Radius of convergence function of p-adic differential equations

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Structure of the talk

Part 1 : Berkovich space of a sub-affinoid of $\mathbb{A}^1_K$

- Settings
- Paths on the Berkovich space and norms of type $|.|_{c,R}$
- Maximal Skeleton and log-properties
- Two examples of functions : $|.| \mapsto \rho_{|.|,X}$ and $|.| \mapsto \rho^{\mathrm{gen}}_{|.|,X}$

Part 2 : Differential Equations and (Elementary) Stratifications

- Analytic functions in a neighborhood of the diagonal
- Differential equations VS (Elementary) Stratification

Part 3 : Def. and properties of the radius of convergence function of an analytic function around the diagonal (RCF)

- Lower semi-continuity of the RCF of a function around the diagonal
- A criterion of continuity
- Dwork-Robba theorem and continuity of the RCF of a stratification.

Part 1 : The Berkovich space of a K-affinoid

Settings

- $(K, |.|)$ =complete ultrametric field of characteristic 0.
- X = 1-dimensional, connected, affinoid sub-space fo $\mathbb{A}^1_K$ of the form :

$$X = \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i) \ .$$

For technical reasons we assume $c_0, \ldots, c_n \in K$, $R_0 < +\infty$, and $\mathrm{D}^-(c_i, R_i) \subset \mathrm{D}^+(c_0, R_0)$ for all $i = 1, \ldots, n$.
We call it a K-affinoid for simplicity.

Analytic functions over X

- Let $H_K^{rat}(X) \subset K(T)$ be the sub-ring of rational functions without poles over X, together with the sup-norm $\|.\|_X$.
- The Banach algebra of analytic functions over X is the completion

$$(\mathcal{H}_K(X), \|.\|_X) = (H_K^{rat}(X), \|.\|_X)\widehat{\ }$$

(also called Krasner's analytic elements over X).

X as a functor

$$X \;=\; \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i)\;.$$

- If Ω/K is a complete valued field extension, then

$$X(\Omega) \;:=\; \{x \in \Omega \mid |x - c_0| \leq R_0,\; |x - c_i| \geq R_i,\; \text{for all } i = 1, \ldots, n\,\}$$

- So, for $f \in \mathcal{H}_K(X)$, the sup-norm is given by

$$\|f\|_X \;:=\; \sup_{\Omega/K} \sup_{x \in X(\Omega)} |f(x)|_\Omega$$

This has a meaning since there exists a field Ω_*, and a finite family of points $t_{c_0,R_0}, \ldots, t_{c_n,R_n} \in X(\Omega_*)$ (Shilov boundary) such that for all Ω/K one has $\sup_{x \in X(\Omega)} |f(x)|_\Omega \leq \max(|f(t_{c_0,R_0})|, \ldots, |f(t_{c_n,R_n})|)$. So that $\|f\|_X = \max(|f(t_{c_0,R_0})|, \ldots, |f(t_{c_n,R_n})|)$. For simplicity we write

$$\|f\|_X \;:=\; \sup_{x \in X} |f(x)|\;.$$

Berkovich space attached to X

Definition

The Berkovich Space $\mathscr{M}(\mathcal{H}_K(X)) = \mathscr{M}(X)$ is the set of all bounded multiplicative semi-norms

$$|.| \ : \ \mathcal{H}_K(X) \to \mathbb{R}_{\geq 0}$$

satisfying $|1| = 1$, $|0| = 0$ and

- $|f \cdot g| \leq |f| \cdot |g|$
- $|f + g| \leq \max(|f|, |g|)$
- There exists $C > 0$ such that $|f| \leq C\|f\|_X$, for all $f \in \mathcal{H}_K(X)$.

Topology of $\mathscr{M}(X)$

The topology of $\mathscr{M}(X)$ is the finest one (i.e. that with less open subset) making continuous each function of the type

$$|.| \mapsto |f| : \mathscr{M}(X) \ \to \ \mathbb{R}_{\geq 0}$$

for all $f \in \mathcal{H}_K(X)$.

Semi-norms of type $|.|_t$. Connection with Dwork's terminology

Let $(\Omega, |.|)/(K, |.|)$ be a complete valued field extension, and let $t \in X(\Omega)$. We set

$$|f|_t := |f(t)|_\Omega \,, \quad f \in \mathcal{H}_K(X).$$

This semi-norm lies in $\mathscr{M}(X)$.

- All semi-norms in $\mathscr{M}(X)$ are of type $|.|_t$ for a convenient Ω/K, and $t \in X(\Omega)$.
- We call such a point $t \in X(\Omega)$ a Dwork generic point for $|.|$. Notice that t is not unique.

Paths in $\mathscr{M}(X)$ and semi-norms of type $|.|_{c,R}$

- Let Ω/K be a c.v.f.e. Let $c \in \Omega$, $R \geq 0$.
- For a polynomial $P(T) \in K[T]$, we write P as $\sum a_n(T-c)^n$, with $a_n \in \Omega$, then we set

$$|P(T)|_{c,R} := \sup_n |a_n|R^n = \sup_{|x-c|=R} |P(x)|$$

- We set $|P/Q|_{c,R} := |P|_{c,R}/|Q|_{c,R}$, then we have a semi-norm on $K(T)$ and hence on $H_K^{rat}(X)$.
- NOTE : $|P/Q|_{c,R} \neq \sup_{|x-c|=R} |P(x)/Q(x)|$.

FACT

$$X = \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i) .$$

The semi-norm $|.|_{c,R}$ extends to a semi-norm of $\mathcal{H}_K(X)$ in $\mathscr{M}(X)$ if and only if one of the following conditions is fulfilled

- $c \in X(\Omega)$ and $R \leq R_0$;
- $c \in \mathrm{D}^-(c_i, R_i)$ and $R_i \leq R \leq R_0$ for some $i = 1, \dots, n$.

Continuous paths

- If $c \in \mathrm{D}^-(c_i, R_i)$ (resp. if $c \in X(\Omega)$), then the path
$$R \mapsto |.|_{c,R} \;:\; [R_i, R_0] \to \mathscr{M}(X)$$
$$(\text{resp. } R \mapsto |.|_{c,R} \;:\; [0, R_0] \to \mathscr{M}(X))$$
is continuous. That is, for all $f \in \mathcal{H}_K(X)$, the function
$$R \mapsto |f|_{c,R}$$
is continuous.
- NOTE : Of course there is a Dwork generic point $t_{c,R}$ for $|.|_{c,R}$ so that
$$|.|_{c,R} = |.|_{t_{c,R}}$$
$$i.e. \quad |f|_{c,R} \;=\; |f(t_{c,R})| \;, \qquad \text{for all } f \in \mathcal{H}_K(X) \;.$$

FACT

$$|c_1 - c_2| \leq R \quad \Longrightarrow \quad |.|_{c_1,R} \;=\; |.|_{c_2,R}$$

FACT

$$|c_1 - c_2| \leq R \quad \Longrightarrow \quad |.|_{c_1,R} \;=\; |.|_{c_2,R}$$

$\mathscr{M}(X)$ is archwise connected

In fact for all $|.| \in \mathscr{M}(X)$ there is a path connecting $|.|$ to $|.|_{c_0,R_0}$:

- First choose a Dwork generic point t for $|.|$ so that $|.| = |.|_t = |.|_{t,0}$
- Then consider the path $R \mapsto |.|_{t,R}$: for $R = R_0$ we have

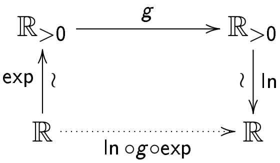
$$|.|_{t,R_0} = |.|_{c_0,R_0}$$

We always have a path connecting $|.|$ to $|.|_{c_0,R_0}$:

$$|.| = |.|_{t,0} \bullet\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\bullet\; |.|_{t,R_0} = |.|_{c_0,R_0}$$

Log-properties

- We say that a given function g has logarithmically a given property if the function $(\ln \circ g \circ \exp)$ has that property.



Piecewise Log-affinity

- For $c \in X(\Omega)$ and $f \in \mathcal{H}_K(X)$ the function $R \mapsto |f|_{c,R}$ is piecewise of the form aR^b i.e. logarithmically affine of the form

$$\rho \mapsto \ln(a) + \ln(b) \cdot \rho \, ,$$

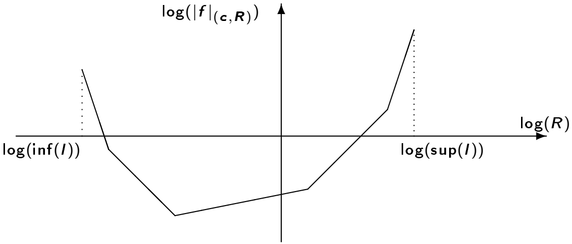
where $\rho = \ln(R)$.

Log-properties

Let $f \in \mathcal{H}_K(X)$. Let $I \subseteq \mathbb{R}$ be an interval.

Let $c \in X(\Omega/K)$, (resp. c lies in a hole of X).

If the path $R \mapsto |.|_{c,R}$, $R \in I$, does not encounter any holes of X (i.e. if there is no holes of X in the annulus $|x - c| \in I$), then $R \mapsto |f|_{c,R}$ is Log-convex :



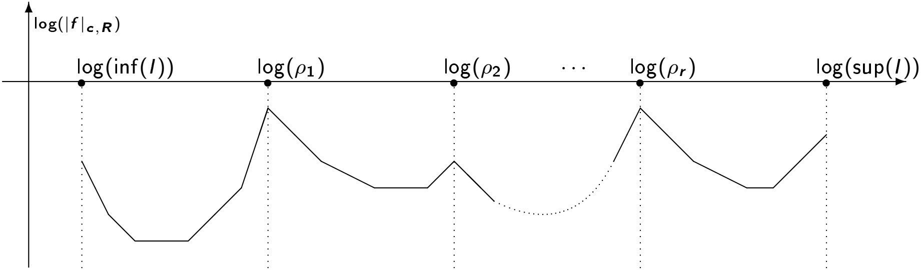
The breaks are in correspondence of the values of R equal to the distance of a zero z of f from c : $R = |z - c|$. The difference of two consecutive slopes is equal to the number of zeros of f "at that distance from c".

Log-properties

Let $f \in \mathcal{H}_K(X)$. Let $I \subseteq \mathbb{R}$ be an interval.

Let $c \in X(\Omega/K)$, (resp. c lies in a hole of X).

If the path $R \mapsto |.|_{c,R}$, $R \in I$, encounters some hole of X (i.e. if there is some hole of X in the annulus $|x - c| \in I$), then $R \mapsto |f|_{c,R}$ has the following shape :



where ρ_i are the distances of the holes of X from c.

Maximal skeleton

There is a natural order relation in $\mathscr{M}(X)$ (induced by the order of $\mathbb{R}$) :

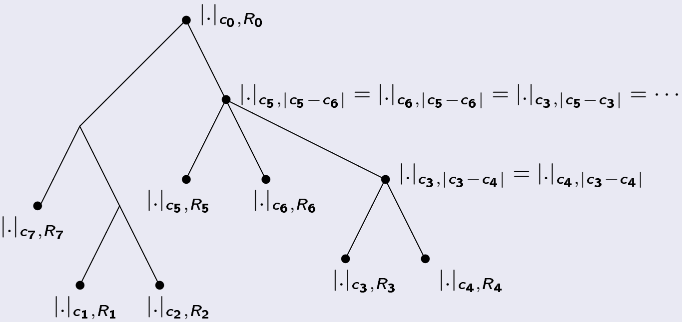
$$|.|_1 \leq |.|_2 \overset{Def}{\Longleftrightarrow} |f|_1 \leq |f|_2 , \text{ for all } f \in \mathcal{H}_K(X) .$$

Maximal skeleton

$$X = \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i) .$$

The set of *maximal points* with respect to the above order is

$$\boxed{\mathscr{S}_X := \cup_{i=1}^n \{ |.|_{c_i,R} \}_{R\in[R_i,R_0]}}$$



For $c \in X(\Omega)$ we possibly have

$$|.|_{c,0} \;=\; |.|_{c,R} \text{ for some } R > 0.$$

As an example if $X = \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i)$, and if t_{c_i,R_i} is a Dwork generic point of a semi-norm $|.|_{c_i,R_i}$ in the Shilov boundary, then

$$|.|_{t_{c_i,R_i},0} \;=\; |.|_{t_{c_i,R_i},R} \;\text{ for all }\; 0 \leq R \leq R_i \;.$$

Definition of $\rho^{gen}_{|.|,X}$.

We call $\rho^{gen}_{|.|,X}$ the supremum of the $R \geq 0$ such that $|.| = |.|_{t,R}$ for some unspecified Dwork's generic point $t \in X(\Omega)$.

Equivalent definitions

Let $t \in X(\Omega)$ be a fixed Dwork generic point for $|.|$, then

- $\rho^{gen}_{|.|,X} = \sup(R \; s.t. \; |.| = |.|_{t,R})$ (i.e.the def.is indep.on the choice of t).
- $\rho^{gen}_{|.|,X} \;=\; \mathrm{dist}(t, K^{\mathrm{alg}}) = \inf(|T - c| \; s.t. \; c \in K^{alg})$.
- Every point in $\mathrm{D}^-(t, \rho^{gen}_{|.|,X})$ is a Dwork generic point for $|.|$.

The function $\rho^{gen}_{|.|,X}$ is not continuous

Proof.

Assume that K is a algebraically closed. Let $x_1, x_2, \dots$ be a sequence in $\mathcal{O}_K$ such that the residual classes are different $\overline{x_i} \neq \overline{x_j}$ for all $i \neq j$.
Let $X = \mathrm{D}^+(0,1)$ be the closed unit disk. Then

1. $\lim_n |.|_{x_n} = |.|_{0,1}$ in $\mathscr{M}(X)$
2. $\rho^{gen}_{|.|_{x_n}} = 0$ for all $n \geq 1$.
3. $\rho^{gen}_{|.|_{0,1}} = 1$.

We now prove these three facts.

1. Indeed each $f \in \mathcal{H}_K(X)$ has a finite number of zeros. If there is no zeros in the disk $\mathrm{D}^-(x_n, 1)$ then $|f|_{x_n} = |f|_{x_n,R}$ for all $R < 1$ (see properties of slide 11), hence by continuity one has
   $|f|_{x_n} = |f|_{x_n,1} = |f|_{0,1}$ (see FACT of slide 9).
   This proves that $\lim_n |.|_{x_n} = |.|_{0,1}$.

□

Continuation of the proof.

2. One has $\rho^{gen}_{|.|_{x_n}} = 0$ because the function $f := (T - c)$ verifies $|f|_{x_n} = |x_n - c|$ so that for all $R > 0$ we choose c such that $|x_n - c| < R$ so

$$|T - c|_{x_n,R} = |(T - x_n) + (x_n - c)_{x_n,R} = \max(R, |x_n - c|) = R$$

so $|f_c|_{x_n} \neq |f_c|_{x_n,R}$. Then $\rho^{gen}_{|.|_{x_n},X} = \max(R\ s.t.\ |.|_{x_n} = |.|_{x_n,R}) = 0$

3. Now let $t_{0,1}$ be a Dwork generic point for $|.|_{0,1}$. The same proof does not hold for $|.|_{0,1}$ because $\rho^{gen}_{|.|_{0,1},X} = \mathrm{dist}(t_{0,1}, K)$, so t is transcendental over K.
   –The equality $|P|_{t_{0,1}} = |P|_{0,1}$ holds for all polynomial P of degree 1 by a similar computation.
   – The equality hence holds for all polynomial because the semi-norm is multiplicative.
   – The equality hence holds for all for all rational fraction and by density it holds also for all $f \in \mathcal{H}_K(X)$.

The function $|.| \mapsto \rho_{|.|,X}$

Let $t \in X(\Omega)$, then we set

$$\rho_{t,X} := \mathrm{dist}(t, \mathbb{A}^1 - X) = \sup(\rho > 0 \ s.t.\ \mathrm{D}^-(t,\rho) \subset X) .$$

Definition of $|.| \mapsto \rho_{|.|,X}$

Let $|.| \in \mathscr{M}(X)$ and let t be a Dwork generic point for $|.| : |.| = |.|_t$. Then

$$\rho_{|.|,X} := \rho_{t,X}$$

Continuity of $|.| \mapsto \rho_{|.|,X}$

If $X = \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i)$, then (clearly)

$$\rho_{t,X} = \min(R_0, |t - c_1|_\Omega, \dots, |t - c_n|_\Omega) .$$

NOTE : $|t - c_i|_\Omega = |T - c_i|$ because t is Dwork generic point. So
– This proves that the def. of $\rho_{|.|,X}$ does not depend on the choice of t.
– The function $|.| \mapsto \rho_{|.|,X}$ is continuous because it is the minimum of a finite family of continuous functions

$$|.| \mapsto R_0 \quad \text{and} \quad |.| \mapsto |T - c_i| , \ i = 1, \dots, n$$

these functions are continuous by the definition of the topology of $\mathscr{M}(X)$.

ρ^{gen} VS ρ

continuity

- The function ρ^{gen} is upper semi-continuous (because it is the infimum of an (infinite) family of continuous functions, see slides 15).
- The function ρ is continuous.

increasing

Assume that $|.|_1, |.|_2 \in \mathscr{M}(X)$, with $|.|_1 \leq |.|_2$, then

- $\rho^{gen}_{|.|_1} \leq \rho^{gen}_{|.|_2}$ (increasing function), and $\rho^{gen}_{|.|_t} = 0$ if $t \in \widehat{K^{alg}}$.
- $\rho_{|.|_1} = \rho_{|.|_2}$: So that $|.| \mapsto \rho_{|.|}$ is completely determined by its values on the maximal skeleton.

Equality on the maximal skeleton

If $|.|$ belongs to the maximal skeleton of $\mathscr{M}(X)$, then

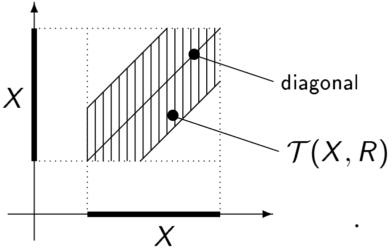
$$\rho^{gen}_{|.|} = \rho_{|.|}\ , \qquad (\rho_{|.|_{c_i,R}} = \rho^{gen}_{|.|_{c_i,R}} = R)$$

Part 2 : Differential equations and (Elementary) Stratifications

Analytic functions on tubes around the diagonal

- *Tube around the diagonal of* $X \times X$:

$$\mathcal{T}(X,R) := \{ (x,y) \in X \times X \mid |x-y| < R \} ,$$



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- $\mathcal{A}_K(\mathcal{T}(X,R)) :=$ Analytic functions on the tube.

If $R > 0$ is small enough then $\mathcal{T}(X,R) \cong X \times \mathrm{D}^-(0,R)$. The isomorphism is given by $(x,y) \mapsto (y, x-y)$.
In term of functions this gives

$$\mathcal{A}_K(\mathcal{T}(X,R)) = \{\textstyle\sum_{n\geq 0} f_n(y)(x-y)^n \ \textit{such that} \ f_n \in \mathcal{H}_K(X) \text{ and } \lim_n \|f_n\|_X \cdot \rho^n = 0 , \forall \rho < R \}$$

A *stratification* over X is the data of a finite free module M over $\mathcal{H}_K(X)$, together with an unspecified $R > 0$, and a $\mathcal{A}_K(\mathcal{T}(X,R))$-linear isomorphism

$$\chi \;:\; p_1^*\mathrm{M} \;\xrightarrow{\sim}\; p_2^*\mathrm{M}$$

where $p_i : \mathcal{H}_K(X) \to \mathcal{A}_K(\mathcal{T}(X,R))$ are the projections $p_1, p_2 : \mathcal{T}(X,R) \rightrightarrows X$, and where $p_i^*\mathrm{M} \;:=\; \mathrm{M} \otimes_{\mathcal{H}_K(X),p_i} \mathcal{A}_K(\mathcal{T}(X,R))$.

–We ask moreover that χ satisfies :

(Δ) $\Delta^*(\chi) = \mathrm{Id}_{\mathrm{M}}$, where $\Delta : X \to \mathcal{T}(X,R)$ is the diagonal.
(C) χ satisfies a certain *cocycle relation* (see slide 26).

Morphisms are $\mathcal{H}_K(X)$-linear maps $\alpha : \mathrm{M} \to \mathrm{N}$ satisfying

$$p_2^*(\alpha) \circ \chi^{\mathrm{M}} \;=\; \chi^{\mathrm{N}} \circ p_1^*(\alpha)\;.$$

$$\begin{array}{ccc} p_2^*\mathrm{M} & \xleftarrow[\sim]{\chi^{\mathrm{M}}} & p_1^*\mathrm{M} \\ {\scriptstyle p_2^*(\alpha)}\downarrow & & \downarrow{\scriptstyle p_1^*(\alpha)} \\ p_2^*\mathrm{N} & \xleftarrow[\chi^{\mathrm{N}}]{\sim} & p_1^*\mathrm{N} \end{array}$$

Differential equations

Differential modules

A differential module over X is a finite free $\mathcal{H}_K(X)$-module together with a connection $\nabla : \mathrm{M} \to \mathrm{M}$, that is a linear map satisfying

$$\nabla(f \cdot m) = f' \cdot m + f \cdot \nabla(m), \quad \textit{for} \quad f \in \mathcal{H}_K(X), \ m \in \mathrm{M}.$$

a morphism between differential equations is a $\mathcal{H}_K(X)$-linear map commuting with the $\nabla's$. We call $d - \mathrm{Mod}(\mathcal{H}_K(X))$ the category of differential modules over $\mathcal{H}_K(X)$.

Once a basis of M is chosen, then the connection defines a diff. eq.

$$Y' = G(T)Y \ , \quad G(T) \in M_n(\mathcal{H}_K(X)) \ .$$

Indeed we have the diagram :

$$\begin{array}{ccc} \mathrm{M} & \xrightarrow{\quad \nabla \quad} & \mathrm{M} \\ \wr \uparrow & & \uparrow \wr \\ \mathcal{H}_K(X)^n & \xrightarrow{\frac{d}{dT} - G(T)} & \mathcal{H}_K(X)^n. \end{array}$$

Differential equations VS (Elementary) Stratifications

The notion of Stratification comes back to A.Grothendieck, P.Berthelot, L.Illusie, N.M.Katz, [...]. They was trying to find a substitute to the notion of Diff. Eq. in characteristic p (see P.Berthelot's Talk).

Theorem :

The category $d-\mathrm{Mod}(\mathcal{H}_K(X))$ is equivalent to $\mathrm{Strat}(\mathcal{H}_K(X))$.

The basic idea of the correspondence is the following :

- To give the connection $\nabla : \mathrm{M} \to \mathrm{M}$ means to give the matrix $G(T) \in M_n(\mathcal{H}_K(X))$ for the differential equation $Y' = G(T)Y$.
- To give the stratification $p_1^*\mathrm{M} \xrightarrow{\sim} p_2^*\mathrm{M}$ means to give the two variable matrix $Y(x,y) \in GL_n(\mathcal{A}_K(\mathcal{T}(X,R)))$ of the stratification.

The relations between them is that $Y(x,y)$ is the solution of the equation : it verifies

$$\frac{d}{dx}Y(x,y) \;=\; G(x)\cdot Y(x,y)\ .$$

Definition of Generic Taylor solution

- Consider a diff. eq. $Y' = G(x)Y,\ G(x) \in M_n(\mathcal{H}_K(X))$.
- We have a Taylor solution at a given point $x_0 \in X$:

$$Y(x) \ := \ \sum_{n\geq 0} Y^{(n)}(x_0)\frac{(x-x_0)^n}{n!}\ , \quad Y(x_0) = \mathrm{Id}\ .$$

- We consider it as a function of x and of x_0.
- We choose the Taylor solution equal to Id at x_0 : $Y(y) = \mathrm{Id},\ \forall\ y \in X$.
- If $Y^{(n)} = G_n(x)Y$, we consider the "*GENERIC*" Taylor solution :

$$Y(x,y) \ := \ \sum_{n\geq 0} G_n(y)\frac{(x-y)^n}{n!}$$

It verifies

$$\frac{d}{dx}Y(x,y) \ = \ G(x)\cdot Y(x,y)\ .$$

$$Y(x,y) \quad := \quad \sum_{n\geq 0} G_n(y)\frac{(x-y)^n}{n!}$$

- FACT : This function converges on a certain tube $\mathcal{T}(X,R)$, for some $R>0$ and it verifies :

$$\boxed{\frac{d}{dx}Y(x,y) = G(x)Y(x,y)}$$

(Δ) $Y(x,x) = \mathrm{Id}$, for all $x \in X$;

(C) $Y(x,y)Y(y,z) = Y(x,z)$ (cocycle relation), for all $(x,y),(y,z) \in \mathcal{T}(X,R)$;

- In fact a matrix $Y(x,y) \in GL_n(\mathcal{A}_K(\mathcal{T}(X,R)))$ is the matrix of a stratification χ if and only if it verifies (Δ) and (C).

From Stratifications to Diff.Eq.

• Reciprocally from a stratification χ with matrix $Y(x, y)$ one gets the differential equation by considering

$$G(x) := \frac{d}{dx}(Y(x,y)) \cdot Y(x,y)^{-1} \; .$$

It happens that $G(x) \in M_n(\mathcal{H}_K(X))$ does not depend on y because of the cocycle relation.

– The idea is the following. Write

$$Y(x,y) = \sum_{n \geq 0} H_n(y)(x-y)^n \;\in\; GL_n(\mathcal{A}_K(\mathcal{T}(X,R)))$$

then :

- $H_1 = G$,
- The cocycle relation implies that H_n, $n \geq 1$, are totally determined by the dominant term $H_1 = G$.

Part 3 : Definition and properties of the radius of convergence function of an analytic function around the diagonal

Radius of convergence function (History)

Historical note

- Firstly Dwork found a p-adic proof of the rationality of the Zeta function of a variety in char. $p > 0$. The proof involved a particular p-adic diff.eq. then he defined a framework for a de Rham cohomology theory (overconvergent functions, differential modules, ...) today called Dwork's cohomology.
- Christol, Dwork, Robba, [...] worked with diff. eq. on annulus $\{r < |x| < r'\}$.
- For a diff.mod. M over the annulus they have defined the Radius of convergence of M at a Dwork generic point t_ρ for $|.|_\rho = |.|_{0,\rho}$, $\rho \in]r, r'[$ as

$$Rad(\mathrm{M}, t_\rho) \;:=\; \min(\rho, Rad(Y(x, t_\rho)))\;.$$

  where $Rad(Y(x, t_\rho))$ is the minimum radius of the entries of the Taylor solution matrix $Y(x, t_\rho)$ at t_ρ.
- ρ was there to make the definition invariant by base changes

Radius of convergence function (History)

Historical note

- In [CD94] Christol and Dwork study the continuity of the function $\rho \mapsto Rad(\mathrm{M}, t_\rho)$ on the segment $]r, r'[$.
- Some years later F.Baldassarri and L.Di Vizio [BV07] defined the radius of convergence function for a differential module over a Berkovich space.
  As observed by them there was a lack of definitions : even the case of a disk was missing in the literature. In this paper they prove
  - the continuity of the radius function in the case of a 1-dim affinoid
  - the upper semi-continuity in the general case.
- Finally F.Baldassarri proved the continuity over a curve in a recent paper (to appears in Inventiones).

Motivation

The radius of convergence function is an invariant (by isomorphisms) of the diff. module. It encodes numerical invariants like the p- adic irregularity (Christol-Mebkhout), and the formal irregularity (B.Malgrange).

Radius of convergence (definition).

- X = K-affinoid
- $\nabla : \mathrm{M} \to \mathrm{M}$ = connection
- $\chi : p_1^*\mathrm{M} \xrightarrow{\sim} p_2^*\mathrm{M}$ = stratification attached to ∇
- $G(T)$ = matrix of ∇ $\quad (Y' = G(T)Y,\ G \in M_n(\mathcal{H}_K(X)))$
- $Y(x,y)$ = matrix of χ $\quad (Y(x,y) \in GL_n(\mathcal{A}_K(\mathcal{T}(X,R))))$
- $Y^{(n)} = G_n Y$ iterated matrices
- Then $$Y(x,y) = \sum_{n \geq 0} G_n(y) \frac{(x-y)^n}{n!}$$

Definition of Radius

For all $|.| \in \mathscr{M}(X)$ we set

$$Rad(\mathrm{M}, |.|) := \min(\ \rho_{|.|,X}\ ,\ Rad(Y(x,t))\ )$$

where t =Dwork generic point for $|.|$, and where

$$Rad(Y(x,t)) = \liminf_n \frac{1}{\sqrt[n]{|G_n(t)|/|n!|}}$$

clearly it does not depends on the choice of t but only on $|.| = |.|_t$.

Comments to the definition

$$Rad(\mathrm{M},|.|) \;:=\; \min(\rho_{|.|,X}, Rad(Y(x,t)))$$

- The use of $\rho_{|.|,X}$ in the definition was introduced by Baldassarri - Di Vizio.
- In the context of Christol-Dwork [CD94] over an annulus we have
$$\rho_{|.|_\rho} = \rho^{gen}_{|.|_\rho}.$$
- Some authors (namely Kedlaya's recent Book) still uses the definition of radius with the $\rho^{gen}_{|.|,X}$ instead of $\rho_{|.|,X}$. This could be, for other reasons, a convenient choice, because in this case the definition of the radius (using ρ^{gen}) results to be equal to a spectral norm of ∇ so the definition seems more intrinsic.

We now list some differences :

- The definition of $Rad(\mathrm{M},|.|)$ using $\rho^{gen}_{|.|,X}$ is not invariant by base field extension of K, because $\rho^{gen}_{|.|,X}$ measures the distance from K^{alg} of a Dwork generic point for $|.|$.
- The definition of $Rad(\mathrm{M},|.|)$ using $\rho_{|.|,X}$ is invariant by base field extension because it measures the distance of $|.|$ from the holes of X.

Invariance by pull-back over a Berkovich point

- As an example consider the case of the annulus $\{r < |x| < r'\}$ and the norm $|.| = |.|_{c,\rho}$, $c \in K$. Let $\mathcal{A}_K(]r,r'[) :=$Tate algebra

$$\mathscr{H}(|.|_{c,\rho}) \ := \ (Frac(\mathcal{A}_K(]r,r'[)), |.|_{c,\rho})^{\widehat{}}$$

be the complete valued field attached to the Berkovich point $|.|_{c,\rho}$. We have a morphism of Berkovich spaces

$$\imath : \mathscr{M}(\mathscr{H}(|.|_{c,\rho})) \ \to \ \mathscr{M}(\mathcal{A}_K(]r,r'[))$$

FACT : If $\rho > 0$ then d/dx extends to $\mathscr{H}(|.|_{c,\rho})$.
We then can consider the pull-back over $\mathscr{H}(|.|_{c,\rho})$ of a differential module over the annulus.
Then
  - The definition of $Rad(\mathrm{M}, |.|)$ using $\rho^{gen}_{|.|,X}$ is invariant by pull-back by $\imath$.
  - The definition of $Rad(\mathrm{M}, |.|)$ using $\rho_{|.|,X}$ is not invariant by pull-back by $\imath$ because $\rho_{|.|,X}$ is the "distance from the holes of X" and it highly depends on the base space X.

Invariance by pull-back by an inclusion of affinoids.

The same phenomena arises for an inclusion of two affinoids :

$$X' \subseteq X$$

- The radius function defined using $\rho^{gen}_{|.|,X}$ is invariant by this pull-back
- The radius function defined using $\rho_{|.|,X}$ is not invariant by this pull-back

The reasons are the same that in the above slide.

See the recent Inventiones paper of F.Baldassarri to see how to obtain a global intrinsic/normalized definition of the Radius function using $\rho_{|.|,X}$ in a way such that it glues.

In this talk we are mainly concerned with a fixed K-affinoid X. In this case

- The radius function defined using $\rho^{gen}_{|.|,X}$ is not continuous,
- The radius function defined using $\rho_{|.|,X}$ is continuous,

Radius of a function around the diagonal

We consider a function $f(x,y) \in \mathcal{A}_K(\mathcal{T}(X,R))$, $R > 0$:

$$f(x,y) \ := \ \sum_{n\geq 0} f_n(y)(x-y)^n, \quad f_n \in \mathcal{H}_K(X).$$

Definition of Radius of $f(x,y)$:

Let $|.| \in \mathscr{M}(X)$, then

$$Rad(f(x,y),|.|) \ = \ \min(\ \rho_{|.|,X} \ , \ \liminf_n \frac{1}{\sqrt[n]{|f_n|}} \ )$$

Transfer theorem

Let $|.|_1, |.|_2 \in \mathscr{M}(X)$, then the radius function is decreasing

$$|.|_1 \leq |.|_2 \quad \Longrightarrow \quad Rad(f(x,y),|.|_1) \geq Rad(f(x,y),|.|_1) \ .$$

Again this theorem is false using ρ^{gen} in the definition. Because $\rho^{gen}_{|.|_1,X} \leq \rho^{gen}_{|.|_2,X}$, while $\rho_{|.|_1,X} \ = \ \rho_{|.|_2,X}$

Convergene locus of $f(x,y)$, x-radius and y-radius

Notice that $f(x,y)$ converges on a tube $\mathcal{T}(X,R)$, but usually the convergence locus of $f(x,y)$ is larger than $\mathcal{T}(X,R)$.

NOTE : Considering the Radius means considering y-sections of the convergence locus : one specializes $y \to t$ (t=Dwork generic point for $|.|$) and one checks the Radius of convergence of the 1-variable function around t

$$f(x,t) = \sum_{n\geq 0} f_n(t)(x-t)^n$$

So we call $Rad_x(f(x,y),|.|) := Rad(f(x,y),|.|)$ the x-radius. One can define the y-radius by specializing x at t. In general

$$Rad_x(f(x,y),|.|) \neq Rad_y(f(x,y),|.|) \ .$$

Proposition

$$Rad_x(f(x,y),|.|) \leq \rho^{gen}_{|.|,X} \iff Rad_y(f(x,y),|.|) \leq \rho^{gen}_{|.|,X}$$

In this case they are equal.

Proposition

If $Y(x,y)$ is the generic Taylor solution of an equation $Y' = GY$, then (without any assumption on the radius) one always has

$$Rad_x(Y(x,y),|.|) \;=\; Rad_y(Y(x,y),|.|)\;.$$

(intended as the minimum radius of the entries). This result is no longer true for its entries.

Examples of radius functions

$$X = \mathrm{D}^+(0,1)$$

$$\rho_{|.|,X} = 1, \text{ for all } |.| \in \mathscr{M}(X)$$

$$f(x,y) = \sum_{n\geq 0} f_n(y)(x-y)^n, \quad f_n(y) = y/p^n$$

$$Rad_x(f(x,y),|.|) = \begin{cases} 1 & \text{if } |.|=|.|_0 \\ |p| & \text{if } |.|\neq|.|_0 \end{cases},$$

$$Rad_y(f(x,y),|.|) = \begin{cases} 1 & \text{if } |.|=|.|_p \\ |p| & \text{if } |.|\neq|.|_p \end{cases}.$$

Examples of radius functions

$$X = \mathrm{D}^+(0,r),\ r > 1$$

$$\rho_{|.|,X} = r, \text{ for all } |.| \in \mathscr{M}(X)$$

$$f(x,y) = \sum_{n\geq 0} f_n(y)(x-y)^n, \quad f_n(y) = y^n$$

$$Rad_x(f(x,y),|.|_t) = \min(\frac{1}{|t|}, r)$$

$$Rad_y(f(x,y),|.|_t) = \begin{cases} \min(\frac{1}{|t|},1) & \text{if } |t|\neq 1 \\ \geq \min(\frac{1}{|t|},1) & \text{if } |t|=1 \end{cases} .$$

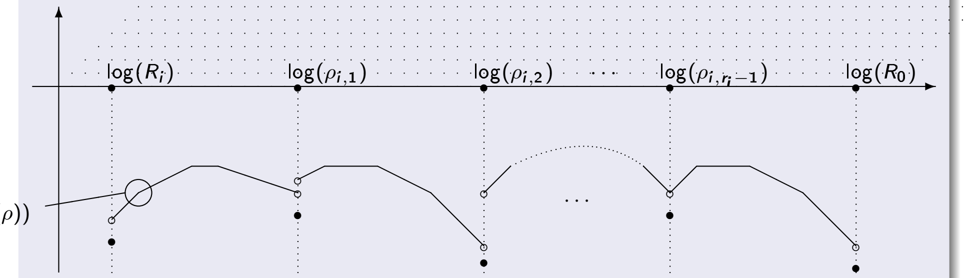
In particular $Rad_x \neq Rad_y$ on the whole $\mathrm{D}^-(0,1)$.

For all $f(x,y) \in \mathcal{A}_K(\mathcal{T}(X,R))$, we consider the radius function

$$|.| \mapsto Rad(f(x,y),|.|) \;:\; \mathscr{M}(X) \to \mathbb{R}_{\geq 0}$$

Behavior on the maximal skeleton

The function $Rad(f(x,y),|.|)$ has the following log-shape on a branch $R \mapsto |.|_{c_i,R}$ of the maximal skeleton



- $\rho_{i,j}$ corresponds to the distances of c_i from the holes of X.
- The function is log-concave on the annuli not intersecting the holes.

This is a general fact about concave functions :

theorem

I =interval of $\mathbb{R}$. A concave function

$$f : I \to \mathbb{R}$$

is continuous on the interior of I.

- Each function $R \mapsto |f_n|_{c,R}$ is log-convex
- Then each $R \mapsto \frac{1}{\sqrt[n]{|f_n|_{c,R}}}$ is log-concave
- inflim of (log-)concave functions is a (log-)concave function hence use the theorem

Theorem

For all $f(x,y) \in \mathcal{A}_K(\mathcal{T}(X,R))$, the radius function

$$|.| \mapsto Rad(f(x,y),|.|) \ : \ \mathscr{M}(X) \to \mathbb{R}_{\geq 0}$$

is LSC on each point $|.| \in \mathscr{M}(X)$ satisfying $\rho^{gen}_{|.|,X} > 0$. This form an open subset of $\mathscr{M}(X)$ containing the skeleton.

Recall that $\rho^{gen}_{|.|,X} = 0$ if and only if a Dwork generic points for $|.|$ lies in $\widehat{K^{\mathrm{alg}}}$.

One distinguishes the points of the maximal skeleton from the others.

Points in the maximal skeleton :

- Log-concavity on the skeleton implies that the restriction of Rad on it is LSC (easy). Actually we have continuity if no holes in the path.
- Transfer theorem implies the LSC on the skeleton (because the values the radius is minimal on the points of the maximal skeleton).

If $|.|$ is another point (not in the skeleton), then $|.| = |.|_{c,R}$ with $R > 0$.

- Then the annulus $A := \{R - \delta < |x - c| < R + \delta\}$ is an open neighborhood of $|.|$,
- If $\delta > 0$ is small, the restriction of $Rad(f, |.|)$ is continuous on the path $\rho \mapsto |.|_{c,\rho}$, $\rho \in]R - \delta, R + \delta[$ because there is "no holes of X"
- So for all $\varepsilon > 0$ there is $\delta > 0$ small such that

$$Rad(f, |.|_{c,\rho}) \geq Rad(f, |.|_{c,R}) - \varepsilon,$$

  for all $\rho \in]R - \delta, R + \delta[$. (this is the LSC on this path).
- by transfer this also holds for the other points of this annulus (the radius is minimal in this path). So the inequality is true for all $|.|$ of A.

proposition

If $Y(x,y)$ is the Taylor solution of a diff.eq. then

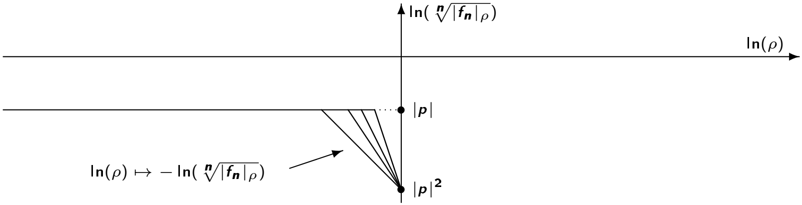
$$|.| \mapsto Rad(Y(x,y),|.|) \ : \ \mathscr{M}(X) \to \mathbb{R}_{\geq 0}$$

is LSC on the whole $\mathscr{M}(X)$.

The reason is that the radius of a differential equation is locally constant on the $\widehat{K^{alg}}$-rational points (i.e. those for which $\rho^{gen}_{|.|,X} = 0$).

Example of non continuous radius

$X := \mathrm{D}^+(0,1)$ be the closed unit disk. Let r_n be a sequence of real numbers satisfying : (a) $r_n < 0$, (b) $\lim_n r_n = 0$, (c) $r_n \in \frac{n\ln(|p|)}{\mathbb{N}}$. Let now $\alpha_n := \frac{\ln(|p|^n)}{r_n} \in \mathbb{N}$. Then the function $f(x,y) := \sum_{n\geq 0} f_n(y)(x-y)^n$, with $f_n := p^{-n} + p^{-2n}x^{\alpha_n}$ verifies $|f_n|_\rho = \sup(|p|^{-n}, |p|^{-2n}\rho^{\alpha_n})$, in particular $|f_n|_1 = |p|^{-2n}$ for all $n \geq 0$. Then $\sqrt[n]{|f_n|_\rho} = \sup(|p|^{-1}, |p|^{-2}\rho^{\frac{\ln(|p|)}{r_n}})$.



Hence $Rad_x(f(x,y), |.|_\rho) := \min(1, \liminf_n \frac{1}{\sqrt[n]{|f_n|_\rho}}) = \liminf_n \frac{1}{\sqrt[n]{|f_n|_\rho}} =$

$$\begin{cases} |p| & \text{if} \quad \rho < 1 \\ |p|^2 & \text{if} \quad \rho = 1 \end{cases}.$$

A criterion for the continuity

The above example proves that we need a sort of UNIFORM CONVERGENCE result in the sense of LSC functions.

- We have to ask to the sequence of functions

$$|.| \mapsto \frac{1}{\sqrt[n]{|f_n|}}$$

to be "*superiorly uniformly convergent*" to $Rad(f(x,y),|.|)$.

Criterion of continuity

If $f(x,y) = \sum_{n\geq 0} f_n(y)(x-y)^n$ verifies

- $Rad(f(x,y),|.|)$ is locally constant on the K^{alg}-rational points (rigid points) (i.e. those for which $\rho^{gen} = 0$),
- There exists a sequence C_n such that $\lim_n C_n = 1$, $C_n \geq 1$ for all large values of n such that

$$|f_n| \cdot Rad(f(x,y),|.|)^n = C_n^n \ , \quad \text{fro all } n \geq 0 \ , \forall |.| \in \mathscr{M}(X)$$

then $Rad(f(x,y),|.|)$ is a continuous function on the whole $\mathscr{M}(X)$.

Dwork-Robba's Theorem

$$Rad(\mathrm{M}, |.|) \ := \ \min\Big(\rho_{|.|,x} \ , \ \liminf_n \frac{1}{\sqrt[s]{\frac{|G_s|}{|s!|}}} \ \Big)$$

where G_s is the matrix of $\nabla^s : Y^{(s)} = G_s Y$. Recall that the Taylor solution is

$$Y(x,y) \ = \ \sum_{s \geq 0} G_s(y) \frac{(x-y)^s}{s!} \ .$$

Dwork-Robbs's Theorem

Let $t \in X(\Omega)$, with Ω / K an arbitrary complete valued field extension. Then for all $0 < \rho \leq Rad(\mathrm{M}, |.|_{t,0})$ one has

$$\frac{|G_s|_{t,\rho}}{|s!|} \leq \rho^{-s} \cdot \{s, n-1\}_p \cdot \max_{0 \leq i \leq n-1} (|G_i|_{t,\rho} \cdot \rho^i)$$

where $\{s,n\}_p := \sup_{1 \leq \lambda_1 < \cdots < \lambda_n \leq s} |\lambda_1 \cdots \lambda_n|^{-1}$

The Dwork-Robba's theorem in their form asserts that the asymptotic growth of the coefficients is controlled by the first n coefficients of the Taylor solution matrix.

Dwork-Robbs's Theorem revisited

There exists constants $\mu > 0$ and $\kappa > 1$ such that

$$\frac{|G_s|}{|s!|} \cdot Rad(\mathrm{M}, |.|)^s \;\leq\; s^{\mu(n-1)}$$

for all $|.| \in \mathscr{M}(X)$.

Corollary

The radius of convergence function of a differential module is continuous :)

The above version of the Dwork-Robba's theorem is exactly the above criterion of continuity (see slide 46).

NOTE : the idea of considering the Dwork-Robba's goes back to Christol-Dwork [CD94]. The same idea have been taken up also by [BV07] who have been able to generalize the Dwork-Robba's theorem in many variables. So the upper semi-continuity holds in higher dimensional spaces, but the lower semi-continuity (that should hold for all funtions around the diagonal) is an open problem since the base space is actually much more complicated.

NOTE : The proof we have presented here is supposed to separate the obstructions to the continuity of topological nature from those coming from differential equation.

Bibliography

NOTE : All the computations of this talk can be found in the last version of [Pu10] which will be available in a few week on arxiv. The older version already present in arxiv does not contain any of these computations.

[CD94] F.Baldassarri and L. Di Vizio, *Continuity of the radius of convergence of p-adic differential equations on Berkovich spaces*, ArXiv :07092008[math.NT],13 September 2007, (2007),1-22.

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