An algorithm computing non solvable spectral radii of p-adic differential equations.

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ABSTRACT

We obtain an algorithm computing explicitly the values of the non solvable spectral radii of convergence of the solutions of a differential module over a point of type 2, 3 or 4 of the Berkovich affine line.

Introduction

By a theorem of Young [You92] the small values of the radii of convergence of the solutions of a differential operator are explicit, and coincides with the small slopes of the Newton polygon of a differential operator attached to the module (cf. Prop.2.2). Larger radii are not immediately readable on the coefficients of the operator. This discrepancy is a peculiarity of the p-adic world: only small radii are "*visible*". To overcome this problem B.Dwork observed that the pull-back by Frobenius functor increase the radii of the solutions, and together with G.Christol [CD94] they constructed an inverse of the Frobenius functor (often called Frobenius antecedent) in order to make the radii of the solutions smaller, and hence explicitly intelligible in a cyclic basis. Although theoretically satisfactory, the inversion of Frobenius is a completely implicit operation. Moreover the antecedent exists only if all the radii of the solutions are not small. So one is obliged to factorize the module by the radii of the solutions if one wants to understand the non minimal radii of the solutions. The factorization is also an implicit operation. Recently in [Ked10] K.Kedlaya observed that the Frobenius Push-forward operation has essentially the same effects as the inversion of the Frobenius on the radii of the solutions, and he is able to control the exact behavior of the radii of the solutions under this operation (even small radii).[1] Frobenius push-forward functor is completely explicit, and it allows to obtain a concrete algorithm to compute the radii of the solutions that are not maximal (i.e. non solvable). The price to pay is that the dimension of the push-forward by Frobenius is p-times that of the original module, so that the complexity of the algorithm is multiplied by p at each application of the push-forward. Hence more the radius is large, more the complexity increase. Moreover the algorithm admits an end if and only if all the radii are not maximal (i.e. non *solvable*). Eventually we provide the algorithm, but we avoid to provide a complete formula, whose complexity seems too great and impossible to write. Explicit examples are quite complicate even in the rank one case. This note is intended to make explicit the computations, and the link between the different results, in view to make them explicitly calculable by a computer.

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[1]The very first reference for this is [Chr77], in which the author introduces the push-forward, and its relation with the pull-back, and use it to prove the inversion of Frobenius (existence of the antecedent) under the condition that the solutions of the module are Bounded.

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1. Radii of convergence

Let $(K, |.|)$ be a complete field with respect to an ultrametric absolute value $|.| : K \to \mathbb{R}_{\geqslant 0}$. Let L/K be a complete valued field extension, let $c \in L$, and $\rho > 0$. For all polynomial $P(T) := \sum_i a_i T^i \in K[T]$ define $|P|_{c,\rho} := \sup_{n \geqslant 0} |P^{(i)}(n)| \rho^n / n!$. The setting $|P_1/P_2|_{c,\rho} := |P_1|_{c,\rho} / |P_2|_{c,\rho}$ defines an absolute value on the field of fraction $K(T) := \mathrm{Frac}(K[T])$, and hence a Berkovich point $\xi_{c,\rho}$ of the affine line $\mathbb{A}_K^{1,\mathrm{an}}$. If $c \in K$ one obtains in this way all the points of type 2 or 3 of $\mathbb{A}_K^{1,\mathrm{an}}$, if one allows $c \notin K$ one also has all points of type 4. The derivative d/dT extends by continuity to the completion $\mathscr{H}_{c,\rho}$ of $(K(T), |.|_{c,\rho})$. A differential module over $\mathscr{H}_{c,\rho}$ is a finite dimensional $\mathscr{H}_{c,\rho}$-vector space M together with a K-linear map $\nabla : \mathrm{M} \to \mathrm{M}$ satisfying $\nabla(fm) = d(f)m + f\nabla(m)$, for all $f \in \mathscr{H}_{c,\rho}$, $m \in \mathrm{M}$. Let $r(\xi_{c,\rho}) \geqslant \rho$ be the radius of the point $\xi_{c,\rho}$. If Ω/K is a complete valued field extension such that $\mathbb{A}_\Omega^{1,\mathrm{an}}$ has an Ω-rational point $t_{c,\rho} \in \Omega$ lifting $\xi_{c,\rho}$, then $r(\xi_{c,\rho})$ is the radius of the largest open disk $\mathrm{D}_\Omega^-(t_{c,\rho}, r(\xi_{c,\rho}))$ satisfying $\mathrm{D}_\Omega^-(t_{c,\rho}, r(\xi_{c,\rho})) \cap K^{\mathrm{alg}} = \emptyset$.

DEFINITION 1.1. *Let $r := \mathrm{rank}(\mathrm{M})$. For $i = 1, \ldots, r$ we denote by $\mathcal{R}_i = \mathcal{R}_i^{\mathrm{M}}(\xi_{c,\rho}) \leqslant r(\xi_{c,\rho})$ the radius of the largest open disk centered at $t_{c,\rho}$ on which M has at least $r - i + 1$ Ω-linearly independent Taylor solutions. One has $\mathcal{R}_1 \leqslant \mathcal{R}_2 \leqslant \cdots \leqslant \mathcal{R}_r$.*

2. Comparison of Newton polygons and computation of small radii

Let $r \geqslant 1$ be an integer. A *slope sequence* is the data of r real numbers $s_1 \leqslant \ldots \leqslant s_r$ in increasing order. Define the i-th partial height as $h_i := s_1 + \cdots + s_i$. A slope sequence defines univocally a convex function $h : [0, r] \to \mathbb{R}$ defined as $h(0) = 0$, $h(i) := h_i$, and $h(x) = s_i x + (h_i - i \cdot s_i)$ for all $x \in]i-1, i]$, $i = 0, \ldots, r$. The function h is called the *Newton polygon with slopes $s_1 \leqslant \ldots \leqslant s_r$.*

DEFINITION 2.1. *The Newton polygon with slopes $s_i := s_i^{\mathrm{M}}(\xi_{c,\rho}) := \ln(\mathcal{R}_i^{\mathrm{M}}(\xi_{c,\rho}))$ is called the spectral Newton polygon of M. We denote by $h_i := h_i^{\mathrm{M}}(\xi_{c,\rho})$ its i-th partial height.*

Let $\mathcal{L} = \sum_{i=0}^r g_{r-i} d^i$, $g_i \in \mathscr{H}_{c,\rho}$, be a differential operator with $g_0 = 1$ and $g_r \neq 0$. Let $v_0 = 0$, and for all $i = 1, \ldots, r$ let $v_i := -\ln(|g_i|_{c,\rho}/\omega^i)$. Let $L_i := \{(x, y) \in \mathbb{R}^2 \text{ such that } x = i, y \geqslant v_i\}$, note that L_i is empty if and only if $g_i = 0$. Define the *spectral Newton polygon $NP(\mathcal{L})$* as the intersection of all upper half planes $H_{a,b} := \{(x, y) \in \mathbb{R}^2 \text{ such that } y \geqslant ax + b\}$ with $\{(i, v_i)\}_{i=0,\ldots,r} \subset H_{a,b}$. Let $h^{\mathcal{L}} : [0, r] \to \mathbb{R}$ be the convex function whose epigraph is $NP(\mathcal{L})$: $h^{\mathcal{L}}(x) = \min\{y \text{ such that } (x, y) \in NP(\mathcal{L})\}$. Explicitly one has $h_i^{\mathcal{L}} := h^{\mathcal{L}}(i) = \sup_{s \in \mathbb{R}}\{s \cdot i + \min_{j=0,\ldots,r} v_j - s \cdot j\}$. Then $NP(\mathcal{L})$ is the Newton polygon with slopes $\{s_i^{\mathcal{L}} := h_i^{\mathcal{L}} - h_{i-1}^{\mathcal{L}}\}_{i=1,\ldots,r}$.

PROPOSITION 2.2 ([You92]). *Let $\mathcal{L}$ be a differential operator as above and let M be the differential module defined by $\mathcal{L}$. Let $C := \ln(\omega \cdot r(\xi_{c,\rho}))$, then for all $i = 1, \ldots, r$ one has*

$$\min(s_i^{\mathrm{M}}, C) = \min(s_i^{\mathcal{L}}, C). \qquad \square \tag{2.1}$$

In order to apply (2.1) we need an algorithm to find a cyclic basis of M (cf. section 3), and a technique (Frobenius push-forward) making all the radii smaller than $\omega r(\xi_{c,\rho})$ (cf. section 4).

3. Explicit Cyclic vector

Let (F, d) be a differential field and let $F\langle d \rangle = \oplus_{i \geqslant 0} F \circ d^i$ be the Weil algebra of differential operators. The multiplication of $F\langle d \rangle$ extends that of $F = F \circ d^0$ by the rule $d \circ f = f \circ d + d(f)$, for all $f \in F$. Finite dimensional differential modules over F are exactly torsion left $F\langle d \rangle$-modules. The so called *cyclic vector theorem* asserts that all differential module are not only torsion modules over $F\langle d \rangle$: they are cyclic modules i.e. of the form $(\mathrm{M}, \nabla) = (F\langle d \rangle / F\langle d \rangle \mathcal{L}, d)$, for some $\mathcal{L} := \sum_{i=0}^r g_{r-i} d^i \in F\langle d \rangle$, with $g_0 = 1$, $g_i \in F$. The image of $\{1, d, d^2, \ldots, d^{r-1}\}$ in the quotient form a

basis of M, and the action of ∇ is given by the multiplication by d in the quotient. In fact the cyclic vector theorem is equivalent to the existence of an element $c \in M$, called cyclic vector, such that $\{c, \nabla(c), \nabla^2(c), \ldots, \nabla^{r-1}(c)\}$ is a basis of M. In this case if $c_i := \nabla^i(c)$, and if $\nabla^r(c_0) = \sum_{i=0}^{r-1} f_i c_i$, then $f_i = -g_{r-i}$. The existence of such a vector is due to [Del70, Ch.II, Lemme 1.3]. Subsequently N.M.Katz provided the following explicit algorithm

THEOREM 3.1 ([Kat87]). *Let* (M, ∇) *be a differential module over* (F, d) *of rank* r, *and let* $\mathbf{e} := \{e_0, \ldots, e_{r-1}\} \subset M$ *be a basis of M. Let* $a_0, \ldots, a_{r(r-1)} \in F$ *be* $r(r-1) + 1$ *distinct constants i.e.* $d(a_i) = 0$. *Then at least one of the following elements of M is a cyclic vector:*

$$c(\mathbf{e}, T - a_i) := \sum_{j=0}^{r-1} \frac{(T - a_i)^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \nabla^k(e_{j-k}) . \qquad \square \qquad (3.1)$$

REMARK 3.2. *The explicit base change matrices to the Katz's cyclic basis are quite complicate and have been explicitly computed in [Pul]. On the other hand the proof of the existence of a cyclic vector of Deligne [Del70, Ch.II, Lemme 1.3] proves that the family of cyclic vectors in a given module is the complement of an hypersurface. The base change matrices of Katz's algorithm are quite involved and hard to find by hand even in small examples. It is often convenient to pick an arbitrary vector and test if it is cyclic.*

4. Frobenius Push-Forward and explicit computation of larger radii

HYPOTHESIS 4.1. $\mathcal{R}_i^M$ *is invariant by scalar extensions of* K, *and by translations. So in the sequel we will assume* $c = 0$ *and replace the indexation* (c, ρ) *by* ρ. *In this case one has* $r(\xi_\rho) = \rho$. *We then work with* $|.|_\rho$, ξ_ρ, $\mathcal{H}_\rho$, $r(\xi_\rho) = \rho$ *with the evident meaning of notation. If the reader needs to preserve the setting* (c, ρ), *the same computations hold replacing the map* $\varphi : T \mapsto T^p$ *by* $T \mapsto (T - c)^p + c$. *Or alternatively one also can preserve* $\varphi : T \to T^p$, *and proceed as in [Pul12, Section 7] to check the behavior of the radii by Frobenius at points that are close enough to the principal branch* $\rho \mapsto |.|_{0,\rho}$ *(this is often necessary if one search the slopes of* $\mathcal{R}_i^M$ *along a Berkovich path* $\rho \mapsto |.|_{c,\rho}$, *with* $c \in K$ *and* ρ *close to* $|c|$*).*

Let $\widetilde{T}, T$ be two variables, and let $\varphi : K(T) \to K(\widetilde{T})$ be the ring morphism sending T into $\widetilde{T}^p$. This extends into a isometric inclusion $\varphi : \mathcal{H}_{\rho^p} \to \mathcal{H}_\rho$ of degree p. One has the rule $\frac{d}{dT}(f(T)) = \frac{d/d\widetilde{T}}{p\widetilde{T}^{p-1}}(f(T))$, for all $f \in K(T)$. We call

$$d_{\rho^p} := \frac{d}{dT} , \qquad d_\rho := \frac{d}{d\widetilde{T}} , \qquad \widetilde{d}_{\rho^p} := (p\widetilde{T}^{p-1})^{-1} \frac{d}{d\widetilde{T}} . \qquad (4.1)$$

Let $(\widetilde{M}, \nabla)$ is a differential module over $(\mathcal{H}_\rho, d_\rho)$ of rank r. Since $(\widetilde{d}_{\rho^p})_{|\mathcal{H}_{\rho^p}} = d_{\rho^p}$, then $(\widetilde{M}, (p\widetilde{T}^{p-1})^{-1}\nabla)$ is a differential module over $(\mathcal{H}_\rho, \widetilde{d}_{\rho^p})$ that can be seen (by restriction of the scalars) as a differential module over $(\mathcal{H}_{\rho^p}, d_{\rho^p})$ of rank pr. We call $(\varphi_*\widetilde{M}, \varphi_*\nabla)$ the differential module so obtained.

4.1 Explicit matrix of $\varphi_*\nabla$.

One has a direct sum decomposition $\mathcal{H}_\rho = \oplus_{k=0}^{p-1} \varphi(\mathcal{H}_{\rho^p}) \cdot \widetilde{T}^k$, so that each $g(\widetilde{T}) \in \mathcal{H}_\rho$ can be uniquely written as $g(\widetilde{T}) = \sum_{k=0}^{p-1} g_k(\widetilde{T}^p)\widetilde{T}^k = \sum_{k=0}^{p-1} g_k(T)\widetilde{T}^k$. The derivation $\widetilde{d}_{\rho^p}$ stabilizes globally each factor and $\widetilde{d}_{\rho^p}(g_k(T)\widetilde{T}^k) = d_{\rho^p}(g_k(T))\widetilde{T}^k$. For all $g(\widetilde{T}) \in \mathcal{H}_\rho$ we define $\varphi_*(g)(T) \in M_{p \times p}(\mathcal{H}_{\rho^p})$ as the matrix of the multiplication by $g(\widetilde{T})/(p\widetilde{T}^{p-1})$, with respect to the basis $1, \widetilde{T}, \ldots, \widetilde{T}^{p-1}$ over

$\mathscr{H}_{\rho^p}$. One has

$$\varphi_*(g)(T) \;=\; (pT)^{-1} \cdot \begin{pmatrix} g_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \cdots & \cdots & \cdots & Tg_0(T) \\ g_0(T) & g_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \cdots & \cdots & Tg_1(T) \\ g_1(T) & g_0(T) & g_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \cdots & Tg_2(T) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{p-2}(T) & g_{p-3}(T) & \cdots & \cdots & & g_1(T) & g_0(T) \; g_{p-1}(T) \end{pmatrix} \tag{4.2}$$

Notice that the terms over the diagonal are multiplied by T. Let $(\widetilde{M}, \widetilde{\nabla})$ be a differential module over $\mathscr{H}_\rho$. Fix a $\mathscr{H}_\rho$-linear isomorphism $\mathscr{H}_\rho^r \xrightarrow{\sim} \widetilde{M}$ (i.e. a basis of $\widetilde{M}$), and let $\frac{d}{d\widetilde{T}} - G(\widetilde{T})$ be the map $\widetilde{\nabla}$ in this basis, where $G(\widetilde{T}) = (g_{i,j}(\widetilde{T}))_{i,j=1,\ldots,r} \in M_{r\times r}(\mathscr{H}_\rho)$. Writing $\mathscr{H}_\rho^r = (\oplus_{k=0}^{p-1}\varphi(\mathscr{H}_{\rho^p}) \cdot \widetilde{T}^k)^r$ one sees that the matrix of $\varphi_*(\widetilde{\nabla})$ is given by the block matrix

$$\varphi_*(G)(T) \;:=\; (\varphi_*(g_{i,j})(T))_{i,j=1,\ldots,r} \;\in\; M_{pr\times pr}(\mathscr{H}_{\rho^p}) \,. \tag{4.3}$$

4.2 Behavior of the radii by Frobenius push-forward

THEOREM 4.2 ([Ked10, Thm.10.5.1]). *Let $\mathcal{R}_1 \leqslant \ldots \leqslant \mathcal{R}_r$ be the radii of $\widetilde{M}$ at ξ_ρ (cf. Def.1.1). Let i_1 be such that $\mathcal{R}_{i_1} \leqslant \omega\rho < \mathcal{R}_{i_1+1}$.[2] Then, up to permutation, the spectral radii of $\varphi_*\widetilde{M}$ at ξ_{ρ^p} are*

$$\bigcup_{i\leqslant i_1}\Big\{\underbrace{|p|\rho^{p-1}\mathcal{R}_i,\ldots,|p|\rho^{p-1}\mathcal{R}_i}_{p\text{-times}}\Big\} \bigcup_{i>i_1}\Big\{\mathcal{R}_i^p, \underbrace{\omega^p\rho^p,\ldots,\omega^p\rho^p}_{p-1\text{-times}}\Big\} \,. \qquad \square \tag{4.4}$$

If $s_1 \leqslant \ldots \leqslant s_r$ is the slope sequence of the spectral Newton polygon of $\widetilde{M}$ at ξ_ρ, and if $i_0 \geqslant i_1$ satisfies $\mathcal{R}_{i_0} < \rho = \mathcal{R}_{i_0+1}$,[3] then by Theorem (4.2) the slope sequence associated to $\varphi_*\widetilde{M}$ at ξ_{ρ^p} is

$$\overbrace{\ln\Big(|p|\rho^{p-1}\Big) + s_1 = \cdots = \ln\Big(|p|\rho^{p-1}\Big) + s_1}^{p\text{-times}} \leqslant \cdots \leqslant \overbrace{\ln\Big(|p|\rho^{p-1}\Big) + s_{i_1} = \cdots = \ln\Big(|p|\rho^{p-1}\Big) + s_{i_1}}^{p\text{-times}} \leqslant$$

$$\leqslant \overbrace{\ln\Big(\omega^p\rho^p\Big) = \cdots = \ln\Big(\omega^p\rho^p\Big)}^{(p-1)(r-i_1)\text{-times}} < ps_{i_1+1} \leqslant \cdots \leqslant ps_{i_0} < \overbrace{\ln(\rho^p) = \cdots = \ln(\rho^p)}^{(r-i_0)\text{-times}}. \tag{4.5}$$

We have two main goals here. Firstly the sequence $s_1 \leqslant \cdots \leqslant s_r$ is perfectly determined by the knowledge of the slope sequence (4.5) of $\varphi_*\widetilde{M}$ (even if some of the s_i are equal to the critical value $\ln(\omega\rho)$), see [Pul12, Prop. 6.17] for a more precise statement. Secondly the values of s_i satisfying $s_i < \ln(\omega^{1/p}\rho)$ corresponds to small radii[4] of $\varphi_*\widetilde{M}$ that are explicitly intelligible by Proposition 2.2. Iterating this construction one obtains an explicit algorithm that computes all the radii $\mathcal{R}_1,\ldots,\mathcal{R}_{i_0}$ in a finite number of steps.

REMARK 4.3. *If one shall avoid the use of a cyclic basis using [Ked10, Lemma 6.7.3, Thm.6.7.4, Conjecture 4.4.9].*

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[2]It is understood that $i_1 = 0$ if $\omega\rho < \mathcal{R}_1$.
[3]It is understood that $i_0 = r$ if $\mathcal{R}_r < \rho$.
[4]i.e. radii that are smaller than $\omega\rho^p$.

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