Infinitesimal deformation of ultrametric differential equations

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Deformation functor

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$$\begin{tikzcd}[column sep=small] & \{Diff.Eq.\} \arrow[dl, dotted, "\mathrm{Def}_{\Gamma_K}"' , "(\Sigma=\Gamma_K)"'] \arrow[dr, dotted, "\mathrm{Def}_q", "(\Sigma=\{q\})"] & \\ \{(\varphi,\Gamma_K)-\mathrm{modules}\} & & \{q-Diff.Eq.\} \end{tikzcd}$$

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$(\Sigma = \Gamma_K)$ $(\Sigma = \{q\})$

Application to Morita's p-adic Γ_p function and to some values at positive integers of Kubota-Leopoldt's L-functons

Notation

- K := complete ultrametric field of characteristic 0
- $\mathrm{D}^-(c,R), \mathrm{D}^+(c,R)$ =open and closed disk
- $\mathcal{A}_K(c,R)$ =ring of analytic functions over $\mathrm{D}^-(c,R)$
- X := sub-K-affinoid of $\mathbb{A}^1_K$ defind by

$$X := \mathrm{D}^+(c_0,R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i,R_i)$$

with $c_0, \ldots c_n \in K$ and $R_0 \geq R_1, \ldots, R_n > 0$

- $\mathcal{H}_K(X)$ =analytic functions over X
- $\|f\|_X := \sup_{\Omega/K,\, x\in X(\Omega)} |f(x)|_\Omega$, for all $f \in \mathcal{H}_K(X)$

Differential equations and Elementary Stratifications

$d - \mathrm{Mod}(\mathcal{H}_K(X)) := \{$finite free $\mathcal{H}_K(X)$-modules with a connection$\}$

Consider $Y' = G(x)Y$, (define G_n by $Y^{(n)} = G_n Y$) then we have the *generic Taylor solution*

$$Y(x,y) := \sum_{n \geq 0} G_n(y)(x-y)^n/n! \,. \tag{1}$$

This function $Y(x,y)$ converges over

$\mathcal{T}(X,R) := \{(x,y) \in X \times X \text{ such that } |x-y| < R\} = \text{TUBE of radius R}$

and $Y(x,y)$ satisfies :

- $Y(x,y)$ converges over $\mathcal{T}(X,R)$ for some $R > 0$
- $Y(x,y)Y(y,z) = Y(x,z)$ for $(x,y),(y,z),(x,z) \in \mathcal{T}(X,R)$
- $Y(x,x) = \mathrm{Id}$.

Differential equations and Elementary Stratifications

One has an equivalence

$$\mathrm{S} \; : \; d - \mathrm{Mod}(\mathcal{H}_K(X)) \xrightarrow{\sim} \mathrm{Strat}(\mathcal{H}_K(X)) \; , \tag{2}$$

where an object of $\mathrm{Strat}(\mathcal{H}_K(X))$ is a finite free $\mathcal{H}_K(X)$-module M together with an isomorphism

$$\chi_{\mathrm{M}} : p_1^*\mathrm{M}_{|_{\mathcal{T}(X,R)}} \xrightarrow{\sim} p_2^*\mathrm{M}_{|_{\mathcal{T}(X,R)}} \; ,$$

(where $p_i : X \times X \to X$ are the projections) satisfying

- R is unspecified
- χ_{M} satisfies $p_{1,2}^*(\chi_{\mathrm{M}}) \circ p_{2,3}^*(\chi_{\mathrm{M}}) = p_{1,3}^*(\chi_{\mathrm{M}})$ (holds where defined)
- $\Delta^*\chi_{\mathrm{M}} = \mathrm{Id}_{\mathrm{M}}$, where $\Delta : X \to X \times X$ = diagonal immersion.

S sends $Y' = GY$ into $(\mathrm{M}, \chi_{\mathrm{M}})$ whose matrix is the Taylor solution $Y(x,y)$
S^{-1} sends $(\mathrm{M}, \chi_{\mathrm{M}})$ with matrix $Y(x,y)$ into $Y' = GY$ where

$$G(x) := d/dx(Y(x,y)) \cdot Y(x,y)^{-1} \text{ belongs to } \mathcal{H}_K(X).$$

Generic Idea of the deformation

Example : Let $R > 0$ fixed (and small). Consider the (easy) case of a

$$\sigma \ : \ X \ \xrightarrow{\sim} \ X$$

satisfying for all $x \in X(\Omega)$, and all Ω/K (uniformly on X)

$$|\sigma(x) - x| < R.$$

Then the map $\Delta_\sigma : X \to X \times X$ sending $\Delta_\sigma(x) = (\sigma(x), x)$ satisfies

$$\Delta_\sigma(X) \ \subset \ \mathcal{T}(X, R)\ .$$

It takes meaning to consider the pull-back of χ_{M} providing that χ_{M} converges over $\mathcal{T}(X, R)$. Then

$$\Delta_\sigma^* p_1^* \mathrm{M} \ = \ \sigma^* \mathrm{M} \ \xrightarrow[\sim]{\Delta_\sigma^* \chi_{\mathrm{M}}} \ \mathrm{M} \ = \ \Delta_\sigma^* p_2^* \mathrm{M}\ .$$

We have a σ-deformation functor $\mathrm{Def}_\sigma := \Delta_\sigma^* \circ \mathrm{S}$:

$$\mathrm{Def}_\sigma \ : \ d - \mathrm{Mod}(\mathcal{H}_K(X))^{[R]} \ \longrightarrow \ \sigma - \mathrm{Mod}(\mathcal{H}_K(X))$$

Analytic functions on tubes

Note 1 : The most part of σ does not verify $\Delta_\sigma(X) \subset \mathcal{T}(X,R)$!

Note 2 : Likely if $Y(x,y)$ converges on $\mathcal{T}(X,R)$ then
the convergence domain of $Y(x,y)$ is usually not reduced to $\mathcal{T}(X,R)$!

About Note 2 : Let $y_0 \in X(\Omega)$, Ω/K. We define

$$\rho_{y_0,X} \ , := , \mathrm{dist}(y_0, \mathbb{A}^1 - X) \ .$$

$$\mathcal{A}_K(\mathcal{T}(X,R)) = \{f(x,y) = \textstyle\sum_{n\geq 0} f_n(y)(x-y)^n, \ \text{s.t. } \lim_n \|f_n\|_X R^n = 0\}$$

It may happen that $f(x,y_0)$ converges outside X, but we define the radius of $f(x,y)$ as

$$Rad(f(x,y),y_0) \ \ := \ \ \min(\ Rad(f(x,y_0))\ ,\ \rho_{y_0,X}\ )$$

If $h(x) \in \mathcal{H}_K(X)$ then $Rad(h(x)f(x,y),y_0) = Rad(f(x,y),y_0)$.

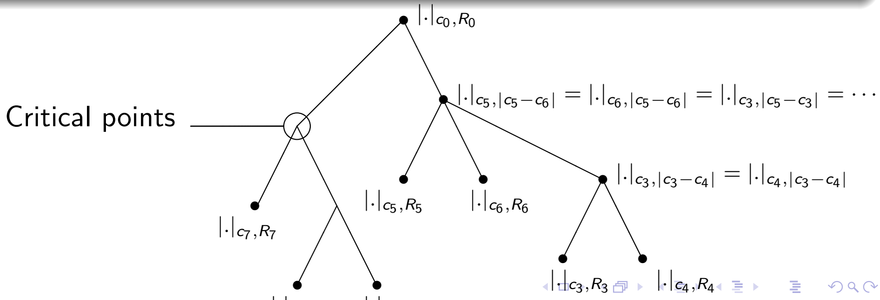
Maximal Skeleton of a Berkovich space

Let $X = \mathrm{D}^+(c_0, R_0) - \cup_{i=1}^n \mathrm{D}^-(c_i, R_i)$.

$$\left.\begin{array}{l} c \in \mathrm{D}^+(c_0, R_0) \\ \rho \leq R_0 \\ R_i \leq \rho \ \text{ if } c \in \mathrm{D}^-(c_i, R_i)\ . \end{array}\right\} \Longrightarrow \quad |f(x)|_{c,\rho} \ \in \ \mathscr{M}(X)$$

Proposition : The set of maximal points of $\mathscr{M}(X)$ is given by

$$\mathscr{S}_X \ := \ \cup_{i=1}^n \{|.|_{c_i,\rho} \ \text{ s.t. } \rho \in [R_i, R_0]\}\ .$$



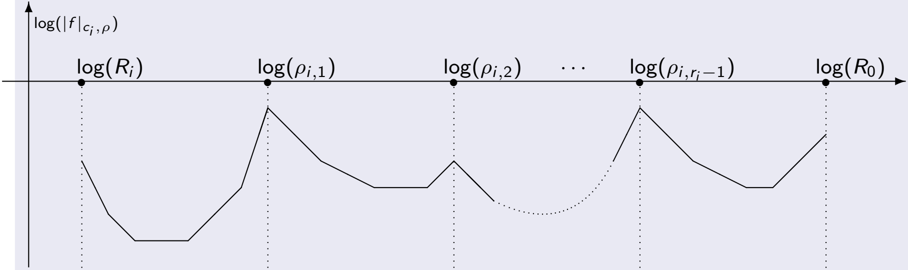
Behavior of a function on the Maximal Skeleton

If $g(x) \in \mathcal{H}_K(X)$, consider the continuous function

$$|.|_* \mapsto |f|_* \ : \ \mathscr{M}(X) \to \mathbb{R}_{\geq 0} \ .$$

$$|.|_* \leq |.|_{**} \iff |f|_* \leq |f|_{**} \text{ for all } \ f \in \mathcal{H}_K(X)$$

The restriction of this function to the branch $[|.|_{c_i,R_i}, |.|_{c_i,R_0}]$ of $\mathscr{S}_X$ gives :



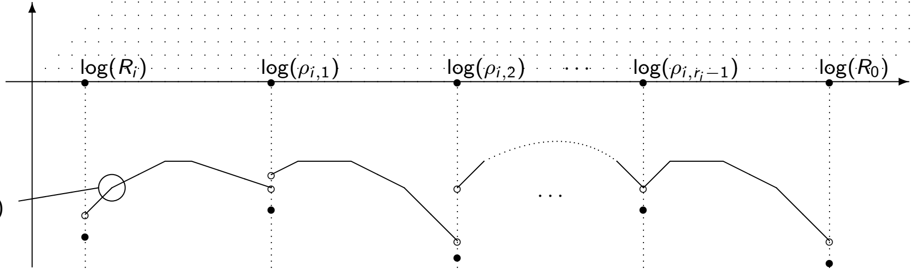
where $R_i < \rho_{c_i,1} < \ldots < R_0$ are the "critical points" of $\mathscr{S}_X$.

Let $f(x,y) \in \mathcal{A}_K(\mathcal{T}(X,R))$, and $|.|_*, |.|_{**} \in \mathscr{M}(X)$, then

$$|.|_* \le |.|_{**} \implies \begin{cases} \rho_{|.|_*,X} = \rho_{|.|_{**},X} \\ Rad(f(x,y),|.|_*) \ge Rad(f(x,y),|.|_{**}) \end{cases}$$

Then $Rad(f(x,y),-)$ is MINIMAL on the maximal points $\mathscr{S}_X$.

$$R(\rho) := Rad(f(x,y),|.|_{c_i,\rho})/\rho_{|.|_{c_i,\rho},X} = Rad(f(x,y),|.|_{c_i,\rho})/\rho\ .$$



Infinitesimal automorphisms

DEFINITION : We say that $\sigma : X \xrightarrow{\sim} X$ is infinitesimal if for all Ω/K, all $y \in X(\Omega)$, one has

$$|\sigma(y) - y| < \rho_{y,X} \ . \qquad (*)$$

PROPOSITION : The following are equivalents

- (*) holds for all points of $\mathscr{M}(X)$
- (*) holds for all points in the maximal Skeleton $\mathscr{S}_X$
- (*) holds for critical points (finite set of conditions)

σ-compatible analytic functions on tubes

DEFINITION : We say that $f(x,y)$ is σ-compatible if for all Ω/K and all $y_0 \in X(\Omega)$

$$|\sigma(y_0) - y_0| < Rad(f(x,y), y_0) \,. \qquad (**)$$

PROPOSITION : The following are equivalents

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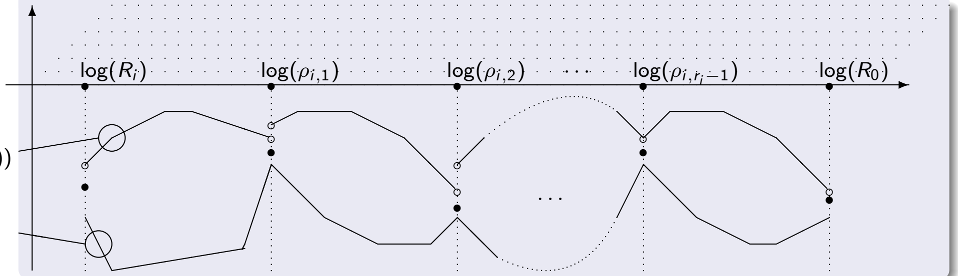


Deformation as Pull Back

Let Σ be a family of infinitesimal automorphisms.

DEFINITION : We denote by

$$d - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}(\Sigma)}$$

the full subcategory of $d - \mathrm{Mod}(\mathcal{H}_K(X))$ formed by equations whose stratification is σ-compatible for all $\sigma \in \Sigma$.

THEOREM : The pull-back functor is well defined and we have a deformation functor

$$\mathrm{Def}_\Sigma : d - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}(\Sigma)} \longrightarrow \Sigma - \mathrm{Mod}(\mathcal{H}_K(X))$$

DEFINITION : We call Taylor admissible Σ-modules the essential image

$$\Sigma - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}} .$$

Non degeneracy and fully faithfulness of Def_Σ

DEFINITION : A family Σ is non degenerate if

1. Σ is infinitesimal
2. There exists Ω/K and $y_0 \in X(\Omega)$ such that
$$\mathcal{A}_\Omega(y_0, R)^\Sigma = \Omega$$
for all $R \leq \rho_{y_0,X}$ for which $\mathrm{D}^-(y_0, R)$ is invariant under all $\sigma \in \Sigma$.

CRITERION OF NON DEGENERACY : If there exists a $|.|_* \in \mathscr{M}(X)$ such that 0 belongs to the closure of the set

$$\{|\sigma(T) - T|_*^n\}_{\sigma\in\Sigma, n\geq 1} - \{0\} \ , \tag{3}$$

then Σ is non degenerate.

THEOREM : If Σ is non degenerate, then Def_Σ is fully faithful and we have an equivalence

$$\mathrm{Def}_\Sigma : d - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}(\Sigma)} \xrightarrow{\ \sim\ } \Sigma - \mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{adm}}$$

Katz-Matsuda's Canonical extension for Taylor admissible Σ-modules

$\mathcal{R}_K := \{f = \sum_{n\in\mathbb{Z}} a_n T^n \mid f \text{ converges for } \varepsilon < |T| < 1 (\varepsilon = \text{unspecified})\}$
It "corresponds" to $k((t))$ where k=residual field of K

$\mathcal{H}_K^\dagger := \{f = \sum_{n\in\mathbb{Z}} a_n T^n \mid f \text{ converges for } \varepsilon_1 < |T| < \varepsilon_2 (\varepsilon_i = \text{unspecified})\}$
Its residual ring is $k[t, t^{-1}]$.

THEOREM : Let Σ be an non degenerate family over $\mathcal{H}_K^\dagger$. Choose a Frobenius φ of $\mathcal{H}_K^\dagger$. Then we have a section of the scalar extension functor

$$\mathrm{M} \mapsto \mathrm{M} \otimes_{\mathcal{H}_K^\dagger} \mathcal{R}_K : \Sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\varphi),\mathrm{adm}} \longrightarrow \Sigma - \mathrm{Mod}(\mathcal{R}_K)^{(\varphi),\mathrm{adm}}$$

called *canonical extension*

$$\mathrm{Can} : \Sigma - \mathrm{Mod}(\mathcal{R}_K)^{(\varphi),\mathrm{adm}} \longrightarrow \Sigma - \mathrm{Mod}(\mathcal{H}_K^\dagger)^{(\varphi),\mathrm{adm}}$$

Break decomposition, and quasi-unipotence of Taylor admissible Σ-modules

THEOREM : Every object in $\Sigma - \mathrm{Mod}(\mathcal{R}_K)^{(\varphi),\mathrm{adm}}$ admits a break decomposition by the slopes of the radius of convergence.

LEMMA : Every infinitesimal automorphism σ of $\mathcal{H}_K^\dagger$ extends uniquely to étale extensions of $\mathcal{H}_K^\dagger$, and $\mathcal{R}_K$.

THEOREM : Every object of $\Sigma - \mathrm{Mod}(\mathcal{R}_K)^{(\varphi),\mathrm{adm}}$ is quasi unipotent.

Recalls about p-adic Representations

K :=finite extension of $\mathbb{Q}_p$ $\quad K_n := K(\boldsymbol{\mu}_{p^n})$ $\quad K_\infty := \cup_n K_n$

k :=residual field of K $\quad k_\infty$:=res.field of K_∞ $\quad F' := \mathrm{Frac}(\mathbf{W}(k_\infty))$

$\mathscr{G}_K := \mathrm{Gal}(K^{\mathrm{alg}}/K)$, $\quad \mathscr{H}_K := \mathrm{Gal}(K^{\mathrm{alg}}/K_\infty)$, $\quad \Gamma_K := \mathrm{Gal}(K_\infty/K)$

THEOREM (L.BERGER) : There exists a functor

$$\mathrm{N}_{dR} : \mathrm{Rep}_{dR}(\mathscr{G}_K) \xrightarrow{\sim} d - \mathrm{Mod}(\mathcal{R}_{F'})^{(\varphi)}$$

SOME RESULTS OF BERGER :

- $\mathrm{N}_{dR}(\mathrm{V})$ is quasi unipotent $\Longleftrightarrow$ V is potentially semi-stable,
- $\mathrm{N}_{dR}(\mathrm{V})$ is unipotent $\Longleftrightarrow$ $\mathrm{V}_{|\mathscr{G}_{K_n}}$ is semi-stable for some $n \geq 0$,
- $\mathrm{N}_{dR}(\mathrm{V})$ is trivial $\Longleftrightarrow$ $\mathrm{V}_{|\mathscr{G}_{K_n}}$ is crystalline for some $n \geq 0$.

A RESULT OF MARMORA :

- $\mathrm{Irr}\ \mathrm{N}_{dR}(\mathrm{V}) = \mathrm{sw}(\mathrm{V}_{|\mathscr{G}_{K_n}})$ for some $n >> 0$.

Recalls

Aim

One expects to read in $\mathrm{N}_{dR}(\mathrm{V})$ every invariant of V whose nature is "potential".
More precisely one expect that

Every invariant of $\mathrm{N}_{dR}(\mathrm{V})$ corresponds to an invariant of $\mathrm{V}_{|\mathscr{G}_{K_n}}$ for $n >> 0$.

Question

Is there some equivalence of categories ?

Recalls about a recent work of L.Berger

FONTAINE+(CHERBONIER-COLMEZ) : We have equivalences

$$\begin{tikzcd} \mathrm{Rep}(\mathscr{G}_K) \arrow[r, "\mathrm{D}", "\sim"'] \arrow[dr, "\sim", "\mathrm{D}^{\dagger}"'] & (\varphi,\Gamma_K)-\mathrm{Mod}(\mathcal{E}_{F'})^{\text{ét}} \\ & (\varphi,\Gamma_K)-\mathrm{Mod}(\mathcal{E}^{\dagger}_{F'})^{\text{ét}} \arrow[u, "\wr"'] \end{tikzcd} \tag{4}$$

L.Berger defined a full sub-category

$$(\varphi,\Gamma_K)-\mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét,LT}}$$

- φ is étale in the sense of Kedlaya (i.e. $Mat(\varphi) \in GL_n(\mathcal{O}_{\mathcal{E}^{\dagger}_{F'}}))$
- Γ_K acts Locally Trivially on the (φ,Γ_K)-module (following Berger)

THEOREM (L.BERGER) : The functor N_{dR} factorizes as follows

$$\begin{tikzcd}
\mathrm{Rep}_{dR}(\mathscr{G}_K) \arrow[r, "\mathrm{D}^\dagger", "\sim"'] \arrow[drr, "\mathrm{N}_{dR}"'] & (\varphi,\Gamma_K)-\mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét,LT}} \arrow[r, "\mathrm{C}_{\Gamma_K}"'] & (\varphi,\nabla)-\mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét}} \arrow[d] \\
& & d-\mathrm{Mod}(\mathcal{R}_{F'})^{(\varphi)}
\end{tikzcd} \tag{5}$$

(Notation are not standard)

Motivated by the above correspondence of invariants we consider

$$\begin{aligned}
\mathrm{Germ\ Rep}_{dR}(\mathscr{G}_K) &:= \varinjlim_n \mathrm{Rep}_{dR}(\mathscr{G}_{K_n}) \\
\mathrm{Germ}(\varphi,\Gamma_K)-\mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét,LT}} &:= \varinjlim_n (\varphi,\Gamma_{K_n})-\mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét,LT}}
\end{aligned}$$

$$\begin{tikzcd}
\mathrm{GermRep}_{dR}(\mathscr{G}_K) \arrow[r, "\mathrm{D}^\dagger", "\sim"'] \arrow[drr, "\mathrm{N}_{dR}"'] & \mathrm{Germ}(\varphi,\Gamma_K)\text{-}\mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét,LT}} \arrow[r, "\mathrm{C}_{\Gamma_K}"'] & (\varphi,\nabla)\text{-}\mathrm{Mod}(\mathcal{R}_{F'} \arrow[d] \\
& & d-\mathrm{Mod}(\mathcal{R}_{F'})^{(}
\end{tikzcd}$$

Berger's functor N_{dR} as a confluence functor

$$\mathrm{GermRep}(\mathscr{G}_K) \xrightarrow[\sim]{\mathrm{D}^\dagger} \mathrm{Germ}(\varphi, \Gamma_K)\text{-}\mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét,LT}} \overset{\mathrm{C}_{\Gamma_K}}{\to} (\varphi, \nabla)\text{-}\mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét}}$$

By Γ_K-deformation we have

$$\mathrm{Def}_{\Gamma_K} \; : \; (\varphi, \nabla) - \mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét}} \xrightarrow{\sim} \mathrm{Germ}(\varphi, \Gamma_K) - \mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét,adm}}$$

THEOREM : We have the inclusion

$$\mathrm{Germ}(\varphi, \Gamma_K) - \mathrm{Mod}(\mathcal{R}_{F'})^{\acute{e}t, LT} \subseteq \mathrm{Germ}(\varphi, \Gamma_K) - \mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét,adm}}$$

and the functor C_{Γ_K} is a quasi inverse of Def_{Γ_K} (and hence coincides with the Γ_K-confluence functor Conf_{Γ_K}). In particular

$$\mathrm{Germ\ Rep}_{dR}(\mathscr{G}_K) \subseteq (\varphi, \nabla) - \mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét}}$$

is quasi-isomorphic to a fully faithful subcategory.

Reconciliating questions

$$\mathrm{Germ}(\varphi,\Gamma_K) - \mathrm{Mod}(\mathcal{R}_{F'})^{\acute{e}t,LT} \subseteq \mathrm{Germ}(\varphi,\Gamma_K) - \mathrm{Mod}(\mathcal{R}_{F'})^{\text{ét,adm}}$$

Is that inclusion an equality ?

If NOT then the above theorem induces to suspect the existence of an extension of the functor N_{dR} to a larger category.

What about *L*-functions in this correspondence ?

Is there a more analytic proof of the results of L.Berger ?

Can we compute the integer "n" for which the differential invariants of $\mathrm{N}_{dR}(\mathrm{V})$ coincides with the Galois invariants of $\mathrm{V}_{|\mathscr{G}_{K_n}}$?

Is there a Taylor formula expressing the stratification directely from the action of Γ_K ?

Finite difference equations

Finite Difference Equations $\sigma = \sigma_{q,h}$

$$\sigma_{q,h}(f(x)) = f(qx+h)$$

we have the q-derivation :

$$\Delta_{q,h} := \frac{\sigma_{q,h}-1}{\sigma_{q,h}(T)-T}$$

If $\sigma_{q,h}(Y) = AY$ ($\Longleftrightarrow \Delta_{q,h}(Y) = GY$, with $G = (A-I)/(\sigma_{q,h}(T)-T)$) is a q-difference equation one has the twisted (q,h)-Taylor formula

$$Y(x,y) = \sum_{n\geq 0} G_{[n]}(y)\frac{(x-y)_{q,h}^{[n]}}{[n]_{q,h}^{!}}$$

where $\Delta_{q,h}^{n} Y = G_{[n]}Y$, and $[n]_q := 1+q+q^2+\cdots+q^{n-1}$,

$$[n]_q^{!} := [n]_q[n-1]_q\cdots[1]_q,$$

$$(x-y)_{q,h}^{[n]} = (x-y)(x-\sigma_{q,h}(y))(x-\sigma_{q,h}^{2}(y))\cdots(x-\sigma_{q,h}^{n-1}(y)).$$

Consequences of the existence of the (q, h)-Taylor formula

We have a characterization of *Taylor admissible $\sigma_{q,h}$-modules*.

Sketch of the results :

- One defines a "Formal Radius" of convergence of the above (q, h)-Taylor expansion (imitating the classical case)

- One proves that the following are equivalent :
  - the "Formal Radius" is bigger than a certain bound on the critical points of the Skeleton
  - the above (q, h)-Taylor expansion converges and gives a Stratification,

  in this case the Formal Radius coincides with the classical radius of the stratification.

- One proves that the following are equivalent :
  - A given $\sigma_{q,h}$-module is Taylor admissible
  - "Its Formal Radius at critical points is big enough"

Comparing with (φ, Γ_K)-modules

FROM BERGER'S TALK : For (φ, Γ_K)-modules one chose an element

$$t := \text{“}\log(1+T)\text{”}$$

and one localizes with the maps

$$\iota_n : \mathcal{R}^\rho \to K_n[\![t]\!]$$

and if $\mathbb{D} = (\varphi, \Gamma_K)$-module attached to V =de Rham, then

$$\mathrm{D}_{dR}(\mathrm{V}) \quad := \quad (\mathbb{D}^\rho \otimes_{\mathcal{R}^\rho, \iota_n} K_n((t)))^{\Gamma_K}$$

HEURISTIC INTERPRETATION : $T \mapsto t$ is a change of variable that transforms the action of Γ_K into a q-difference operator ! In fact

$$\begin{array}{rcl} \Gamma_K & \xrightarrow{\sim} & \mathbb{Z}_p^\times \\ \gamma_q & := & \chi^{-1}(q) \\ \gamma_q(T) & = & (1+T)^q - 1 \\ \gamma_q(t) & = & qt \,. \end{array}$$

Then γ_q is a q-difference operator with respect to the variable t.

Suggestion/Idea :

- Taking the pull back of the q-Taylor formula for q-differences equations by the above change of variable, can we consider a Twisted Taylor formula adapted to the action of Γ_K ?
(the change of variable presents poles)

- Can we re-prove the theorem of Berger in a more analytic way just checking the convergence locus of that Taylor expansion ?

- Can we generalize the domain of definition of N_{dR} by this method ?

A result about the roots of unity

THEOREM : One has an equivalence

$$d-\mathrm{Mod}(\mathcal{R}_{K_\infty})^{\mathrm{sol}} \xrightarrow[\sim]{\mathrm{Def}_{\boldsymbol{\mu}_{p^\infty}}} \boldsymbol{\mu}_{p^\infty}-\mathrm{Mod}(\mathcal{R}_{K_\infty})^{\mathrm{adm}} \tag{6}$$

and

$$d-\mathrm{Mod}(\mathcal{R}_{K_\infty})^{(\varphi)} \xrightarrow[\sim]{\mathrm{Def}_{\boldsymbol{\mu}_{p^\infty}}} \boldsymbol{\mu}_{p^\infty}-\mathrm{Mod}(\mathcal{R}_{K_\infty})^{(\varphi),\mathrm{adm}} \tag{7}$$

By composing with the result of Y.André

$$d-\mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\varphi)} \xrightarrow{\sim} \mathrm{Rep}_{K^{\mathrm{alg}}}(\mathcal{I}_{k((x))^{\mathrm{alg}}}) \tag{8}$$

we then have an equivalence

$$\mathrm{Rep}_{K^{\mathrm{alg}}}(\mathcal{I}_{k((x))^{\mathrm{alg}}}) \xrightarrow{\sim} \boldsymbol{\mu}_{p^\infty}-\mathrm{Mod}(\mathcal{R}_{K_\infty})^{(\varphi),\mathrm{adm}} . \tag{9}$$

Morita's p-adic Gamma function as solution of a differential equation

THEOREM(MORITA) : There exists an unique continuous function $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p$ satisfying

$$\Gamma_p(0) = 1 \,, \qquad \Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & \text{if} \quad |x| = 1 \\ -\Gamma_p(x) & \text{if} \quad |x| < 1 \end{cases}$$

Moreover Γ_p is locally analytic and its expansion at zero verifies

$$\log(\Gamma_p^{(0)}(T)) = \lambda_0 T + \sum_{m\geq 1} L_p(1+2m, \omega_p^{2m}) \frac{T^{2m+1}}{2m+1}$$

$$\begin{array}{rcl} \omega_p & = & \text{Teichmüller Dirichelt cheracter corresponding to } p \neq 2 \\ L_p(s, \omega_p^{2m}) & = & \text{Kubota-Leopoldt } L\text{-function corresponding to } \omega_p^{2m} \\ \lambda_0 & = & \text{the constant coefficient of} \\ & & \text{p-adic K-L zeta function } \xi_p(s) = \sum_{n\geq -1} \lambda_n (s-1)^n \end{array}$$

THEOREM (DWORK) : $\Gamma_p^{(0)}(T)$ converges exactly for $|T| < |p|^{\frac{1}{p(p-1)}}$.

By using the *family* of finite difference equations
$$\{\Gamma_p(T+p^n) = P_n(T)\Gamma(T)\}_{n\geq 1}, \text{ where } P_n(T)\in\mathbb{Z}[T]$$

- we analyze every single equation in detail,
- we find "by confluence" a differential equation :

THEOREM : The function $\Gamma_p^{(0)}(T)$ is the Taylor solution at $T=0$ of a differential equation
$$Y' = g_0(T)Y$$
satisfying :

- $g_0(T) = \sum_{n\geq 0} a_n T^n$ is analytic on $\mathrm{D}^-(0,1)$, $g_0(T)\in\mathcal{A}_{\mathbb{Q}_p}(0,1)$;
- $Rad(Y_0(x,y),|\cdot|_\rho) = \begin{cases} \omega|p|^{1/p} & \text{if} \quad 0 \leq \rho \leq \omega \\ \frac{\omega|p|}{\rho^{p-1}} & \text{if} \quad \omega \leq \rho \leq \omega^{\frac{1}{p-1}} \\ \frac{\omega|p|^{n+1}}{\rho^{p^n(p-1)}} & \text{if} \quad \omega^{\frac{1}{p^{n-1}(p-1)}} \leq \rho \leq \omega^{\frac{1}{p^n(p-1)}}, \quad n\geq 1 \end{cases}$

  where $\omega := |p|^{1/(p-1)}$;

The radius of convergence of this equation is intimately (directly) related to the Newton polygon of $g_0(T)$, so we have the following corollary

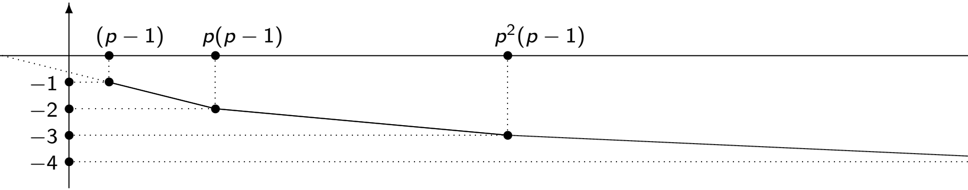
Corollary :

The wedges of the Newton polygon of $g_0(T)$ having horizontal coordinate greater than $p-1$ are the points $\{(p^n(p-1), -n-1)\}_{n\geq 0}$. In particular

1. $v_p(a_{p^n(p-1)}) = -n-1$, for all $n \geq 0$,
2. Moreover for all $k \geq 0$ (and $n \geq 0$) one has

$$v_p(a_k) \geq \begin{cases} 0 & \text{if} \quad 0 \leq k \leq p-2\,, \\ -n-1 & \text{if} \quad p^n(p-1) \leq k \leq p^{n+1}(p-1)\,. \end{cases}$$

as illustrated in the following picture :



Values at positive integers of some K.-L.'s L-functions

Taking the log-derivative of the Morita's formula

$$\log(\Gamma_p^{(0)}(T)) = \lambda_0 T + \sum_{m\geq 1} L_p(1+2m, \omega_p^{2m}) \frac{T^{2m+1}}{2m+1}$$

one has

$$\frac{\Gamma_p^{(0)}(T)'}{\Gamma_p^{(0)}(T)} = g_0(T) = \lambda_0 + \sum_{m\geq 1} L_p(1+2m, \omega_p^{2m})\, T^{2m}$$

Corollary : For all $m \geq 1$ one has

$$v_p(\, L_p(1+p^m(p-1), \omega_p^{p^m(p-1)})\,) = v_p(\, \zeta_p(1+p^m(p-1))\,) = -m-1\,,$$

and moreover

$$v_p(L_p(1+2m, \omega_p^{2m})) \geq \begin{cases} 0 & \text{if} \quad 0 \leq 2m \leq (p-1) \\ -n-1 & \text{if} \quad p^n(p-1) \leq 2m \leq p^{n+1}(p-1) \end{cases}$$

By using the fact that $\Gamma_p^{(0)}(T)$ is simultaneously solution of the family

$$\{\Gamma_p^{(0)}(x+p^n) = P_n(x)\Gamma_p^{(0)}(x)\}_{n\geq 1}$$

and at the same time of the differential equation

$$\Gamma_p^{(0)}(T)' = g_0(T)\Gamma_p^{(0)}(T)$$

one has congruences between the coefficients of the power series $\Gamma_p^{(0)}$

For $\ell, k \geq 1$ we set

$$S_\ell(k) := \sum_{i=1,(i,p)=1}^{k-1} 1/i^\ell$$

Corollary :

- One has

$$\log(\Gamma_p(p^n)) = p^n\lambda_0 + \sum_{m\geq 1} p^{n(1+2m)}\frac{L_p(1+2m,\omega_p^{2m})}{1+2m}\,;$$

- for $\ell = 1$ one has

$$S_1(p^n) \ := \ \sum_{i=1,(i,p)=1}^{p^n-1}\frac{1}{i} \ = \ \sum_{m\geq 1} p^{2mn}\cdot L_p(1+2m,\omega_p^{2m})\,;$$

- for $\ell \geq 2$ one has

$$\frac{(-1)^{\ell-1}}{\ell}\cdot S_\ell(p^n) = \sum_{m\geq \ell/2}\binom{1+2m}{\ell}p^{n(1+2m-\ell)}\cdot\frac{L_p(1+2m,\omega_p^{2m})}{1+2m}.$$