Lubin-Tate Groups

in

$p-$adic Differential Equations

Muenster, February 2005

SUMMARY

- **Notations in Lubin-Tate theory**
- **Introduction: Dwork, Robba and Matsuda**
- $P-$**exponentials**
- **Frobenius structure**
- **Generalized Dwork's θ function**
- **Description of the group** $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty})$

Applications:

- **Lifting Galois characters of** $\mathrm{G_E} := Gal(\mathrm{E}^{\mathrm{sep}}/\mathrm{E})$**,** **from** $\mathrm{E} = k((t))$ **to** $\mathcal{E}^\dagger$
- **Explicit computation of the Fontaine-Katz differential equation attached to Artin-Schreier character**

NOTATIONS

- $(K,|.|)$ =complete ultrametric field, $\mathcal{O}_K$ =its valuation ring, k =its residue field (not necessary perfect), of char. $p>0$.
- $\mathcal{R}_K$ =Robba ring.
- $\mathcal{E}_K = \{f(T) = \sum_{-\infty}^{+\infty} a_i T^i \mid \sup_i |a_i| < +\infty,\ \lim_{i\to-\infty} |a_i| = 0\}$.
- $\mathcal{E}^{\dagger}_{K,T} = \mathcal{R}_K \cap \mathcal{E}_K$.
- For all L/K finite. Let $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_L)$ be the group (under $\otimes$) of isomorphism classes of rank one solvable diff. eq. over $\mathcal{R}_L$.
- For all algebraic H/K we put $\mathcal{R}_H := \mathcal{R}_K \otimes_K H$. Then all diff. eq. over $\mathcal{R}_H$ come by scalar extension from an equation over $\mathcal{R}_L$, with L/K fnite, $L \subseteq H$. Then we set

$$\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_H) := \bigcup_{L/K \text{ finite },\ L\subseteq H} \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_L).$$

NOTATIONS IN LUBIN-TATE GROUPS

A L-T series over $\mathbb{Q}_p$ is a series

$$P(X) = \mathrm{w}X + \cdots + X^p + \cdots \in \mathbb{Z}_p[[X]]$$

where $(\cdots)$ are things smaller than $|p|$ and $\mathrm{w} = pu$, with $u \in \mathbb{Z}_p^\times$.

Theorem 0.1 (L-T) *There exists a unique formal group law $G_P(X,Y) \in \mathbb{Z}_p[[X,Y]]$ such that $P(G_P(X,Y)) = G_P(P(X),P(Y))$. This group is endowed with an action of $\mathbb{Z}_p$. Namely for all $a \in \mathbb{Z}_p$ there exists a unique $[a]_P(X) \in \mathbb{Z}_p[[X]]$ such that*

1. $[a](X) = aX + \cdots$;
2. $G_P([a](X),[a](X)) = [a](G_P(X,Y))$.

- By unicity we have $P(X) = [\mathrm{w}](X)$.
- **Examples:** $P(X) = pX + X^p$. $P(X) = (X+1)^p - 1 \Rightarrow G_P = \hat{\mathbb{G}}_m$.

DWORK-ROBBA-MATSUDA EXPONENTIALS

When $\exp(\pi_m T + \pi_{m-1}\frac{T^p}{p} + \cdots + \pi_0 \frac{T^{p^m}}{p^m})$ is convergent for $|T| < 1$?

Examples:

Dwork:	$\exp(\pi T)$,	$\theta(T) = \exp(\pi(T - T^p))$,
	$\boldsymbol{\pi^{p-1} = -p}$;	
Robba:	$\exp(\pi(\alpha_1 \frac{T^{p^m}}{p^m} + \cdots + \alpha_{m-1}\frac{T^p}{p} + \alpha_m T))$,	
	$|\pi| = |p|^{1/p-1}$, but $\boldsymbol{\alpha_i}$ **= unknown**;	
Matsuda:	$\exp((\xi^{p^m} - 1)\frac{T^{p^m}}{p^m} + \cdots + (\xi^p - 1)\frac{T^p}{p} + (\xi - 1)T)$,	
	$\boldsymbol{\pi_j = \xi^{p^{m-j}} - 1}$, with $\xi^{p^{m+1}} = 1$ (primitive).	

$P-$sequences and $P-$exponentials

Let P a L-T series. A $P-$*sequence* is a family $\{\pi_j\}_{j\geq 0}$ such that

$$P(\pi_{j+1}) = \pi_j \ , \qquad P(\pi_0) = 0 \ , \qquad , \qquad \pi_0 \neq 0.$$

Theorem 0.2 *The following formal series converges for $|T| < 1$:*

$$E_m(T) = \exp(\pi_m T + \pi_{m-1}\frac{T^p}{p} + \cdots + \pi_0 \frac{T^{p^m}}{p^m}).$$

Proof : Let $\phi_j(\lambda_0, \lambda_1, \ldots) := \lambda_0^{p^j} + p\lambda_1^{p^{j-1}} + \cdots + p^j\lambda_j$ (phantom comp.). Let $E(T) = \exp(T + T^p/p + \cdots)$ be the Artin-Hasse exp.

• Then:

$$\prod_{j\geq 0} E(\lambda_j T^{p^j}) = \exp(\phi_0 T + \phi_1 T^p/p + \cdots).$$

$\Rightarrow$We need $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots)$ such that $\phi(\boldsymbol{\lambda}) = \langle \pi_m, \ldots, \pi_0, 0, 0, \ldots \rangle$.

• There exists a morphism $s_P : \mathbb{Z}_p[[X]] \hookrightarrow \mathbf{W}(\mathbb{Z}_p[[X]])$, satisfing

$$\phi_j(s_P(h(X))) = h(P(P(\cdots(P(X))))), \quad (j - \text{times}).$$

$P-$sequences and $P-$exponentials - 2

- Then $\boldsymbol{\lambda} = s_P(X)|_{X=\pi_m}$. Now, to have convergence for $|T| < 1$, we need that $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots)$ satisfies $|\lambda_j| \leq 1$, $\forall\ j \geq 0$.

Lemma 0.1 *Let* $h(X) = \sum_{i\geq 0} a_i X^i \in \mathbb{Z}_p[[X]]$. *Let* $\boldsymbol{\lambda} := (\lambda_0, \lambda_1, \ldots) = s_P(h(X))|_{X=\pi_m}$. *Then*

$$\{|\lambda_0|, \ldots, |\lambda_{r-1}| < 1 \textit{ and } |\lambda_r| = 1\} \Longleftrightarrow |a_0| = |p|^r.$$

- This proves the theorem. But leads us to understand the situation

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Lemma 0.2 *Let* $d = np^m > 0$,$(n,p) = 1$,$(\lambda_0, \ldots, \lambda_m) \in \mathbf{W}_m(\mathcal{O}_K)$.

$$\mathrm{e}_d(\boldsymbol{\lambda}, T) := \exp(\pi_m \phi_0 T^{-n} + \pi_{m-1}\phi_1 \frac{T^{-np}}{p} + \cdots + \pi_0 \phi_m \frac{T^{-d}}{p^m})$$

converges at least for $|T| > 1$, *where* $\phi_j = \phi_j(\boldsymbol{\lambda})$.

Proof : $\mathrm{e}_d(\boldsymbol{\lambda}, T) = \prod_{j=0}^{m} E_{m-j}(\lambda_j T^{-np^j})$ and $|\lambda_j| \leq 1.\Box$

$P-$sequences and $P-$exponentials - 3

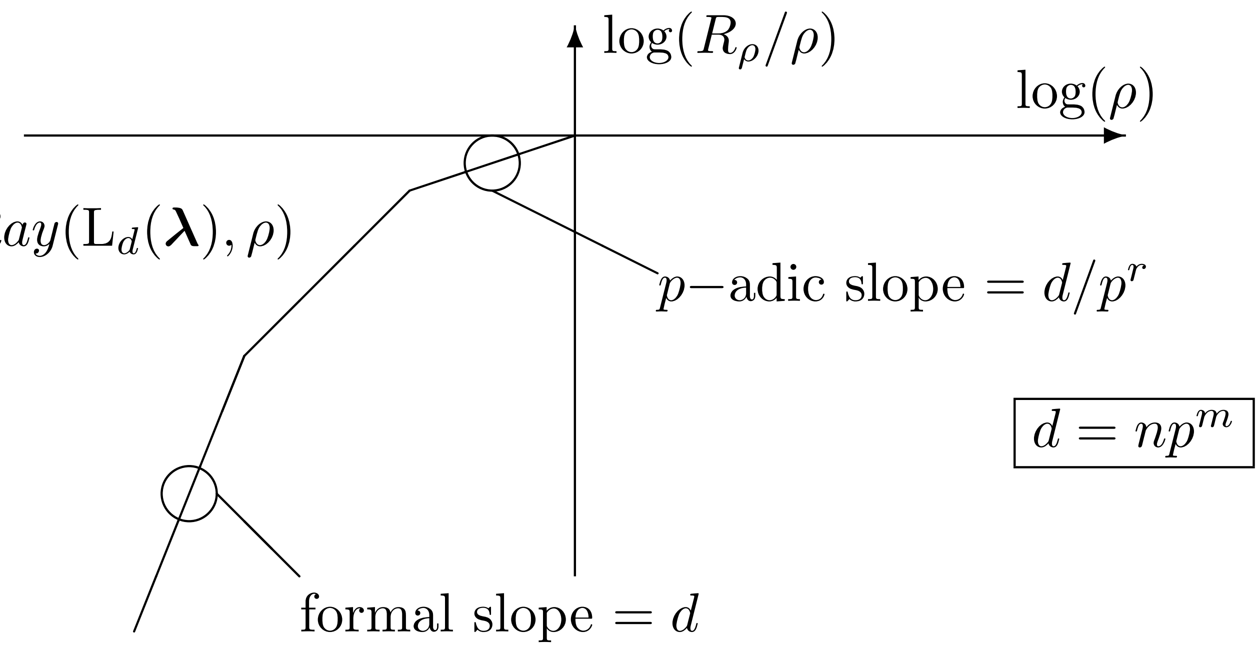
- Since $\mathrm{e}_d(\boldsymbol{\lambda},T)$ converges for $|T|>1$, hence $\mathrm{L}_d(\boldsymbol{\lambda})$ is **solvable**:

$$\mathrm{L}_d(\boldsymbol{\lambda}) := \partial_T + n\cdot(\pi_m\phi_0 T^{-n} + \pi_{m-1}\phi_1 T^{-np} + \cdots + \pi_0\phi_m T^{-d}).$$

Theorem 0.3 $\mathrm{L}_d(\boldsymbol{\lambda})$ *is trivial if and only if* $|\lambda_0|,|\lambda_1|,\ldots,|\lambda_m|<1$. *If* $|\lambda_0|,\ldots,|\lambda_{r-1}|<1$ *and* $|\lambda_r|=1$, $r\leq m$, *then its irregularity is*

$$\underline{\mathrm{Irr}(\mathrm{L}_d(\boldsymbol{\lambda})) = d/p^r = np^{m-r}.}$$

$R_\rho := Ray(\mathrm{L}_d(\boldsymbol{\lambda}),\rho)$



Classification of rk 1 differential modules

- Let $\mathrm{M}_d(\boldsymbol{\lambda})$ be the isomorphism class of $\mathrm{L}_d(\boldsymbol{\lambda})$ over $\mathcal{R}_{K(\pi_m)}$. We get then, in an evident way, a group morphism

$$\mathrm{M}_d : \mathbf{W}_m(\mathcal{O}_K) \xrightarrow[\boldsymbol{\lambda} \mapsto \mathrm{M}_d(\boldsymbol{\lambda})]{} \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K(\pi_m)}).$$

By the precedent theorem this map factorizes as follows:

$$\begin{array}{ccc} \mathbf{W}_m(\mathcal{O}_K) & \xrightarrow{\mathrm{M}_d} & \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K(\pi_m)}) \; . \\ \downarrow & \nearrow_{\mathrm{M}_d} & \\ \mathbf{W}_m(k) & & \end{array}$$

- Observe that this process produces only differental equations of

$$\textit{formal} \text{ slope } = d.$$

Now we want all formal slopes.

Classification of rk 1 differential modules - 2

• Let $K_\infty = \cup_m K(\pi_m)$. Write $v_p(d)$ instead of m, then

$$\oplus_{d>0}\mathbf{W}_{v_p(d)}(k) \xrightarrow{\otimes_{d>0}\mathrm{M}_d} \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty}).$$

What are its image and cokernel? <u>Answer:</u> If $K = \overline{K}$, then

$$\begin{array}{rcl} \mathrm{Coker} & \cong & \{\text{Isomorphism class of } \partial_T + a_0 \mid a_0 \in \mathbb{Z}_p\} \\ \mathrm{Im} & \cong & (\oplus_{d>0}\mathbf{W}_{v_p(d)}(\mathcal{O}_{\overline{K}}))/\sim \end{array}$$

The equivalence "$\sim$" identifies $(\lambda_0, \ldots, \lambda_m)$ with $(0, \lambda_0^p, \ldots, \lambda_m^p)$:

$$\mathbf{W}_{v_p(d)}(\bar{k}) \xrightarrow{(\lambda_0,\ldots,\lambda_m)\mapsto(0,\lambda_0^p,\ldots,\lambda_m^p)} \mathbf{W}_{v_p(pd)}(\bar{k}).$$

• We have $\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{\overline{K}}) = [(\oplus_{d>0}\mathbf{W}_{v_p(d)}(\mathcal{O}_{\overline{K}}))/\sim] \oplus \mathbb{Z}_p/\mathbb{Z}$, then, by Galois descent

$$\boxed{\mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty}) \cong [(\oplus_{d>0}\mathbf{W}_{v_p(d)}(\mathcal{O}_{K_\infty}))/\sim] \oplus \mathbb{Z}_p/\mathbb{Z}}$$

- Actually: what is the Frobenius functor in terms of Witt vectors?

$$\begin{array}{ccc} \oplus_{d>0}\mathbf{W}_{v_p(d)}(k) & \xrightarrow{\otimes \mathrm{M}_d} & \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty}) \\ \Big\downarrow {\scriptstyle ?} & & \Big\downarrow {\scriptstyle \mathbf{Frob}} \\ \oplus_{d>0}\mathbf{W}_{v_p(d)}(k) & \xrightarrow{\otimes \mathrm{M}_d} & \mathrm{Pic}^{\mathrm{sol}}(\mathcal{R}_{K_\infty}) \end{array}$$

- The frobenius functor sends

$$(\lambda_0, \ldots, \lambda_m) \mapsto (0, \lambda_0^p, \ldots, \lambda_m^p) : \mathbf{W}_{v_p(d)}(\bar{k}) \to \mathbf{W}_{v_p(pd)}(\bar{k}).$$

- We prove that

$$\mathrm{M}_d(\lambda_0, \ldots, \lambda_m) \xrightarrow{\sim} \mathrm{M}_{pd}(0, \lambda_0^p, \ldots, \lambda_m^p)$$

FROBENIUS STRUCTURE

Example(Dwork): Let $P(X) = pX + X^p$ (w $= p$). Then $\mathrm{e}_1(1, T) = \exp(\pi_0 T^{-1})$. Dwork showed that $\theta(T)$ is overconvergent:

$$\theta(T) = \frac{\exp(\pi_0 T^{-1})}{\exp(\pi_0 T^{-p})}.$$

Hence $\mathbf{M_1(1)} \xrightarrow{\sim} \mathbf{M_{\boldsymbol{p}}(0, 1)}$, since $\exp(\pi_0 T^{-1}) = \mathrm{e}_1((1), T)$ and

$$\exp(\pi_0 T^{-p}) = \exp(\pi_1 \cdot 0 \cdot T^{-1} + \pi_0 p \frac{T^{-p}}{p}) = \mathrm{e}_p((0, 1), T).$$

Indeed $\boldsymbol{\lambda} = (0, 1) \iff \phi(\boldsymbol{\lambda}) = \langle 0, p \rangle$.
Example(Matsuda): Let $P(X) = (X + 1)^p - 1$, (w $= p$), then $\pi_j = \xi_j - 1$, with $\xi_j^{p^{j+1}} = 1$ and $\xi_j^{p^i} = \xi_{j-i}$. Then, $\underline{\text{(if } p \neq 2)}$,

$$\theta_m(T) := \frac{\exp(\pi_m T + \cdots + \pi_0 T^{p^m}/p^m)}{\exp(\pi_m T^p + \cdots + \pi_0 T^{p^{m+1}}/p^m)}$$

is overconv. Hence $\mathbf{M_{\boldsymbol{p^m}}(1, 0, \ldots, 0)} \xrightarrow{\sim} \mathbf{M_{\boldsymbol{p^{m+1}}}(0, 1, 0, \ldots, 0)}$.

FROBENIUS STRUCTURE - 2

Theorem 0.4 *If* $\mathrm{w} = p$, *then* $\theta_d(\boldsymbol{\lambda}, T)$ *is overconvergent for all* $\boldsymbol{\lambda}$

$$\boxed{\theta_d(\boldsymbol{\lambda}, T) := \frac{\mathrm{e}_d(\boldsymbol{\lambda}, T^{-1})}{\mathrm{e}_d(\sigma(\boldsymbol{\lambda}), T^{-p})}}$$

Here $\sigma : \mathcal{O}_K \to \mathcal{O}_K$ *is a lifting of the* $p-$*th pwer map of* k

Example of proof: Take $\mathrm{e}_p((1,0), T^{-1}) = \exp(\pi_1 T + \pi_0 T^p/p)$:

$$\frac{\exp(\pi_1 T + \pi_0 \frac{T^p}{p})}{\exp(\pi_1 T^p + \pi_0 \frac{T^{p^2}}{p^2})} = \exp\Big((\frac{\pi_1}{\pi_2})\pi_2 T + (\frac{\pi_0}{\pi_1} - p)\pi_1 \frac{T^p}{p} - p\pi_0 \frac{T^{p^2}}{p^2}\Big)$$

$$= \overbrace{\exp(p\pi_2 T)}^{\text{overconvergent}} \cdot \exp\Big(\overbrace{(\frac{\pi_1}{\pi_2} - p)}^{\phi_0^*}\pi_2 T + \overbrace{(\frac{\pi_0}{\pi_1} - p)}^{\phi_1^*}\pi_1 \frac{T^p}{p} + \overbrace{-p}^{\phi_2^*}\pi_0 \frac{T^{p^2}}{p^2}\Big)$$

• **IDEA:** If there exists $\boldsymbol{\lambda}^* = (\lambda_0^*, \lambda_1^*, \lambda_2^*)$, such that $\phi_j^* = \phi_j(\boldsymbol{\lambda}^*)$ and $|\lambda_j^*| < 1$ for all j, then θ is overconvergent. Because

$$\theta_{p^2}(\boldsymbol{\lambda}^*, T)'' ='' E_2(\lambda_0^* T) \cdot E_1(\lambda_1^* T^p) \cdot E_0(\lambda_2^* T^{p^2}).$$

FROBENIUS STRUCTURE - 3

- Recall that we have a ring morphism

$$s_P : \mathbb{Z}_p[[X]] \to \mathbf{W}(\mathbb{Z}_p[[X]])$$

such that the phantom component of $s_P(h(X))$ is

$$\langle h(X), h(P(X)), h(P(P(X))), \ldots \rangle.$$

- Then $h(X) = P(X)/X - p$. Indeed specialize $X \mapsto \pi_2$, then

$$\begin{aligned} h(\pi_2) &= (\frac{\pi_1}{\pi_2} - p) = \phi_0^* \\ h(P(\pi_2)) &= h(\pi_1) = (\frac{\pi_0}{\pi_1} - p) = \phi_1^* \\ h(P(P(\pi_2))) &= h(\pi_0) = (\frac{0}{\pi_0} - p) = -p = \phi_2^* \end{aligned}$$

- Moreover, by lemma 0.1, we can control the value $|\lambda_j^*|$. Precisely

$$|\lambda_0^*|, |\lambda_1^*|, |\lambda_2^*| < 1 \iff |a_0| = |\mathrm{w} - p| \leq |p|^3 \ \Box$$

NOTATIONS:

- $\Lambda/\mathbb{Q}_p$ =finite extension with residue field $\mathbb{F}_q$. Let $k/\mathbb{F}_q$ be perfect.
- A $P-$sequence $\{\pi_j\}_{j\geq 0}$ is fixed. Suppose $\pi_s \in \Lambda$. Then set

$$K = \Lambda \otimes_{\mathbb{F}_q} \mathbf{W}(k), \qquad \sigma = \mathrm{Id}_\Lambda \otimes \mathrm{F}, \qquad \mathrm{E} = k((t))$$

- Let $\alpha : \mathrm{G_E} \to \mathbb{Z}/p^{s+1}\mathbb{Z}$. Denote by V_α the repr. of $\mathrm{G_E}$ defined by

$$g(\mathbf{e}) := \xi_s^{\alpha(g)} \cdot \mathbf{e}\ , \quad \text{for all } g \in \mathrm{G_E} = Gal(\mathrm{E}^{\mathrm{sep}}/\mathrm{E}),$$

where ξ_s is the unique $p^{s+1}-$th root of 1 s.t. $|(\xi_s - 1) - \pi_s| < |\pi_s|$.

$$\left\{ \begin{array}{c} \text{Rank one repr.} \\ \text{of } \mathrm{E} := k((t)) \end{array} \right\} \xrightarrow{\text{Local monodromy}} \left\{ \begin{array}{c} \text{Rank one diff. eq.} \\ \text{over } \mathcal{R}_K \end{array} \right\}$$

$$\{\mathbf{?}\} \longleftrightarrow \{P - \text{exponentials}\}$$

- The Artin-Schreier theory says that we have a sequence:

$$0 \to \mathbb{Z}/p^{s+1}\mathbb{Z} \to \mathbf{W}_s(\mathrm{E}) \xrightarrow{\bar{\mathrm{F}}-1} \mathbf{W}_s(E) \xrightarrow{\delta} \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G_E}, \mathbb{Z}/p^{s+1}\mathbb{Z}) \to 0.$$

- Then V_α is given by a Witt vector in $\mathbf{W}_s(k((t)))$.

APPLICATION: Lifting Artin-Schreier charachters

- Can we relate Witt vectors in $\mathbf{W}_s(k((t)))$ and in $\mathbf{W}_m(k)$?
- What are the vectors corresponding to $P-$exponentials?

Lemma 0.3 *We have the following symmetry ($m \leq s$):*

$$\mathrm{e}_d((a_0, \ldots, a_m), T) = \mathrm{e}_{p^s}((0, \ldots, 0, a_0 T^{-n}, \ldots, a_{m-1} T^{-d/p}, a_m T^{-d}), 1)$$

Proof : Let $\boldsymbol{m}_d(T) := (0, \ldots, 0, a_0 T^{-n}, \ldots, a_{m-1} T^{-d/p}, a_m T^{-d})$.
Then $\phi(\boldsymbol{m}_d(T)) = p^{s-m} \langle 0, \ldots, 0, \phi_0 T^{-n}, \ldots, \phi_m T^{-d} \rangle$.$\square$

Examples:

- $\mathrm{e}_1((a), T) = \exp(\pi_0(a) T^{-1}) = \exp(\pi_0(a T^{-1})) = \mathrm{e}_1((a T^{-1}), 1)$

$$\begin{aligned} \bullet\ \mathrm{e}_{np}((a_0, a_1), T) &= \exp(\pi_1 \phi_0 T^{-n} + \pi_0 \phi_1 \frac{T^{-np}}{p}) \\ &= \exp(\pi_1 \phi_0(a_0 T^{-n}) \cdot 1 + \phi_1(a_0 T^{-n}, a_1 T^{-np}) \frac{1}{p}) \\ &= \mathrm{e}_{p^2}((a_0 T^{-n}, a_1 T^{-np}), 1) \end{aligned}$$

- Then $P-$exponentials correspond to "monomials" $\boldsymbol{m}_d(T)$.
- In general we can write $\mathrm{e}_d(\boldsymbol{a}_d, T) = \mathrm{e}_{p^s}(\boldsymbol{m}_d(T), 1)$, $(s \geq m)$.
- Nice property:

$$\prod_d \mathrm{e}_d(\boldsymbol{a}_d, T) = \prod_d \mathrm{e}_{p^s}(\boldsymbol{m}_d(T), 1) = \mathrm{e}_{p^s}(\sum_d \boldsymbol{m}_d(T), 1)$$

Lemma 0.4 *For all $s \geq 0$ we have a decomposition*

$$\mathbf{W}_s(\mathrm{E}) = \oplus_{d>0} \mathbf{W}_s^{(-d)}(k) \oplus \mathbf{W}_s(k) \oplus \mathbf{W}_s(tk[[t]])$$

$$\mathbf{W}_s^{(-d)}(k) = \{Monomials : \boldsymbol{m}_d(T) = (\ldots, a_{m-1}t^{-d/p}, a_m t^{-d})\}.$$

- We get the usual one for $s = 0$: $k((t)) = \oplus_{d>0} kt^{-d} \oplus k \oplus tk[[t]]$.

$$\left\{ \begin{array}{c} \text{Every solution of diff.eq.} \\ \text{is product of } P-\text{exp.} \end{array} \right\} \quad \stackrel{\sim}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \text{Every } \boldsymbol{f}(t) \in \mathbf{W}_s(\mathrm{E}) \\ \text{is sum of monomials} \end{array} \right\}$$

- For all $\boldsymbol{f}(T) = \sum_{d<0} \boldsymbol{m}_d(T)$, write

$$\mathrm{e}_{p^s}(\boldsymbol{f}(T), 1) = \prod_d \mathrm{e}_{p^s}(\boldsymbol{m}_d(T), 1) = \prod_d \mathrm{e}_d(\boldsymbol{a}_d, T)$$

• Keep attention to the fact that $\mathrm{e}_{p^s}(\boldsymbol{f}(T),1)$ has meaning if and only if $\boldsymbol{f}(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_K[T^{-1}])$.

Example: 1) $Z \mapsto \theta_{p^s}(\boldsymbol{\lambda}, Z)$ converges for $|Z| < 1+\varepsilon$.

One can consider its value at 1. Then

Lemma 0.5 *For all complete ring* $(\mathrm{A}, |.|)$ *and all Witt vector* $\boldsymbol{\lambda} \in \mathbf{W}_s(\mathrm{A})$,

$$\theta_{p^s}(\boldsymbol{\lambda}, 1)$$

has a meaning.

2) The same is true for $Z \mapsto \mathrm{e}_{p^s}(\boldsymbol{\lambda}, Z)^{p^{s+1}}$ (overconvergent).

3) But we can not take for example

$$\mathrm{e}_{p^s}((1),1) = \exp(\pi_0) = \text{has no meaning.}$$

But for example $\mathrm{e}_1(T^{-1},1) = \exp(\pi_0 T^{-1})$ has a meaning.

Theorem 0.5 *Let $\varphi : \mathcal{O}^\dagger \to \mathcal{O}^\dagger$ be a Frobenius.*

$$
\begin{array}{ccccccc}
0 \to \mathbb{Z}/p^{s+1}\mathbb{Z} & \longrightarrow & \mathbf{W}_s(\mathrm{E}) & \xrightarrow{\bar{\mathrm{F}}-1} & \mathbf{W}_s(\mathrm{E}) & \overset{\delta}{\twoheadrightarrow} & \mathrm{H}^1(\mathrm{G}_{\mathrm{E}}, \mathbb{Z}/p^{s+1}\mathbb{Z}) \\
\big\uparrow & & \big\uparrow & & \big\uparrow & & \vdots\ \overline{e} \\
\wr\ \mathbf{W}_s(\mathcal{O}_K^{\varphi=1}) & \hookrightarrow & \mathbf{W}_s(\mathcal{O}^\dagger) & \xrightarrow{\varphi-1} & \mathbf{W}_s(\mathcal{O}^\dagger) & & \\
\big\downarrow & & \big\downarrow {\scriptstyle \theta^{(\varphi)}_{p^s}(-,1)} & & \big\downarrow {\scriptstyle \mathrm{e}_{p^s}(-,1)^{p^{s+1}}} & & \downarrow \\
1 \to \boldsymbol{\mu}_{p^{s+1}} & \longrightarrow & (\mathcal{E}^\dagger)^\times & \xrightarrow[f \mapsto f^{p^{s+1}}]{} & (\mathcal{E}^\dagger)^\times & \underset{\delta_{\mathrm{Kum}}}{\twoheadrightarrow} & \mathrm{H}^1(\mathrm{G}_{\mathcal{E}^\dagger}, \boldsymbol{\mu}_{p^{s+1}})
\end{array}
$$

where $\mathcal{O}_K^{\sigma=1} := \{a \in \mathcal{O}_K \mid a^\sigma = a\}$ *and* $\mathrm{G}_{\mathcal{E}^\dagger} := Gal(\overline{\mathcal{E}^\dagger}/\mathcal{E}^\dagger)$. *More explicitely* $\theta^{(\varphi)}_{p^s}(-,1)$ *induces the identification*

$$1 \mapsto \xi_s^{-1} : \mathbb{Z}/p^{s+1}\mathbb{Z} \xrightarrow{\sim} \mu_{p^{s+1}}$$

where ξ_s *is the unique* p^{s+1}*−th root of* 1 *such that*

$$|(\xi_s - 1) - \pi_s|.$$

Explicit computation of Local monodromy functor

- This gets us an explicit section of the henselian correspondence:

$$\left\{\text{Unramified ext.of } \mathcal{O}^\dagger\right\} \rightsquigarrow \{\text{Separable ext.of } \mathrm{E} = k((t))\}$$

$$0 \to \mathbb{Z}/p^{s+1}\mathbb{Z} \to \mathbf{W}_s(\mathrm{E}) \xrightarrow{\bar{\mathrm{F}}-1} \mathbf{W}_s(E) \xrightarrow{\delta} \mathrm{Hom}^{\mathrm{cont}}(\mathrm{G}_{\mathrm{E}}, \mathbb{Z}/p^{s+1}\mathbb{Z}) \to 0.$$

1) Let $\boldsymbol{f}(T) \in \mathbf{W}_s(\mathcal{O}^\dagger)$, let $\alpha = \delta(\overline{\boldsymbol{f}(T)})$, $\alpha : \mathrm{G}_{\mathrm{E}} \to \mathbb{Z}/p^{s+1}\mathbb{Z}$.

2) Let F/E be the extension defined by the kernel of α.

3) Let $\mathcal{F}^\dagger = \mathcal{E}^\dagger_{L,\widetilde{T}}/\mathcal{E}^\dagger$ be the corresponding unramified extension.

4) Let $\boldsymbol{g}(\widetilde{T}) \in \mathcal{F}^\dagger$ be such that $\bar{\mathrm{F}}\left(\overline{\boldsymbol{g}(\widetilde{T})}\right) - \overline{\boldsymbol{g}(\widetilde{T})} = \overline{\boldsymbol{f}(T)}$.

5) Then $$\boxed{\mathcal{F}^\dagger = \mathcal{E}^\dagger\left(\theta_{p^s}(\boldsymbol{g}(\widetilde{T}), 1)\right)}$$

- Exemple of Dwork: $\exp(\pi_0 T^{-1}) = \mathrm{e}_1((1), T) = \mathrm{e}_1((T^{-1}), 1)$,

$s = 0$, $\mathbf{W}_0(\mathrm{E}) = \mathrm{E}$, $\boldsymbol{f}(T) = T^{-1}$, $g(\widetilde{T}) = \mathbf{?}$, $g(\widetilde{T})^p - g(\widetilde{T}) = T^{-1}$,

$$\theta_{p^s}(g(\widetilde{T}), 1) = \mathbf{?}.$$

Explicit computation of differential equation

• Let $\mathbf{D}^\dagger(\mathrm{V}_\alpha) = (\mathrm{V}_\alpha \otimes_\Lambda \mathcal{O}^{\dagger,\mathrm{unr}})^{\mathrm{G_E}}$ the φ−module attached to V_α. Then

$$\mathbf{e} \otimes \theta_{p^s}(\boldsymbol{g}(\widetilde{T}), 1)$$

is a basis of $\mathbf{D}^\dagger(\mathrm{V}_\alpha) = (\mathrm{V}_\alpha \otimes_\Lambda \mathcal{O}^{\dagger,\mathrm{unr}})^{\mathrm{G_E}}$.

• Actually let $g \in \mathrm{G_E}$, then, by the big diagram

$$\begin{aligned} g(\mathbf{e} \otimes \theta_{p^s}(\boldsymbol{g}(\widetilde{T}))) &= g(\mathbf{e}) \otimes g\left(\theta_{p^s}(\boldsymbol{g}(\widetilde{T}), 1)\right) \\ &= \xi_s^{\alpha(g)} \cdot \mathbf{e} \otimes \xi_s^{-\alpha(g)} \theta_{p^s}(\boldsymbol{g}(\widetilde{T}), 1) \end{aligned}$$

• Can we describe the differential equation $\partial_T - \frac{\partial_T(\theta_{p^s}(\boldsymbol{g}(\widetilde{T}),1))}{\theta_{p^s}(\boldsymbol{g}(\widetilde{T}),1)}$?

• We observe that

$$\theta_{p^s}(\boldsymbol{g}(\widetilde{T}), 1)^{p^{s+1}} = \mathrm{e}_{p^s}(\boldsymbol{f}(T), 1)^{p^{s+1}}$$

But $\mathrm{e}_{p^s}(\boldsymbol{f}(T), 1)$ has no meaning!

• If $\boldsymbol{f}(T) \in \mathbf{W}_s(T^{-1}\mathcal{O}_K[T])$, actually $\mathrm{e}_{p^s}(\boldsymbol{f}(T), 1)$ has a meaning.

- Write

$$\boldsymbol{f}(T) = \boldsymbol{f}_-(T) + \boldsymbol{f}_0 + \boldsymbol{f}_+(T),$$

$\boldsymbol{f}_- \in \mathbf{W}_s(T^{-1}\mathcal{O}_K[T^{-1}])$, $\boldsymbol{f}_0 \in \mathbf{W}_s(\mathcal{O}_K)$, $\boldsymbol{f}_+(T) \in \mathbf{W}_s(\mathcal{O}_K[[T]])$.

- We can suppose $\boldsymbol{f}_+(T) = 0$, because $\delta(\overline{\boldsymbol{f}_+(T)}) = 0$.
- Then separe the Problem:

$$y^{p^{s+1}} = \mathrm{e}_{p^s}(\boldsymbol{f}_0, 1)^{p^{s+1}}$$

$$\theta_{p^s}(\boldsymbol{g}(\widetilde{T}), 1) = \underbrace{y}_{\text{constant}} \cdot \underbrace{\mathrm{e}_{p^s}(\boldsymbol{f}_-(T), 1)}_{\text{has meaning}}$$

- Then

$$\partial_T - \frac{\partial_T(\theta_{p^s}(\boldsymbol{g}(\widetilde{T}), 1))}{\theta_{p^s}(\boldsymbol{g}(\widetilde{T}), 1)} = \partial_T - \frac{\partial_T\left(\mathrm{e}_{p^s}(\boldsymbol{f}_-(T), 1)\right)}{\mathrm{e}_{p^s}(\boldsymbol{f}_-(T), 1)}$$

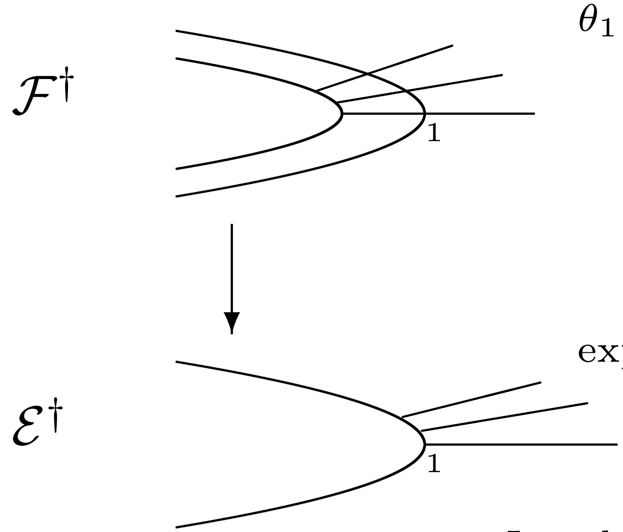
This is the Fontaine-Katz equation!

Example: $s = 0$, $\mathbf{W}_s(\mathrm{E}) = \mathrm{E}$, $\alpha = \delta(\bar{a}t^{-1})$, then the diff.eq. is

$$\partial_T + \pi_0 a T^{-1}, \qquad (\partial_T = T\frac{d}{dT})$$

its solution at infinity is $\exp(\pi_0 a T^{-1})$. (a =arbitrary lift of $\bar{a}$). And

$$\mathcal{F}^\dagger = \mathcal{E}^\dagger(\mathrm{e}_1(aT^{-1}, 1)) = \mathcal{E}^\dagger(\exp(\pi_0 a T^{-1})).$$



$\theta_1(\boldsymbol{g}(\widetilde{T}), 1) = \exp(\pi_0 T^{-1})$ converge for $|\widetilde{T}| > 1 - \varepsilon$

$\exp(\pi_0 T^{-1}) = \mathrm{e}_1(T^{-1}, 1)$ converge for $|T| > 1$

In other words the analytic function $\exp(\pi_0 T^{-1})$ become overconvergent and equal to $\theta_1(\boldsymbol{g}(\widetilde{T}), 1)$ in the new variable $\widetilde{T}$.

• **General algorithm:**

1)Start from $\overline{\boldsymbol{f}}(t) \in \mathbf{W}_s(\mathrm{E})$. $\alpha = \delta(\overline{\boldsymbol{f}}(t))$.
2)Lift it into $\boldsymbol{f}(T) \in \mathbf{W}_s(\mathcal{O}_K[[T]][T^{-1}])$ in an arbitrary way.
3)Take $\phi(\boldsymbol{f}(T)) = \langle \phi_0(T), \ldots, \phi_s(T) \rangle$, then separe the degrees:

$$\phi_j(T) = \phi_j^-(T) + \phi_j^0(T) + \phi_j^+(T) \ ,$$

with $\phi_j^- \in T^{-1}\mathcal{O}_K[T^{-1}]$, $\phi_j^0 \in \mathcal{O}_K$, $\phi_j^+ \in \mathcal{O}_K[[T]]$, for all $j \geq 0$.
4) The Fontaine-Katz differential equation is then

$$\boxed{\partial_T + \sum_{j=0}^{s} \pi_{s-j} \partial_T(\phi_j^-(T)) \frac{1}{p^j}.}$$

THE END