$p-$Adic Confluence of q-Difference Equations

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Summary

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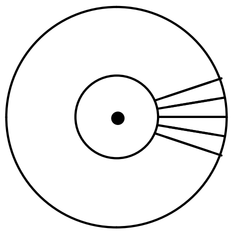
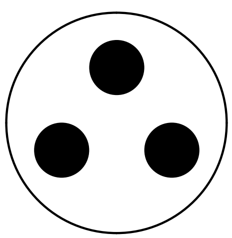
NOTATIONS

- $(K, |.|)$ =ultrametric field of characteristic 0, containing $\mathbb{Q}_p$
- k =residual field of K, $\mathrm{char}(k) = p > 0$,
- <u>*Analytic Functions*</u> on the annulus $\{r_1 < |x| < r_2\}$ are defined as



$$\mathcal{A}_K(]r_1, r_2[) = \{\sum_{-\infty}^{+\infty} a_i T^i, \text{ s.t. } a_i \in K, \text{ and } \lim_{i\to\pm\infty} |a_i|\rho^i = 0, \;\; r_1 < \forall\rho < r_2\}$$

- The <u>*Robba ring*</u> is formed by *germs of analytic functions at the wedge* : $\mathcal{R}_K := \bigcup_{\varepsilon>0} \mathcal{A}_K(]1-\varepsilon, 1[)$.
- A <u>*affinoïd*</u> will be always a set of the following type : let $c_0 \in K$, $R_0 > 0$, $c_1, \ldots, c_n \in \mathrm{D}_K^-(c_0, R_0)$, and $0 < R_1, \ldots, R_n < R_0$.



$$X := \mathrm{D}_K^+(c_0, R_0) - \cup_{i=1,\ldots,n} \mathrm{D}_K^-(c_i, R_i).$$

$\mathcal{H}_K(X) :=$ <u>*analytic elements*</u> on X.

$\mathcal{H}_K(X)$ is the completion, with respect to the sup-norm, of the ring of rational fractions without poles on X.

DIFFERENCE AND DIFFERENTIAL MODULES

• Let $\mathcal{Q}_1 := \mathcal{Q}_1(X)$ be the *open subgroup* of $\mathrm{D}^-_K(1,1)$ of elements q s.t. $f(T) \mapsto f(qT)$ is an automorphism of $\mathcal{H}_K(X)$. We set :

$$\delta_1 := T \cdot \tfrac{d}{dT} \;, \qquad \sigma_q(f(T)) := f(qT) \;, \qquad \Delta_q(f(T)) = \tfrac{\sigma_q(f)-f}{q-1}$$

• Let $G(T) \in M_n(\mathcal{H}_K(X))$, $A(T) \in GL_n(\mathcal{H}_K(X))$, and let $H(T) := (A(t) - \mathrm{I})/(q-1)$. We shall study the equations

$$\delta_1(Y) = G(T) \cdot Y \;, \quad \sigma_q(Y) = A(T) \cdot Y \;, \quad \Delta_q(Y) = H(T) \cdot Y \;.$$

This corresponds to give a free and of finite type $\mathcal{H}_K(X)-$module, together with an action of δ_1 (resp. σ_q, Δ_q) :

$$\mathrm{M} \xrightarrow[\vec{v} \mapsto \delta_1(\vec{v}) + G(T)\vec{v}]{\delta_1^{\mathrm{M}}} \mathrm{M} \;, \quad \mathrm{M} \xrightarrow[\vec{v} \mapsto A(T)\sigma_q(\vec{v})]{\sigma_q^{\mathrm{M}}} \mathrm{M} \Longleftrightarrow \mathrm{M} \xrightarrow[\vec{v} \mapsto \Delta_q(\vec{v}) + H(T)\vec{v}]{\Delta_q^{\mathrm{M}}} \mathrm{M} \;.$$

$$\delta_1 - \mathrm{Mod}(\mathcal{H}_K(X)) \;, \quad \sigma_q - \mathrm{Mod}(\mathcal{H}_K(X)) = \Delta_q - \mathrm{Mod}(\mathcal{H}_K(X)) \;.$$

Differences between the complex and the $p-$adic frameworks

complex : $\mathbb{C}$	$p-$**adic :** K (e.g. $K = \mathbb{Q}_p$)
Constants are "big" i.e. $\neq \mathbb{C}$	Constants are reduced to K
Known cases $\lvert q\rvert > 1$ or $\lvert q\rvert < 1$	Studied case is $\lvert q\rvert = 1$ and more precisely $\lvert q-1\rvert < 1$
If $c \in \mathbb{C}$, $c \neq 0$, $c \neq \infty$, $q \neq 1$ $\mathrm{D}(c,R)$ =disk, with $R > 0$ $\boldsymbol{\Longrightarrow q \cdot \mathrm{D}(c,R) \neq \mathrm{D}(c,R)}$	If $c \in K$, $\mathrm{D}(c,R)$ =disk, with $R > \lvert q-1\rvert \lvert c\rvert$ $\boldsymbol{\Longrightarrow q\mathrm{D}(c,R) = \mathrm{D}(c,R)}$
$\Rightarrow$ The notion of "solution on a disk" has no meaning	If the disk is q invariant, then some kind of equations admits a ***"Taylor solution"*** in that disk .

Heuristic Idea of confluence

Let $\{\ \Delta_q(Y_q) = G(q,T)\cdot Y_q\ \}_{q\in \mathrm{D}^-(1,\varepsilon), q\neq 1}$ be a family of $q-$differences equations. Suppose that there exists the limit

$$\lim_{q\to 1} G(q,T) = G(1,T)\ , \quad \text{in} \quad M_n(\mathcal{H}_K(X))\ .$$

Consider the differential equation

$$\delta_1(Y_1) = G(1,T)\cdot Y_1\ .$$

The confluence consists heuristically in the study of the **conditions to have**

$$\lim_{q\to 1} Y_q = Y_1\ .$$

in a convenable meaning.

We will show that in the $p-$adic framework one can have

$$Y_q = Y_1\ .$$

Example-1

Let X be the annulus $\varepsilon < |T| < 1$. Let $\delta_1 := Td/dT$, let $\pi^{p-1} = -p$, and let (M, ∇) be the differential module defined by the equation

$$(*) \qquad \delta_1(y) = G(1,T) \cdot y \ ,$$

with

$$G(1,T) = -\pi T^{-1} \ .$$

Clearly $G(1,T) \in \mathcal{H}(X)$. The Taylor solution at ∞ is

$$y = \exp(\pi T^{-1}) \ ,$$

it converges for $|T| > 1$. Let $q \in K^\times$, then the q-deformation if $(*)$ is

$$y(qT) = A(q,T) \cdot y(T)$$

with

$$A(q,T) = \exp(\pi(q^{-1} - 1)T^{-1}) \ .$$

Hence if $|q-1| < \varepsilon$, then $A(q,T) \in \mathcal{H}(X)^\times$.

Example-2

We compute the q-deformation of (U_m, ∇_{U_m}) given by :

$$\delta_1(Y_{U_m}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \cdot Y_{U_m} \ , \qquad Y_{U_m}(T) = \begin{pmatrix} 1 & \ell_1 & \cdots & \ell_{m-2} & \ell_{m-1} \\ 0 & 1 & \ell_1 & \cdots & \ell_{m-2} \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & 1 & \ell_1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \ , \tag{1}$$

where $\ell_n := \log(T)^n/n!$. One has

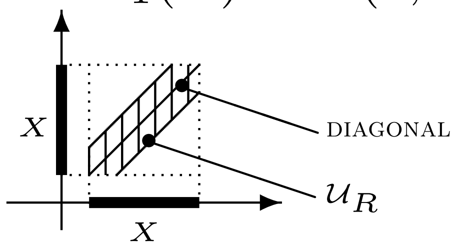
$$\sigma_q(\ell_n(T)) \ = \ \log(qT)^n/n! \ = \ \sum_{i=0}^{n} \frac{\log(q)^{n-k}}{(n-k)!} \cdot \ell_k(T) \ .$$

Then $Y_{U_m}(qT) = A(q,T) \cdot Y_m(T)$, with

$$A(q,T) = \begin{pmatrix} 1 & \log(q) & \frac{\log(q)^2}{2} & \cdots & \frac{\log(q)^{m-1}}{(m-1)!} \\ 0 & 1 & \log(q) & \cdots & \frac{\log(q)^{m-2}}{(m-2)!} \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & \log(q) \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} \ . \tag{2}$$

The unique computation of our paper :

Let $\delta_1(Y) = G(1,T)Y(T)$ be a differential equation over $\mathcal{H}(X)$. Let



$$Y(x,y) := \sum_{n\geq 0} G_n(y)\frac{(x-y)^n}{n!}$$

be its *generic Taylor solution*, where $G_0 = \mathrm{Id}$, $G_1 = T^{-1}G(1,T)$, and $G_{n+1} = \frac{d}{dT}(G_n) + G_n G_1$. Then $Y(x,y)$ is a function on $\mathcal{U}_R := \{(x,y) \in X \times X \mid |x-y| < R\}$ with values in $GL_n(K)$, s.t.

$$Y(x,y)Y(y,z) = Y(x,z)\ , \quad Y(x,y)^{-1} = Y(y,x)\ .$$

Hence $Y(qT,y) = A(q,T)Y(T,y)$, with

$$A(q,T) = Y(qT,y)Y(T,y)^{-1} = Y(qT,y)Y(y,T) = Y(qT,T)\ .$$

If $|q-1|$ =small, then $A(q,T) \in GL_n(\mathcal{H}(X))$:

$$T \mapsto A(q,T) : X \xrightarrow[T\mapsto(qT,T)]{} \mathcal{U}_R \xrightarrow[(x,y)\mapsto Y(x,y)]{} GL_n(K)\ .$$

The Confluence of André-Di Vizio

- They study the case in which :

– The ring of functions is the Robba ring $\mathcal{R}_K$

– $|q-1| < |p|^{\frac{1}{p-1}}$ in particular q *is <u>not</u> a root of unity.*

– Both differential and $q-$difference equations are supposed to have a so called *Frobenius Structure.*

- By the $p-$adic local monodromy theorem

$$\delta_1 - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} \xrightarrow[T_1]{\sim} \mathrm{Rep}_{K^{\mathrm{alg}}}(\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a)$$

- André-Di Vizio proved the (not less hard) $q-$analogue of this theorem :

$$\Delta_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} \xrightarrow[T_q]{\sim} \mathrm{Rep}_{K^{\mathrm{alg}}}(\mathcal{I}_{k^{\mathrm{alg}}((t))} \times \mathbb{G}_a) \ ,$$

and hence, by composition, they deduce an equivalence

$$\mathrm{Def}_{q,1} : \delta_1 - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} \xrightarrow[T_q^{-1}\circ T_1]{\sim} \Delta_q - \mathrm{Mod}(\mathcal{R}_{K^{\mathrm{alg}}})^{(\phi)} \ .$$

What is a "Deformation theory" ?

• Our contributions are :

– We describe construct $\mathrm{Def}_{1,q}$ in a *very elementary way* ,

– This construction works over *every ring of "functions"*,

– This construction works for a *much more large class of equations*,

– We describe what happens if q is *root of unity*.

• We show these facts by introducing a **category of "sheaves"** $\boxed{\sigma-\mathrm{Mod}(\mathcal{H}_K(X))^{\mathrm{an}}}$ over the topological space $\mathcal{Q}_1$ such that :

– The $\underline{\text{stalk at } q=1}$ is a *differential equation* $(\mathrm{M},\delta_1^{\mathrm{M}})$;

– If $\underline{q\notin\cup_n\boldsymbol{\mu}_n}$, the stalk at q is a $q-$*difference equation* $(\mathrm{M},\sigma_q^{\mathrm{M}})$;

– If $\underline{q\in\cup_n\boldsymbol{\mu}_n}$, the stalk at q is *triplet* $(\mathrm{M},\sigma_q^{\mathrm{M}},\delta_q^{\mathrm{M}})$, consisting in a $q-$difference module together with a ***"q-tangent operator*** δ_q^{M}***"***.

• A *"<u>deformation theory</u>"* is a **sub-category** $\mathcal{D}\subset\sigma-\mathrm{Mod}(\mathcal{H}_K(X))$ such that $\forall q\in\mathcal{Q}_1$ the "stalk-functor" is an equivalence :

$$\begin{array}{ccc} & \mathcal{D} & \\ \cong\swarrow & & \searrow\cong \\ \delta_1-\mathrm{Mod}(\mathcal{H}_K(X)) & \xrightarrow[\cong]{\mathrm{Def}_{1,q}} & \Delta_q-\mathrm{Mod}(\mathcal{H}_K(X))\ . \end{array}$$

THE CATEGORY OF ANALYTIC σ–MODULES

Définition 0.1 *Let* $U \subset \mathcal{Q}_1 := \{|q-1| < 1\}$ *be an open subset. Let* $\langle U \rangle$ *be the* open *subgroup of* $\mathcal{Q}$ *generated by* U*. An* analytic σ–module *on* U *is a free and of finite type* $\mathcal{H}(X)$*–module* M*, together with a group morphism*

$$\sigma^{\mathrm{M}} : \langle U \rangle \xrightarrow[q \mapsto \sigma_q^{\mathrm{M}}]{} \mathrm{Aut}_K^{\mathrm{cont}}(\mathrm{M}) \tag{3}$$

such that :

– *For all* $q \in \langle U \rangle$, $f \in \mathcal{H}(X)$, $m \in \mathrm{M}$ *one has*

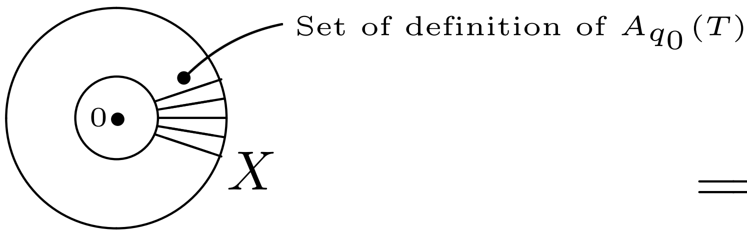
$$\sigma_q^{\mathrm{M}}(fm) = \sigma_q(f) \cdot \sigma_q^{\mathrm{M}}(m)$$

– *For all* $q \in U$ *there exists* $\tau_q > 0$ *such that (in an arbitrary basis of* M*) the matrix* $A(q,T)$ *of* σ_q^{M} *is "analytic" in the set*

$$(Q,T) \quad \in \quad \mathrm{D}^-(q,\tau_q) \times X \ . \tag{4}$$

CONSTANT DEFORMATION

Let $q_0 \in \mathcal{Q}_1 = \{|q-1|<1\}$, $\underline{q_0 \notin \boldsymbol{\mu}_{p^\infty}}$, $A_{q_0}(T) \in GL_n(\mathcal{H}(X))$,



$$\sigma_{q_0}(Y) = A_{q_0}(T)\cdot Y \qquad (*)$$

$$\Longrightarrow (\Delta_{q_0})^n(Y) = H_{q_0,n}(T)\cdot Y$$

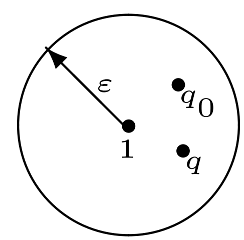
Définition 0.2 *The equation* $(*)$ *is said* **Taylor admissible**, *if there exists* $R>0$ *such that the "generic Taylor solution"*

$$Y_q(x,y) := \sum_{n\geq 0} H_{q_0,n}(y)\frac{(x-y)_{q_0,n}}{[n]_q^!} \qquad (5)$$

converges in the open $\mathcal{U}_R := \{(x,y)\in X\times X \mid |x-y|<R\}$.

Theorem 0.1 *If* $(*)$ *is Taylor admissible, then there exists* $\varepsilon > |q_0 - 1|$ *such that for all* $q \in \mathrm{D}^-(q_0,\varepsilon)$ *the matrix*

$$\boxed{A(q,T) := \sigma_q(Y_q(T,y))\cdot Y_q(T,y)^{-1}}$$



lies in $GL_n(\mathcal{H}(X))$.

CONSTANT ANALYTIC $\sigma-$MODULES

We define a "deformation theory" by giving the following sub-category called $\boxed{\sigma-\mathrm{Mod}(\mathcal{H}(X))_U^{\mathrm{adm}}}$:

• An analytic $\sigma-$module $(\mathrm{M},\sigma^{\mathrm{M}})$ is said **constant** if the generic Taylor solution $Y(T,y)$ is *simultaneously* solution of every equation

$$Y(qT,y)=A(q,T)\cdot Y(T,y) \tag{6}$$

defined by M. The previous theorem provides that if $q\notin\boldsymbol{\mu}_{p^\infty}$, then

Every *Taylor admissible* differential or q_0-difference equation *extends* to a constant $\sigma-$module defined in an open sub-group $\mathrm{D}^-(1,\varepsilon)$ of $\mathcal{Q}_1$, with $\varepsilon>|q_0-1|$.

• In other words if $\sigma_{q_0}^{\mathrm{M}}:\mathrm{M}\to\mathrm{M}$ is a q_0-difference module, then M is *canonically endowed* with an action of $\sigma_q^{\mathrm{M}}:\mathrm{M}\to\mathrm{M}$, for all

$$q\in\langle\mathrm{D}^-(1,\varepsilon)\rangle\ ,\qquad \text{with }\ \varepsilon>|q_0-1|.$$

- If q is not a root of 1, then the forgetful functor is faithful

$$\sigma - \mathrm{Mod}(\mathcal{H}(X))_U^{\mathrm{const}} \xrightarrow{\mathrm{Res}_q^U} \sigma_q - \mathrm{Mod}(\mathcal{H}(X))^{\mathrm{adm}} \ .$$

Let $\sigma_q - \mathrm{Mod}(\mathcal{H}(X))_U^{\mathrm{adm}}$ be the essential image of this functor, then one obtain an equivalence

$$\sigma - \mathrm{Mod}(\mathcal{H}(X))_U^{\mathrm{const}} \xrightarrow{\sim} \sigma_q - \mathrm{Mod}(\mathcal{H}(X))_U^{\mathrm{adm}} \ .$$

- We obtain the so called "<u>constant deformation</u>" for all $q, q' \in U$:

$$\begin{array}{ccc} & \sigma - \mathrm{Mod}(\mathcal{H}(X))_{U=\mathrm{D}^-(1,\varepsilon)}^{\mathrm{const}} & \\ \cong \swarrow & & \searrow \cong \\ \sigma_q - \mathrm{Mod}(\mathcal{H}(X))_U^{\mathrm{adm}} & \overset{\mathrm{Def}_{q,q'}}{\underset{\sim}{\dashrightarrow}} & \sigma_{q'} - \mathrm{Mod}(\mathcal{H}(X))_U^{\mathrm{adm}}. \end{array}$$

where $\varepsilon > |q-1|, |q'-1|$.

Situation on the roots of unity ?

Problems :

If $q = \xi$ is a root of unity, then :

• The operator σ_ξ is of finite order ;

• The sub-ring of $\mathcal{H}_K(X)$ of σ_ξ-constants is not reduced to K ;

• Since $\mathrm{End}(\mathbb{I}) \xrightarrow{\sim} \{\sigma_q - \text{constants}\}$, then we can not obtain an equivalence with other values of $q \neq \xi$.

• The idea is to

"**Replace the category of ξ−differences with another one**".

In fact, for $q = 1$, the category we expect is the differential eq.

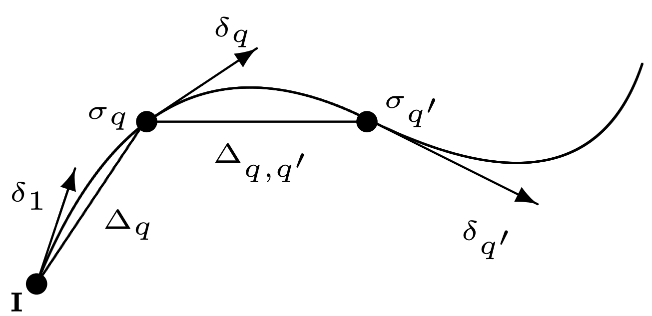
• The question is "*What kind of objects one finds over ξ ?*" in other words "*What is the* STALK *at $q = \xi$ of a constant module ?*"

Situation on the roots of 1 ?

• If $q = 1$, then "1−differences equation" has no meaning.

For $q = 1$ we expect to obtain a differential equation.

$\mathrm{End}_K^{\mathrm{cont}}(\mathcal{H}(X))$



$\Delta_q := \frac{\sigma_q - 1}{q-1}$; $\Delta_{q,q'} := \frac{\sigma_q - \sigma_{q'}}{q-q'}$

$\delta_1 := \lim_{q\to 1} \frac{\sigma_q - 1}{q-1} = T\frac{d}{dT}$;

$$\delta_q := q \cdot \lim_{q'\to q} \frac{\sigma_{q'} - \sigma_q}{q'-q} = \sigma_q \circ \delta_1$$

Let $\mathrm{M} \in \sigma - \mathrm{Mod}(\mathcal{H}(X))_U^{\mathrm{const}}$. For all $q \in U$ we set

$$\delta_q^{\mathrm{M}} := q \cdot \lim_{q'\to q} \frac{\sigma_{q'}^{\mathrm{M}} - \sigma_q^{\mathrm{M}}}{q' - q} \quad .$$

Then, for all $f \in \mathcal{H}(X)$, $m \in \mathrm{M}$, one has :

$$\Longrightarrow \qquad \delta_q^{\mathrm{M}}(fm) = \delta_q(f) \cdot \sigma_q^{\mathrm{M}}(m) + \sigma_q(f) \cdot \delta_q^{\mathrm{M}}(m) \ .$$

The analytic and constant $(\sigma,\delta)-$modules

- We define the category $(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}(X))^{\mathrm{const}}_U$. If q is not a root of unity then we find

$$\sigma-\mathrm{Mod}(\mathcal{H}(X))^{\mathrm{const}}_U \underset{\text{Forget }\delta}{\overset{\text{Construction }\delta}{\rightleftarrows}} (\sigma,\delta)-\mathrm{Mod}(\mathcal{H}(X))^{\mathrm{const}}_U .$$

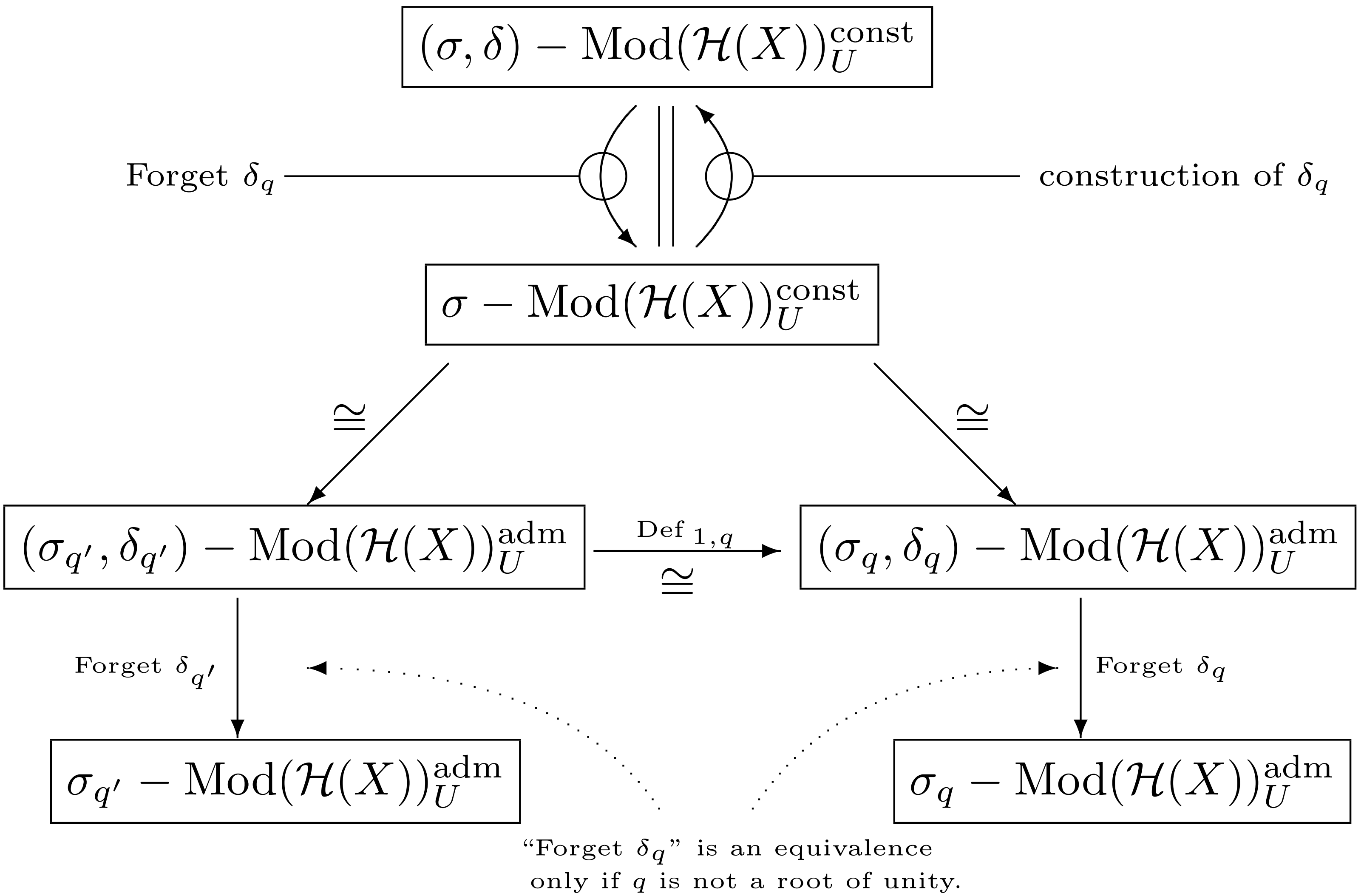
for example if $q=1$ this shows the convergence of the limit in $\mathrm{M}_n(\mathcal{H}(X))$ (highly non trivial)

$$G(1,T)=\lim_{q\to 1}\frac{A(q,T)-\mathrm{Id}}{q-1}$$

Without hypothesis on $q\in U=\mathrm{D}^-(1,\varepsilon)$, the restriction is faithful

$$(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}(X))^{\mathrm{const}}_{\mathrm{D}^-(1,\varepsilon)} \xrightarrow{\mathrm{Res}^U_q} (\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{H}(X))^{\mathrm{adm}}$$

• Let $U \subseteq \mathcal{Q}_1(X) := \{|q-1| < 1\}$. If $(\sigma_q, \delta_q) - \mathrm{Mod}(\mathcal{H}(X))_U$ is the essential image of Res_q^U, then $\forall q, q' \in U$ one has :



Proposition : The restriction

$$\bigcup_{\varepsilon>0}(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}(X))^{\mathrm{const}}_{<\mathrm{D}(q,\varepsilon)>} \xrightarrow[\sim]{\cup_{\varepsilon>0}\mathbf{Res}^{<\mathrm{D}(q,\varepsilon)>}_{\{q\}}} (\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{H}(X))^{\mathrm{adm}}$$

is an **equivalence** (even if q is a root of 1).

- If q =root of unity we find

$$\begin{array}{ccc}
\bigcup_{\varepsilon>0}\sigma-\mathrm{Mod}(\mathcal{H}(X))^{\mathrm{const}}_{<\mathrm{D}(q,\varepsilon)>} & = & \bigcup_{\varepsilon>0}(\sigma,\delta)-\mathrm{Mod}(\mathcal{H}(X))^{\mathrm{const}}_{<\mathrm{D}(q,\varepsilon)>} \\
\Big\downarrow {\scriptstyle \mathrm{Res}^{U}_{\{q\}}} & & \wr\Big\downarrow {\scriptstyle \mathrm{Res}^{U}_{\{q\}}} \\
\sigma_q-\mathrm{Mod}(\mathcal{H}(X))^{\mathrm{adm}} & \xleftarrow[\text{Forget } \delta_q^{\mathrm{M}}]{} & (\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{H}(X))^{\mathrm{adm}}
\end{array}$$

The right hand restriction is always an equivalence while the left hand restriction is an equivalence only for q not root of 1. The operator <u>δ_q *contains the information in the neighborhood of* q</u>. The functor "Forget δ_q" is an equivalence when q is not a root of 1.

- If q is a root of 1, then $(\sigma_q,\delta_q)-\mathrm{Mod}(\mathcal{H}(X))^{\mathrm{adm}}$ is the "<u>good notion of stalk</u>".