}^1(X(*Z), \mathcal{F}) \longrightarrow H_{\text{dR}}^1(W, \mathcal{F}) \quad (4.20)$$

is surjective.

Proof. We may assume that X is connected. If X is projective and Z is empty, the density hypothesis implies that $W = X$ and the result is obvious. We now assume that we are not in this case. It then follows from Corollary 4.6 that W is cohomologically Stein.

Set $\mathcal{F}' := \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1$. By Lemma 4.16, there exists a locally free \mathcal{O}_X -module of finite rank G such that $G \otimes_{\mathcal{O}_X} \mathcal{O}_X[*Z] \simeq \mathcal{F}'$. If X is projective, then Z is not empty. Let z be a (rigid) point of Z . The sheaf $\mathcal{O}_X(z)$ is ample, hence there exists $n \in \mathbb{N}$ such that $G(n \cdot z)$ is generated by global sections. Since X is compact, there exist a positive integer q and a surjective morphism $\mathcal{O}_X(-n \cdot z)^q \rightarrow G$. If X is not projective, then, by Corollary 4.6, it is cohomologically Stein, hence, by Corollary 4.9, there exist a positive integer q and a surjective morphism $\mathcal{O}_X^q \rightarrow G$. In any case, we have a surjective morphism

$$\mathcal{O}_X[*Z]^q \rightarrow \mathcal{F}'. \quad (4.21)$$

Using the same arguments as in the proof of Corollary 4.10, one can prove that the map $\mathcal{F}'(X) \rightarrow \mathcal{F}'(W)$ has dense image. One now concludes as in the proof of Lemma 4.15. \square

5. The Christol-Mebkhout limit formula for analytic cohomology

In this last section, we prove a statement of commutation of cohomology with projective limits. As an application, we show that if X is a quasi-Stein curve that can be conveniently “approximated” by a family of quasi-Stein curves $\{X_n\}_n$, then the de Rham cohomology of a differential equation \mathcal{F} over X can be recovered as the limit of the de Rham cohomologies of its restrictions to the X_n 's.

The fundamental assumption here is the finite-dimensionality of the cohomology of \mathcal{F} on X_n . Indeed, the fact that $\mathcal{O}(X_n)$ is Fréchet implies that $H_{\text{dR}}^1(X_n, \mathcal{F})$ is separated, and hence that the connection is a strict map, which is the crucial point (see [CM95, Theorem 4], see also Proposition 3.23). This technique has been introduced by Christol and Mebkhout for open annuli (see [CM00, Proof of 8.3-1]), following an original idea of Grothendieck (see [Gro61, Chap.0, 13.2.4] and even [Gro54]).

The fundamental assumption is a Mittag–Leffler property (see [Bou71, II, §3, n° 5, Théorème 1]). Results of this kind already appear in the work of Grothendieck (see [Gro57, Proposition 3.10.2] and [Gro61, 0, Proposition 13.2.3]).

Definition 5.1. *Let (I, \leq) be a directed partially ordered set that admits a cofinal countable subset. We say that a projective system $((X_i)_{i \in I}, (f_{i,j})_{i \leq j \in I})$ of topological spaces satisfies the Mittag–Leffler condition if, for each $i \in I$, there exists $j \geq i$ such that, for each $j' \geq j$, $f_{i,j'}(U_{j'})$ is dense in $f_{i,j}(U_j)$.*

Proposition 5.2. *Let (I, \leq) be a directed partially ordered set that admits a cofinal countable subset. Let $((U_i, u_i)_{i \in I}, (a_{i,j})_{i \leq j \in I})$ and $((V_i, v_i)_{i \in I}, (b_{i,j})_{i \leq j \in I})$ be inverse systems of normoid Fréchet spaces and let $(f_i : U_i \rightarrow V_i)_{i \in I}$ be an inverse system of contractions between them. We consider the set-theoretical inverse limits $U := \varprojlim_{i \in I} U_i$, $V := \varprojlim_{i \in I} V_i$ and $f := \varprojlim_{i \in I} f_i$.*

Assume that

- i) the projective systems $(U_i, a_{i,j})$ and $(\text{Ker}(f_i), a_{i,j})$ satisfy the Mittag–Leffler condition;*
- ii) for each $i \in I$, f_i is topologically strict.*

Then, we have a K -linear isomorphism

$$\text{Coker}(f) \xrightarrow{\sim} \varprojlim_{i \in I} \text{Coker}(f_i). \quad (5.1)$$

In particular, the K -vector space $\text{Coker}(f)$ is finite-dimensional if, and only if, the net of dimensions $(\dim_K \text{Coker}(f_j))_{j \in J}$ is eventually constant. In this case, we have

$$\dim_K \text{Coker}(f) = \lim_{j \in J} \dim_K \text{Coker}(f_j). \quad (5.2)$$

Proof. Let $i \in I$. Denote the image of f_i by W_i . By assumption, it is a closed subspace of V_i , hence it is naturally endowed with a structure of normoid Fréchet space. The space $\text{Coker}(f_i)$ is naturally a normoid Fréchet space too and we have an exact sequence

$$0 \rightarrow W_i \rightarrow V_i \rightarrow \text{Coker}(f_i) \rightarrow 0. \quad (5.3)$$

Since the projective system $(U_i, a_{i,j})$ satisfies the Mittag–Leffler condition, for each $i \in I$, there exists $j \geq i$ such that, for every $j' \geq j$, $a_{i,j'}(U_{j'})$ is dense in $a_{i,j}(U_j)$. Since f_i is topologically strict, it follows that $b_{i,j'}(W_{j'})$ is dense in $b_{i,j}(W_j)$. In other words, the projective system $(W_i, b_{i,j})$ satisfies the Mittag–Leffler condition. By [Gro61, 0, Proposition 13.2.2 and Remarque 13.2.4], we get an exact sequence

$$0 \rightarrow \varprojlim_{i \in I} W_i \rightarrow \varprojlim_{i \in I} V_i \rightarrow \varprojlim_{i \in I} \text{Coker}(f_i) \rightarrow 0. \quad (5.4)$$

Similarly, starting with the system of exact sequences

$$0 \rightarrow \text{Ker}(f_i) \rightarrow U_i \rightarrow W_i \rightarrow 0, \quad (5.5)$$

we get an exact sequence

$$0 \rightarrow \varprojlim_{i \in I} \text{Ker}(f_i) \rightarrow \varprojlim_{i \in I} U_i \rightarrow \varprojlim_{i \in I} W_i \rightarrow 0, \quad (5.6)$$

which shows that the image of f coincides with $\varprojlim_{i \in I} W_i$. The result follows.

□

Let us now write down an explicit situation where Proposition 5.2 can be applied.

Lemma 5.3. *Let X be a quasi-smooth K -analytic curve with countably many connected components, none of them being proper. Let \mathcal{F} be a locally free sheaf of finite rank endowed with a connection ∇ . Set $\mathcal{F}' := \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X^1$. Let $(X_n)_{n \in \mathbb{N}}$ be a non-decreasing sequence of analytic domains of X forming a covering of X for the G -topology such that, for each $n \in \mathbb{N}$, the restriction map $\mathcal{O}(X_{n+1}) \rightarrow \mathcal{O}(X_n)$ has dense image (for the topology of compact convergence).*

Then, there exist a set M and admissible M -normoid Fréchet structures on $\mathcal{F}(X)$, $\mathcal{F}'(X)$, the $\mathcal{F}(X_n)$'s and the $\mathcal{F}'(X_n)$'s such that

- i) *for each $n, m \in \mathbb{N}$ with $n \geq m$, the maps $\mathcal{F}(X_n) \rightarrow \mathcal{F}(X_m)$ and $\mathcal{F}'(X_n) \rightarrow \mathcal{F}'(X_m)$ are contractions with dense images;*
- ii) *for each $n \in \mathbb{N}$, the maps $\mathcal{F}(X) \rightarrow \mathcal{F}(X_n)$ and $\mathcal{F}'(X) \rightarrow \mathcal{F}'(X_n)$ are contractions with dense images;*
- iii) *for each $n \in \mathbb{N}$, the map $\nabla(X_n): \mathcal{F}(X_n) \rightarrow \mathcal{F}'(X_n)$ is a contraction;*
- iv) *the map $\nabla(X): \mathcal{F}(X) \rightarrow \mathcal{F}'(X)$ is a contraction;*
- v) *the restriction maps induce the following isomorphisms both in $\text{Ban}_{M,K}$ and in the category of sets*

$$\mathcal{F}(X) \xrightarrow{\sim} \varprojlim_{n \in \mathbb{N}} \mathcal{F}(X_n) \quad \text{and} \quad \mathcal{F}'(X) \xrightarrow{\sim} \varprojlim_{n \in \mathbb{N}} \mathcal{F}'(X_n) \quad (5.7)$$

and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\nabla} & \mathcal{F}'(X) \\ \downarrow & & \downarrow \\ \varprojlim \mathcal{F}(X_n) & \xrightarrow{\varprojlim \nabla(X_n)} & \varprojlim \mathcal{F}'(X_n) . \end{array} \quad (5.8)$$

Proof. Denote by \mathcal{V} the set of affinoid domains V of X such that

- i) there exists a finite étale morphism from V to an affinoid domain of the affine line;
- ii) \mathcal{F}_V is free.

It is a covering of X for the G -topology and, for every $n \in \mathbb{N}$, the set $\mathcal{V}_n := \{V \in \mathcal{V} \mid V \subseteq X_n\}$ is a covering of X_n for the G -topology.

By Lemma 4.12, there exist basic admissible \mathcal{V} -Banachoid structures u and u' associated to \mathcal{V} on $\mathcal{F}(X)$ and $\mathcal{F}'(X)$ respectively such that the map $\nabla(X)$ is a contraction.

For every $n \in \mathbb{N}$, endow $\mathcal{F}(X_n)$ and $\mathcal{F}'(X_n)$ with the induced \mathcal{V} -Banachoid structures v_n and v'_n respectively (in the sense of item iii) of Definition 1.11). Then, for every $n \in \mathbb{N}$, the restriction maps $\mathcal{F}(X) \rightarrow \mathcal{F}(X_n)$ and $\mathcal{F}'(X) \rightarrow \mathcal{F}'(X_n)$ and the map $\nabla(X_n): \mathcal{F}(X_n) \rightarrow \mathcal{F}'(X_n)$ are contractions. Note also that, by Lemma 1.12, the structures u_n and u'_n are admissible.

By assumption, the space X has countably many connected components. We deduce that the same result holds for the X_n 's. It now follows from Remark 3.20 that the structures u, u', u_n and u'_n , for every $n \in \mathbb{N}$, are normoid Fréchet structures.

The density properties follow from Corollary 4.10.

The isomorphisms in $\text{Ban}_{\mathcal{V},K}$ and in the category of sets follow from the constructions we have just made and the results in Section 2.3. □

We can now state the main result of this section.

Theorem 5.4. *Let X be a quasi-smooth K -analytic curve with finitely many connected components, none of them being proper. Assume moreover that there exists a non-decreasing sequence of analytic domains $(X_n)_{n \in \mathbb{N}}$ of X forming a covering of X for the G -topology and an integer n_0 such that, for each $n \geq n_0$,*

- i) the natural map $\pi_0(X_n) \rightarrow \pi_0(X)$ is bijective;*
- ii) the restriction map $\mathcal{O}(X_{n+1}) \rightarrow \mathcal{O}(X_n)$ has dense image;*
- iii) there exists a complete non-trivially valued extension L of K such that the de Rham cohomology group $H_{\text{dR}}^1(X_n, L, (\mathcal{F}_L)|_{X_n, L})$ is a finite-dimensional L -vector space.*

Then,

- (a) the natural map

$$H_{\text{dR}}^0(X, \mathcal{F}) \xrightarrow{\sim} \varprojlim_n H_{\text{dR}}^0(X_n, \mathcal{F}|_{X_n}) \quad (5.9)$$

is an isomorphism, $H_{\text{dR}}^0(X, \mathcal{F})$ is a finite-dimensional K -vector space and there exists an integer n_1 such that, for each $n, m \in \mathbb{N}$ satisfying $n \geq m \geq n_1$, the natural map $H_{\text{dR}}^0(X_n, \mathcal{F}|_{X_n}) \rightarrow H_{\text{dR}}^0(X_m, \mathcal{F}|_{X_m})$ is an isomorphism;

- (b) for each $n, m \geq n_0$, the natural map $H_{\text{dR}}^1(X_n, \mathcal{F}|_{X_n}) \rightarrow H_{\text{dR}}^1(X_m, \mathcal{F}|_{X_m})$ is surjective;
- (c) the natural map

$$H_{\text{dR}}^1(X, \mathcal{F}) \xrightarrow{\sim} \varprojlim_n H_{\text{dR}}^1(X_n, \mathcal{F}|_{X_n}) \quad (5.10)$$

is an isomorphism and, for each $n \geq n_0$, the natural map $H_{\text{dR}}^1(X, \mathcal{F}) \rightarrow H_{\text{dR}}^1(X_n, \mathcal{F}|_{X_n})$ is surjective.

In particular, $H_{\text{dR}}^1(X, \mathcal{F})$ is finite-dimensional if, and only if, the sequence of dimensions $(h_{\text{dR}}^1(X_n, \mathcal{F}|_{X_n}))_{n \in \mathbb{N}}$ (or equivalently the sequence of indexes $(\chi_{\text{dR}}(X_n, \mathcal{F}|_{X_n}))_{n \in \mathbb{N}}$) is eventually constant. In this case, the natural map $H_{\text{dR}}^1(X, \mathcal{F}) \rightarrow H_{\text{dR}}^1(X_n, \mathcal{F}|_{X_n})$ is an isomorphism for all n large enough and we have

$$h_{\text{dR}}^1(X, \mathcal{F}) = \lim_{n \rightarrow +\infty} h_{\text{dR}}^1(X_n, \mathcal{F}|_{X_n}) \quad \text{and} \quad \chi_{\text{dR}}(X, \mathcal{F}) = \lim_{n \rightarrow +\infty} \chi_{\text{dR}}(X_n, \mathcal{F}|_{X_n}). \quad (5.11)$$

Proof. By Corollary 4.6, X is cohomologically Stein, hence the cohomology groups $H_{\text{dR}}^0(X, \mathcal{F})$ and $H_{\text{dR}}^1(X, \mathcal{F})$ may be respectively identified with the kernel and cokernel of the map $\nabla(X): \mathcal{F}(X) \rightarrow \Omega^1(X) \otimes_{\mathcal{O}(X)} \mathcal{F}(X)$. Similar result holds for the X_n 's and after extending the scalars.

(a) The first part of the statement holds since inverse limits commute with kernels. It is also well-known that all the H_{dR}^0 's are finite-dimensional since the spaces have finitely many connected components.

Moreover, since \mathcal{F} is locally free, it follows from analytic continuation (see [Ber90, Corollary 3.3.21]) that, for every $n \geq n_0$, the map $\mathcal{F}(X_{n+1}) \rightarrow \mathcal{F}(X_n)$ is injective, hence the map $H_{\text{dR}}^0(X_{n+1}, \mathcal{F}|_{X_{n+1}}) \rightarrow H_{\text{dR}}^0(X_n, \mathcal{F}|_{X_n})$ is injective too. We deduce that the sequence of dimensions $(h_{\text{dR}}^0(X_n, \mathcal{F}|_{X_n}))_{n \in \mathbb{N}}$ is eventually non-increasing, hence eventually constant. It follows that the maps $H_{\text{dR}}^0(X_n, \mathcal{F}|_{X_n}) \rightarrow H_{\text{dR}}^0(X_m, \mathcal{F}|_{X_m})$ are isomorphisms for all n, m large enough.

For the rest of the argument, we use the notation of Lemma 5.3 and endow the $\mathcal{F}(X_n)$'s and the $\mathcal{F}'(X_n)$'s with normoid Fréchet structures satisfying the properties stated there.

(b) Let $n \geq m \geq n_0$. The map $\mathcal{F}'(X_n) \rightarrow \mathcal{F}'(X_m)$ has dense image, hence the image $F_{m,n}$ of the map $H_{\text{dR}}^1(X_n, \mathcal{F}|_{X_n}) \rightarrow H_{\text{dR}}^1(X_m, \mathcal{F}|_{X_m})$ is a dense subspace of $H_{\text{dR}}^1(X_m, \mathcal{F}|_{X_m})$. On the other hand, since $H_{\text{dR}}^1(X_m, \mathcal{F}|_{X_m})$ is a normoid Fréchet space that is finite-dimensional over K , by Lemma 1.10, the image $F_{m,n}$ is closed. We deduce that it is equal to $H_{\text{dR}}^1(X_m, \mathcal{F}|_{X_m})$ itself.

(c) By Corollary 3.24, for each $n \geq n_0$, the map $\nabla(X_n): \mathcal{F}(X_n) \rightarrow \mathcal{F}'(X_n)$ is topologically strict. By (a), the projective system of the $H_{\text{dR}}^0(X_n, \mathcal{F}|_{X_n})$'s satisfies the Mittag-Leffler condition. Lemma 5.3 ensures that the other conditions required by Proposition 5.2 are satisfied. The isomorphism (5.10) follows. The surjectivity property is a consequence of (b). □

REFERENCES

- Ber90 Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.
- Ber93 Vladimir G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Inst. Hautes Études Sci. Publ. Math.*, (78):5–161 (1994), 1993.
- BGR84 S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.
- Bou71 N. Bourbaki. *Éléments de mathématique. Topologie générale I*. Chap. 1 à 4. Nouvelle édition. Paris: Hermann. XV, 351 p., 1971.
- Bou74 N. Bourbaki. *Éléments de mathématique. Topologie générale*. Chap. 5 à 10. Nouvelle édition. Paris: Hermann. 336 p., 1974.
- CM95 Gilles Christol and Zoghman Mebkhout. Topological p -adic vector spaces and index theory. In *Proceedings of the 3rd International Conference on p -adic Functional Analysis (Aubière, 1994)*, volume 2, pages 93–98, 1995.
- CM00 G. Christol and Z. Mebkhout. Sur le théorème de l'indice des équations différentielles p -adiques. III. *Ann. of Math. (2)*, 151(2):385–457, 2000.
- CM01 G. Christol and Z. Mebkhout. Sur le théorème de l'indice des équations différentielles p -adiques. IV. *Invent. Math.*, 143(3):629–672, 2001.
- Del70 Pierre Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin, 1970.
- Duc Antoine Ducros. La structure des courbes analytiques. <http://www.math.jussieu.fr/~ducros/livre.html>.
- FvdP04 Jean Fresnel and Marius van der Put. *Rigid analytic geometry and its applications*, volume 218 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2004.
- Gro54 A. Grothendieck. *Espaces vectoriels topologiques*. Instituto de Matemática Pura e Aplicada, Universidade de São Paulo, São Paulo, 1954.
- Gro57 A. Grothendieck. Sur quelques points d'algèbre homologique. *Tohoku Math. J. (2)*, 9:119–221, 1957.
- Gro61 A. Grothendieck. *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.
- Gru66 Laurent Gruson. Théorie de Fredholm p -adique. *Bull. Soc. Math. Fr.*, 94:67–95, 1966.
- Ked10 Kiran S. Kedlaya. *p -adic Differential Equations*, volume 125 of *Cambridge Studies in Advanced Mathematics*. Cambridge Univ. Press, 2010.
- Kie67 Reinhardt Kiehl. Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie. *Invent. Math.*, 2:256–273, 1967.
- LvdP95 Q. Liu and M. van der Put. On one-dimensional separated rigid spaces. *Indag. Math. (N.S.)*, 6(4):439–451, 1995.
- ML98 Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- Sch02 Peter Schneider. *Nonarchimedean functional analysis*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002.

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