Overview On Some Recent Results about p-Adic Differential Equations over Berkovich curves

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Introduction

Introduction : some major historical landmarks

1960 B.Dwork,
Rationality of the Zeta function *of a variety of positive characteristic.*

1980 B.Dwork-P.Robba,
Affine line, ***neighborhood of a point*** *(of Berkovich)*, ***Robba ring****.*

2000 G.Christol-Z.Mebkhout, (André, Berger, Crew, Kedlaya, ...)
Differential Equations over the ***Robba ring****, application over* ***rigid curves****, link with* p***-adic representations.***

2010 F.Baldassarri, K.S.Kedlaya, J.Poineau, A.P.
Differential Equations over ***Berkovich curves*** *(global theory).*

Introduction

Several different languages can be used to describe the geometry of the space underling the equation (Tate, Raynaud, Berkovich, Huber, Meredith,...).
We mention the language of ***rigid cohomology*** whose one of aims consists in associating a “good” category of coefficients with all variety in positive characteristic.

- These coeficients are ”essentially” some *differential equations*, defined over the **generic fiber** of a certain model of the variety.
- The generic fiber has, in a natural way, a structure of **Berkovich analytic space.**
- The equations of rigid cohomology usually have certain operators (**Frobenius**) plus some other **restrictions.**
- In comparison with ℓ-adic sheaves, p-adic differential equations are more “*explicit*”, and allow sometimes **direct computations.**

The view point of our recent works is somehow **orthogonal** to that of rigid cohomology.

- We do not start from a problem in positive characteristic. Instead, we directly consider
  1. a quasi-smooth **Berkovich curve** X,
  2. a **differential equation** $\mathscr{F}$ **over** X,
  3. with **no restrictions**.

- The study of differential equations with such a degree of generality, in particular the finiteness of their de Rham cohomology groups, was an essentially **unexplored problem until 2013.**

  For instance, even for a curve as simple as an open disk or annulus, there was ***no criteria*** *describing the finiteness of the cohomology.*

- Results in this direction are essentially due (among other actors) to Dwork and Robba, then Christol and Mebkhout, and are (up to some exceptions) of **local nature** in the sense of Berkovich.

Introduction

Since Dwork and Robba, a particular attention began to be devoted to a serious difference between the complex theory and the p-adic one :

Triviality over a disk

Over an open disk there are *non singular* differential equations with solutions that **do not converge on the whole disk.**

Example

The equation $y' = y$ has solution

$$y \;=\; \exp(x) \;=\; \sum_{n\geq 0} \frac{x^n}{n!} \;. \tag{1}$$

Now, this series has a finite p-adic radius of convergence. However, the equation shows no singularities.

Dwork and Robba

I their pioneer work, Dwork and Robba introduced several key notions as (among others)

1. Overconvergence ;
2. Frobenius structure ;
3. Solvability ;
4. Decomposition by the radii ;
5. Relation between the radii and the cohomology.

These ideas influenced large part of the literature.

After the pioneer work of Dwork and Robba, several authors have pursued an agenda of analysis whose aim was to prove the following important relations :

1. Factorization/decomposition theorems by the radii ;
2. Finite dimensionality of the de Rham cohomology and index theorems.

From 2010 on we assist at some important results of **global nature** :

- Kedlaya's book 2010 :
  - important improvements of classical results,
  - new decomposition results of global nature over disks and annuli ;
- F.Baldassarri (following a common work with L.Di Vizio) :

  2010 generalizes to curves Christol-Dwork's proof of the continuity of the smaller radius of convergence.
- In a sequence of works (most of them with J.Poineau) we have

  2012 Generalized the **continuity** : new property **finiteness** ;

  2013 Established some **decomposition** theorems of global nature ;

  2013 Proved the **finiteness** of the de Rham cohomology ;

  2017 Established the link with the **Riemann-Hurwitz formula**.
- Kedlaya 2013 :
  - generalized the **p-adic local monodromy thm** ;
  - reproved our result about the **finiteness of the radii** ;
  - showed that the controlling graph contain **no points of type** 4 ;
  - Improvement of **exponent theory** (see his talk).
- Poineau-Bojkovic :

  2016 **Behavior of the radii by push-forward**+relation with **ramification**

Notation on Berkovich curves

Notation

$(K, |.|)$ is a complete valued field of **characteristic** 0.

To simplify, in this talk we assume that K is **algebraically closed.**

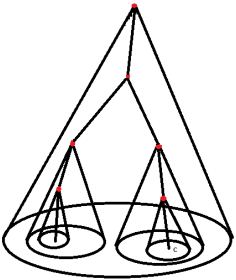
Berkovich curves

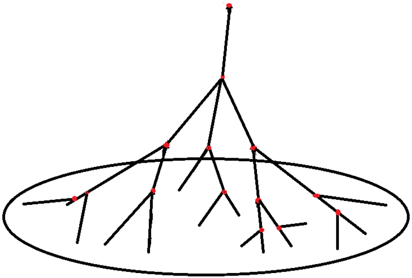
A K-analytic Berkovich curve is said **rig-smooth** or **quasi-smooth** if Ω^1_X is a **locally free** $\mathcal{O}_X$-module of rank one.

- This definition allows boundary.

Open disk

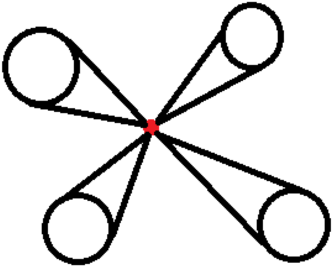
As an analytic space, an open disk is the union of its closed sub-disks.





It is a arcwise connected space

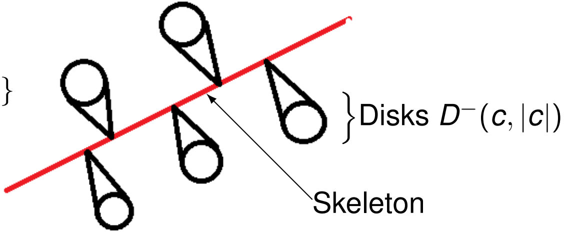
Closed disk



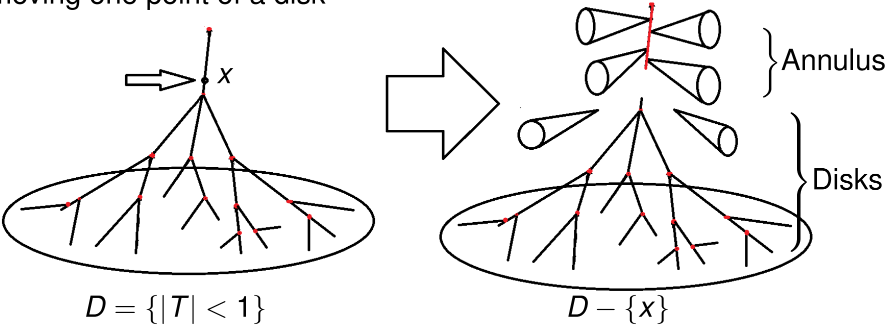
1. The union of all open sub-disks is an open, but **not a covering**
2. The space is **connected**
3. The red-point is the **boundary**

Open annuli

Annulus $\{r < |T| < 1\}$



Removing one point of a disk



Type of a point

We can classify points in 4 types.
Informally speaking this translated in the following topological notions :

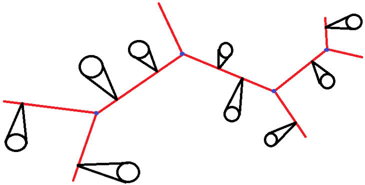
Type 1	Type 2	Type 3	Type 4
Rational pts	Bifurcation	2 directions	Final points non rational

Type 4 points are absent if the base field is spherically complete.

Semi-stable reduction

Theorem (V.Berkovich - A.Ducros)

Let X be a quasi-smooth curve. There exists a ***locally finite*** *subset $S \subseteq X$ formed by points of type 2 or 3 such that $X - S$ is a disjoint union of open* ***disks*** *and* ***annuli****. We call S a* ***weak triangulation****.*

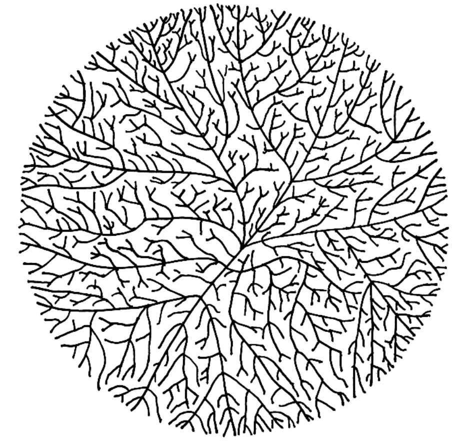


Skeleton

The union Γ_S of the skeletons of the annuli that are connected components of $X - S$ together with the points of S is a locally finite graph in X. Called the **skeleton** of S.

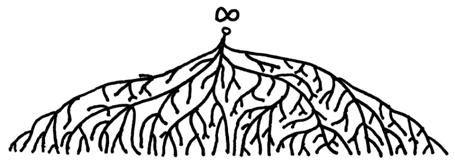
Projective line

Projective line $\mathbb{P}^{1,\mathrm{an}}_K$



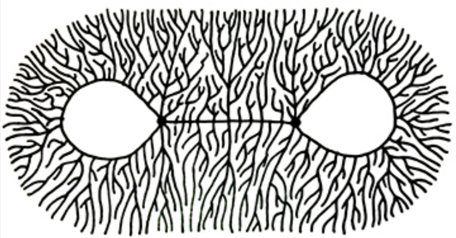
Droite Affine

Droite Affine $\mathbb{A}_K^{1,\mathrm{an}}$



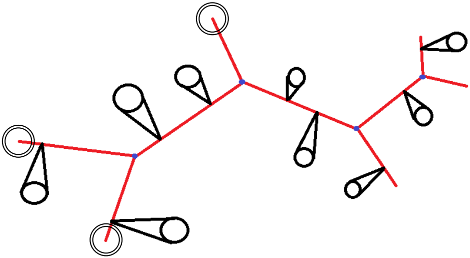
Une courbe

Une courbe



Open boundary

The **open boundary** is formed by the germs of open segments that are not relatively compact in X.



Residual curve and genus of a point

- $\widetilde{K}$ = **residual field** of K ;
- $x \in X$ be a point of **type 2** (bifurcation) ;
- $\mathscr{H}(x)$ = field of the point x ;
- $\widetilde{\mathscr{H}(x)}$ = **residual field** of $\mathscr{H}(x)$.

Then, $\widetilde{\mathscr{H}(x)}/\widetilde{K}$ is a field with degree of transcendence 1.

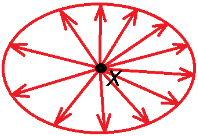
It is the field of functions of a unique **smooth projective curve** $\mathcal{C}_x$ over $\widetilde{K}$. Set

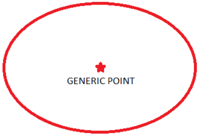
$$g(x) := g(\mathcal{C}_x) \ .$$

- If x is a point of **type** $\neq 2$ we set $g(x) = 0$. In this case x has a neighborhood in X isomorphic to a domain of $\mathbb{A}_K^{1,an}$.
- If $g(x) > 0$, there is **no neighborhood** of x isomorphic to a domain of the line.
- Points of positive genus form a **locally finite set** in the curve.

There exists an **injective map**

$$\psi_x \ : \ \{\text{Directions out of } x\} \longrightarrow \{\widetilde{K}\text{-rational Pts of } \mathcal{C}_x\}\ . \tag{2}$$





Genus and Euler characteristic of X

Genus

The **genus** $g(X)$ of the quasi-smooth curve X is by definition

$$g(X) \;=\; 1-\chi_{top}(X)+\sum_{x\in X} g(x) \;\geq\; 0 \tag{3}$$

where $\chi_{top}(X)\leq 1$ is the Euler characteristic of the topological space underling X in the sense of **singular homology**.

Characteristic of X

The **Euler characteristic of** X (following Q.Liu) is by definition

$$\chi_c(X) \;=\; 2-2g(X)-N(X) \tag{4}$$

where $N(X)$ is the number of germ of segments at the **open boundary** of X.

If $X = C^{an}$ is the analytification of a smooth algebraic curve C then

$$g(X) \;=\; \text{algebraic genus of } C\;. \tag{5}$$

Differential equations over quasi-smooth Berkovich curves

Differential equations

Let X be a quasi-smooth curve

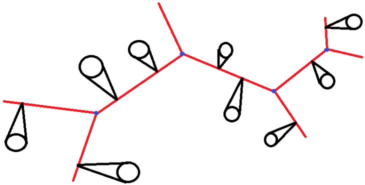
Definition (Differential equation)

A **differential** equation over X is a locally free $\mathcal{O}_X$-module of finite rank $\mathscr{F}$, endowed with a **connection**

$$\nabla : \mathscr{F} \to \mathscr{F} \otimes \Omega^1_{X/K} . \tag{6}$$

Contrary to the complex case, a differential equation of this type is not always analytically trivial over an open disk. The reason is that the radii of convergence of the Taylor solutions are not always **maximal**.

Radii of convergence and convergence Newton polygon



Let us fix a weak triangulation S. Let Γ_S be its skeleton.
Let $x \in X(K)$ be a K-rational point.
Then $x \notin \Gamma_S$ and we denote by $D(x, S)$ **the largest open disk** in X centered at x such that $\Gamma_S \cap D(x, S) = \emptyset$.

Definition

Let $r := \mathrm{rang}_x(\mathscr{F})$. Denote by $D_{S,i}(x, \mathscr{F}) \subseteq D(x, S)$ the **largest open sub-disk** on which $\mathscr{F}$ has at least $r - i + 1$ linearly independent solutions :

$$\{x\} \neq D_{S,1}(x, \mathscr{F}) \subseteq D_{S,2}(x, \mathscr{F}) \subseteq \cdots \subseteq D_{S,r}(x, \mathscr{F}) \subseteq D(x, S) . \qquad (7)$$

Radii of convergence

$$\{x\} \neq D_{S,1}(x,\mathscr{F}) \subseteq D_{S,2}(x,\mathscr{F}) \subseteq \cdots \subseteq D_{S,r}(x,\mathscr{F}) \subseteq D(x,S)\ . \tag{8}$$

Radii of convergence

Fix a coordinate on $D(x,S)$.

$R =$ radius of $D(x,S)$ in this coordinate.

$R_i =$ radius of $D_{S,i}(x,\mathscr{F}) \subseteq D(x,S)$

Then, set

$$\mathcal{R}_{S,i}(x,\mathscr{F})\ :=\ \frac{R_i}{R}\ \leq\ 1\ . \tag{9}$$

Radius at an arbitrary point

If $x \in X$ is an arbitrary point, it is possible to define $\mathcal{R}_{S,i}(x, \mathscr{F})$.

Namely, there exists a complete valued field extension Ω/K such that x becomes Ω-rational.

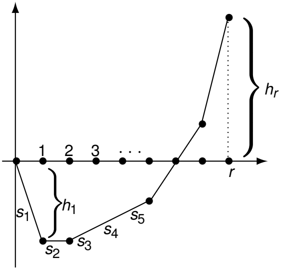
We then define the radii after base change to X_Ω.

General polygons

A **polygon** is the datum of a sequence of slopes

$$-\infty < \quad s_1 \leq s_2 \leq \cdots \leq s_r \quad < +\infty \ .$$

This defines a unique convex function that is $= 0$ at 0 and which that is affine of slope s_i over $[i-1, i]$:



(10)

Define the **partial heights** as $h_0 = 0$ and

$$h_i := s_1 + s_2 + \cdots + s_i \ .$$

We set

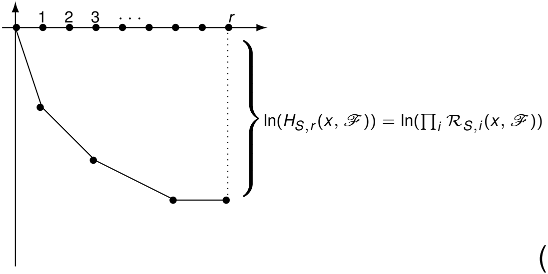
$$s_i := \ln(\mathcal{R}_{S,i}(x, \mathscr{F})) \,. \tag{11}$$

We consider the sequence of slopes

$$\ln(\mathcal{R}_{S,1}(x, \mathscr{F})) \leq \cdots \leq \ln(\mathcal{R}_{S,r}(x, \mathscr{F})) \,.$$

The corresponding polygon is called the **convergence Newton polygon** of $(\mathscr{F}, \nabla)$.

$$\ln(\boldsymbol{\mathcal{R}}_S(x, \mathscr{F})) : \tag{12}$$



Continuity and finiteness of the radii

Continuity of the first radius

Remember that $\mathcal{R}_{S,1}(x, \mathscr{F})$ is the radius of the largest disk "at x" where **all the solutions converge.**
It is a function

$$\mathcal{R}_{S,1} \ : \ X \to \mathbb{R}_{>0} \ .$$

The continuity of the first radius $\mathcal{R}_{S,1}(x, \mathscr{F})$ has been proved in several steps

1994 Christol-Dwork : **skeletons of annuli**

2007 Baldassarri-Di Vizio : **affinoid domains of the line + semi-continuity in more variables**

2010 Baldassarri : **Berkovich curves.**

Continuity of higher radii

Let us now consider $i \geq 1$

$$\mathcal{R}_{S,i} \;:\; X \to \mathbb{R}_{>0} \;.$$

2010 Kedlaya's book : **Continuity along skeletons of annuli**

2012 Pulita-Poineau : **Continuity and finiteness over Berkovich curves**

Finiteness

Theorem (Poineau-P., 2012)

For all i, the function $\mathcal{R}_{S,i}(-,\mathscr{F})$ verifies :

1. It is **continuous** ;
2. It **factorizes** through a locally finite graph $\Gamma_{S,i}(\mathscr{F})$ containing Γ_S ;
3. Along any edge $\Gamma_{S,i}(\mathscr{F})$ the function $\ln(\mathcal{R}_{S,i}(x,\mathscr{F}))$ is piecewise **affine**, with a **locally finite number of breaks.**

The graph $\Gamma_{S,i}(\mathscr{F})$ is called **the controlling graph** of $\mathscr{F}$.

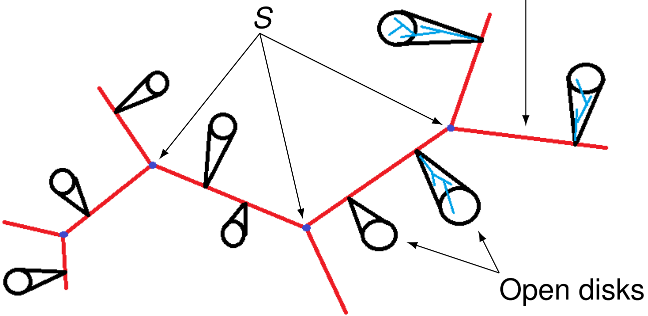
In 2013 Kedlaya

- re-proved the same result (similar methods)
- showed that $\Gamma_{S,i}(\mathscr{F})$ has **no points of type** 4 (Tannakian methods).

The locally finite graph $\Gamma_{S,i}(\mathscr{F})$

$X :=$ Curve

Γ_S = skeleton of S



S = weak triangulation

$\Gamma_S \subseteq \Gamma_S(\mathscr{F})$

{ blue + red } = Controlling graph

Global decomposition by the radii

Major historical landmarks

The informal idea is that the filtration by the radii of the space of the solutions implies a **decomposition** of the differential equation itself by sub-differential equations.

1975 Robba : decomposition over $\mathscr{H}(x)$, where $x = \eta_{0,1} \in \mathbb{A}_K^{1,an}$ (Gauss norm) ;

1977 Dwork-Robba : decomposition over $\mathscr{O}_x$, where $x = \eta_{0,1} \in \mathbb{A}_K^{1,an}$;

2000 Christol-Mebkhout : decomposition over the **Robba ring** ;

2010 Kedlaya's Book : decomposition over **disks and annuli** ;

2013 Poineau-P. : **global decomposition** over curves.

Global decomposition/factorization by the radii

Theorem (Poineau-P.)

Let $i \in \{1, \dots, r\}$ be a fixed index. Assume that for all $x \in X$ we have

$$\mathcal{R}_{S,i-1}(x, \mathscr{F}) \; < \; \mathcal{R}_{S,i}(x, \mathscr{F}) \; . \tag{13}$$

Then $\mathscr{F}$ decomposes as

$$0 \to \mathscr{F}_{\geq i} \to \mathscr{F} \to \mathscr{F}_{<i} \to 0 \; , \tag{14}$$

where $\mathrm{rank}(\mathscr{F}_{<i}) = i - 1$ and for all $x \in X$ we have

$$\mathcal{R}_{S,k}(-, \mathscr{F}) \; = \; \begin{cases} \mathcal{R}_{S,k}(x, \mathscr{F}_{<i}) & \text{if} \quad k < i \\ \mathcal{R}_{S,k-i+1}(x, \mathscr{F}_{\geq i}) & \text{if} \quad k \geq i \end{cases} \tag{15}$$

For differential equations over $\mathbb{C}((T))$ this "***is***" the classical decomposition of B.Malgrange.

Direct sum decomposition

In literature decompositions by the radii all are in direct sum. There are examples where the global decomposition is not always in direct sum.

Theorem

The exact sequence

$$0 \to \mathscr{F}_{\geq i} \to \mathscr{F} \to \mathscr{F}_{<i} \to 0\,, \tag{16}$$

splits in the following cases

1. For all $x \in X$ one has $\mathcal{R}_{S,i-1}(x,\mathscr{F}^*) < \mathcal{R}_{S,i}(x,\mathscr{F}^*)$;
2. We have the following inclusion of graphs

$$\Big(\Gamma_{S,1}(\mathscr{F}) \cup \cdots \cup \Gamma_{S,i-1}(\mathscr{F})\Big) \subseteq \Gamma_{S,i}(\mathscr{F})\,. \tag{17}$$

It result important to **measure** the size of these graphs.

A measure of the complexity of the graphs

A measure of the complexity of the graphs

- We have been able to provide an explicit **upper bound** for the number of **vertex** and **edges** of the controlling graphs.
- The bound is given in term of the **slopes of the radii** at the boundary of X and the open boundary of X.
- Over a **projective curve**, the bound **only depends on the rank** of the differential equation.
- Under appropriate conditions on the exponents, the bound is related to the de Rham **index.**

Global irregularity and index theorem

Global irregularity

Set

1. $\partial^o X$ = **open boundary** of X ;
2. b = **germ of segment** in X ;
3. $\partial_b H_{S,r}$ = **slope** along b of the **total height** $H_{S,r}$ of the convergence Newton polygon ;

We define the **global irregularity** of $\mathscr{F}$ as

$$\mathrm{Irr}_X(\mathscr{F}) \ := \ \sum_{b\in\partial^o X} \partial_b H_{S,r}(-,\mathscr{F}) + [\sum_{x\in\partial X} dd^c H_{S,r}(x,\mathscr{F}) + \chi(x,S)]$$

where

1. ∂X=**boundary** of X ;
2. $dd^c H_{S,r}$ is the **sum of all slopes** $\partial_b H_{S,r}$ of $H_{S,r}$ out of x, (b is a germ of segment out of x) ;
3. $\chi(x,S) := 2-2g(x)-N_S(x)$, where $N_S(x)$ is the number of directions of Γ_S out of x. It is a certain **characteristic** related to the residual curve of x.

Index theorem

Let X be a **quasi-smooth** Berkovich curve, with

1. a finite **genus**,
2. a finite **boundary**,
3. admitting a finite **skeleton** Γ_S.

Theorem (Poineau-P.)

Let $\mathscr{F}$ be a differential equation over X, such that

(1) *$\mathscr{F}$ is free of* ***Liouville*** *numbers (technical assumption) ;*

(2) *The radii of $\mathscr{F}$ are* ***affine*** *functions at the* ***open boundary*** *of X ;*

(3) *The radii of $\mathscr{F}$ are* ***not maximal*** *at the* ***boundary*** *of X.*

Then,

$$\dim \mathrm{H}^{\bullet}_{\mathrm{dR}}(X, \mathscr{F}) \; < \; +\infty$$

and we have the following index formula

$$\chi_{\mathrm{dR}}(X, \mathscr{F}) \; = \; \chi_c(X) \cdot \mathrm{rank}(\mathscr{F}) - \mathrm{Irr}_X(\mathscr{F}) \; . \tag{18}$$

Index Theorem

The major assumption is the assumption (2) about the **affinity of the radii.**

- Assumption (2) is **automatically satisfied** in the following situations
  1. Around a **meromorphic singularity** ;
  2. If X is **relatively compact** in a larger curve Y and $\mathscr{F}$ is the restriction of an equation defined over Y ;
  3. In particular, in the **overconvergent** case ;
- If X is a general quasi-smooth curve, under assumptions analogous to (1) and (3), we provide a **necessary and sufficient criterion** for the finite dimensionality of the de Rham cohomology.
- We also treat the case of **meromorphic singularities.**