Arithmetic and Differential Swan conductors. The Rank one case via $\boldsymbol{\pi}$-exponentials.

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SUMMARY

- **Arithmetic Swan conductor**
  - **Kato's definition in the non perfect case**
- **Differential Swan conductor**
  - **Kedlaya's definition in the non perfect case**
- **Co-monomials and explicit description of**

$$\mathrm{H}^1(k(\!(t)\!), \mathbb{Q}_p/\mathbb{Z}_p)$$

- **$\boldsymbol{\pi}$−exponentials as solutions of Differential equations**
  - **Explicit computation of the monodromy functor in the rank one case**

$Proof:$
- **Decomposition in *pure* co-monomials**
- **Radius of the differential equation attached to a pure co-monomial**

NOTATIONS

K :=finite extension of $\mathbb{Q}_p$, $\mathbb{F}_q$:=residue field of K,

k :=field containing $\mathbb{F}_q$, E :=c.d.v.f. with residue field k,

$\mathcal{O}_{L_0}$:=a Cohen ring of k, $\mathcal{O}_{\mathcal{E}_{L_0}}$:=a Cohen ring of E

$\mathcal{O}_L := \mathcal{O}_{L_0} \otimes_{\mathbf{W}(\mathbb{F}_q)} \mathcal{O}_K$, $\mathcal{O}_{\mathcal{E}_L} := \mathcal{O}_L \otimes_{\mathcal{O}_{L_0}} \mathcal{O}_{\mathcal{E}_{L_0}}$

$$\begin{array}{ccccccc}
 & & \mathcal{O}_K = \mathcal{O} & \longrightarrow & \mathcal{O}_L & \longrightarrow & \mathcal{O}_{\mathcal{E}_L} \\
 & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{Z}_p & \longrightarrow & \mathbf{W}(\mathbb{F}_q) & \longrightarrow & \mathcal{O}_{L_0} & \longrightarrow & \mathcal{O}_{\mathcal{E}_{L_0}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{F}_p & \longrightarrow & \mathbb{F}_q & \longrightarrow & k & \longrightarrow & \mathrm{E} \cong k(\!(t)\!)
\end{array} .$$

- $\mathrm{E} \cong k(\!(t)\!)$ (once we have chosen an uniformizer element t)
- $\mathcal{O}_{\mathcal{E}_{L_{(0)}}} \cong \{\sum a_i T^i \mid \lim_{i \to -\infty} a_i = 0, \forall i\ |a_i| \leq 1, a_i \in L_{(0)}\}$

(Amice-Fontaine ring)

NOTATIONS

- $\mathcal{A}_L(I) = \{\sum_{-\infty}^{+\infty} a_i T^i,\ \text{s.t. } \lim_{i\to\pm\infty} |a_i|\rho^i = 0, \forall\ \rho \in I\}$
  (= Analytic Funct. on the annulus $|T| \in I$)
- $\mathcal{R}_L := \bigcup_{\varepsilon>0} \mathcal{A}_L(]1-\varepsilon, 1[)$ (= Robba ring)
- $\mathcal{E}^{\dagger}_{L,T} = \mathcal{E}^{\dagger}_L := \mathcal{R}_L \cap \mathcal{E}_L$ (= Bounded Robba ring).
- φ :=Lifting of the Frobenius $x \mapsto x^q$
- $\mathrm{G_E} := \mathrm{Gal}(\mathrm{E}^{\mathrm{sep}}/\mathrm{E})$, $\mathcal{I}_{\mathrm{E}}$:=inertia, $\mathcal{P}_{\mathrm{E}}$:=wild inertia
- $\mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E}) = \{\alpha : \mathrm{G_E} \to GL(V) \mid V =$ finite free $\mathcal{O}_K-$module,
  such that $\alpha(\mathcal{I}_{\mathrm{E}})$ is finite$\}$

FONTAINE's EQUIVALENCE: THE PERFECT CASE

- Assume k =perfect.
- (Fontaine-Tsuzuki's) classical case (1998):

$$\boxed{\mathbf{D}^{\dagger} : \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E}) \xrightarrow{\ \sim\ } (\varphi,\nabla) - \mathrm{Mod}(\mathcal{O}_{\mathcal{E}^{\dagger}_L}/\mathcal{O}_L)}$$

where:

- $(\varphi,\nabla) - \mathrm{Mod}(\mathcal{O}_{\mathcal{E}^{\dagger}_L}/\mathcal{O}_L) := \{(\mathrm{D},\varphi^{\mathrm{D}},\nabla) \mid \mathrm{D} :=$ finite free$/\mathcal{O}_{\mathcal{E}^{\dagger}_L}$,
$\varphi^{\mathrm{D}} : \mathrm{D} \to \mathrm{D}$ is φ-semilinear
$\nabla : \mathrm{D} \to \mathrm{D} \otimes \Omega^1_{\mathcal{O}_{\mathcal{E}^{\dagger}_L}/\mathcal{O}_L}$ connection$\}$

NOTE: If t =uniformizer of $\mathrm{E} \cong k((t))$, T =lifting of t in $\mathcal{O}_{\mathcal{E}^{\dagger}_L}$, then

$$\Omega^1_{\mathcal{O}_{\mathcal{E}^{\dagger}_L}/\mathcal{O}_L} \cong \mathcal{O}_{\mathcal{E}^{\dagger}_L} \cdot dT\ , \qquad \mathrm{D} \otimes \Omega^1_{\mathcal{O}_{\mathcal{E}^{\dagger}_L}/\mathcal{O}_L} \cong \mathrm{D} \cdot dT$$

the data of $\nabla : \mathrm{D} \to \mathrm{D} \otimes \Omega^1_{\mathcal{O}_{\mathcal{E}^{\dagger}_L}/\mathcal{O}_L}$ is equivalent to a connection

$$\nabla_T : \mathrm{D} \to \mathrm{D}\ .$$

FONTAINE's EQUIVALENCE REVISITED: THE NON PERFECT CASE

- Assume k =arbitrary.
- Kedlaya's generalization (December 2006):

$$\boxed{\mathbf{D}^{\dagger} : \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E}) \xrightarrow{\sim} (\varphi,\nabla)-\mathrm{Mod}(\mathcal{O}_{\mathcal{E}^{\dagger}_L}/\mathcal{O}_K)}$$

where:

- $(\varphi,\nabla)-\mathrm{Mod}(\mathcal{O}_{\mathcal{E}^{\dagger}_L}) := \{(\mathrm{D},\varphi^{\mathrm{D}},\nabla) \mid$ with $\mathrm{D} :=$ finite free$/\mathcal{O}_{\mathcal{E}^{\dagger}_L}$,
  $\varphi^{\mathrm{D}} : \mathrm{D} \to \mathrm{D}$ is φ-semilinear
  $\nabla : \mathrm{D} \to \mathrm{D} \otimes \Omega^1_{\mathcal{O}_{\mathcal{E}^{\dagger}_L}/\mathcal{O}_K}$ integrable$\}$

- If t =uniformizer of $\mathrm{E} \cong k((t))$, if $\{\bar{u}_1,\ldots,\bar{u}_r\} = p$-basis of k,

$T, u_1,\ldots,u_r \in \mathcal{O}_{\mathcal{E}^{\dagger}_L}$ are lifting of $t,\bar{u}_1,\ldots,\bar{u}_r$, then

$$\Omega^1_{\mathcal{O}_{\mathcal{E}^{\dagger}_L}/\mathcal{O}_K} \cong \mathcal{O}_{\mathcal{E}^{\dagger}_L}\cdot dT \oplus \left(\oplus_{i=1,\ldots,r}\mathcal{O}_{\mathcal{E}^{\dagger}_L}\cdot du_i\right).$$

The data of $\nabla : \mathrm{D} \to \mathrm{D} \otimes \Omega^1_{\mathcal{O}_{\mathcal{E}^\dagger_L}/\mathcal{O}_K}$ is equivalent to a family of connections

$$\begin{aligned} \nabla_T &: \mathrm{D} \to \mathrm{D} \\ \nabla_{u_1} &: \mathrm{D} \to \mathrm{D} \\ \nabla_{u_2} &: \mathrm{D} \to \mathrm{D} \\ &\cdots\cdots\cdots \\ \nabla_{u_r} &: \mathrm{D} \to \mathrm{D} \end{aligned}$$

commuting with φ, and commuting between them.

• NOTE: If k is perfect, then $\Omega^1_{\mathcal{O}_{\mathcal{E}^\dagger_L}/\mathcal{O}_L} = \Omega^1_{\mathcal{O}_{\mathcal{E}^\dagger_L}/\mathcal{O}_K}$, hence this theory refines that of Fontaine-Tsuzuki.

• We are interested only to these differential equations. We set

$$\mathrm{MCF}(\mathcal{O}_{\mathcal{E}^\dagger_L}/\mathcal{O}_K) = \{(\mathrm{D},\nabla) \mid \nabla : \mathrm{D} \to \mathrm{D} \otimes \Omega^1_{\mathcal{O}_{\mathcal{E}^\dagger_L}/\mathcal{O}_K}, + \text{commutations}, \text{ with an (unspecified) Frobenius } \varphi^{\mathrm{D}} : \varphi^*(\mathrm{D}) \to \mathrm{D}\}$$

Perfect case VS non perfect case

Assume k non necessarily perfect. We notice that

$$\begin{array}{ccc} \mathrm{Gal}(k^{\mathrm{perf}}(\!(t)\!)^{\mathrm{sep}}/k^{\mathrm{perf}}(\!(t)\!)) & \xrightarrow[can]{\sim} & \mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!)) \\ \cup & & \cup \\ \mathcal{I}_{k^{\mathrm{perf}}(\!(t)\!)} & \xrightarrow[can]{\sim} & \mathcal{I}_{k(\!(t)\!)} \end{array}$$

hence:

$$\begin{array}{ccc} \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}\Big(\mathrm{Gal}(k^{\mathrm{perf}}(\!(t)\!)^{\mathrm{sep}}/k^{\mathrm{perf}}(\!(t)\!))\Big) & \xleftarrow[\sim]{can} & \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}\Big(\mathrm{Gal}(k(\!(t)\!)^{\mathrm{sep}}/k(\!(t)\!))\Big) \\ \wr\downarrow \mathbf{D}^{\dagger} & \odot & \wr\downarrow \mathbf{D}^{\dagger} \\ (\varphi,\nabla)-\mathrm{Mod}(\mathcal{O}_{\mathcal{E}^{\dagger}_{L^{\mathrm{perf}}}}/\mathcal{O}_{L^{\mathrm{perf}}}) & \xleftarrow[\sim]{can} & (\varphi,\nabla)-\mathrm{Mod}(\mathcal{O}_{\mathcal{E}^{\dagger}_{L}}/\mathcal{O}_{K}) \end{array}$$

where the last horizontal functor is given by

$$(\mathrm{D},\varphi^{\mathrm{D}},\nabla_T,\{\nabla_{u_i}\}_i) \longmapsto (\mathrm{D}\otimes_{\mathcal{O}_{\mathcal{E}^{\dagger}_L}}\mathcal{O}_{\mathcal{E}^{\dagger}_{L^{\mathrm{perf}}}},\varphi^{\mathrm{D}}\otimes\varphi,\nabla_T\otimes 1)$$

List of main results

$$\mathbf{D}^{\dagger} : \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E}) \quad \xrightarrow{\sim} \quad (\varphi, \nabla) - \mathrm{Mod}(\mathcal{O}_{\mathcal{E}^{\dagger}_L}/\mathcal{O}_K)$$

- *On the left hand side:* One has a complete description of

$$\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}/\mathbb{Z})_{p-\mathrm{tor}} = \mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p) \qquad \text{(Pulita 2006).}$$

- *On the right hand side:* We obtain a complete description of the group $\mathrm{Pic}^{\mathrm{Frob}}(\mathcal{O}_{\mathcal{E}^{\dagger}_L})$ =group, under tensor product, of isomorphism classes of rank one objects in $\mathrm{MCF}(\mathcal{O}_{\mathcal{E}^{\dagger}_L}/\mathcal{O}_K)$.

- After choosing an identification $(\mathbb{Q}_p/\mathbb{Z}_p) \supset (\mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\sim} \boldsymbol{\mu}_{p^\infty}(\mathcal{O}_K)$ we make explicit the isomorphism

$$\mathrm{Hom}^{\mathrm{fin}}(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p) \supset \mathrm{Hom}^{\mathrm{fin}}(\mathrm{G_E}, (\mathcal{O}_K)^{\times})_{p\text{-tor}} \xrightarrow{\sim} \mathrm{Pic}^{\mathrm{Frob}}(\mathcal{O}_{\mathcal{E}^{\dagger}_L})_{p\text{-tor}},$$

induced by the functor $\mathbf{D}^{\dagger}$ (Pulita 2006).

- Understanding of the *Kato's filtration* on the L.H.S., and of the *Kedlaya's Irregularity* on the R.H.S. proof of the equality

$$\text{(Kato)} \quad \mathrm{Swan}_{\mathrm{Arithm}} \; = \; \mathrm{Swan}_{\mathrm{Diff}} \quad \text{(Kedlaya).}$$

Abbes-Saito's filtration in rank one case: Kato's filtration

The Artin-Schreier-Witt theory describes $\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p)$:

$$\begin{array}{ccccccccc}
0 \to \mathbb{Z}/p^{m+1}\mathbb{Z} & \longrightarrow & \mathbf{W}_m(\mathrm{E}) & \xrightarrow{\bar{\mathrm{F}}-1} & \mathbf{W}_m(\mathrm{E}) & \xrightarrow{\delta} & \mathrm{Hom}(\mathrm{G_E}, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to 0 \\
\downarrow \imath & \odot & \downarrow \mathrm{V} & \odot & \downarrow \mathrm{V} & \odot & \downarrow \jmath \\
0 \to \mathbb{Z}/p^{m+2}\mathbb{Z} & \longrightarrow & \mathbf{W}_{m+1}(\mathrm{E}) & \xrightarrow{\bar{\mathrm{F}}-1} & \mathbf{W}_{m+1}(\mathrm{E}) & \xrightarrow{\delta} & \mathrm{Hom}(\mathrm{G_E}, \mathbb{Z}/p^{m+2}\mathbb{Z}) \to 0
\end{array}$$

$$\downarrow \ \downarrow \ \downarrow$$

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathbf{CW}(\mathrm{E}) \xrightarrow{\bar{\mathrm{F}}-1} \mathbf{CW}(\mathrm{E}) \xrightarrow{\delta} \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p) \to 0$$

where

$$\mathbf{CW}(\mathrm{E}) = \varinjlim_m \Big(\mathbf{W}_m(\mathrm{E}) \xrightarrow{\mathrm{V}} \mathbf{W}_{m+1}(\mathrm{E}) \xrightarrow{\mathrm{V}} \cdots \Big),$$

and where $\mathrm{Hom}^{\mathrm{cont}}$ means that a character $\alpha : \mathrm{G_E} \to \mathbb{Q}_p/\mathbb{Z}_p$ factorizes by a finite quotient of $\mathrm{G_E}$.

NOTE: Elements in $\mathbf{CW}(\mathrm{E})$ are $(\cdots, 0, 0, \overline{f}_0, \ldots, \overline{f}_m)$, $\overline{f}_i \in \mathrm{E}$.

Abbes-Saito's filtration in rank one case: Kato's filtration

- K.Kato (1989) defined a filtration on $\mathrm{H}^1(\mathrm{G}_\mathrm{E},\mathbb{Q}/\mathbb{Z})$ in 3 steps:

1) The setting

$$v(\cdots,0,0,\overline{f}_0,\ldots,\overline{f}_m) := \min(v_t(\overline{f}_0)/p^m, v_t(\overline{f}_1)/p^{m-1},\cdots,v_t(\overline{f}_m))$$

defines a valuation on $\mathbf{CW}(\mathrm{E})$. Define a filtration on $\mathbf{CW}(\mathrm{E})$ as:

$$\mathrm{Fil}_d(\mathbf{CW}(\mathrm{E})) = \{\boldsymbol{c} := (\cdots,0,0,\overline{f}_0,\ldots,\overline{f}_m) \mid v(\boldsymbol{c}) \geq -d\}.$$

2) Thank to the A-S-W sequence

$$0 \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbf{CW}(\mathrm{E}) \xrightarrow{\bar{\mathrm{F}}-1} \mathbf{CW}(\mathrm{E}) \xrightarrow{\delta} \mathrm{H}^1(\mathrm{G}_\mathrm{E},\mathbb{Q}_p/\mathbb{Z}_p) \to 0.$$

we define: $\mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E},\mathbb{Q}_p/\mathbb{Z}_p)) := \delta(\mathrm{Fil}_d(\mathbf{CW}(\mathrm{E})))$.

3) $$\mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\mathrm{E},\mathbb{Q}/\mathbb{Z}) = \oplus_{\ell=\mathrm{prime}} \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_\mathrm{E},\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

$$\Longrightarrow \mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E},\mathbb{Q}/\mathbb{Z})) := \text{Inverse image of } \mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E},\mathbb{Q}_p/\mathbb{Z}_p)).$$

4) $$\boxed{\mathrm{Swan}_{\mathrm{Arithm}}(\alpha) = \min\Big\{d \geq 0 \mid \alpha \in \mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E},\mathbb{Q}/\mathbb{Z}))\Big\}}$$

Description of the Kato's filtration of $\mathrm{H}^1(\mathrm{G_E}, \mathbb{Q}_p/\mathbb{Z}_p)$

- Once chosen a uniformizer element $t \in \mathrm{E}$, one has $\mathrm{E} \cong k(\!(t)\!)$ and

$$\mathbf{CW}(k(\!(t)\!)) = \mathbf{CW}(t^{-1}k[t^{-1}]) \oplus \mathbf{CW}(k) \oplus \mathbf{CW}(tk[\![t]\!])$$

- One has

$$\begin{aligned}
\mathrm{H}^1(\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p) \quad &\cong \quad \frac{\mathbf{CW}(\mathrm{E})}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(\mathrm{E}))} \\
&= \quad \frac{\mathbf{CW}(t^{-1}k[t^{-1}])}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(t^{-1}k[t^{-1}]))} \oplus \overbrace{\frac{\mathbf{CW}(k)}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(k))}}^{\mathrm{H}^1(\mathrm{G}_k, \mathbb{Q}_p/\mathbb{Z}_p)} .
\end{aligned}$$

$$\begin{cases}
\frac{\mathbf{CW}(t^{-1}k[t^{-1}])}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(t^{-1}k[t^{-1}]))} = \text{Pontriagyn dual of } \mathcal{P}_{\mathrm{G}_{\mathrm{E}}^{\mathrm{ab}}} \\
\frac{\mathbf{CW}(k)}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(k))} = \text{Pontriagyn dual of } (\mathrm{G}_k^{\mathrm{ab}})_{p\text{-tor}} \\
\qquad = \mathrm{H}^1(\mathrm{G}_k, \mathbb{Q}_p/\mathbb{Z}_p)
\end{cases}$$

Description of the Kato's filtration of $\mathbf{CW}(\mathrm{E})$

DEFINITION (PULITA 2006): A *co-monomial of degree $-d$* is a co-vector of the form

$$\boldsymbol{\lambda} t^{-d} := (\cdots, 0, 0, \lambda_0 t^{-n}, \lambda_1 t^{-np}, \ldots, \lambda_m t^{-np^m}) \in \mathbf{CW}(k(\!(t)\!))$$

where $d = np^m$, $(n,p) = 1$, $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(k)$. We call $\mathbf{CW}^{(-d)}(k)$ the sub-group of $\mathbf{CW}(\mathrm{E})$ formed by such elements.

PROPOSITION: $\mathbf{CW}(\mathrm{E})$ together with the Kato's filtration is graduated:

$$\mathbf{CW}(\mathrm{E}) := \oplus_{d \geq 0} \mathrm{Gr}_d(\mathbf{CW}(\mathrm{E})) \ .$$

Moreover:

$$\mathrm{Gr}_d(\mathbf{CW}(\mathrm{E})) = \begin{cases} \mathbf{CW}(k[\![t]\!]) & \text{if} \quad d = 0 \ , \\ \\ \mathbf{CW}^{(-d)}(k) & \text{if} \quad d > 0 \ . \end{cases}$$

NOTE: For all char.p ring R (not necessarily with unit el.t) one has

$$\frac{\mathbf{CW}(R)}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(R))} = \varinjlim\Big(\mathbf{CW}(R) \xrightarrow{\bar{\mathrm{F}}} \mathbf{CW}(R) \xrightarrow{\bar{\mathrm{F}}} \cdots\Big)$$

• *Hence we are interested to the action of* $\bar{\mathrm{F}}$. $\mathrm{Gr}_0(\mathbf{CW}(\mathrm{E}))$ is a sub-$\bar{\mathrm{F}}$-module, while

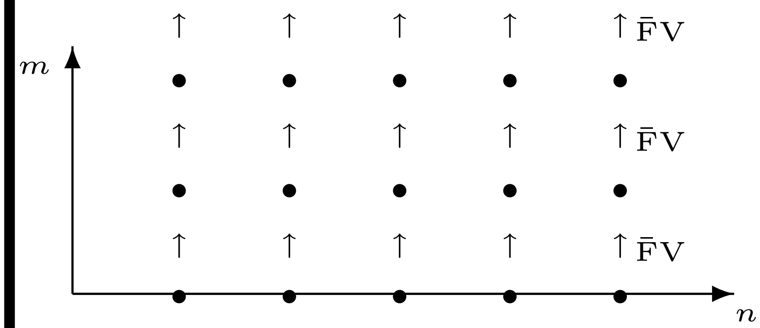
$$\bar{\mathrm{F}}(\mathbf{CW}^{(-d)}(k)) \;\subset\; \mathbf{CW}^{(-pd)}(k),$$

in fact

$$(\cdots,0,0,\lambda_0 t^{-n},\ldots,\lambda_m t^{-d}) \overset{\bar{\mathrm{F}}}{\longmapsto} (\cdots,0,0,\lambda_0^p t^{-pn},\ldots,\lambda_m^p t^{-pd})$$

• If $d = np^m$, $(n,p)=1$, one has a isomorphism:

$$\begin{array}{ccc} \mathbf{CW}^{(-d)}(k) & \xrightarrow{\sim} & \mathbf{W}_m(k) \\ \downarrow \bar{\mathrm{F}} & & \downarrow \mathrm{V}\bar{\mathrm{F}}=p \\ \mathbf{CW}^{(-pd)}(k) & \xrightarrow{\sim} & \mathbf{W}_{m+1}(k) \end{array} \qquad \begin{array}{ccc} (\cdots,0,0,\lambda_0 t^{-n},\ldots,\lambda_m t^{-d}) & \longrightarrow & (\lambda_0,\ldots,\lambda_m) \\ \downarrow \bar{\mathrm{F}} & & \downarrow \cdot p \\ (\cdots,0,0,\lambda_0^p t^{-pn},\ldots,\lambda_m^p t^{-pd}) & \longrightarrow & (0,\lambda_0^p,\ldots,\lambda_m^p) \end{array}$$



$d = np^m$

$$\begin{array}{ccc} \mathbf{CW}^{(-np^{m+1})}(k) & \xrightarrow{\sim} & \mathbf{W}_{m+1}(k) \\ \uparrow{\scriptstyle \bar{\mathrm{F}}} & & \uparrow{\scriptstyle \mathrm{V}\bar{\mathrm{F}}=p} \\ \mathbf{CW}^{(-np^{m})}(k) & \xrightarrow{\sim} & \mathbf{W}_{m}(k) \end{array}$$

Hence for all $(n,p) = 1$, $n > 0$ one has:

$$\varinjlim_m \Big(\mathbf{CW}^{(-np^m)}(k) \xrightarrow{\bar{\mathrm{F}}} \mathbf{CW}^{(-np^{m+1})}(k) \xrightarrow{\bar{\mathrm{F}}} \cdots \Big) \ = $$

$$= \ \varinjlim_m \Big(\mathbf{W}_m(k) \xrightarrow{p} \mathbf{W}_{m+1}(k) \xrightarrow{p} \cdots \Big)$$

<u>DEFINITION:</u> We set

$$\widetilde{\mathbf{CW}}(k) = \varinjlim_{m} \Big(\mathbf{W}_m(k) \xrightarrow{p} \mathbf{W}_{m+1}(k) \xrightarrow{p} \cdots \Big)$$

It has a natural filtration:

$$\mathrm{Fil}_m(\widetilde{\mathbf{CW}}(k)) := \mathbf{W}_m(k) \ \subset \ \widetilde{\mathbf{CW}}(k)$$

Theorem: Let $\mathrm{J}_p := \{n > 0, (n,p) = 1\}$. Then

- $$\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p) \cong \widetilde{\mathbf{CW}}(k)^{(\mathrm{J}_p)} \oplus \mathrm{H}^1(\mathrm{G}_k, \mathbb{Q}_p/\mathbb{Z}_p)$$

where $\widetilde{\mathbf{CW}}(k)^{(\mathrm{J}_p)}$ =direct sum of copies of $\widetilde{\mathbf{CW}}(k)$, indexed by J_p.

- Recall that $\mathrm{Fil}_d(\mathrm{H}^1) = \delta(\mathrm{Fil}_d(\mathbf{CW}(\mathrm{E}))$. For all $d \geq 0$, one has ([a])

$$\mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p)) = \oplus_{n\in \mathrm{J}_p} \Big(\mathrm{Fil}_{m_{n,d}}(\widetilde{\mathbf{CW}}(k))\Big) \oplus \mathrm{H}^1(\mathrm{G}_k, \mathbb{Q}_p/\mathbb{Z}_p) \ ,$$

$$\mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p)) = \begin{cases} \mathbf{W}_{v_p(d)}(k)/p\mathbf{W}_{v_p(d)}(k) & \text{if} \quad d > 0 \ , \\ \\ \mathrm{H}^1(\mathrm{G}_k, \mathbb{Q}_p/\mathbb{Z}_p) & \text{if} \quad d = 0 \ , \end{cases}$$

where $m_{n,d} := \max\{m \geq 0 \mid np^m \leq d\}$, and $v_p(d)$ is the p-adic valuation of d.

[a] Recall that $\mathrm{Fil}_m(\widetilde{\mathbf{CW}}(k)) \cong \mathbf{W}_m(k)$.

The epimorphism $\mathrm{Proj}_d : \mathrm{Gr}_d(\mathbf{CW}(\mathrm{E})) \to \mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p))$ corresponds via the isomorphism

$$\mathrm{Gr}_d(\mathbf{CW}(\mathrm{E})) = \mathbf{CW}^{(-d)}(k) \xrightarrow{\sim} \mathbf{W}_{v_p(d)}(k)$$

to the following:

$$\mathrm{Proj}_d = \begin{cases} \mathbf{W}_{v_p(d)}(k) \longrightarrow \mathbf{W}_{v_p(d)}(k)/p\mathbf{W}_{v_p(d)}(k) & \text{if} \quad d > 0\,, \\ \\ \mathbf{CW}(k[[t]]) \underset{t \mapsto 0}{\longrightarrow} \dfrac{\mathbf{CW}(k)}{(\bar{\mathrm{F}}-1)(\mathbf{CW}(k))} & \text{if} \quad d = 0\,. \end{cases}$$

<u>Corollary:</u> One has

$$\mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}/\mathbb{Z})) = \mathrm{Fil}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p)) \oplus \Big(\oplus_{\ell \neq p} \mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \Big)\,,$$

and

$$\mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}/\mathbb{Z})) = \begin{cases} \mathrm{Gr}_d(\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p)) & \text{if} \quad d > 0\,, \\ \\ \mathrm{Fil}_0(\mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_p/\mathbb{Z}_p)) \oplus \Big(\oplus_{\ell \neq p} \mathrm{H}^1(\mathrm{G}_\mathrm{E}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \Big) & \text{if} \quad d = 0\,. \end{cases}$$

Genesis of the irregularity: Formal irregularity

Differential modules over $L((T))$ are cyclic (Cyclic vector theorem) i.e. defined by an operator

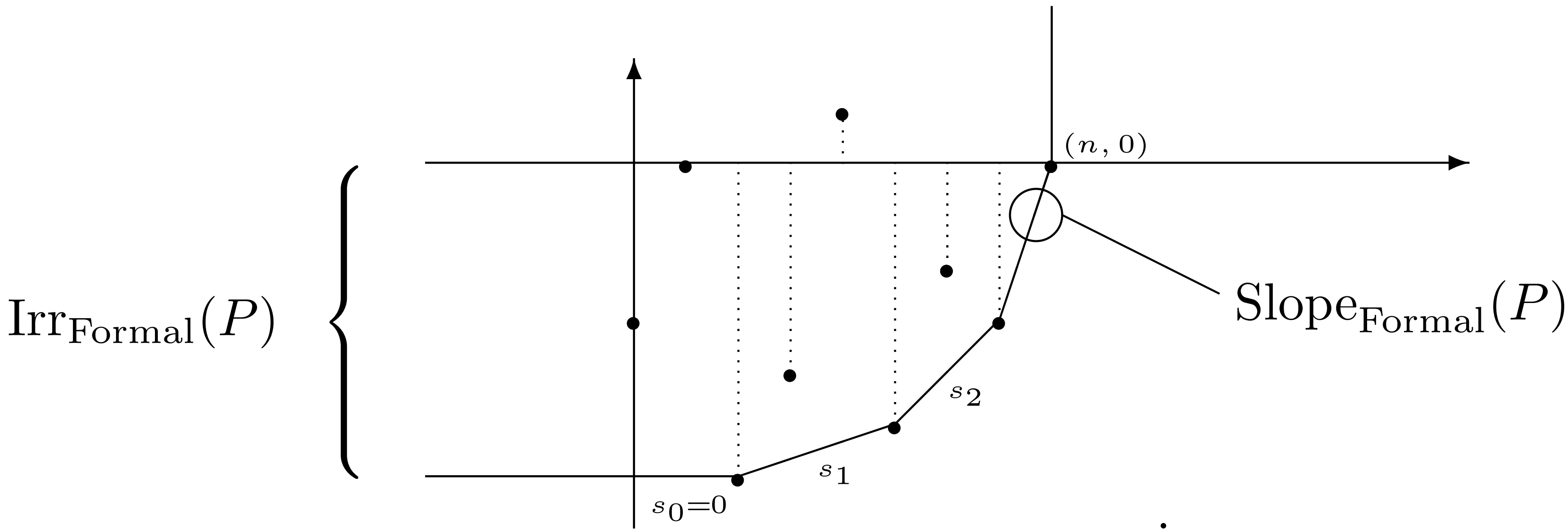
$$P(T, \frac{d}{dT}) \;:=\; \sum_{k=0}^{n} g_k(T)(\frac{d}{dT})^k \;,$$

with $g_k(T) \in L((T))$, $g_n(T) = 1$. The *Formal Irregularity*, and the *Formal Slope* of P are defined as (Malgrange)

$$\mathrm{Irr}_{\mathrm{Formal}}(P) \;:=\; \max_{0\leq k\leq m}\{k - v_T(g_k)\} - (n - v_T(g_n)) \;,$$

$$\mathrm{Slope}_{\mathrm{Formal}}(P) \;=\; \max\Big(0\;,\; \max_{k=0,\dots,n}\Big(\frac{v_T(g_n) - v_T(g_k)}{n-k} - 1\Big)\Big)\;,$$

The *Formal Newton polygon* of P is the convex hull in $\mathbb{R}^2$ of the set $\{(\;k\;,\;(v_T(g_k) - k) - (v_T(g_n) - n)\;)\}_{k=0,\dots,n}$ together with the extra points $\{(-\infty, 0)\}$ and $\{(0, +\infty)\}$.



- *Formal break decomposition* of $\mathrm{M} = L(\!(T)\!)[\frac{d}{dT}]/L(\!(T)\!)[\frac{d}{dT}]\cdot P$:

$$\mathrm{M} = \oplus_{s\geq 0}\mathrm{M}(s)$$

into $L(\!(T)\!)[\frac{d}{dT}]$-submodules. $\mathrm{M}(s)$ =unique sub-module of M whose Newton polygon consists in a single slope s (counted with multiplicity) of the Newton polygon of M.

$$\boxed{\mathrm{Irr}_{\mathrm{formal}}(\mathrm{M}) = \textstyle\sum_{s>0} s\cdot \dim_{L(\!(T)\!)}\mathrm{M}(s)\ .}$$

<u>NOTE:</u> The Formal Slope is <u>***explicitly given***</u> by the coeff. $\{g_k\}_k$!

p-adic Irregularity : the perfect case.

• Let $\nabla_T : \mathrm{M} \to \mathrm{M}$ be a $\mathcal{R}_L$-differential module, defined over some annulus $]1-\varepsilon, 1[$. Let M_ρ be the restriction of M to $[\rho, \rho]$. We set

$$|\nabla_T^n|_\rho := \sup_{m \in \mathrm{M}_\rho} \frac{|\nabla_T^n(m)|_{\mathrm{M}_\rho}}{|m|_{\mathrm{M}_\rho}} ,$$

$$|\nabla_T|_{\mathrm{Sp},\rho} := \limsup_{n\to\infty} |(\nabla_T)^n|_{\mathrm{M}_\rho}^{1/n} ,$$

$$\|(d/dT)^n\|_\rho = \sup_{f \in \mathrm{Frac}(\mathcal{A}_L[\rho,\rho])} \frac{|(d/dT)^n(f)|_\rho}{|f|_\rho} = |n!|\rho^{-1} ,$$

$$\|d/dT\|_{\mathrm{Sp},\rho} := \limsup_{n\to\infty} |(d/dT)^n|_\rho^{1/n} = \omega\cdot\rho^{-1}, \quad \omega := |p|^{\frac{1}{(p-1)}}$$

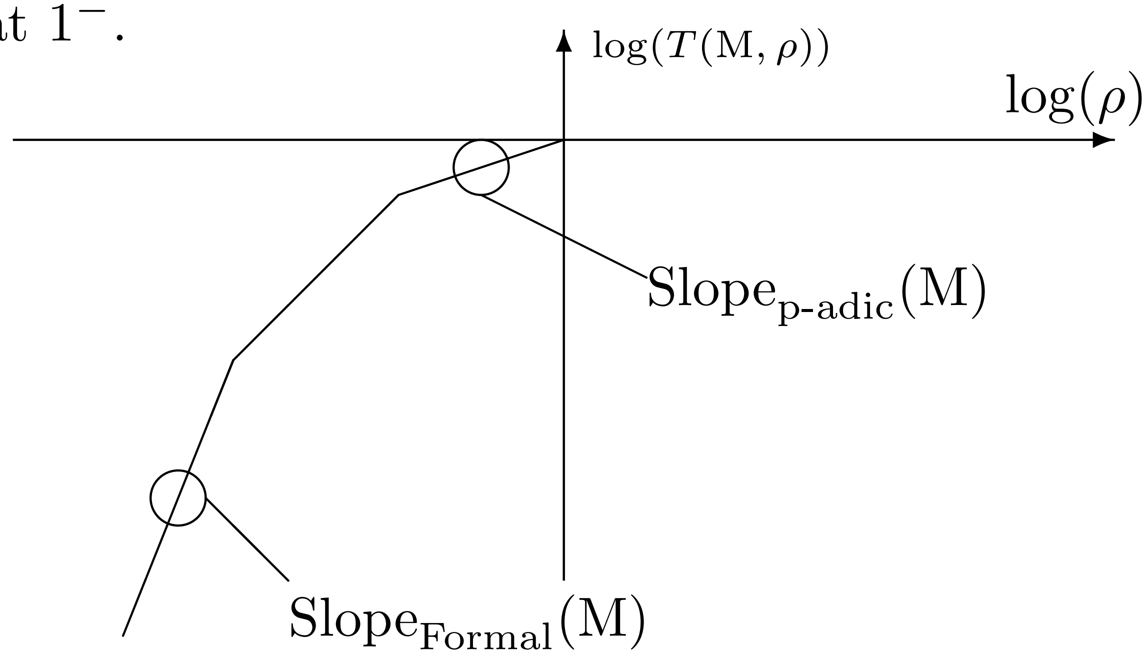
Toric radius of convergence: (Christol-Dwork-Robba $\sim$ 1994)

$$\boxed{T(\mathrm{M},\rho) := \frac{\|d/dT\|_{Sp,\rho}}{|\nabla_T|_{\mathrm{Sp},\rho}}}$$

Geometric interpretation: ("*à la Christol-Mebkhout*") One has $T(\mathrm{M},\rho) = Ray(\mathrm{M},\rho)/\rho$ where $Ray(\mathrm{M},\rho)$ is the radius of convergence of a generic Taylor solution, bounded by ρ, at the generic point t_ρ.

p-adic slope: Link with the formal theory.

DEFINITION (ROBBA): The *p-adic slope* of M is the log-slope of $T(\mathrm{M},\rho)$ at 1^-.



The above definition is motivated by the following:

LEMMA(CHRISTOL-DWORK-ROBBA): If $P(\frac{d}{dT},T) \in (L(\!(t)\!) \cap \mathcal{R}_L)[\frac{d}{dT}]$ then the log-slope at 0^+ of the *Toric radius of convergence* $T(\mathrm{M},\rho)$ is equal to $\mathrm{Slope}_{\mathrm{Formal}}(\mathrm{M})$.

Break decomposition

THEOREM (CHRISTOL-MEBKHOUT-2000): One has a break decomposition:

$$\mathrm{M} = \oplus_{s\geq 0}\mathrm{M}(s) \ .$$

$\mathrm{M}(s)$ =greatest submodule of M of "*pure*" slope s. That is there exists $\varepsilon > 0$ such that

1. For all $\rho \in]1-\varepsilon, 1[$, $\mathrm{M}(s)$ is (the biggest submodule of M) trivialized by $\mathcal{A}_K(t_\rho, \rho^{s+1})$, (where t_ρ is the Berkovich point corresponding to $|.|_\rho$);
2. For all $\rho \in]1-\varepsilon, 1[$, and for all $s' < s$, $\mathrm{M}(s)$ has no solutions in $\mathcal{A}_K(t_\rho, \rho^{s'+1})$.

One has:

$$\mathrm{Irr}_{p-adic}(\mathrm{M}) := \sum_{s>0} s \cdot \dim_{\mathcal{R}_L} \mathrm{M}(s) \ .$$

Kedlaya's refined definition in the non perfect case

DEFINITION (KEDLAYA DEC.2006): Let $(\mathrm{M}, \nabla_T, \nabla_{u_1}, \ldots, \nabla_{u_r}) \in \mathrm{MCF}(\mathcal{O}_{\mathcal{E}_L^\dagger}/\mathcal{O}_K)$. We define the *Toric radius of convergence* of M as:

$$T(\mathrm{M}, \rho) := \min\Big(\frac{\|d/dT\|_{Sp,\rho}}{|\nabla_T|_{\mathrm{Sp},\rho}} \;,\; \frac{\|d/du_1\|_{Sp,\rho}}{|\nabla_{u_1}|_{\mathrm{Sp},\rho}} \;,\; \ldots \;,\; \frac{\|d/du_r\|_{Sp,\rho}}{|\nabla_{u_r}|_{\mathrm{Sp},\rho}} \Big)$$

We set

$$\mathrm{Slope}_{\mathrm{p}-adic}(\mathrm{M}) := \log\text{-slope of } T(\mathrm{M}, \rho) \text{ at } 1^-.$$

PROPOSITION (KEDLAYA): Break decomposition holds:

$$\mathrm{M} = \oplus_{s \geq 0} \mathrm{M}(s)$$

$M(s)$ =greatest submodule of M of "*pure*" slope s.

DEFINITION (KEDLAYA): We set

$$\mathrm{Irr}_{\mathrm{Kedlaya}}(\mathrm{M}) := \sum_{s>0} s \cdot \dim_{\mathcal{R}_L} \mathrm{M}(s) \;.$$

Arithmetic and differential Swan conductors

$$\mathbf{D}^\dagger : \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E}) \xrightarrow{\sim} (\varphi,\nabla)-\mathrm{Mod}(\mathcal{O}_{\mathcal{E}^\dagger_L}/\mathcal{O}_K)$$

DEFINITION (KEDLAYA): Let $\mathrm{V} \in \mathrm{Rep}^{\mathrm{fin}}_{\mathcal{O}_K}(\mathrm{G_E})$. Let $\mathrm{M} \in \mathrm{MCF}(\mathcal{R}_L/\mathcal{O}_K)$ be the differential module $\mathbf{D}^\dagger(\mathrm{V})$ together with $\nabla_T, \nabla_{u_1}, \ldots, \nabla_{u_r}$. We set

$$\boxed{\mathrm{Swan}_{\mathrm{Diff}}(\mathrm{V}) := \mathrm{Irr}_{p\text{-adic}}(\mathrm{M})\ .}$$

CONJECTURE (KEDLAYA): One has

$$\boxed{\mathrm{Swan}_{\mathrm{Diff}}(\mathrm{V}) := \mathrm{Swan}_{\mathrm{Arithm}}(\mathrm{V})\ .}$$

- We have proved this conjecture in Rank one case in three steps:

Proof:
- **1)** We compute the functor $\mathbf{D}^\dagger$ in rank one case
- **2)** We reduce the conjecture to the case of a co-monomial
- **3)** We compute the differential Swan conductor in this case

Computation of the functor $\mathbf{D}^\dagger$ in rank one case

- Fix an isomorphism $1 \mapsto \xi_{p^{m+1}} : \mathbb{Z}/p^{m+1}\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_{p^\infty}(\mathcal{O}_K)$. Then:

$$\alpha : \mathrm{G_E} \to \boldsymbol{\mu}_{p^\infty}(\mathcal{O}_K) \qquad \Longleftrightarrow \qquad \alpha : \mathrm{G_E} \to \mathbb{Z}/p^{m+1}\mathbb{Z}\ .$$

- We classify differential equations, and describe $\mathbf{D}^\dagger$ simultaneously.
- The Tame case is easy.
- The wild case is obtained as follows:

$$\begin{array}{ccccc}
 & & \mathbf{W}_m(T^{-1}\mathcal{O}_L[T^{-1}]) & \xrightarrow{[\mathrm{e}_{p^m}(-,1)]} & \mathrm{Pic}^{\mathrm{Frob}}(\mathcal{O}_{\mathcal{E}_L^\dagger}/\mathcal{O}_K)_{p-\mathrm{tor}} \\
 & & \big\downarrow\!\!\downarrow & & \wr \big\uparrow \text{Induced by } \mathbf{D}^\dagger \\
\mathbf{W}_m(t^{-1}k[t^{-1}]) & \xrightarrow{\bar{\mathrm{F}}-1} & \mathbf{W}_m(t^{-1}k[t^{-1}]) & \xrightarrow{\delta} \!\!\!\to & \mathrm{Hom}(\mathcal{P}_{\mathrm{G_E^{ab}}}, \mathbb{Z}/p^{m+1}\mathbb{Z})
\end{array}$$

where:

- $\mathrm{Pic}^{\mathrm{Frob}}(\mathcal{O}_{\mathcal{E}_L^\dagger}/\mathcal{O}_K) :=$ Group, under $\otimes$, of Isom.classes of rk 1 objects.
- For $\boldsymbol{f}^-(T) \in \mathbf{W}_m(T^{-1}\mathcal{O}_L[T^{-1}])$, $[\mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1)]$ is the isom.class of the diff.eq. whose solution is $\mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1)$, that we will define now.

$\longrightarrow \longrightarrow \longrightarrow$

$\boldsymbol{\pi}$−exponentials

- Fix a Lubin-Tate group law $\mathfrak{G}(X,Y)$, with multiplication by p given by $[p]_{\mathfrak{G}}(X) = (pX + \cdots + a_p X^p + \cdots) \in X\mathbb{Z}_p[X]$.
- Let $\boldsymbol{\pi} := (\pi_j)_{j\geq 0}$, $[p]_{\mathfrak{G}}(\pi_0) = 0$, $[p]_{\mathfrak{G}}(\pi_j) = \pi_{j-1}$, $\forall j$, be the generator of the Lubin-Tate group of $\mathfrak{G}$ such that

$$|\pi_m - (\xi_{p^{m+1}} - 1)| < |\pi_m|$$

Definition (Pulita 2006): Let $\boldsymbol{f}^-(T) = (f_0^-(T), \ldots, f_m^-(T)) \in \mathbf{W}_m(T^{-1}\mathcal{O}_L[T^{-1}])$. We set

$$\mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1) := \exp\Big(\pi_m \phi_0^-(T) + \pi_{m-1}\frac{\phi_1^-(T)}{p} + \cdots + \pi_0 \frac{\phi_m^-(T)}{p^m}\Big) ,$$

where $\phi_j^-(T) := \sum_{k=0}^{j} p^k \cdot f_k^-(T)^{p^{j-k}} \in T^{-1}\mathcal{O}_L[T^{-1}]$ is the j-th phantom component of $\boldsymbol{f}^-(T)$.

Theorem (Pulita 2006): If V is the representation defined by $\boldsymbol{f}^-(T)$, then $\mathrm{e}_{p^m}(\boldsymbol{f}^-(T), 1)$ is the solution of the diff. eq. $\mathbf{D}^\dagger(\mathrm{V})$.

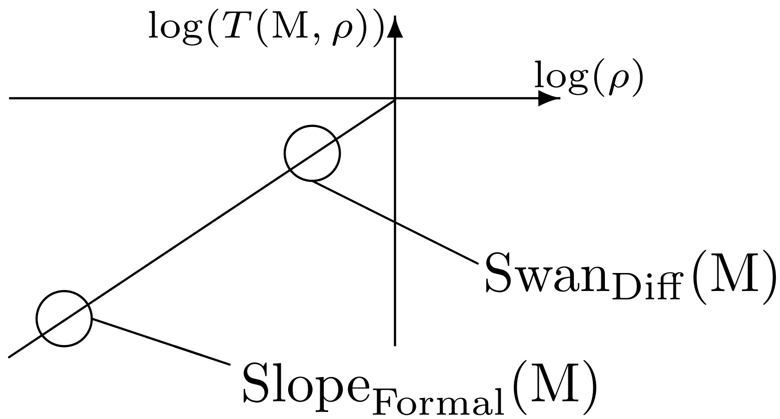
Sketch of the proof of $\mathrm{Swan}_{\mathrm{Arithm}} = \mathrm{Swan}_{\mathrm{Diff}}$

- "Sum of characters" $\Longleftrightarrow$ "Tensor product of corresponding repr."
- We know that every $\boldsymbol{f}(t) \in \mathbf{CW}(\mathrm{E})$ can be uniquely written as $\boldsymbol{f} = \sum_{d>0} \boldsymbol{\lambda}_d t^{-d} + \boldsymbol{f}^0 + \boldsymbol{f}^+$, with

$$\begin{aligned} \boldsymbol{\lambda}_d t^{-d} &\in \mathbf{CW}^{(-d)}(k)\ , \\ \boldsymbol{f}^0 &\in \mathbf{CW}(k)\ , \\ \boldsymbol{f}^+ &\in \mathbf{CW}(tk[\![t]\!])\ . \end{aligned}$$

Then we are reduced to *the case of a co-monomial* $\boldsymbol{\lambda}_d t^{-d}$. We know "its $\mathrm{Swan}_{\mathrm{Arith}}$" by the previous description of the Kato's filtration.

- In this case the log-graphic of $T(\mathrm{M},\rho)$ has no breaks in $]-\infty, 0[$:



$\mathrm{Swan}_{\mathrm{Diff}}(\mathrm{M}) = \mathrm{Slope}_{\mathrm{Formal}}(\mathrm{M})$

$\Downarrow$

The Formal slope is **explicitly** computable from the equation.

The End