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Spectre d'équations différentielles p-adiques

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Résumé : Les équations différentielles constituent un important outil pour l'étude des variétés algébriques et analytiques, sur les nombres complexes et p-adiques. Dans le cas p-adique, elles présentent des phénomènes qui n'apparaissent pas dans le cas complexe. En effet, le rayon de convergence des solutions d'une équation différentielle linéaire peut être fini, et cela même en l'absence de pôles.

La connaissance de ce rayon permet d'obtenir de nombreuses informations intéressantes sur l'équation. Plus précisément, depuis les travaux de F. Baldassarri, on sait associer un rayon de convergence a tout point d'une courbe p-adique au sens de Berkovich munie d'une connexion. Des travaux récents de F. Baldassarri, K. Kedlaya, J. Poineau et A. Pulita ont révélé que ce rayon se comporte de manière très contrainte. Afin de pousser l'étude, on introduit un objet géométrique qui raffine aussi ce rayon, le spectre au sens de Berkovich d'une équation différentielle.

Dans ce mémoire de thèse, nous définissons le spectre d'un module différentiel et donnons ses premières propriétés. Nous calculons aussi les spectres de quelques classes de modules différentiels: modules différentiels d'une équations différentielles à coefficients constants, modules différentiels singuliers réguliers et enfin modules différentiels sur un corps de séries de Laurent.

Mots clés : Équations differentielles p-adiques, Espace de Berkovich, Theorie spéctrale.

Abstract : Differential equations constitute an important tool for the investigation of algebraic and analytic varieties, over the complex and the p-adic numbers. In the p-adic setting, they present phenomena that do not appear in the complex case. Indeed, the radius of convergence of the solutions of a linear differential equation may be finite, even without presence of poles.

The knowledge of that radius permits to obtain several interesting informations about the equation. More precisely, since the works of F. Baldassarri, we know how to associate a radius of convergence to all points of a p-adic curve in the sense of Berkovich endowed with a connection. Recent works of F. Baldassarri, K.S. Kedlaya, J. Poineau, and A. Pulita have proved that this radius behaves in a very controlled way. The radius of convergence can be refined using subsidiary radii, that are known to have similar properties. In order to push forward the study, we introduce a geometric object that refines also this radius, the spectrum in the sense of Berkovich of a differential equation.

In the present thesis, we define the spectrum of a differential equation and provide its first properties. We also compute the spectra of some classes of differential modules: differential modules of a differential équation with constant coefficients, singular regular differential modules and at last differential modules over the field of Laurent power series.

Keywords : p-adic differential equations, Berkovich spaces, spectral Theory.

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Introduction (version française)

Les équations différentielles constituent un important outil pour l'étude des variétés algébriques et analytiques, sur les nombres complexes et p-adiques. Notamment, la cohomologie de de Rham est l'un des moyens les plus puissants pour obtenir des informations algébriques et analytiques. Par ailleurs, les phénomènes ultramétriques apparaissent naturellement en étudiant les solutions formelles de Taylor d'une équation autour des points singuliers et réguliers.

Aussitôt que la théorie des équations différentielles est devenue un sujet d'étude central autour des années 1960, après les travaux de B. Dwork, P. Robba (et al.), plusieurs phénomènes intéressants sont apparus: dans le cas ultramétrique, les solutions d'une équation différentielle linéaire, contrairement au cas complexe, peuvent diverger, même si les coefficients de l'équation sont des fonctions entières. Par exemple, sur le corps de base $\mathbb{Q}_p$ des nombres p-adiques, la série entière exponentielle $\exp(T) = \sum_{n\geq 0} \frac{T^n}{n!}$ qui est solution de l'équation $y' = y$ a un rayon de convergence égal à $|p|^{\frac{1}{p-1}}$ même si l'équation ne présente pas de singularités. Cependant, le comportement du rayon de convergence est très contraint et sa connaissance permet d'obtenir plusieurs informations sur l'équation. Notamment, il contrôle la *finitude de la dimension* de la cohomologie de de Rham.

L'usage du langage de Berkovich introduit dans [Ber90] et [Ber93] a permis de montrer que la variation de ce rayon est bien contrôlée par un sous-graphe fini d'une courbe, au sens de Berkovich. Plus précisément, depuis les travaux de F. Baldassarri et L. Di Vizio [BD08] et F. Baldassarri dans [Bal10] on sait associer un rayon de convergence à un point d'une courbe p-adique au sens de Berkovich munie d'une connexion. Dans la publication [Bal10] F. Baldassarri démontre que le rayon de convergence dans le cadre d'une courbe est une fonction continue. Il conjecture également qu'il se factorise à travers un graphe localement fini sur lequel la courbe se rétracte, graphe qui est appelé *graphe contrôlant* dans les travaux J. Poineau et A. Pulita. C'est dans l'article [Pul15] que A. Pulita démontre cette conjecture pour le rayon de convergence d'un module différentiel sur un domaine affinoïde de la droite analytique affine $\mathbb{A}_k^{1,\mathrm{an}}$. Il généralise aussi ce résultat aux rayons subsidiaires, c'est-à-dire aux rayons correspondant à chacune des solutions de l'équation. Ces résultats sont généralisés au cas des courbes quasi-lisses au sens de Berkovich et approfondis dans les articles de J. Poineau et A. Pulita [PP15], [PP13a]. Dans le papier [PP13b] ils fournissent un critère nécessaire et suffisant pour la finitude de la cohomologie de de Rham. Ce critère est vérifié dans le cas où le graphe contrôlant est fini.

Dans notre travail, nous nous intéressons à un invariant plus fin que le rayon de convergence, le spectre au sens de Berkovich d'un module différentiel. Notre objectif principal est de pouvoir relier cet invariant aux les résultats mentionnés ci-dessus, et d'en déduire des informations supplémentaires sur l'équation. Dans ce mémoire, nous introduisons le spectre associé à un module différentiel et donnons ses premières propriétés. Nous calculons aussi les spectres de quelques classes de modules différentiels: modules différentiels d'équations différentielles à coefficients constants, modules différentiels singuliers réguliers et enfin modules différentiels sur le corps des séries de Laurent.

1 Motivation

Le point de départ de ce mémoire est l'intéressante relation entre le rayon de convergence et la notion de *spectre* (au le sens de Berkovich). En effet, soit A un anneau muni d'une dérivation $d : A \to A$. Un *module differentiel* sur (A, d) est un A-module libre de rang fini muni d'une application $\mathbb{Z}$-linéaire

$$\nabla : M \to M$$

vérifiant la relation $\nabla(fm) = d(f)m + f\nabla(m)$ pour tous $m \in M$ et $f \in A$. Si l'on se donne un isomorphisme $M \simeq A^\nu$, alors ∇ coïncide avec un opérateur de la forme

$$d + G : A^\nu \to A^\nu$$

où d opère sur A^ν composante par composante et G est une matrice carrée à coefficients dans A.

Si de plus A est une algèbre de Banach pour une norme donnée $\|.\|$ et A^ν est munie la norme max, alors on peut munir l'algèbre des endomorphismes $\mathbb{Z}$-linéaires bornés de la norme d'opérateur, qu'on notera aussi par $\|.\|$. La norme spectrale de ∇ est donnée par

$$\|\nabla\|_{\mathrm{Sp}} = \lim_{n\to\infty} \|\nabla^n\|^{\frac{1}{n}}.$$

Le lien entre le rayon de convergence de ∇ et son spectre est le suivant: d'une part, dans le cas où $\|.\|$ correspond à un point de Berkovich x de la droite affine analytique affine et A est le corps correspondant à ce point, la norme spectrale $\|\nabla\|_{\mathrm{Sp}}$ coïncide avec l'inverse du rayon de convergence de M en x multiplié par une certaine constante (cf. [CD94, p. 676] or [Ked10, Definition 9.4.4]). D'autre part, la norme spectrale $\|\nabla\|_{\mathrm{Sp}}$ est aussi égale au rayon du plus petit disque centré en zéro et contenant le spectre de ∇ au le sens de Berkovich (cf. [Ber90, Theorem 7.1.2]).

Le spectre apparaît alors comme un nouvel invariant de la connexion ∇, qui généralise et raffine le rayon de convergence. Cela était notre première motivation. Cependant l'étude du spectre d'un opérateur a son propre intérêt et mérite d'avoir sa propre théorie.

2 Théorie spectrale au sens de Berkovich

Soient k un corps arbitraire et E une k-algèbre unitaire non nulle. On rappelle que classiquement, le spectre $\Sigma_{f,k}(E)$ d'un élément f de E est l'ensemble des éléments λ de k tels que $f-\lambda.1_E$ n'est pas inversible dans E (cf. [Bou07, §1.2. Définition 1]). Dans le cas où $k=\mathbb{C}$ et $(E,\|.\|)$ est une $\mathbb{C}$-algèbre de Banach, le spectre de f vérifie les propriétés suivantes:

- Il est non vide et compact.
- Le plus petit disque centré en 0 et contenant $\Sigma_{f,k}(E)$ a un rayon égal à $\|f\|_{\mathrm{Sp}}=\lim\limits_{n\to+\infty}\|f^n\|^{\frac{1}{n}}$.
- La fonction résolvante $R_f:\mathbb{C}\setminus\Sigma_{f,k}(E)\to E$, $\lambda\mapsto(f-\lambda.1_E)^{-1}$ est une fonction analytique à valeurs dans E.

Malheureusement, cela n'est plus vrai dans le cas ultramétrique. Dans [Vis85] M. Vishik fournit un exemple d'opérateur avec un spectre vide et une résolvante non globalement analytique. Nous illustrons cette pathologie avec un exemple de notre contexte, où des connexions avec un spectre vide apparaissent.

Exemple 2.1. Soit k un corps algébriquement clos de caractéristique zero muni de la valeur absolue triviale. Soit $A:=k(\!(S)\!)$ le corps des séries formelles de Laurent muni de la valeur absolue S-adique donnée par $|\sum_{n\geq n_0}a_nS^n|=r^{n_0}$, si $a_{n_0}\neq 0$, où $|S|=r<1$ est un nombre réel non nul. On considère un module différentiel irrégulier de rang un sur A défini par l'opérateur $\frac{\mathrm{d}}{\mathrm{dS}}+g:A\to A$, où $g\in A$ a une valuation S-adique $n_0\leq -2$. On considère la connexion $\nabla=\frac{\mathrm{d}}{\mathrm{dS}}+g$ comme un élément de la k-algèbre de Banach $E:=\mathcal{L}_k(A)$ des applications k-linéaires bornées de A pour la norme d'opérateur usuelle: $\|\varphi\|=\sup_{f\in A\setminus\{0\}}\frac{|\varphi(f)|}{\|f\|}$, pour tout $\varphi\in E$. Le spectre classique de l'opérateur différentiel $\frac{\mathrm{d}}{\mathrm{dS}}+g$ est vide. En effet, $\|\frac{\mathrm{d}}{\mathrm{dS}}\|=\frac{1}{r}$ et $|(g-a)^{-1}|\leq r^2$ pour tout $a\in k$, donc $\|(g-a)^{-1}\circ(\frac{\mathrm{d}}{\mathrm{dS}})\|\leq r$ (resp. $\|(\frac{\mathrm{d}}{\mathrm{dS}})\circ(g-a)^{-1}\|\leq r$) et la série $\sum_{n\geq 0}(-1)^n\cdot((g-a)^{-1}\circ(\frac{\mathrm{d}}{\mathrm{dS}}))^n$ (resp. $\sum_{n\geq 0}(-1)^n((\frac{\mathrm{d}}{\mathrm{dS}})\circ(g-a)^{-1})^n$) converge dans $\mathcal{L}_k(k(\!(S)\!))$. Par conséquent, pour tout $a\in k$, $(\sum_{n\geq 0}(-1)^n\cdot((g-a)^{-1}\circ(\frac{\mathrm{d}}{\mathrm{dS}}))^n)\circ(g-a)^{-1}$ (resp. $(g-a)^{-1}\circ(\sum_{n\geq 0}(-1)^n\cdot((\frac{\mathrm{d}}{\mathrm{dS}})\circ(g-a)^{-1})^n)$ est un inverse à gauche (resp. droite) de $\frac{\mathrm{d}}{\mathrm{dS}}+(g-a)$ dans $\mathcal{L}_k(k(\!(S)\!))$ et a n'appartient pas au spectre. Donc le spectre classique est vide.

Pour faire face à ce problème V. Berkovich a compris qu'au lieu de définir le spectre comme un sous-ensemble du corps de base k, il fallait le définir comme un sous-ensemble de la droite affine analytique $\mathbb{A}_k^{1,\mathrm{an}}$, qui un espace plus grand. Sa théorie des espaces analytiques (cf. [Ber90], [Ber93]) vérifie plusieurs bonnes propriétés topologiques locales telles la compacité, la connexité par arcs... Dans ce cadre Berkovich a développé une théorie spectrale pour les opérateurs ultramétriques [Ber90, Chapter 7]. La définition du spectre donnée par Berkovich est la suivante: Soit $(k,|.|)$ un corps ultramétrique complet pour $|.|$ et soit $\mathbb{A}_k^{1,\mathrm{an}}$ la droite affine de Berkovich. Pour un point $x \in \mathbb{A}_k^{1,\mathrm{an}}$ on note $\mathscr{H}(x)$ le corps résiduel complet associé. On fixe sur $\mathbb{A}_k^{1,\mathrm{an}}$ une fonction coordonnée T. Le spectre $\Sigma_{f,k}(E)$ d'un élément f d'une k-algèbre de Banach E est l'ensemble des points $x \in \mathbb{A}_k^{1,\mathrm{an}}$, tels que $f\otimes 1 - 1\otimes T(x)$ n'est pas inversible dans $E\hat{\otimes}_k\mathscr{H}(x)$. On peut montrer (cf. [Ber90, Proposition 7.6]) que cela est équivalent à dire qu'il existe un corps valué complet Ω contenant isométriquement k et une constante $c\in\Omega$ tel que

- l'image de c par la projection canonique $\mathbb{A}_\Omega^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ est x;
- $f\otimes 1 - 1\otimes c$ n'est pas inversible dans $E\hat{\otimes}_k\Omega$.

Dans un certain sens $\Omega = \mathscr{H}(x)$ et $c = T(x)$ est le choix possible minimal. Ce spectre est non vide si E est non nulle, compact et vérifie les propriétés mentionnées ci-dessus (cf. [Ber90, Theorem 7.1.2]. Remarquons que si $f=\nabla$ est une connexion et si $E=\mathcal{L}_k(M)$ est la k-algèbre de Banach des endomorphismes k-linéaires bornés de M, alors $f\otimes 1 - 1\otimes c$ n'est plus un opérateur. En effet, $\mathcal{L}_k(M)\hat{\otimes}_k\Omega$ ne coïncide pas avec $\mathcal{L}_\Omega(M\hat{\otimes}_k\Omega)$, sauf dans le cas où Ω est une extension finie. En particulier, *le théorème de l'indice* ne peut être utilisé dans l'étude de la bijectivité ensembliste de $\nabla\otimes 1 - 1\otimes c$.

Revenant à notre exemple précédant, en utilisant cette définition on peut montrer que le spectre de $\frac{\mathrm{d}}{\mathrm{dS}} + g$ est réduit maintenant à un seul point *non rationel* $x_{0,r^{n_0}} \in \mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}}$ (cf. Chapitre 6). La dépendance en r montre que le spectre de Berkovich *dépend du choix de la valeur absolue sur* k, contrairement au spectre classique qui est une notion purement algébrique.

3 Description des chapitres

Dans cette partie nous décrivons le contenu et les résultats principaux de chaque chapitre. On donne avant quelques notations:

Notation 3.1. Soit $(k,|.|)$ un corps ultramétrique complet de caractéristique zero. Soient $c\in k$ et $r\in\mathbb{R}_+$. On note $D^+(c,r)$ le disque fermé de $\mathbb{A}_k^{1,\mathrm{an}}$ centré en c et de rayon r, et $D^-(c,r)$ le disque ouvert de $\mathbb{A}_k^{1,\mathrm{an}}$ centré en c et de rayon r. Le point $x_{c,r}\in\mathbb{A}_k^{1,\mathrm{an}}$ correspond au bord de Shilov du disque $D^+(c,r)$.

3.1 Chapitre 1: Basic notions

Le premier chapitre concerne les notions de bases. Nous rappelons les définitions et propriétés concernant les espaces de Banach. Nous rappelons aussi la définition général des espaces analytiques au sens de Berkovich et de leurs morphismes. Nous décrivons précisément la droite affine analytique $\mathbb{A}_k^{1,\mathrm{an}}$.

3.2 Chapitre 2: Spectral theory in the sense of Berkovich

Le deuxième chapitre est consacré à l'étude du spectre au sens de Berkovich. Nous nous intéressons en particulier à la variation de ce spectre. Pour cela nous définissons en premier lieu une topologie sur $K(\mathbb{A}_k^{1,\mathrm{an}})$, l'ensemble des sous-ensembles compacts non vides de $\mathbb{A}_k^{1,\mathrm{an}}$. Nous montrons que, dans le cas où $\mathbb{A}_k^{1,\mathrm{an}}$ est un espace métrique, cette topologie coïncide avec la topologie induite par la métrique d'Hausdorff sur $K(\mathbb{A}_k^{1,\mathrm{an}})$. Nous étudions ensuite la continuité de l'application

$$\begin{array}{rcl}\Sigma_{.,k}(E) : E & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ f & \mapsto & \Sigma_{f,k}(E),\end{array}$$

où $\Sigma_{f,k}(E)$ est le spectre de f vu comme élément de E. Nous remarquons qu'en général cette application n'est pas continue. Cependant, nous obtenons les résultats de continuité suivants:

Théorème 3.2. *Soit E une k-algèbre de Banach commutative. L'application*

$$\Sigma_{.k}(E) : E \to K(\mathbb{A}_k^{1,\mathrm{an}})$$

est continue.

Théorème 3.3. *Soient E une k-algèbre de Banach et $f \in E$. Si $\Sigma_{f,k}(E)$ est totalement discontinu, alors l'application $\Sigma_{.,k}(E) : E \to K(\mathbb{A}_k^{1,\mathrm{an}})$ est continue en f.*

Ces résultats existent déjà dans le cas complexe. Les preuves sont inspirées par le cas complexe. Nous donnerons aussi un exemple où cette application n'est pas continue.

3.3 Chapitre 3: Differential modules and Spectra

Dans le troisième chapitre, on introduit et étudie le spectre associé à un module différentiel. Le chapitre est organisé comme suit. Dans la première partie, nous donnons les notions de bases concernant les modules différentiels.

Dans la deuxième partie, nous introduisons le spectre associé à un module différentiel. Nous procédons comme suit. Soient X un domaine affinoïde de $\mathbb{A}_k^{1,\mathrm{an}}$ et x un point de type (2), (3) ou (4). Pour éviter les confusions nous fixons une autre fonction coordonnée S de X. Posons $A := \mathcal{O}(X)$ ou $\mathscr{H}(x)$ et $d = g(S)\frac{\mathrm{d}}{\mathrm{dS}}$ avec $g(S) \in A$. Soit (M, ∇) un module différentiel sur (A, d). Comme $M \simeq A^\nu$ pour un certain $\nu \in \mathbb{N} \setminus \{0\}$, on peut munir M de la structure de k-espace de Banach de A^ν (c'est un espace de Banach pour la norme max). Dans ce cas, $\nabla \in \mathcal{L}_k(M)$ (la k-algèbre de Banach des applications k-linéaires bornées de M pour la norme d'opérateur). Le spectre associé à (M, ∇) est le spectre $\Sigma_{\nabla,k}(\mathcal{L}_k(M))$ de ∇ vu comme élément de $\mathcal{L}_k(M)$. Ce spectre ne dépend pas du choix de l'isomorphisme $M \simeq A^\nabla$. Nous montrons comment il se comporte dans une suite exacte, et démontrons que pour des modules différentiels (M, ∇), (M_1, ∇_1) et (M_2, ∇_2) tels que $(M, \nabla) = (M_1, \nabla_1) \oplus (M_2, \nabla_2)$ dans la catégorie des modules différentiels sur (A, d), nous avons:

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{\nabla_1,k}(\mathcal{L}_k(M_2)) \cup \Sigma_{\nabla_2,k}(\mathcal{L}_k(M)). \tag{1}$$

Le résultat le plus important de cette partie est le suivant:

Proposition 3.4. *On suppose que k est algébriquement clos. Soient X un domaine affinoïde de $\mathbb{A}_k^{1,\mathrm{an}}$ et $x \in \mathbb{A}_k^{1,\mathrm{an}}$ un point de type (2), (3) ou (4). On pose $A := \mathcal{O}(X)$ ou $\mathscr{H}(x)$ et $d = g(S)\frac{\mathrm{d}}{\mathrm{dS}}$ avec $g \in A$. Soit (M, ∇) un module différentiel sur (A, d) tel qu'il existe une base pour laquelle la matrice associée G est à coefficients constants (i.e. $G \in \mathcal{M}_n(k)$). Alors le spectre de ∇ est $\Sigma_{\nabla,k}(\mathcal{L}_k(M)) := \bigcup_{i=1}^n (a_i + \Sigma_{d,k}(\mathcal{L}_k(A)))$, où $\{a_1, \cdots, a_n\} \subset k$ est l'ensemble des valeurs propres de G.*

En particulier le spectre dépend fortement du choix de la dérivation d. Cette proposition montre l'importance du calcul de $\Sigma_{d,k}(\mathcal{L}_k(A))$. C'est pour cela qu'une grande partie des Chapitres 4 et 5 est dédiée au calcul du spectre de $\frac{\mathrm{d}}{\mathrm{dS}}$ et $S\frac{\mathrm{d}}{\mathrm{dS}}$. Dans la troisième partie, nous démontrons la version spectrale en rang un du théorème de Young [You92], [Ked10, Theorem 6.5.3], [CM02, Theorem 6.2].

Théorème 3.5 (Young). *Soit $x \in \mathbb{A}_k^{1,\mathrm{an}}$ un point de type (2), (3) ou (4). Soit $\mathcal{L} = \sum\limits_{i=0}^{n} g_{n-i}\frac{\mathrm{d}}{\mathrm{dS}}^i$ avec $g_0 = 1$ et $g_i \in \mathscr{H}(x)$, et soit (M, ∇) le module différentiel associé sur $(\mathscr{H}(x), d)$. On pose $|\mathcal{L}|_{Sp} := \max\limits_{0 \leq i \leq n} |g_i|^{\frac{1}{i}}$. Si $|\mathcal{L}|_{Sp} > \|d\|$ alors $\|\nabla\|_{\mathrm{Sp}} = |\mathcal{L}|_{Sp}$.*

Notre énoncé est le suivant:

Théorème 3.6. *On suppose que k est algébriquement clos. Soient Ω une extension non-algébrique complète de k et $\pi_{\Omega/k} : \mathbb{A}_\Omega^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ la projection canonique. Soient $d : \Omega \to \Omega$ une k-dérivation bornée et (Ω, ∇) un module différentiel sur(Ω, d) avec $\nabla := d + f$ et $f \in \Omega$. Si $r_k(\pi_{\Omega/k}(f)) > \|d\|$ où $r_k(\pi_{\Omega/k}(f))$ est le rayon du point $\pi_{\Omega/k}(f)$, alors on a*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(\Omega)) = \{\pi_{\Omega/k}(f)\}.$$

Nous essayons d'étendre actuellement ce résultat à un rang quelconque. Nous conjecturons l'énoncé suivant:

Conjecture 3.7. *Soit $x \in \mathbb{A}_k^{1,\mathrm{an}}$un point de type (2), (3) ou (4). Soit d une dérivation bornée sur $\mathscr{H}(x)$ et soit (M,∇) module différentiel sur $(\mathscr{H}(x),d)$. Soit $\{m,\nabla(m),\cdots,\nabla^{n-1}(m)\}$ une base cyclique et soit G la matrice associée dans cette base. Soit $\Sigma_{G,k}(\mathcal{M}_n(\mathscr{H}(x))) := \{y_1,\cdots,y_N\}$. Alors pour chaque il existe $C_G := (C_{y_1},\cdots,C_{y_N}) \in \mathbb{R}_+^N$ qui ne dépend que de G, tel que*

$$r_k(y_i) > C_{y_i}\|d\| \Rightarrow y_i \in \Sigma_{\nabla,k}(\mathcal{L}_k(M)).$$

Si de plus, pour tout $y_i \in \Sigma_{G,k}(\mathcal{M}_n(\mathscr{H}(x)))$ on a $r_k(y_i) > C_y\|d\|$. Alors

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{G,k}(\mathcal{M}_n(\mathscr{H}(x)).$$

Dans la dernière partie, nous observons que le spectre est un invariant par le push-forward par un morphisme étale. Plus précisément: soit $\varphi : Y \to X$ un morphisme étale. Soit $y \in Y$ un point de type différent de (1) et $x := \varphi(y)$. L'application φ induit une extension finie $\mathscr{H}(x) \hookrightarrow \mathscr{H}(y)$. Soit d une dérivation bornée sur $\mathscr{H}(x)$, et soit $\varphi^* d$ le pull-back de d par φ, qui est bornée et prolonge d. Pour tout module différentiel (M,∇) sur $(\mathscr{H}(y),\varphi^* d)$ le push-forward $(\varphi_* M,\varphi_*\nabla)$ (obtenu par restriction de scalaires) est un module différentiel sur $(\mathscr{H}(x),d)$. Comme (M,∇) et $(\varphi_* M,\varphi_*\nabla)$ coïncident comme k-espaces de Banach, leurs spectres respectifs vérifient:

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{\varphi_*\nabla,k}(\mathcal{L}_k(\varphi_* M)). \quad (2)$$

Même si il n'est pas difficile de voir cela, cette propriété nous permet de calculer les spectres d'une classe importante de modules différentiels (cf. Chapters 5 and 6).

3.4 Chapitre 4: Spectrum of a differential equation with constant coefficients

Le quatrième chapitre est dédié au calcul du spectre d'une équation différentielle à coefficients constants et à l'étude de sa variation. Nous distinguerons la notion d'équation différentielle dans le sens d'un polynôme différentiel $P(d)$ opérant sur A de la notion de module différentiel (M,∇) sur (A,d) associé à $P(d)$ dans une base cyclique. En effet, il n'est pas difficile de montrer que le spectre de $P(\frac{\mathrm{d}}{\mathrm{dS}}) = (\frac{\mathrm{d}}{\mathrm{dS}})^n + a_{n-1}(\frac{\mathrm{d}}{\mathrm{dS}})^{n-1} + \cdots + a_0$ comme élément de $\mathcal{L}_k(A)$, où $a_i \in k$, est donné par la formule (cf. [Bou07, p. 2] and Lemma 2.1.16)

$$\Sigma_{P(\frac{\mathrm{d}}{\mathrm{dS}})}(\mathcal{L}_k(A)) = P(\Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}(\mathcal{L}_k(A))). \quad (3)$$

Cet ensemble est toujours soit un disque fermé soit l'adhérence d'un disque ouvert (cf. Lemma 2.1.16, Remark 4.2.4). De façon assez surprenante, il est différent du spectre $\Sigma_\nabla(\mathcal{L}_k(M))$ du module différentiel (M,∇) associé à $P(\frac{\mathrm{d}}{\mathrm{dS}})$ dans une base cyclique (i.e le spectre de ∇ comme élément de $\mathcal{L}_k(M)$). En effet, ce dernier est une union finie soit de disques fermés soit d'adhérences de disques ouverts (cf. Theorem 3.8) centrés en les racines du polynôme (commutatif) $Q = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in k[X]$ associé à P. Afin d'introduire notre résultat suivant, on note $\tilde{k}$ le corps résiduel de k et on pose:

$$\omega = \begin{cases} |p|^{\frac{1}{p-1}} & \text{si } \mathrm{car}(\tilde{k}) = p \\ 1 & \text{si } \mathrm{car}(\tilde{k}) = 0 \end{cases}. \tag{4}$$

Le résultat principal de ce chapitre est le suivant:

Théorème 3.8. *On suppose que k est algébriquement clos. Soient X un domaine affinoïde connexe de $\mathbb{A}_k^{1,\mathrm{an}}$ et $x \in \mathbb{A}_k^{1,\mathrm{an}}$ un point de type (2), (3) ou (4). On pose $A := \mathcal{O}(X)$ ou $\mathscr{H}(x)$. Soit (M, ∇) un module différentiel sur $(A, \frac{\mathrm{d}}{\mathrm{dS}})$ tel qu'il existe une base pour laquelle la matrice associée G est à coefficients constants (i.e. $G \in \mathcal{M}_n(k)$). Soit $\{a_1, \dots, a_N\} \subset k$ l'ensemble des valeurs propres de G. Le comportement du spectre $\Sigma_\nabla(\mathcal{L}_k(M))$ de ∇ comme élément de $\mathcal{L}_k(M)$ est récapitulé dans le tableau suivant:*

$A =$	$\mathcal{O}(X)$		$\mathscr{H}(x)$	
	$X = D^+(c, r)$	$X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$	*x de type (2) ou (3)*	*x de type (4)*
$\mathrm{car}(\tilde{k}) = p > 0$	$\bigcup_{i=1}^{N} D^+(a_i, \frac{\omega}{r})$	$\bigcup_{i=1}^{N} D^+(a_i, \frac{\omega}{\min_j r_j})$	$\bigcup_{i=1}^{N} D^+(a_i, \frac{\omega}{r(x)})$	
$\mathrm{car}(\tilde{k}) = 0$	$\bigcup_{i=1}^{N} (D^-(a_i, \frac{1}{r}) \cup \{x_{a_i, \frac{1}{r}}\})$	$\bigcup_{i=1}^{N} D^+(a_i, \frac{1}{\min_j r_j})$	$\bigcup_{i=1}^{N} D^+(a_i, \frac{1}{r(x)})$	$\bigcup_{i=1}^{N} (D^-(a_i, \frac{1}{r(x)}) \cup \{x_{a_i, \frac{1}{r(x)}}\})$

La preuve de tout ces cas consiste à calculer le spectre de $\frac{\mathrm{d}}{\mathrm{dS}}$ et appliquer la Proposition 3.4.

En utilisant ce résultat nous démontrons que la variation du spectre vérifie une propriété de continuité:

Soit X un domaine affinoïde de $\mathbb{A}_k^{1,\mathrm{an}}$. Soit (M, ∇) un module différentiel sur $(\mathcal{O}(X), \frac{\mathrm{d}}{\mathrm{dS}})$ tel qu'il existe une base pour laquelle la matrice associée G est à coefficients constants. Pour un point $x \in X$ de type différent de (1), le module différentiel (M, ∇) s'étend à un module différentiel sur $(\mathscr{H}(x), \frac{\mathrm{d}}{\mathrm{dS}})$. Dans la base correspondante de (M_x, ∇_x) la matrice associée est G.

Théorème 3.9. *On suppose que k est algébriquement clos. Soient $X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$ un domaine affinoïde et $x \in X$ un point de type (2), (3) ou (4). Soit (M, ∇) un module différentiel sur $(\mathcal{O}(X), \frac{\mathrm{d}}{\mathrm{dS}})$ tel qu'il existe une base pour laquelle la matrice associée G est à coefficients constants. On pose:*

$$\begin{array}{rccc} \Psi : & [x, x_{c_0, r_0}] & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ & y & \mapsto & \Sigma_{\nabla_y}(\mathcal{L}_k(M_y)) \end{array}.$$

Alors on a

- *Pour tout $y \in [x, x_{c_0, r_0}]$, la restriction de Ψ à $[x, y]$ est continue en y.*

- *Si* $y \in [x, x_{c_0,r_0}]$ *est un point de type (3), alors* Ψ *est continue en* y.
- *Si* $\mathrm{char}(\tilde{k}) = 0$ *et* $y \in [x, x_{c_0,r_0}]$ *est un point de type (4), alors* Ψ *est continue en* y.

3.5 Chapitre 5: Spectrum of a regular singular differential module

Le cinquième chapitre est consacré au calcul du spectre d'un module différentiel singulier régulier (M, ∇) sur $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$, avec $x \in \mathbb{A}_k^{1,\mathrm{an}}$ un point de type (2), (3) ou (4). Nous entendons par module différentiel singulier régulier sur $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$, un module différentiel (M, ∇) tel qu'il existe une base pour laquelle la matrice associée G est à coefficients constants (i.e. $G \in \mathcal{M}_n(k)$). Soit $\mathrm{Log}_c : D^-(c, |c|) \to \mathbb{A}_k^{1,\mathrm{an}}$ la fonction analytique associée au morphisme d'anneaux

$$\begin{array}{ccc} k[T] & \longrightarrow & \mathcal{O}(D^-(c,|c|)) \\ T & \mapsto & \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{(-1)^n}{c^n n}(T-c)^n. \end{array} \tag{5}$$

Le résultat principal de ce chapitre est le suivant:

Théorème 3.10. *On suppose que* k *est algébriquement clos. Soit* $x \in \mathbb{A}_k^{1,\mathrm{an}}$ *un point de type (2), (3) ou (4). Soit* (M, ∇) *un module différentiel sur* $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$ *tel qu'il existe une base pour laquelle* G *est à coefficients constants (i.e.* $G \in \mathcal{M}_n(k)$*). Soit* $\{a_1, \dots, a_N\} \subset k$ *l'ensemble des valeurs propres* G*. Le comportement du spectre* $\Sigma_\nabla(\mathcal{L}_k(M))$ *de* ∇ *vu comme élément de* $\mathcal{L}_k(M)$ *est récapitulé dans le tableau suivant:*

	$x = x_{0,r}$		$x \in D^-(c,\|c\|)$ *avec* $c \in k \setminus \{0\}$	
	x *de type (2)*	x *de type (3)*	x *de type (2) ou (3)*	x *de type (4)*
$\mathrm{car}(\tilde{k}) = p > 0$	$\bigcup_{i=1}^N a_i + \mathbb{Z}_p$		$\bigcup_{i=1}^N \bigcup_{j\in\mathbb{N}} D^+(a_i + j, \frac{\omega}{r_k(\mathrm{Log}_c(x))})$	
$\mathrm{car}(\tilde{k}) = 0$	$\bigcup_{i=1}^N D^+(a_i, 1)$	$\bigcup_{i=1}^N a_i + (\mathbb{Z} \cup \{x_{0,1}\})$	$\bigcup_{i=1}^N D^+(a_i, \frac{\omega}{r_k(\mathrm{Log}_c(x))})$	$\bigcup_{i=1}^N (D^-(a_i, \frac{\omega}{r_k(\mathrm{Log}_c(x))}) \cup \{x_{a_i, \frac{\omega}{r_k(\mathrm{Log}_c(x))}}\})$

La stratégie de la preuve consiste à calculer d'abord le spectre de la dérivation $S\frac{\mathrm{d}}{\mathrm{dS}}$. Ensuite, comme déjà mentionné, nous appliquons la proposition 3.4. Par conséquent, les cas récapitulés ci-dessus découlent des différentes formes du spectre de $S\frac{\mathrm{d}}{\mathrm{dS}}$. Dans le cas où $x = x_{0,r}$ pour $r > 0$, les méthodes employées dans les trois cas pour calculer $\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x)))$ sont complètement différentes. En effet, dans le cas où $\mathrm{car}(\tilde{k}) = p > 0$, nous obtenons le spectre grâce au push-forward par la fonction Frobenius. En revanche, pour les autres cas nous calculons le spectre directement. Dans le cas où $x \in D^-(c, |c|)$ où $c \in k \setminus \{0\}$, à l'aide de la formule (2), le push-forward par Log_c nous permet de réduire le calcul au calcul du spectre d'un module différentiel sur $(\mathscr{H}(\mathrm{Log}_c(x)), \frac{\mathrm{d}}{\mathrm{dS}})$ à coefficients constants. Comme le degré de Log_c dans le cas où $\mathrm{car}(\tilde{k}) = p > 0$ diffère du cas où $\mathrm{car}(\tilde{k}) = 0$, les spectres sont légèrement différents.

Ce résultat montre que: contrairement au cas traité dans le chapitre 4, dans le cas car$(\tilde{k}) = 0$, la variation du spectre peut ne pas être du tout continue. Néanmoins, nous avons le resultat de continuité:

Soit X un domaine affinoïde de $\mathbb{A}_k^{1,\mathrm{an}}$. Soit (M, ∇) un module différentiel sur $(\mathcal{O}(X), S\frac{\mathrm{d}}{\mathrm{dS}})$ tel qu'il existe une base pour laquelle la matrice associée G est à coefficients constants. Pour un point $x \in X$ de type différent de (1), le module différentiel (M, ∇) s'étend à un module différentiel sur $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$. Dans la base correspondante de (M_x, ∇_x) la matrice associée est G.

Théorème 3.11. *On suppose que k est algébriquement clos et* $\mathrm{car}(\tilde{k}) = p > 0$ *ou* $|k| = \mathbb{R}_+$. *Soient* $X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$ *un domaine affinoïde et* $x \in X$ *un point de type (2), (3) ou (4). Soit* (M, ∇) *un module différentiel sur* $(\mathcal{O}_X(X), S\frac{\mathrm{d}}{\mathrm{dS}})$ *tel qu'il existe une base pour laquelle G est à coefficients constants. On pose:*

$$\begin{array}{rcl} \Psi : [x, x_{c_0,r_0}] & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ y & \mapsto & \Sigma_{\nabla_y,k}(\mathcal{L}_k(M_y)) \end{array}.$$

Alors on a:

- *pour* $y \in [x, x_{c_0,r_0}]$, *la restriction de* Ψ *à* $[x, y]$ *est continue en* y.
- *Si* $y \in [x, x_{c_0,r_0}]$ *est un point de la forme* $x_{0,R}$, *alors* Ψ *est continue en* y.
- *Si* $y \in [x, x_{c_0,r_0}]$ *est un poin de type (3), alors* Ψ *est continue en* y.

3.6 Chapitre 6: Spectrum of a linear differential equation over a field of formal power series

Dans le sixième chapitre nous calculons le spectre d'un module différentiel (M, ∇) sur $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$, où k est trivialement valué et algébriquement clos, et $k((S))$ est le corps des séries formelles de Laurent et muni de la valeur absolue S-adique donnée par $|\sum_{i\geq N} a_i S^i| := r^N$, si $a_N \neq 0$, avec $r < 1$. On fait remarquer que le choix de r identifie (non canoniquement) $(k((S)), |.|)$ avec le corps $\mathscr{H}(x_{0,r})$ du point $x_{0,r} \in \mathbb{A}_k^{1,\mathrm{an}}$. Le résultat de ce chapitre dépend fortement de cette identification. Il semblerait, en particulier, que le spectre de Berkovich n'est pas indépendent de r, même si la notion classique du spectre est purement algébrique.

Théorème 3.12. *Soit* (M, ∇) *un module différentiel sur* $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$. *Soit* $\{\gamma_1, \cdots, \gamma_{\nu_1}\}$ *l'ensemble des pentes de* (M, ∇). *Soit* $\{a_1, \cdots, a_{\nu_2}\}$ *l'ensemble des exposants de la partie régulière de*(M, ∇) . *Le spectre de* ∇ *vu comme élément de* $\mathcal{L}_k(M)$ *est:*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \{x_{0,r^{-\gamma_1}}, \cdots, x_{0,r^{-\gamma_{\nu_1}}}\} \cup \bigcup_{i=1}^{\nu_2}(a_i + \mathbb{Z}).$$

Nous exposons maintenant l'idée de la preuve. La décomposition selon les pentes du polygone de Newton d'un module différentiel nous permet de décrire (M, ∇) comme une somme directe de modules différentiels sur $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$:

$$(M, \nabla) = (M_{\text{reg}}, \nabla_{\text{reg}}) \oplus (M_{\text{irr}}, \nabla_{\text{irr}})$$

où $(M_{\text{reg}}, \nabla_{\text{reg}})$ est un module différentiel singulier régulier et $(M_{\text{irr}}, \nabla_{\text{irr}})$ est un module différentiel irrégulier sans partie régulière. Par la formule (cf. (1)), nous avons $\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{\nabla_1,k}(\mathcal{L}_k(M_1)) \cup \Sigma_{\nabla_2,k}(\mathcal{L}_k(M_2))$.

Par conséquent, afin d'obtenir $\Sigma_{\nabla,k}(\mathcal{L}_k(M))$, il suffit de connaître le spectre d'un module singulier régulier et le spectre d'un module irrégulier sans partie régulière. Le premier cas est déjà traité dans le Chapitre 5. Il reste à traiter le cas purement irrégulier. Pour ce cas nous procédons comme suit. On sait que, tout module différentiel peut s'écrire comme extension de modules différentiels de rang un après extension des scalaires (cf. [Tur55]). En particulier, pour $(M_{\text{irr}}, \nabla_{\text{irr}})$, les modules différentiels de rang un obtenus vérifient les conditions du Théorème 3.6. Par conséquent, le spectre d'un tel module différentiel de rang un est $\{x_{0,r^\gamma}\}$, où γ correspond à une pente de $(M_{\text{irr}}, \nabla_{\text{irr}})$. Nous montrons par des techniques de push-forward et pull-back que, le spectre de $(M_{\text{irr}}, \nabla_{\text{irr}})$ coïncide avec le spectre du module différentiel obtenu par une extension de scalaires et nous concluons la preuve.

4 Perspectives

Notre stratégie pour calculer le spectre d'un module différentiel quelconque défini sur un domaine affinoïde de $\mathbb{A}_k^{1,\text{an}}$ est la suivante. On a besoin de démontrer d'abord la Conjecture 3.7. Après, on doit montrer que le push-forward par la fonction Frobenius (ou un autre morphisme étale convenable) du module différentiel vérifie les conditions de la conjecture. Nous nous attendons à ce que le spectre ait la forme suivante:

Conjecture 4.1. *Supposons que k est algébriquement clos. Soit $x \in \mathbb{A}_k^{1,\text{an}}$ un point de type (2), (3) ou (4). Soit $d : \mathscr{H}(x) \to \mathscr{H}(x)$ une dérivation k-linéaire bornée. Soit (M, ∇) un module différentiel sur $(\mathscr{H}(x), d)$ de rang n. Si $\nabla - a$ est injective pour tout $a \in k$ alors $\Sigma_{\nabla,k}(\mathcal{L}_k(M))$ est un ensemble fini de points de type différent de (1). Plus généralement, le spectre est de la forme*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = (\bigcup_{i=1}^{N} a_i + \Sigma_{d,k}(\mathcal{L}_k(M))) \cup \{x_1, \cdots, x_{N'}\}$$

avec $a_i \in k$, $x_j \in \mathbb{A}_k^{1,\text{an}}$ sont des points de type différent de (1) et $N \leq n$. Notons que N' peut être plus grand que n.

Introduction (english version)

Differential equations constitute an important tool for the investigation of algebraic and analytic varieties, over the complex and the p-adic numbers. Notably, de Rham cohomology is one of the most powerful way to obtain algebraic and analytic informations. Besides, ultrametric phenomena appeared naturally studying formal Taylor solutions of the equation around singular and regular points.

As soon as the theory of ultrametric differential equations became a central topic of investigation around 1960, after the work of B. Dwork, P. Robba (et al.), the following interesting phenomena appeared. In the ultrametric setting, the solutions of a linear differential equation may fail to converge as expected, even if the coefficients of the equation are entire functions. For example, over the ground field $\mathbb{Q}_p$ of p-adic numbers, the exponential power series $\exp(T) = \sum_{n\geq 0} \frac{T^n}{n!}$ which is solution of the equation $y' = y$ has radius of convergence equal to $|p|^{\frac{1}{(p-1)}}$ even though the equation shows no singularities. However, the behaviour of the radius of convergence is well controlled, and its knowledge permits to obtain several informations about the equation. Namely it controls the *finite dimensionality* of the de Rham cohomology.

The use of Berkovich language introduced in [Ber90] and [Ber93] allowed to show that the variation of this radius is well controlled by a finite sub-graph of an analytic curve, in the sense of Berkovich. More precisely, since the works of F. Baldassari and L. Di Vizio in [BD08] and F. Baldassari in [Bal10], we know in which way we associate a radius of convergence to a point that belongs to a p-adic curve endowed with a connection, in the sense of Berkovich. In the paper [Bal10], F. Baldassari shows that the radius of convergence, in the setting of a curve, is a continuous function. He also conjectures that it factorizes trough a locally finite graph, moreover the curve retracts on this graph. In the work of J. Poineau and A. Pulita, this graph is called *controlling graph*. In paper [Pul15], A. Pulita proves the conjecture for the case of a differential module defined over an affinoid domain of the analytic line $\mathbb{A}_k^{1,\mathrm{an}}$. He also generalizes these results to the subsidiary radii, i.e. the respective radii of each solutions of the equation. These results are generalized and investigated in the setting of a quasi-smooth curve in the papers of J. Poineau and A. Pulita [PP15] and [PP13a]. In [PP13b], they provide necessary and sufficient criterion for the finiteness of the de Rham cohomology. This criterion is fulfilled if the controlling graph is finite.

In our work, we focus on the study of an invariant more finer then the radius of convergence, the spectrum in the sense of Berkovich of a differential module. Our main goal is to connect the above results with the spectrum of the connection in the sense of Berkovich. In the present thesis, we introduce the spectrum associated to a differential module, and provide some properties. We also compute the spectra of some classes of differential modules: differential modules of a differential equation with constant coefficients, singular regular modules and at last differential modules over the field of Laurent power series.

1 Motivation

The starting point of the thesis is an interesting relation between this radius and the notion of *spectrum* (in the sense of Berkovich). Indeed, consider a ring A together with a derivation $d : A \to A$. A *differential module* on (A, d) is a finite free A-module M together with a $\mathbb{Z}$-linear map

$$\nabla : M \to M$$

satisfying for all $m \in M$ and all $f \in A$ the relation $\nabla(fm) = d(f)m + f\nabla(m)$. If an isomorphism $M \cong A^\nu$ is given, then ∇ coincides with an operator of the form

$$d + G : A^\nu \to A^\nu$$

where d acts on A^ν componentwise and G is a square matrix with coefficients in A.

If A is moreover a Banach algebra with respect to a given norm $\|.\|$ and A^ν is endowed with the max norm, then we can endow the algebra of bounded $\mathbb{Z}$-linear endomorphisms with the operator norm, that we still denote by $\|.\|$. The spectral norm of ∇ is given by

$$\|\nabla\|_{\mathrm{Sp}} = \lim_{n\to\infty} \|\nabla^n\|^{\frac{1}{n}}.$$

The link between the radius of convergence of ∇ and its spectrum is then the following: on the one hand, when $\|.\|$ is a Berkovich point x of type other than 1 of the analytic affine line and A is the field of this point, the spectral norm $\|\nabla\|_{\mathrm{Sp}}$ coincides with the inverse of the radius of convergence of M at x multiplied by some constant (cf. [CD94, p. 676] or [Ked10, Definition 9.4.4]). On the other hand the spectral norm $\|\nabla\|_{\mathrm{Sp}}$ is also equal to the radius of the smallest disk centred at zero and containing the spectrum of ∇ in the sense of Berkovich (cf. [Ber90, Theorem 7.1.2]).

The spectrum appears then as a new invariant of the connection ∇ generalizing and refining the radius of convergence. This has been our first motivation, however the study of the spectrum of an operator has its own interest and it deserves its own independent theory.

2 Spectral theory in the sense of Berkovich

Let k be an arbitrary field and E be a non-zero k-algebra with unit. Recall that classically, the spectrum $\Sigma_{f,k}(E)$ of an element f of E (cf. [Bou07, §1.2. Defintion 1]) is the set of elements λ of k such that $f-\lambda.1_E$ is not invertible in E. In the case where $k=\mathbb{C}$ and $(E,\|.\|)$ is a $\mathbb{C}$-Banach algebra, the spectrum of f satisfies the following properties:

- It is not empty and compact.
- The smallest disk centred at 0 and containing $\Sigma_{f,k}(E)$ has radius equal to $\|f\|_{\mathrm{Sp}}=\lim\limits_{n\to+\infty}\|f^n\|^{\frac{1}{n}}$.
- The resolvent function $R_f:\mathbb{C}\setminus\Sigma_{f,k}(E)\to E$, $\lambda\mapsto(f-\lambda.1_E)^{-1}$ is an analytic function with values in E.

Unfortunately, this may fail in the ultrametric case. In [Vis85] M. Vishik provides an example of operator with empty spectrum and with a resolvent which is only locally analytic. We illustrate this pathology with an example in our context, where connections with empty classical spectrum abound.

Example 2.1. Let k be an algebraically closed field of characteristic zero endowed with the trivial absolute value. Let $A:=k((S))$ be the field of Laurent power series endowed with the S-adic absolute value given by $|\sum_{n\geq n_0}a_nS^n|=r^{n_0}$, if $a_{n_0}\neq 0$, where $|S|=r<1$ is a nonzero real number [1]. We consider a rank one *irregular* differential module over $A=k((S))$ defined by the operator $\frac{\mathrm{d}}{\mathrm{dS}}+g:A\to A$, where $g\in A$ has S-adic valuation n_0 that is less than or equal to -2. We consider the connection $\nabla=\frac{\mathrm{d}}{\mathrm{dS}}+g$ as an element of the k-Banach algebra $E=\mathcal{L}_k(k((S)))$ of bounded k-linear maps of $k((S))$ with respect to the usual operator norm: $\|\varphi\|=\sup\limits_{f\in A\setminus\{0\}}\frac{|\varphi(f)|}{\|f\|}$, for all $\varphi\in E$. Then, the classical spectrum of the differentiel operator $\frac{\mathrm{d}}{\mathrm{dS}}+g$ is empty. Indeed, since $\|\frac{\mathrm{d}}{\mathrm{dS}}\|=\frac{1}{r}$ and $|(g-a)^{-1}|\leq r^2$ for all $a\in k$, then $\|(g-a)^{-1}\circ(\frac{\mathrm{d}}{\mathrm{dS}})\|\leq r$ (resp. $\|(\frac{\mathrm{d}}{\mathrm{dS}})\circ(g-a)^{-1}\|\leq r$) and the series $\sum_{n\geq 0}(-1)^n\cdot((g-a)^{-1}\circ(\frac{\mathrm{d}}{\mathrm{dS}}))^n$ (resp. $\sum_{n\geq 0}(-1)^n((\frac{\mathrm{d}}{\mathrm{dS}})\circ(g-a)^{-1})^n$) converges in $\mathcal{L}_k(k((S)))$. Hence, for all $a\in k$, $(\sum_{n\geq 0}(-1)^n\cdot((g-a)^{-1}\circ(\frac{\mathrm{d}}{\mathrm{dS}}))^n)\circ(g-a)^{-1}$ (resp. $(g-a)^{-1}\circ(\sum_{n\geq 0}(-1)^n\cdot((\frac{\mathrm{d}}{\mathrm{dS}})\circ(g-a)^{-1})^n)$ is a left (resp. right) inverse of $\frac{\mathrm{d}}{\mathrm{dS}}+(g-a)$ in $\mathcal{L}_k(k((S)))$ and a does not belong to the spectrum. [2] Thus the classical spectrum is empty.

1. In the language of Berkovich, this field can be naturally identified with the complete residual field of the point $x_{0,r}\in\mathbb{A}_k^{1,\mathrm{an}}$.

2. Notice that it is relatively easy to show that any non trivial rank one connection over $k((S))$ is set theoretically bijective. This follows from the classical index theorem of B. Malgrange [Mal74]. However, the set theoretical inverse of the connection may not be automatically bounded. This is due to the fact that the base field k is trivially valued and the Banach open mapping theorem does not hold in general. However, it is possible to prove that any such set theoretical inverse is bounded (cf. Chapter 6).

To deal with this issue V. Berkovich understood that it was better not to define the spectrum as a subset of the base field k, but as a subset of the analytic line $\mathbb{A}_k^{1,\mathrm{an}}$, which is a bigger space[3]. His theory of analytic spaces (cf. [Ber90], [Ber93]) enjoys several good local topological properties such as compactness, path connectedness ... In this setting Berkovich developed a spectral theory for ultrametric operators [Ber90, Chapter 7]. The definition of the spectrum given by Berkovich is the following: Let $(k, |.|)$ be complete field with respect to an ultrametric absolute value and let $\mathbb{A}_k^{1,\mathrm{an}}$ be the Berkovich affine line. For a point $x \in \mathbb{A}_k^{1,\mathrm{an}}$ we denote by $\mathscr{H}(x)$ the associated completed residue field. We fix on $\mathbb{A}_k^{1,\mathrm{an}}$ a coordinate function T. The spectrum $\Sigma_{f,k}(E)$ of an element f of a k-Banach algebra E is the set of points $x \in \mathbb{A}_k^{1,\mathrm{an}}$, such that $f \otimes 1 - 1 \otimes T(x)$ is not invertible in $E\hat{\otimes}_k\mathscr{H}(x)$. It can be proved (cf. [Ber90, Proposition 7.1.6]) that this is equivalent to say that there exists a complete valued field Ω containing isometrically k and a constant $c \in \Omega$ such that

- the image of c by the canonical projection $\mathbb{A}_\Omega^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ is x;
- $f \otimes 1 - 1 \otimes c$ is not invertible as an element of $E\hat{\otimes}_k\Omega$.

In some sense $\Omega = \mathscr{H}(x)$ and $c = T(x)$ are the minimal possible choices. This spectrum is compact, non-empty if E is non-zero algebra and it satisfies the properties listed above (cf. [Ber90, Theorem 7.12]). Notice that if $f = \nabla$ is a operator and if $E = \mathcal{L}_k(M)$ is the k-Banach algebra of bounded k-linear endomorphisms of M, then $f \otimes 1 - 1 \otimes c$ is no more a connection as an element of $E\hat{\otimes}_k\Omega$. Indeed, $\mathcal{L}_k(M)\hat{\otimes}_k\Omega$ does not coincide with $\mathcal{L}_\Omega(M\hat{\otimes}_k\Omega)$, unless Ω is a finite extension. In particular, *no index theorem can be used* to test the set theoretical bijectivity of $\nabla \otimes 1 - 1 \otimes c$.

Coming back to the above example, using this definition it can be proved that the spectrum of $\frac{\mathrm{d}}{\mathrm{dS}} + g$ is now reduced to the individual *non-rational* point $x_{0,r^{n_0}} \in \mathbb{A}_{\mathbb{C}}^{1,\mathrm{an}}$(cf. Chapter 6). The dependence on r shows that the Berkovich spectrum *depends on the chosen absolute value on k*; whereas, instead, the classical spectrum is a completely algebraic notion.

3 Description of chapters

In this part we describe the contain and main statements of each chapter. Before, we provide some notation:

Notation 3.1. Let $(k, |.|)$ be an ultrametric complete field of characteristic zero. Let $c \in k$ and $r \in \mathbb{R}_+$. We denote by $D^+(c, r)$ the closed disk of $\mathbb{A}_k^{1,\mathrm{an}}$ centred at c and with radius r, and $D^-(c, r)$ the open disk of $\mathbb{A}_k^{1,\mathrm{an}}$ centred at c and with radius r. The point $x_{c,r} \in \mathbb{A}_k^{1,\mathrm{an}}$ will corespond to the Shilov boundary of the disk $D^+(c, r)$.

3. This can be motivated by the fact that the resolvent is an analytic function on the complement of the spectrum.

3.1 Chapter 1: Basic notions

The first chapter is devoted to providing setting and notation. We recall definitions and properties related to Banach spaces. We also recall the general definition of analytic spaces and maps in the sense of Berkovich, and more specifically the analytic affine line $\mathbb{A}_k^{1,\mathrm{an}}$.

3.2 Chapter 2: Spectral theory in the sense of Berkovich

The second chapter is devoted to the study of the spectrum in the sense of Berkovich. We are interested, more precisely, to the variation of the spectrum. For that we first define a topology on $K(\mathbb{A}_k^{1,\mathrm{an}})$ the set of nonempty compact subsets. We prove that, in the case where $\mathbb{A}_k^{1,\mathrm{an}}$ is metrisable, this topology coincides with the topology induced by the Hausdorff metric on $K(\mathbb{A}_k^{1,\mathrm{an}})$. We then study the continuity of the map

$$\begin{array}{rcl}\Sigma_{.,k}(E) : E & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ f & \mapsto & \Sigma_{f,k}(E),\end{array}$$

where $\Sigma_{f,k}(E)$ is the spectrum of f as an element of E. We observe that in general this map is not continuous. Nevertheless, we have the following continuity results.

Theorem 3.2. *Let E be a commutative k-Banach algebra. The spectrum map*

$$\Sigma_{.,k} : E \to K(\mathbb{A}_k^{1,\mathrm{an}})$$

is continuous.

Theorem 3.3. *Let E be a k-Banach algebra and $f \in E$. If $\Sigma_{f,k}(E)$ is totally disconnected, then the map $\Sigma_{.,k} : E \to K(\mathbb{A}_k^{1,\mathrm{an}})$ is continuous at f.*

These results already exist in the complex case. The proofs are inspired by the complex case. We provide also an example where this function is not continuous.

3.3 Chapter 3: Differential modules and Spectra

The third chapter is devoted to introducing and studying the spectrum associated to a differential module. The chapter is organized as follows. In the first part, we provide setting and notations related to differential modules.

In the second part, we introduce the spectrum associated to a differential module. We proceed as follows. Let X be an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$ and x a point of type (2), (3) or (4). To avoid confusion we fix another coordinate function S over X. We set

$A = \mathcal{O}(X)$ or $\mathscr{H}(x)$ and $d = g(S)\frac{\mathrm{d}}{\mathrm{dS}}$ with $g(S) \in A$. Let (M, ∇) be a differential module over (A, d). Since $M \simeq A^\nu$ for some $\nu \in \mathbb{N} \setminus \{0\}$, we can endow M with the structure of k-Banach space of A^ν. In this setting, ∇ is an element of $\mathcal{L}_k(M)$ (the k-Banach algebra of bounded k-linear map of M with respect to the usual operator norm). Then the spectrum associated to (M, ∇) is the spectrum $\Sigma_{\nabla,k}(\mathcal{L}_k(M))$ of ∇ as an element of $\mathcal{L}_k(M)$. This spectrum does not depend on the choice of the isomorphism $M \simeq A^\nu$. Then we show how it behaves under exact sequences, and prove that for differential modules (M, ∇), (M_1, ∇_1) and (M_2, ∇_2) such that $(M, \nabla) = (M_1, \nabla_1) \oplus (M_2, \nabla_2)$ in the category of differential modules over (A, d), we have:

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{\nabla_1,k}(\mathcal{L}_k(M_2)) \cup \Sigma_{\nabla_2,k}(\mathcal{L}_k(M)). \tag{6}$$

The most important result of this part is the following:

Proposition 3.4. *We suppose that k is algebraically closed. Let X be an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$ and $x \in \mathbb{A}_k^{1,\mathrm{an}}$ a point of type (2), (3) or (4). We set $A = \mathcal{O}(X)$ or $\mathscr{H}(x)$, and let $d = g(S)d/dS$, with $g \in A$. Let (M, ∇) be a differential module over (A, d) such that there exists a basis for which the associated matrix G has constant entries (i.e $G \in \mathcal{M}_n(k)$). Then the spectrum of ∇ is $\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \bigcup_{i=1}^{n}(a_i + \Sigma_{d,k}(\mathcal{L}_k(A)))$, where $\{a_1, \dots, a_n\} \subset k$ is the multiset of the eigenvalues of G.*

In particular the spectrum highly depends on the choice of the derivation d. This claim shows the importance of computing the spectrum of $\Sigma_{d,k}(\mathcal{L}_k(A))$. Therefore, in Chapters 4 and 5, a large part is devoted to the computation of the spectrum of $\frac{\mathrm{d}}{\mathrm{dS}}$ and $S\frac{\mathrm{d}}{\mathrm{dS}}$.

In the third part, we prove a spectral version in rank one of Young's theorem [You92], [Ked10, Theorem 6.5.3], [CM02, Theorem 6.2], which states the following :

Theorem 3.5 (Young). *Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2), (3) or (4). Let $\mathcal{L} = \sum_{i=0}^{n} g_{n-i}\frac{\mathrm{d}}{\mathrm{dS}}^i$ with $g_0 = 1$ and $g_i \in \mathscr{H}(x)$, and let (M, ∇) be the associated differential module over $(\mathscr{H}(x), \frac{\mathrm{d}}{\mathrm{dS}})$. We set $|\mathcal{L}|_{Sp} = \max_{0 \le i \le n} |g_i|^{\frac{1}{i}}$. If $|\mathcal{L}|_{Sp} > \|\frac{\mathrm{d}}{\mathrm{dS}}\|$ then $\|\nabla\|_{\mathrm{Sp}} = |\mathcal{L}|_{Sp}$.*

Our statement is the following:

Theorem 3.6. *Assume that k is algebraically closed. Let Ω be a non-algebraic complete extension of k and $\pi_{\Omega/k} : \mathbb{A}_\Omega^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ be the canonical projection. Let (Ω, ∇) be the differential module over (Ω, d) with $\nabla := d + f$ and $f \in \Omega$. If $r_k(\pi_{\Omega/k}(f)) > \|d\|$ where $r_k(\pi_{\Omega/k}(f)$ is the radius of the point $\pi_{\Omega/k}(f)$, then we have*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(\Omega)) = \{\pi_{\Omega/k}(f)\}.$$

We are currently trying to extend the statement to higher rank. We conjecture the following:

Conjecture 3.7. *Let $\mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2), (3) or (4). Let d be a bounded derivation on $\mathscr{H}(x)$ and let (M,∇) be a differential module over $(\mathscr{H}(x),d)$. Let $\{m,\nabla(m),\cdots,\nabla^{n-1}(m)\}$ be a cyclic basis and let G be the associated matrix in this basis. Let $\Sigma_{G,k}(\mathcal{M}_n(\mathscr{H}(x))) := \{y_1,\cdots,y_N\}$. Then there exists $C_G := (C_{y_1},\cdots,C_{y_N}) \in \mathbb{R}_+^N$ that depends only on G, such that*

$$r_k(y_i) > C_{y_i}\|d\| \Rightarrow y_i \in \Sigma_{\nabla,k}(\mathcal{L}_k(M)).$$

If moreover, for all $y_i \in \Sigma_{G,k}(\mathcal{M}_n(\mathscr{H}(x)))$ we have $r_k(y_i) > C_{y_i}\|d\|$, then

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{G,k}(\mathcal{M}_n(\mathscr{H}(x)).$$

In the last part, we observe that the spectrum is invariant by a push-forward by an étale map. By that, we mean the following: let $\varphi : Y \to X$ an étale morphism. Let $y \in Y$ be a point not of type (1) and $x := \varphi(y)$. The map φ induces a finite extension $\mathscr{H}(x) \hookrightarrow \mathscr{H}(y)$. Let d be a bounded derivation on $\mathscr{H}(x)$, and let φ^*d be the pull-back of d by φ, which is a bounded derivation that extends d. For a differential module (M,∇) over $(\mathscr{H}(y),\varphi^*d)$ the push-forward $(\varphi_*M,\varphi_*\nabla)$ (obtained by restriction of scalars) is a differential module over $(\mathscr{H}(x),d)$. Since (M,∇) and $(\varphi_*M,\varphi_*\nabla)$ coincides as k-Banach space, their respective spectra satisfy

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{\varphi_*\nabla,k}(\mathcal{L}_k(\varphi_*M)). \tag{7}$$

Even though it is not hard to see this, it allows us to compute spectra of important classes of differential modules (cf. Chapters 5 and 6).

3.4 Chapter 4: Spectrum of a differential equation with constant coefficients

The fourth chapter is devoted to the computation of the spectrum of a differential equation with constant coefficients and the study of its variation. We will distinguish the notion of differential equation in the sense of a differential polynomial $P(d)$ acting on A from the notion of differential module (M,∇) over (A,d) associated to $P(d)$ in a cyclic basis. Indeed, it is not hard to prove that the spectrum of $P(\frac{\mathrm{d}}{\mathrm{dS}}) = (\frac{\mathrm{d}}{\mathrm{dS}})^n + a_{n-1}(\frac{\mathrm{d}}{\mathrm{dS}})^{n-1}+\cdots+a_0$ as an element of $\mathcal{L}_k(A)$, where $a_i \in k$, is given by the easy formula (cf. [Bou07, p. 2] and Lemma 2.1.16)

$$\Sigma_{P(\frac{\mathrm{d}}{\mathrm{dS}})}(\mathcal{L}_k(A)) = P(\Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}(\mathcal{L}_k(A))). \tag{8}$$

This set is always either a closed disk or the topological closure of an open disk (cf. Lemma 2.1.16, Remark 4.2.4). On the other hand, surprisingly enough, this differs form the spectrum $\Sigma_\nabla(\mathcal{L}_k(M))$ of the differential module (M,∇) associated to $P(\frac{\mathrm{d}}{\mathrm{dS}})$ in a cyclic basis (i.e the spectrum of ∇ as an element of $\mathcal{L}_k(M)$). Indeed, this latter is a finite union of either closed disks or topological closures of open disks (cf. Theorem 3.8) centered on the roots of the (commutative) polynomial $Q = X^n + a_{n-1}X^{n-1}+\cdots+a_0 \in$

$k[X]$ associated to P. In order to introduce our next result, we denote by $\tilde{k}$ the residual field of k and we set:

$$\omega = \begin{cases} |p|^{\frac{1}{p-1}} & \text{if } \operatorname{char}(\tilde{k}) = p \\ 1 & \text{if } \operatorname{char}(\tilde{k}) = 0 \end{cases}. \tag{9}$$

The main statement of this chapter is then the following:

Theorem 3.8. *We suppose that k is algebraically closed. Let X be a connected affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$ and $x \in \mathbb{A}_k^{1,\mathrm{an}}$ a point of type (2), (3) or (4). We set $A = \mathcal{O}(X)$ or $\mathscr{H}(x)$. Let (M, ∇) be a differential module over $(A, \frac{\mathrm{d}}{\mathrm{dS}})$ such that there exists a basis for which the associated matrix G has constant entries (i.e. $G \in \mathcal{M}_n(k)$). Let $\{a_1, \dots, a_N\} \subset k$ be the set of eigenvalues of G. Then the behaviour of the spectrum $\Sigma_\nabla(\mathcal{L}_k(M))$ of ∇ as an element of $\mathcal{L}_k(M)$ is summarized in the following table:*

$A =$	$\mathcal{O}(X)$		$\mathscr{H}(x)$	
	$X = D^+(c,r)$	$X = D^+(c_0,r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i,r_i)$	x *of type (2) or (3)*	x *of type (4)*
$\operatorname{char}(\tilde{k}) = p > 0$	$\bigcup_{i=1}^{N} D^+(a_i, \frac{\omega}{r})$	$\bigcup_{i=1}^{N} D^+(a_i, \frac{\omega}{\min_j r_j})$	$\bigcup_{i=1}^{N} D^+(a_i, \frac{\omega}{r(x)})$	
$\operatorname{char}(\tilde{k}) = 0$	$\bigcup_{i=1}^{N} (D^-(a_i, \frac{1}{r}) \cup \{x_{a_i,\frac{1}{r}}\})$	$\bigcup_{i=1}^{N} D^+(a_i, \frac{1}{\min_j r_j})$	$\bigcup_{i=1}^{N} D^+(a_i, \frac{1}{r(x)})$	$\bigcup_{i=1}^{N} (D^-(a_i, \frac{1}{r(x)}) \cup \{x_{a_i,\frac{1}{r(x)}}\})$

The proof in all cases consists of the computation of $\frac{\mathrm{d}}{\mathrm{dS}}$ and application of Proposition 3.4.

Using this result we prove that the variation of the spectrum satisfies a continuity property:

Let X be an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$. Let (M, ∇) be a differential module over $(\mathcal{O}(X), \frac{\mathrm{d}}{\mathrm{dS}})$ such that there exists a basis for which the associated matrix G has constant entries. For a point $x \in X$ not of type (1), the differential module (M, ∇) extends to a differential module (M_x, ∇_x) over $(\mathscr{H}(x), \frac{\mathrm{d}}{\mathrm{dS}})$. In the corresponding basis of (M_x, ∇_x) the associated matrix is G.

Theorem 3.9. *Assume that k is algebraically closed. Let $X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$ be a connected affinoid domain and $x \in X$ be a point of type (2), (3) or (4). Let (M, ∇) be a differential module over $(\mathcal{O}(X), \frac{\mathrm{d}}{\mathrm{dS}})$ such that there exists a basis for which the corresponding matrix G has constant entries. We set:*

$$\begin{array}{rcl} \Psi : [x, x_{c_0,r_0}] & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ y & \mapsto & \Sigma_{\nabla_y}(\mathcal{L}_k(M_y)) \end{array}.$$

Then we have:

- *for each $y \in [x, x_{c_0,r_0}]$, the restriction of Ψ to $[x, y]$ is continuous at y.*
- *If $y \in [x, x_{c_0,r_0}]$ is a point of type (3), then Ψ is continuous at y.*
- *If $\operatorname{char}(\tilde{k}) = 0$ and $y \in [x, x_{c_0,r_0}]$ is a point of type (4), then Ψ is continuous at y.*

3.5 Chapter 5: Spectrum of a regular singular differential module

The fifth Chapter is devoted to the computation of the spectrum of a regular singular differential module (M,∇) over $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$, with $x\in\mathbb{A}_k^{1,\mathrm{an}}$ a point of type (2), (3) or (4). By regular differential module over $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$, we mean a differential module (M,∇) such that there exists a basis for which the associated matrix G has constant entries (i.e. $G\in\mathcal{M}_n(k)$). We set $\mathrm{Log}_c: D^-(c,|c|)\to\mathbb{A}_k^{1,\mathrm{an}}$ to be the analytic map associated with the ring morphism:

$$\begin{array}{ccc} k[T] & \longrightarrow & \mathcal{O}(D^-(c,|c|)) \\ T & \mapsto & \sum_{n\in\mathbb{N}\setminus\{0\}}\frac{(-1)^n}{c^n n}(T-c)^n. \end{array} \tag{10}$$

The main result of the chapter is the following:

Theorem 3.10. *We suppose that k is algebraically closed. Let $x\in\mathbb{A}_k^{1,\mathrm{an}}$ a point of type (2), (3) or (4). Let (M,∇) be a differential module over $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$ such that there exists a basis for which the associated matrix G has constant entries (i.e. $G\in\mathcal{M}_n(k)$). Let $\{a_1,\dots,a_N\}\subset k$ be the set of eigenvalues of G. Then the behaviour of the spectrum $\Sigma_\nabla(\mathcal{L}_k(M))$ of ∇ as an element of $\mathcal{L}_k(M)$ is summarized in the following table:*

	$x=x_{0,r}$		$x\in D^-(c,\lvert c\rvert)$ *with* $c\in k\setminus\{0\}$	
	x of type (2)	*x of type (3)*	*x of type (2) or (3)*	*x of type (4)*
$\mathrm{char}(\tilde{k})=p>0$	$\bigcup_{i=1}^N a_i+\mathbb{Z}_p$		$\bigcup_{i=1}^N\bigcup_{j\in\mathbb{N}} D^+(a_i+j,\frac{\omega}{r_k(\mathrm{Log}_c(x))})$	
$\mathrm{char}(\tilde{k})=0$	$\bigcup_{i=1}^N D^+(a_i,1)$	$\bigcup_{i=1}^N a_i+(\mathbb{Z}\cup\{x_{0,1}\})$	$\bigcup_{i=1}^N D^+(a_i,\frac{\omega}{r_k(\mathrm{Log}_c(x))})$	$\bigcup_{i=1}^N (D^-(a_i,\frac{\omega}{r_k(\mathrm{Log}_c(x))})\cup\{x_{a_i,\frac{\omega}{r_k(\mathrm{Log}_c(x))}}\})$

The strategy of the proof consists in computing firstly the spectrum of the derivation $S\frac{\mathrm{d}}{\mathrm{dS}}$. After, as we have mentioned before, we apply Proposition 3.4. Therefore, the cases summarized in this table follow from the different forms of the spectrum of $S\frac{\mathrm{d}}{\mathrm{dS}}$. In the case where $x=x_{0,r}$ for some $r>0$, the methods used in the three cases to compute $\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x)))$ are completely different. Indeed, in the case where $\mathrm{char}(\tilde{k})=p>0$, we obtain the spectrum by using the push-forward by the Frobenius map. However, for the other cases, as in the constant case, we compute directly the spectrum. In the case where $x\in D^-(c,|c|)$ with $c\in k\setminus\{0\}$, since we have the equality in the formula (7), the push-forward by Log_c allows us to reduce our computation to the computation of a spectrum of a differential module over $(\mathscr{H}(\mathrm{Log}_c(x)),\frac{\mathrm{d}}{\mathrm{dS}})$ with constant coefficients. As the degree of Log_c in the case where $\mathrm{char}(\tilde{k})=p>0$ differs from $\mathrm{char}(\tilde{k})=0$, the spectra in these cases are slightly different.

We observe from this result that: contrary to the case treated in Chapter 4, in the case $\mathrm{char}(\tilde{k})=0$, the continuity of the variation of the spectrum may completely fail. Nonetheless, we have the following continuity result:

Let X be an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$. Let (M,∇) be a differential module over $(\mathcal{O}(X), S\frac{\mathrm{d}}{\mathrm{dS}})$ such that there exists a basis for which the associated matrix G has constant entries. For a point $x \in X$ not of type (1), the differential module (M,∇) extends to a differential module (M_x,∇_x) over $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$. In the corresponding basis of (M_x,∇_x) the associated matrix is G.

Theorem 3.11. *We assume that k is algebraically closed and $\mathrm{char}(\tilde{k}) = p > 0$ or $|k| = \mathbb{R}_+$. Let $X = D^+(c_0,r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i,r_i)$ be a connected affinoid domain and $x \in X$ be a point of type (2), (3) or (4). Let (M,∇) be a differential module over $(\mathcal{O}_X(X), S\frac{\mathrm{d}}{\mathrm{dS}})$ such that there exists a basis for which the corresponding matrix G has constant entries. We set:*

$$\begin{array}{rcl} \Psi : [x, x_{c_0,r_0}] & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ y & \mapsto & \Sigma_{\nabla_y,k}(\mathcal{L}_k(M_y)) \end{array}.$$

Then we have:

- *for each $y \in [x, x_{c_0,r_0}]$, the restriction of Ψ to $[x,y]$ is continuous at y.*
- *If $y \in [x, x_{c_0,r_0}]$ is a point of the form $x_{0,R}$, then Ψ is continuous at y.*
- *If $y \in [x, x_{c_0,r_0}]$ is a point of type (3), then Ψ is continuous at y.*

3.6 Chapter 6: Spectrum of a linear differential equation over a field of formal power series

The sixth chapter is devoted to the computation of the spectrum of a differential module (M,∇) over $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$, where k is trivially valued and algebraically closed, and $k((S))$ is the field of Laurent power series endowed with the S-adic absolute value given $|\sum_{i\geq N} a_i S^i| := r^N$, if $a_N \neq 0$, with $r < 1$. Before discussing our result, we point out that the choice of r identifies (non canonically) $(k((S)), |.|)$ with the field $\mathscr{H}(x_{0,r})$ of the point $x_{0,r} \in \mathbb{A}_k^{1,\mathrm{an}}$. The results of this chapter will depend highly on this identification. It will appears, in particular, that the Berkovich Spectrum is not independent on r while the classical notion of spectrum is a completely algebraic notion. The main statement is the following:

Theorem 3.12. *Let (M,∇) be a differential module over $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$. Let $\{\gamma_1, \cdots, \gamma_{\nu_1}\}$ be the set of the slopes of (M,∇) counted without multiplicity. Let $\{a_1, \cdots, a_{\nu_2}\}$ be the set of the exponents of the regular part of (M,∇) again counted without multiplicity. Then the spectrum of ∇ as an element of $\mathcal{L}_k(M)$ is:*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \{x_{0,r^{-\gamma_1}}, \cdots, x_{0,r^{-\gamma_{\nu_1}}}\} \cup \bigcup_{i=1}^{\nu_2} (a_i + \mathbb{Z}).$$

We now discribe the idea of the proof. The decomposition according to the slopes of Newton polygon of a differential module allows us to write (M, ∇) as a direct sum of differential modules over $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$:

$$(M, \nabla) = (M_{\text{reg}}, \nabla_{\text{reg}}) \oplus (M_{\text{irr}}, \nabla_{\text{irr}})$$

where $(M_{\text{reg}}, \nabla_{\text{reg}})$ is regular singular differential module and $(M_{\text{irr}}, \nabla_{irr})$ is irregular differential module without regular part. As we have seen above (cf. (6)), we have $\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{\nabla_1,k}(\mathcal{L}_k(M_1)) \cup \Sigma_{\nabla_2,k}(\mathcal{L}_k(M_2))$. Therefore, in order to obtain $\Sigma_{\nabla,k}(\mathcal{L}_k(M))$, it is enough to know the spectrum of a regular singular module and the spectrum of a differential module without regular part. The first case is already done in Chapter 5. It just remains to treat the case of an irregular differential module without regular part. For this case, we proceed as follows. We know that, any differential module by some extension of scalars is an extension of differential modules of rank one (cf. [Tur55]). In particular, for $(M_{\text{irr}}, \nabla_{\text{irr}})$, the differential modules of rank one so obtained fulfill the condition of Theorem 3.6. Therefore, more precisely, the spectrum of such a rank one module is $\{x_{0,r^\gamma}\}$, where γ corresponds to a slope of $(M_{\text{irr}}, \nabla_{\text{irr}})$. We prove in this situation by using push-forward and pull back techniques that, the spectrum of $(M_{\text{irr}}, \nabla_{\text{irr}})$ coincides with the spectrum of the differential module obtained by extension of scalars. Hence, we conclude.

4 Perspectives

Our strategy to compute the spectrum of a general differential module defined over an affinoid domain of $\mathbb{A}_k^{1,\text{an}}$ is the following. We need to prove first Conjecture 3.7. Then, we need to show that push-froward by the Frobenius (or another suitable étale morphism) of the differential module fulfills the condition of the conjecture. Then, we expect that the spectrum has the form:

Conjecture 4.1. *Assume that k is algebraically closed. Let $x \in \mathbb{A}_k^{1,\text{an}}$ be a point of type (2), (3) or (4). Let (M, ∇) be a differential module over $(\mathscr{H}(x), d)$, with rank n. if $\nabla - a$ is injective for all $a \in k$ then $\Sigma_{\nabla,k}(\mathcal{L}_k(M))$ is a finite set of point not of type (1). More generally, the spectrum has the form:*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = (\bigcup_{i=1}^{N} a_i + \Sigma_{d,k}(\mathcal{L}_k(M))) \cup \{x_1, \cdots, x_{N'}\}$$

with $a_i \in k$, $x_j \in \mathbb{A}_k^{1,\text{an}}$ are points not of type (1) and $N \leq n$. Note that, N' can be greater then n.

Part I

Spectral Theory

I

Basic notions

Contents

This Chapter is mainly devoted to introducing the objects involved in the next chapters. In the first section, we recall the definitions and properties related to Banach spaces. In the second section, we recall generally the construction of analytic spaces in sense of Berkovich, in order to describe more precisely the analytic line.

Notation 1.0.1. We denote by $\mathbb{R}$ the field of real numbers, and $\mathbb{R}_+ = \{r \in \mathbb{R}|\ r \geq 0\}$.

Convention 1.0.2. We will assume that all the rings are with unit element.

1.1 Banach spaces and the analytic spectrum

This section is organized as follows. In the first part, we provide settings and notations related to modules over an ultrametric ring. The second part is devoted to recalling the definition and properties of the analytic spectrum. In the last part we will state and prove some additional properties related to Banach spaces.

1.1.1 Ultrametric normed rings and modules

Definition 1.1.1. Let M be an abelian group. An *ultrametric semi-norm* on M is a map $\|.\| : M \to \mathbb{R}_+$ satisfying the following properties:

- $\|0\| = 0$.
- $\forall m, n \in M$; $\|m - n\| \leq \max\{\|m\|, \|n\|\}$.

If moreover we have: $\forall m \in M$; $\|m\| = 0 \Rightarrow m = 0$, we say that $\|.\|$ is a *norm*.

A semi-norm (resp. norm) $\|.\| : M \to \mathbb{R}_+$ induces a pseudometric (resp. metric) over M. In this condition, we will say that $(M, \|.\|)$ is either a semi-normed or normed abelian group. We say that $(M, \|.\|)$ is complete if it is complete with respect to its pseudometric.

Definition 1.1.2. Let $\|.\|_1$ and $\|.\|_2$ be two semi-norms on an abelian group M. We say that $\|.\|_1$ *bounds* $\|.\|_2$, if we have:

$$\exists C \in \mathbb{R}_+^*; \quad \forall m \in M, \|m\|_2 \leq C\|m\|_1$$

Properties 1.1.3. *Let $\|.\|_1$ and $\|.\|_2$ be two semi-norms an abelian group M. If they bound each other then the respective induced topologies are equivalent.*

Definition 1.1.4. Let $(M, \|.\|_M)$ and $(N, \|.\|_N)$ be two semi-normed abelian groups, and let $\varphi : M \to N$ be a groups morphism. We say that φ is *bounded* if we have:

$$\exists C \in \mathbb{R}_+^*; \quad \forall m \in M, \|\varphi(m)\|_N \leq C\|m\|_M.$$

In the case where $C = 1$ we say that φ is a *contracting* map. If moreover we have $\|\varphi(m)\|_N = \|m\|_M$ we say that φ is an *isometry*.

Definition 1.1.5. Let $(M, \|.\|_M)$ and $(N, \|.\|_N)$ be two semi-normed abelian groups, and let $\varphi : M \to N$ b a bounded group morphism. If φ is an isomorphism, we say that it is a *bi-bounded* isomorphism if φ^{-1} is bounded too.

Definition 1.1.6. Let $(M, \|.\|)$ be a semi-normed abelian group and N be a subgroup of M. Let $\pi : M \to M/N$ be the canonical projection. The semi-norm $\|.\|$ induces a *residual semi-norm* on the quotient group M/N as follows:

$$\begin{array}{rcl} \|.\|_{res} : M/N & \longrightarrow & \mathbb{R}_+ \\ f & \mapsto & \|f\|_{\text{res}} := \inf\{\|g\| | g \in \pi^{-1}(f)\} \end{array}.$$

Properties 1.1.7. *The residual semi-norm is a norm if and only if N is closed in M. In the case where $(M, \|.\|)$ is a complete normed abelian group and N a closed subgroup, then $(M/N, \|.\|_{res})$ is a complete normed group.*

Definition 1.1.8. Let $(M, \|.\|_M)$ and $(N, \|.\|_N)$ be semi-normed groups and let $\varphi : M \to N$ be a bounded morphism of groups. We say that φ is *admissible* if the induced map $M/\operatorname{Ker}(\varphi) \to \operatorname{Im}(\varphi)$ is a bi-bounded isomorphism.

Definition 1.1.9. Let A be a ring. An ultrametric semi-norm on A is an ultrametric semi-norm $\|.\| : A \to \mathbb{R}_+$ on the abelian group $(A, +)$, satisfying the additional properties:

- $\|1\| = 1$ if $1 \neq 0$.
- $\forall m, n \in A$; $\|mn\| \leq \|m\| \cdot \|n\|$.

If moreover, for all $m, n \in A$ we have $\|mn\| = \|m\| \cdot \|n\|$, we say that $\|.\|$ is a *multiplicative* semi-norm.

Definition 1.1.10. A *normed ring* $(A, \|.\|)$ is a ring A endowed with an ultrametric norm $\|.\| : A \to \mathbb{R}_+$. If in addition it is complete, we say that it is a *Banach ring*. If the norm is multiplicative, we will call $\|.\|$ an *absolute value* and $(A, \|.\|)$ a *valued ring*.

Definition 1.1.11. Let $(A, \|.\|)$ be a normed ring. The *spectral semi-norm* associated to $\|.\|$ is the map:

$$\begin{array}{rcl} \|.\|_{\mathrm{Sp},A} : A & \longrightarrow & \mathbb{R}_+ \\ f & \mapsto & \lim\limits_{n \to +\infty} \|f^n\|^{\frac{1}{n}} \end{array} . \tag{1.1}$$

In the case where for all $f \in A$ we have $\|f^n\| = \|f\|^n$, in other words $\|.\| = \|.\|_{\mathrm{Sp}}$, we say that $(A, \|.\|)$ is *uniform*.

Lemma 1.1.12. *Let $(A, \|.\|)$ and $(B, \|.\|')$ be two normed rings. Let $\varphi : A \to B$ be a bounded ring morphism. If B is uniform then φ is a contracting map.*

Proof. Since φ is bounded, there exists $C > 0$ such that for all $f \in A$ we have

$$\|\varphi(f)\| \leq C\|f\|.$$

Hence, for all $n \in \mathbb{N}$ we have

$$\|\varphi(f)\| = \|\varphi(f^n)\|^{\frac{1}{n}} \leq C^{\frac{1}{n}} \|f^n\|^{\frac{1}{n}} \leq C^{\frac{1}{n}} \|f\|.$$

Consequently, we obtain

$$\|\varphi(f)\| \leq \|f\|.$$

□

Definition 1.1.13. Let $(A, |.|)$ be a normed ring. Let M be an A-module and $\|.\|$ an ultrametric semi-norm (resp. norm) on M. We say that $(M, \|.\|)$ is a *semi-normed* (resp. *normed*) A-module, if $\|.\|$ satisfies the additional property:

$$\exists C \in \mathbb{R}_+^*; \quad \forall m \in M, \forall f \in A, \|fm\| \leq C|f|\cdot\|m\|. \tag{1.2}$$

In the case where $(M, \|.\|)$ is complete we say that it is a *Banach A-module*.

Remark 1.1.14. If the property above is satisfied, we can always find an equivalent norm $\|.\|'$ such that:

$$\forall m \in M, \forall f \in A, \|fm\|' \leq |f|\cdot\|m\|'. \tag{1.3}$$

Remark 1.1.15. In the case where $(A, |.|)$ is a valued field, the inequality in (1.3) is an equality. Indeed, for each $f \in A \setminus \{0\}$, $m \in M$, we have:

$$\|m\| = \|f^{-1}fm\| \leq |f^{-1}|\cdot\|fm\| = |f|^{-1}\cdot\|fm\|.$$

Then,

$$|f|\cdot\|m\| \leq \|fm\|.$$

Notation 1.1.16. We will denote by $(\hat{A}, |.|)$ (resp. $(\hat{M}, \|.\|)$) the completion $(A, |.|)$ (resp. $(M, \|.\|)$) with respect to $|.|$ (resp. $\|.\|$).

Given two normed A-modules $(M, \|.\|)$ and $(N, \|.\|')$. We denote by $\mathcal{L}_A(M, N)$ the A-module of bounded A-linear maps $\varphi : M \to N$. If $(N, \|.\|') = (M, \|.\|)$ we will denote it by $\mathcal{L}_A(M)$. We endow $\mathcal{L}_A(M, N)$ with the *operator norm*:

$$\varphi \mapsto \|\varphi\|_{\mathrm{op}} := \sup_{m \in M \setminus \{0\}} \frac{\|\varphi(m)\|'}{\|m\|}. \tag{1.4}$$

We set:

$$\mathcal{L}_{A,c}(M, N) := \{\varphi \in \mathcal{L}_A(M, N) |\ \|\varphi\|_{\mathrm{op}} \leq 1\}$$

Lemma 1.1.17. *If $(N, \|.\|')$ is a Banach A-module, then $(\mathcal{L}_A(M, N), \|.\|_{op})$ is a Banach A-module.*

Let $\{(M_i, \|.\|_i)\}_{i \in I}$ be a family of normed A-modules. We endow the direct sum $\bigoplus_{i \in I} M_i$ with the following norm

$$\begin{array}{rcl} \|.\| : \bigoplus\limits_{i \in I} M_i & \longrightarrow & \mathbb{R}_+ \\ (m_i)_{i \in I} & \mapsto & \max\limits_{i \in I} \|m_i\|_i \end{array}. \tag{1.5}$$

The completion of $\bigoplus_{i \in I} M_i$ will be denoted by $\widehat{\bigoplus_{i \in I}} M_i$.

Remark 1.1.18. In the case where A is a Banach ring, I is a finite set and the M_i's are Banach A-modules, we have $\bigoplus_{i \in I} M_i = \widehat{\bigoplus}_{i \in I} M_i$.

Let $(M, \|.\|_M)$ and $(N, \|.\|_N)$ be two ultrametric normed A-modules. We endow the tensor product $M \otimes_A N$ with the following semi-norm

$$\begin{array}{rcl} \|.\| : M \otimes_A N & \longrightarrow & \mathbb{R}_+ \\ f & \mapsto & \inf\{\max_i\{\|m_i\|_M \cdot \|n_i\|_N\} \mid f = \sum_i m_i \otimes n_i\} \end{array} . \tag{1.6}$$

The completion of $M \otimes_A N$ with respect to this norm, denoted by $M \hat{\otimes}_A N$, is a Banach $\hat{A}$-module.

Definition 1.1.19. Let $(A, \|.\|)$ be a Banach ring. The category $\mathbf{Ban}_A$ (resp. $\mathbf{Ban}_A^{\leq 1}$) is the category whose objects are Banach A-modules and whose morphisms are bounded A-linear maps (resp. A-linear contracting maps).

Lemma 1.1.20. *Let $\{(M_i, \|.\|)\}_{i \in I}$ be a set of objects in $\mathbf{Ban}_A^{\leq 1}$. The coproduct of this family exists in the category $\mathbf{Ban}_A^{\leq 1}$ and is equal to:*

$$c(\prod_{i \in I} M_i) = \{(m_i)_{i \in I} \in \prod_{i \in I} M_i \mid \lim_{i \to \infty} m_i = 0\}.$$

endowed with the norm:

$$(m_i)_{i \in I} \mapsto \sup_{i \in I} \|m_i\|_i$$

where $\prod_{i \in I} M_i$ is the set-theoretic product. We mean by $\lim_{i \to \infty} m_i = 0$; m_i converges to zero with respect to the filter of complements of finite subsets of I, i.e for all $\varepsilon > 0$, we have $\|m_i\|_i < \varepsilon$ for almost all $i \in I$.

Lemma 1.1.21. *Let $\{(M_i, \|.\|_i)\}_{i \in I}$ be a family of Banach A-modules. Then we have:*

$$\widehat{\bigoplus}_{i \in I} M_i = c(\prod_{i \in I} M_i).$$

Lemma 1.1.22. *Let $\{(M_i, \|.\|)\}_{i \in I}$ be a set of objects in $\mathbf{Ban}_A^{\leq 1}$. The product of the M_i's exists in the category $\mathbf{Ban}_A^{\leq 1}$ and is equal to:*

$$b(\prod_{i \in I} M_i) = \{(m_i)_{i \in I} \in \prod_{i \in I} M_i \mid \sup_{i \in I} \|m_i\|_i < \infty\}$$

endowed with the norm:

$$(m_i)_{i \in I} \mapsto \sup_{i \in I} \|m_i\|_i.$$

Definition 1.1.23. Let $(A, \|.\|)$ be a normed ring. A *normed A-algebra* is a normed ring $(B, \|.\|')$ equipped with a bounded ring morphism $\varphi : (A, \|.\|) \to (B, \|.\|')$. If moreover $(B, \|.\|')$ is complete, we say that it is a *Banach A-algebra*.

Let $(B, \|.\|')$ and $(C, \|.\|'')$ be two normed A-algebras. The tensor norm (cf. (3.5)) induces a structure of normed A-algebra on $B \otimes_A C$.

1.1.2 Analytic spectrum of ring

In this part, we refer the reader to [Ber90, Chapter 1] for the proofs.

Definition 1.1.24. Let $(A, \|.\|)$ be a commutative Banach ring. The *analytic spectrum of* A, denoted by $\mathcal{M}(A)$, is the set of multiplicative semi-norms bounded by $\|.\|$.

For a point x of $\mathcal{M}(A)$ we can associate a residue field as follows. The point x is associated to a multiplicative semi-norm $|.|_x$. The set:

$$\mathfrak{p}_x = \{f \in A | \ |f|_x = 0\}$$

is a prime ideal of A. Therefore, the semi-norm $|.|_x$ extends to a multiplicative norm on the fraction field of $\mathrm{Frac}(A/\mathfrak{p}_x)$. We will denote by $\mathscr{H}(x)$ the completion of $\mathrm{Frac}(A/\mathfrak{p}_x)$ with respect to $|.|_x$, and by $|.|$ the valuation on $\mathscr{H}(x)$ induced by $|.|_x$. We have the natural bounded ring morphism:

$$\chi_x : (A, \|.\|) \to (\mathscr{H}(x), |.|).$$

For all $f \in A$, write $f(x)$ instead of $\chi_x(f)$.

A bounded morphism $\chi : (A, \|.\|) \to (K, |.|)$ where $(K, |.|)$ is a valued complete field is called character of $(A, \|.\|)$. Two characters $\chi' : (A, \|.\|) \to (K', |.|')$ and $\chi'' : (A, \|.\| \to (K'', |.|'')$ are said to be equivalent if there exists a character $\chi : (A, \|.\|) \to (K, |.|)$ and isometric embeddings $(K, |.|) \to (K', |.|')$ and $(K, |.|) \to (K'', |.|'')$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & (K', |.|') \\
(A, \|.\|) \arrow[r] \arrow[urr] \arrow[drr] & (K, |.|) \arrow[ur] \arrow[dr] & \\
 & & (K'', |.|'')
\end{tikzcd}$$

Remark 1.1.25. The map $x \mapsto \chi_x$ identifies $\mathcal{M}(A)$ with the set of equivalence classes of characters of A.

Topology on $\mathcal{M}(A)$

We endow $\mathcal{M}(A)$ with the initial topology with respect to the map

$$\begin{array}{rccc} \psi : \mathcal{M}(A) & \longrightarrow & \mathbb{R}_+^A \\ x & \mapsto & (|f(x)|)_{f \in A} \end{array}.$$

Let $f = (f_1, \cdots, f_n)$ where $f_i \in A$. For every i, let I_i be an interval of $\mathbb{R}_+$ open in $\mathbb{R}_+$. Then

$$\{x \in \mathcal{M}(A) |\, |f_i(x)| \in I_i,\ 1 \leq i \leq n\}$$

is an open subset, and subsets of this form generate the topology of $\mathcal{M}(A)$.

Let $(A, \|.\|)$ and $(B, \|.\|')$ be two commutative Banach rings, let $\varphi : (A, \|.\|) \to (B, \|.\|')$ be a bounded morphism of rings. Then φ induces a continuous map defined as follows:

$$\begin{array}{rcl} \varphi^* : \mathcal{M}(B) & \longrightarrow & \mathcal{M}(A) \\ x & \mapsto & f \mapsto |\varphi(f)(x)|. \end{array} \tag{1.7}$$

Proposition 1.1.26. *Let A be a nonzero commutative Banach ring. The analytic spectrum $\mathcal{M}(A)$ is a nonempty, compact Hausdorff space.*

Corollary 1.1.27. *An element $f \in A$ is invertible if and only if $f(x) \neq 0$ for all $x \in \mathcal{M}(A)$.*

The spectral semi-norm defined on a commutative Banach ring $(A, \|.\|)$ (c.f (1.1)) satisfies the following property:

Properties 1.1.28. *For all element f in A we have:*

$$\|f\|_{\mathrm{Sp},A} = \max_{x \in \mathcal{M}(A)} |f(x)|.$$

Corollary 1.1.29. *Let A be a commutative Banach ring. Then the spectral semi-norm satisfies:*

- $\forall f, g \in A$; $\|fg\|_{\mathrm{Sp}} \leq \|f\|_{\mathrm{Sp}} \cdot \|g\|_{\mathrm{Sp}}$.
- $\forall f, g \in A$; $\|f + g\|_{\mathrm{Sp}} \leq \|f\|_{\mathrm{Sp}} + \|g\|_{\mathrm{Sp}}$.

Lemma 1.1.30. *Let $(A, \|.\|)$ be a Banach ring, let $(B, \|.\|')$ and $(C, \|.\|'')$ be two Banach A-algebras. Let $f \in B\hat{\otimes}_A C$. Then f is not invertible in $B\hat{\otimes}_A C$ if and only if there exists $x \in \mathcal{M}(C)$ such that the image of f by the natural map $B\hat{\otimes}_A C \to B\hat{\otimes}_A \mathscr{H}(x)$ is not invertible.*

Proof. It is obvious that if the image of f is not invertible in $B\hat{\otimes}_A \mathscr{H}(x)$ for some $x \in \mathcal{M}(C)$, then it is the same for f in $B\hat{\otimes}_A C$. We suppose now that f is not invertible in $B\hat{\otimes}_A C$. By Corollary 1.1.27 there exists $z \in \mathcal{M}(B\hat{\otimes}_A C)$ such that $f(z) = 0$ in $\mathscr{H}(z)$. We have the following commutative diagram:

$$\begin{tikzcd} & B\hat{\otimes}_A C \arrow[dr] & \\ C \arrow[ur] \arrow[rr] & & \mathscr{H}(z) \end{tikzcd}$$

By remark 1.1.25 there exists $x \in \mathcal{M}(C)$ such that we have the following diagram:

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & B\hat{\otimes}_A C \arrow[d] \\
\mathscr{H}(x) \arrow[r] & \mathscr{H}(z)
\end{tikzcd}$$

Therefore, we obtain the commutative diagram

$$\begin{tikzcd}
 & B\hat{\otimes}_A C \arrow[dr] \arrow[drr] & & \\
C \arrow[ur] \arrow[dr] & & B\hat{\otimes}_A \mathscr{H}(x) \arrow[r] & \mathscr{H}(z) \\
 & \mathscr{H}(x) \arrow[ur] \arrow[urr] & &
\end{tikzcd}.$$

Then, $f(z) = 0$ implies that the image of f in $B \otimes_A \mathscr{H}(x)$ is not invertible.

□

Corollary 1.1.31. *Let $(A, \|.\|)$ be a Banach ring, let $(B, \|.\|')$ and $(C, \|.\|'')$ be two Banach A-algebras. Let $f \in B\hat{\otimes}_A C$. If f is not invertible in $B\hat{\otimes}_A C$ then there exists $x \in \mathcal{M}(C)$ and $y \in \mathcal{M}(B)$ such that the image of f by the natural map $B\hat{\otimes}_A C \to \mathscr{H}(y)\hat{\otimes}_A \mathscr{H}(x)$ is not invertible.*

1.1.3 Banach spaces

Notation 1.1.32. We fix a valued field $(k, |.|)$ of characteristic zero, complete with respect to the absolute value $|.|$. We set $|k| := \{r \in \mathbb{R}_+ | \ \exists t \in k; r = |t|\}$, $|k^*| := |k| \setminus \{0\}$, $k^\circ := \{a \in k| \ |a| \leq 1\}$, $k^{\circ\circ} := \{a \in k| \ |a| < 1\}$ and $\tilde{k} := k^\circ / k^{\circ\circ}$.

Notation 1.1.33. Let $E(k)$ be the category whose objects are $(\Omega, |.|)$, where Ω is an isometric field extension of k complete with respect to $|.|$, and whose morphisms are the isometric rings morphisms. For $(\Omega, |.|) \in E(k)$, we set Ω^{alg} to be an algebraic closure of Ω, $|.|$ extends uniquely to an absolute value defined on Ω^{alg}. We denote by $\widehat{\Omega^{alg}}$ the completion of Ω^{alg} with respect its absolute value.

The categories of Banach k-spaces

Notation 1.1.34. We will denote by $\mathbf{Ban}_k$ the category whose objects are Banach k-spaces and whose arrows are the bounded k-linear maps, and by $\mathbf{Ban}_k^{\leq 1}$ the subcategory of $\mathbf{Ban}_k$ whose objects are Banach k-spaces and whose arrows are the k-linear contracting maps.

Lemma 1.1.35. *The category* $\mathbf{Ban}_k$ *admits finite products and co-products.*

Lemma 1.1.36. *The category* $\mathbf{Ban}_k^{\leq 1}$ *admits small (indexed by a set) products and co-products.*

Remark 1.1.37. The category $\mathbf{Ban}_k$ has no infinite products and co-products of any collection of nonzero objects.

Definitions and basic properties:

Lemma 1.1.38. *Let* $\Omega \in E(k)$ *and let* M *be a* k*-Banach space. Then, the inclusion* $M \hookrightarrow M \otimes_k \Omega$ *is an isometry. In particular, for all* $v \in M$ *and* $c \in K$ *we have* $\|v \otimes c\| = |c| \cdot \|v\|$.

Proof. Since Ω contains isometrically k, the morphism $M \to M\hat{\otimes}_k\Omega$ (resp. $\Omega \to M\hat{\otimes}_k\Omega$) is an isometry (cf. [Poi13, Lemma 3.1]). By definition $\|v \otimes c\| \leq |c| \cdot \|v\| = |c| \cdot \|v \otimes 1\|$ for all $v \in M$ and $c \in \Omega$. Therefore, by Remark 1.1.15 the tensor norm is a norm on $M\hat{\otimes}_k\Omega$ as an Ω-vector space. Consequently, we obtain $\|v \otimes c\| = |c| \cdot \|v\|$ for all $v \in M$ and $c \in \Omega$. $\square$

Proposition 1.1.39. *Let* M *be a* k*-Banach space, and* B *be a uniform* k*-Banach algebra. Then we have in* $M\hat{\otimes}_k B$ *:*

$$\forall m \in M, \ \forall f \in B, \ \|m \otimes f\| = \|m\| \cdot \|f\|$$

Proof. Let $\mathcal{M}(B)$ be the analytic spectrum of B. For all x in $\mathcal{M}(B)$ the canonical map $B \to \mathscr{H}(x)$ is a contracting map, therefore the map $M\hat{\otimes}_k B \to M\hat{\otimes}_k\mathscr{H}(x)$ is a contracting map too. Then, by Lemme 1.1.38 we have:

$$\forall x \in \mathcal{M}(B) : \forall m \in M, \ \forall f \in B, \ \|m\| \cdot |f(x)| = \|m \otimes f(x)\| \leq \|m \otimes f\| \leq \|m\| \cdot \|f\|$$

thus,

$$\forall m \in M, \ \forall f \in B, \ \|m\| \max_{x \in \mathcal{M}(B)} |f(x)| \leq \|m \otimes f\| \leq \|m\| \cdot \|f\|.$$

Since B is uniform, we have $\max_{x \in \mathcal{M}(B)} |f(x)| = \|f\|_{\mathrm{Sp}} = \|f\|$ (cf. [Ber90, Theorem 1.3.1]), and $\|m\| \cdot \|f\| \leq \|m \otimes f\| \leq \|m\| \cdot \|f\|$.

$\square$

The algebra of bounded linear operators $\mathcal{L}_k(M)$ of M

Lemma 1.1.40. *Let $\Omega \in E(k)$. There exists an isometric k-linear map*

$$\begin{array}{rcl} \Upsilon : \mathcal{L}_k(M) & \longrightarrow & \mathcal{L}_\Omega(M\hat{\otimes}_k\Omega) \\ \varphi & \mapsto & \varphi\hat{\otimes}1 \end{array}$$

which extends to an Ω-linear contracting map

$$\begin{array}{rcl} \Upsilon_\Omega : \mathcal{L}_k(M)\hat{\otimes}_k\Omega & \longrightarrow & \mathcal{L}_\Omega(M\hat{\otimes}_k\Omega) \\ \varphi \otimes a & \mapsto & \varphi\hat{\otimes}a \end{array}.$$

Proof. Let $\varphi \in \mathcal{L}_k(M)$. We have the bilinear map:

$$\begin{array}{rcl} \varphi \times 1 : M \times \Omega & \longrightarrow & M \otimes_k \Omega \\ (x, a) & \mapsto & \varphi(x) \otimes a \end{array}$$

where $M \times \Omega$ is endowed with product norm and $M \otimes_k \Omega$ with tensor norm (cf. (3.5)). The map $\varphi \times 1$ is bounded. Then we have the following commutative diagram:

$$\begin{tikzcd} M \times \Omega \arrow[r, "\varphi\times 1"] \arrow[d, "\otimes"'] & M \otimes_k \Omega \\ M \otimes_k \Omega \arrow[ur, "\exists!\varphi\otimes 1"'] & \end{tikzcd}$$

where $\varphi \otimes 1$ is bounded k-linear map. By construction, the map $\varphi \otimes 1$ is Ω-linear and extends to a bounded Ω-linear map $\varphi\hat{\otimes}1 : M\hat{\otimes}_k\Omega \to M\hat{\otimes}_k\Omega$. So we obtain a k-linear map

$$\begin{array}{rcl} \Upsilon : \mathcal{L}_k(M) & \longrightarrow & \mathcal{L}_k(M\hat{\otimes}_k\Omega) \\ \varphi & \mapsto & \varphi\hat{\otimes}1 \end{array}.$$

We now prove that it is an isometry. Indeed, let $m \in M$ and $a \in \Omega$ then by Lemma 1.1.38:

$$\|\varphi\hat{\otimes}1(m \otimes a)\| = \|\varphi(m) \otimes a\| = \|\varphi(m)\| \cdot |a| \leq \|\varphi\| \cdot \|m\| \cdot |a| = \|\varphi\| \cdot \|m \otimes a\|.$$

Then for $x = \sum_i m_i \otimes a_i \in M \otimes_k \Omega$ we have:

$$\|\varphi\hat{\otimes}1(x)\| \leq \inf\{\max_i(\|\varphi(m_i)\| \cdot |a_i|) \mid x = \sum_i m_i \otimes a_i\} \leq \|\varphi\| \inf\{\max_i \|m_i\| \cdot |a_i|| \; x = \sum_i m_i \otimes a_i\}.$$

By density we obtain

$$\varphi\hat{\otimes}1(x) \leq \|\varphi\| \cdot \|x\|$$

For any $x \in M\hat{\otimes}_k\Omega$. Consequently,

$$\|\varphi\hat{\otimes}1\| \leq \|\varphi\|.$$

On the other hand, since $M \hookrightarrow M\hat{\otimes}_k\Omega$ is an isometry and $\varphi\hat{\otimes}1_{|_M} = \varphi$, we have $\|\varphi\| \leq \|\varphi\hat{\otimes}1\|$.
The map $\Upsilon : \mathcal{L}_k(M) \hookrightarrow \mathcal{L}_\Omega(M\hat{\otimes}_k\Omega)$ extends to a bounded Ω-linear map

$$\begin{array}{rcl}\Upsilon_\Omega : \mathcal{L}_k(M)\hat{\otimes}_k\Omega & \longrightarrow & \mathcal{L}_\Omega(M\hat{\otimes}_k\Omega) \\ \varphi\otimes a & \mapsto & \varphi\hat{\otimes}a\end{array},$$

with $\varphi\hat{\otimes}a := a\varphi\hat{\otimes}1$. We need to prove now that it is a contracting map. Let $\psi = \sum_i \varphi_i \otimes a_i$ be an element of $\mathcal{L}_k(M)\otimes_k\Omega$, its image in $\mathcal{L}_\Omega(M\hat{\otimes}_k\Omega)$ is the element $\sum_i a_i\varphi_i\hat{\otimes}1$. we have:

$$\|\sum_i a_i\varphi_i\hat{\otimes}1\| \leq \max_i \|a_i\varphi_i\hat{\otimes}1\| = \max_i\|\varphi_i\| \cdot |a_i|.$$

By density, this inequality extends to any element $\psi = \sum_i \varphi_i \otimes a_i$ of $\mathcal{L}_k(M)\hat{\otimes}_k\Omega$ (the sum is infinite and converges). Consequently,

$$\|\sum_i a_i\varphi_i\hat{\otimes}1\| \leq \inf\{\max_i\|\varphi_i\| \cdot |a_i|| \ \psi = \sum_i \varphi_i\otimes a_i\} = \|\psi\|.$$

Hence we obtain the result.

□

Lemma 1.1.41. *Let M be a k-Banach space and L be a finite extension of k. Then we have a bi-bounded isomorphism of k-algebras:*

$$\mathcal{L}_k(M)\hat{\otimes}_k L \simeq \mathcal{L}_L(M\hat{\otimes}_k L).$$

Proof. As L may be written as a sequence of finite intermediate extensions generated by one element, by induction we can assume that $L = k(\alpha)$, where α is an algebraic element over k. By Lemma 1.1.40 we have a bounded morphism

$$\begin{array}{rcl}\mathcal{L}_k(M)\hat{\otimes}_k L & \longrightarrow & \mathcal{L}_L(M\hat{\otimes}_k L) \\ \sum_{i=0}^{n-1}\varphi_i\otimes\alpha^i & \mapsto & \sum_{i=0}^{n-1}\varphi_i\hat{\otimes}\alpha^i\end{array}.$$

As $L = k(\alpha)$, we have a k-isomorphism

$$M\hat{\otimes}_k L \simeq \bigoplus_{i=0}^{n-1} M\otimes(\alpha^i\cdot k).$$

Let $\psi \in \mathcal{L}_L(M\hat{\otimes}L)$. The restriction is of the form

$$\begin{array}{rcl}\psi_{|_{M\otimes 1}} : M\otimes 1 & \longrightarrow & M\hat{\otimes}_k L \\ m\otimes 1 & \mapsto & \sum_{i=0}^{n-1}\varphi_i(m)\otimes\alpha^i\end{array},$$

where $\varphi_i \in \mathcal{L}_k(M)$ and uniquely determined .This means that $\psi = \sum_{i=0}^{n-1}\varphi_i\hat{\otimes}\alpha^i$. Therefore, we obtain the map:

$$\begin{array}{rcl}\mathcal{L}_L(M\hat{\otimes}_k L) & \longrightarrow & \mathcal{L}_k(M)\hat{\otimes}_k L \\ \psi & \mapsto & \sum_{i=0}^{n-1}\varphi_i\otimes\alpha^i\end{array}$$

which is the inverse of $\mathcal{L}_k(M)\hat{\otimes}_k L \to \mathcal{L}_L(M\hat{\otimes}_k L)$. In the case where k is not trivially valued, by the open mapping theorem (see [BGR84, Section 2.8 Theorem of Banach]) the last map is bounded. Otherwise, the extension L is trivially valued. Consequently, we have an isometric k-isomorphism: $L \simeq \bigoplus_{i=0}^{n-1} k$ equipped with the max norm. Therefore, we have the isometric isomorphisms $\mathcal{L}_k(M)\hat{\otimes}_k L \simeq \bigoplus_{i=0}^{n-1} \mathcal{L}_k(M)$ and $M\hat{\otimes}_k L \simeq \bigoplus_{i=0}^{n-1} M$ with respect to the max norm. Then for $\psi = \sum_{i=0}^{n-1} \varphi_i \hat{\otimes} \alpha^i$ we obtain:

$$\|\sum_{i=0}^{n-1} \varphi_i \otimes \alpha^i\| = \max_i \|\varphi_i\| \leq \|\psi_{|M\otimes 1}\| \leq \|\psi\|.$$

Hence, we obtain the result.

□

Let M_1 and M_2 be two k-Banach spaces, let $M = M_1 \oplus M_2$ endowed with the max norm (cf. (1.5)). We set:

$$\mathcal{M}(M_1, M_2) = \left\{ \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} | L_1 \in \mathcal{L}_k(M_1), L_2 \in \mathcal{L}_k(M_2, M_1), L_3 \in \mathcal{L}_k(M_1, M_2), L_4 \in \mathcal{L}_k(M_2) \right\}$$

We define the multiplication in $\mathcal{M}(M_1, M_2)$ as follows:

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_4 & A_3B_2 + A_4B_4 \end{pmatrix}.$$

Then $\mathcal{M}(M_1, M_2)$ endowed with the max norm is a k-Banach algebra.

Lemma 1.1.42. *We have a bi-bounded isomorphism of k-Banach algebras:*

$$\mathcal{L}_k(M) \simeq \mathcal{M}(M_1, M_2).$$

Proof. Let p_j be the projection of M onto M_j and i_j be the inclusion of M_j into M, where $j \in \{1, 2\}$. We define the following two k-linear maps:

$$\begin{array}{rcl} \Psi_1 : \mathcal{L}_k(M) & \longrightarrow & \mathcal{M}(M_1, M_2) \\ \varphi & \mapsto & \begin{pmatrix} p_1\varphi i_1 & p_1\varphi i_2 \\ p_2\varphi i_1 & p_2\varphi i_2 \end{pmatrix} \end{array}$$

$$\begin{array}{rcl} \Psi_2 : \mathcal{M}(M_1, M_2) & \longrightarrow & \mathcal{L}_k(M) \\ \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} & \mapsto & i_1L_1p_1 + i_1L_2p_2 + i_2L_3p_1 + i_2L_4p_2 \end{array}.$$

Since the projections and inclusions are bounded maps, the maps Ψ_1 and Ψ_2 are bounded too. It is easy to show that $\Psi_1 \circ \Psi_2 = id_{\mathcal{M}(M_1,M_2)}$ and $\Psi_2 \circ \Psi_1 = id_{\mathcal{L}_k(M)}$. Hence we have an isomorphism of k-Banach spaces.

□

1.2 Berkovich analytic line

This section is organized as follows. The first and the second part are highly inspired by [Ber90, Chapter 2, 3]. They are devoted to recalling the definition of analytic spaces and maps in the sense of Berkovich. The last one is devoted to describing more precisely the analytic line $\mathbb{A}_k^{1,\mathrm{an}}$.

Convention 1.2.1. We fix here $(k, |.|)$ to be a valued field, complete with respect to its absolute value.

1.2.1 Affinoid spaces

For $r_1, \cdots, r_n > 0$, we set:

$$k\{r_1^{-1}T_1, \cdots, r_n^{-1}T_n\} = k\{\underline{r}^{-1}\underline{T}\} = \{f = \sum_{\nu=0}^{\infty} a_\nu \underline{T}^\nu |\ a_\nu \in k,\ \lim_{\nu \to +\infty} |a_\nu| r^\nu = 0\}$$

where $\nu = (\nu_1, \cdots, \nu_n)$, $|\nu| = \nu_1 + \cdots + \nu_n$, $\underline{T}^\nu = T_1^{\nu_1} \cdots T_n^{\nu_n}$ and $\underline{r}^\nu = r_1^{\nu_1} \cdots r_n^{\nu_n}$. Equipped with the usual addition and multiplication $k\{\underline{r}^{-1}\underline{T}\}$ is a k-algebra. The norm:

$$\|f\| = \max_\nu |a_\nu| r^\nu$$

induces a structure of k-Banach algebra on $k\{\underline{r}^{-1}\underline{T}\}$.

Definition 1.2.2. A *k-affinoid algebra $\mathcal{A}$* is a k-Banach algebra, such that there exists a surjective admissible morphism of k-algebra $k\{\underline{r}^{-1}\underline{T}\} \to \mathcal{A}$. We will say that it is strict if we may find such morphism with $r = (1, \cdots, 1)$. An *affinoid algebra over k* is a Ω-affinoid algebra for some $\Omega \in E(k)$.

The *category of k-affinoid algebras* will be the category whose objects are k-affinoid algebras and morphisms are bounded morphisms of k-algebras. The *category of affinoid algebras over k* will be the category whose objects are affinoid algebras over k and morphisms are bounded morphisms of k-algebras.

Remark 1.2.3. Note that, for $\Omega \in E(k)$ we have a natural functor from the category of k-affinoid algebras to the category of Ω-affinoid algebras, which associates $\mathcal{A}\hat{\otimes}_k\Omega$ to $\mathcal{A}$.

Properties 1.2.4. *An affinoid algebra is noetherian, and all its ideals are closed.*

Let $\mathcal{A}$ be an affinoid k-algebra. A Banach A-module is said to be *finite* if there exists a surjective admissible morphism $\mathcal{A}^n \to M$. We will denote the category of finite Banach $\mathcal{A}$-modules with bounded $\mathcal{A}$-linear as morphisms by $\mathbf{Mod}_b^f(\mathcal{A})$. The category of finite $\mathcal{A}$-modules is denoted by $\mathbf{Mod}^f(\mathcal{A})$.

Proposition 1.2.5. *The forgetful functor* $\mathbf{Mod}_b^f(\mathcal{A}) \to \mathbf{Mod}^f(\mathcal{A})$ *induce an equivalence of categories.*

Definition 1.2.6. Let $\mathcal{A}$ be an affinoid k-algebra, we set $X = \mathcal{M}(\mathcal{A})$. A closed subset $V \subset \mathcal{M}(\mathcal{A})$ is called an *affinoid domain*, if there exists a k-affinoid algebra $\mathcal{A}_V$ and a bounded k-algebra morphism $\mathcal{A} \to \mathcal{A}_V$ such that we have the following properties:

- The image of $\mathcal{M}(\mathcal{A}_V)$ in X lies in V.
- Let $\mathcal{B}$ be an affinoid algebra over k and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a bounded k-algebra morphism, such that $\varphi^*(\mathcal{M}(\mathcal{B})) \subset V$. There exists a unique bounded morphism of k-algebra $\mathcal{A}_V \to \mathcal{B}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[dr] & \mathcal{A}_V \arrow[d] \\ & \mathcal{B} \end{tikzcd}$$

Example 1.2.7.
- **Laurent domain:** Let $f = (f_1, \cdots, f_n) \in \mathcal{A}^n$ and $g = (g_1, \cdots, g_m) \in \mathcal{A}^m$ and let $R = (R_1, \cdots, R_n)$ and $R' = (R'_1, \cdots, R'_m)$ be real positive numbers tuples. Then the set:

$$X(R^{-1}f, R'g) := \{x \in X| \, |f_i(x)| \leq R_i, \ |g_j(x)| \geq R'_j, \ 1 \leq i \leq n, \ 1 \leq j \leq m\}$$

  is an affinoid domain in X represented by the morphism:

$$\mathcal{A} \to \mathcal{A}\{R^{-1}f, R'g\} = \mathcal{A}\{R_1^{-1}T_1, \cdots R_n^{-1}T_n, R'_1S_1^{-1}, \cdots, R'_mS_m^{-1}\}/(T_i - f_i, g_jS_j - 1)$$

  Affinoid domains of this form are called *Laurent domains*. If moreover $m = 0$, they are called *Weierstrass domains*. The Laurent neighbourhoods of a point $x \in X$ form a basis of closed neighbourhoods of the point.

- **Rational domain:** Let $g, f_1, \cdots, f_n \in \mathcal{A}$ be elements without common zeros in X. Let $R = (R_1, \cdots, R_n)$ be a real positive numbers tuple. Then the set:

$$X(R^{-1}\frac{f}{g}) := \{x \in X||f_i(x)| \leq R_i|g(x)|, \ 1 \leq i \leq n\}$$

  is an affinoid domain in X represented by the morphism:

$$\mathcal{A} \to \mathcal{A}\{R^{-1}\frac{f}{g}\} := \mathcal{A}\{R_1^{-1}T_1, \cdots R_n^{-1}T_n\}/(gT_i - f_i)$$

  Affinoid domains of this form are called *rational domains*.

Properties 1.2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two k-affinoid algebras. We have the following properties:*

- *If V is an affinoid domain of X then the map $\mathcal{A} \to \mathcal{A}_V$ induces an homeomorphism between $\mathcal{M}(\mathcal{A}_V)$ and V.*

- *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a bounded k-algebra morphism. For an affinoid (resp. Weierstrass, resp. Laurent, resp. rational) domain $V \subset \mathcal{M}(\mathcal{A})$, the set $\varphi^{*-1}(V)$ is an affinoid (resp. Weierstrass, resp. Laurent, resp. rational) domain in $\mathcal{M}(\mathcal{B})$. The associated affinoid k-algebra is $\mathcal{B}\hat{\otimes}_{\mathcal{A}}\mathcal{A}_V$.*
- *For U and V two affinoid (resp. Weirestrass, resp. Laurent, resp. rational) domains, $U \cap V$ is an affinoid domain in $\mathcal{M}(\mathcal{A})$; the respective k-affinoid algebra is $\mathcal{A}_U\hat{\otimes}_{\mathcal{A}}\mathcal{A}_V$.*

Theorem 1.2.9 (Tate's Acyclicity Theorem)**.** *Let $\mathcal{A}$ be a k-affinoid algebra. If $\mathcal{V} = \{V_i\}_{i\in I}$ is a finite cover of $X := \mathcal{M}(\mathcal{A})$ by affinoid domains and M is a finite $\mathcal{A}$-module, then the Čech complex*

$$0 \to M \to \prod_i M \otimes_{\mathcal{A}} \mathcal{A}_{V_i} \to \prod_{i,j} M \otimes_{\mathcal{A}} \mathcal{A}_{V_{ij}} \to \cdots$$

is exact.

Definition 1.2.10. A subset V of X is said to be *special* if it is a finite union $V = \bigcup V_i$ of affinoid domains. We set then $\mathcal{A}_V = \mathrm{Ker}(\prod_i \mathcal{A}_{V_i} \to \prod_{i,j} \mathcal{A}_{V_{ij}})$. By Tate's acyclicity theorem A_V does not depend on the choice of the cover.

Let $\mathcal{A}$ be a k-affinoid algebra and $X := \mathcal{M}(\mathcal{A})$. Using Tate's Acyclicity Theorem, we can define a sheaf of functions over X as follows. For an open $\mathcal{U} \subset X$, we set:

$$\mathcal{O}_X(\mathcal{U}) = \varprojlim_{V \subset U} \mathcal{A}_V$$

where V is a special subset of X and $\mathcal{A}$ is the associated k-Banach algebra. The stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at x is a local noetherian ring.

Definition 1.2.11. A *k-affinoid space* is the data of a locally ringed space $(X, \mathcal{O}_X)$, a k-affinoid algebra$\mathcal{A}$ and an homeomorphism between X and $\mathcal{M}(\mathcal{A})$. An *affinoid space over k* is an Ω-affinoid space for some $\Omega \in E(k)$.

The *category of k-affinoid spaces* is the category whose objects are k-affinoid spaces and morphisms are the morphisms of locally ringed spaces induced by morphisms of k-affinoid algebras. The *category of affinoid spaces over k* is the category whose objects are affinoid spaces over k and morphisms are the morphisms of locally ringed spaces induced by morphisms of affinoid algebras over k.

Remark 1.2.12. Note that the functor $\mathcal{A} \to \mathcal{M}(\mathcal{A})$ induce an anti-equivalence of categories between the category of k-affinoid algebras and k-affinoid spaces. Therefore, for $\Omega \in E(k)$ we have a natural functor from the category k-affinoid spaces to the category Ω-affinoid spaces. For a k-affinoid space $(X, \mathcal{O}_X)$ we denote by $(X_\Omega, \mathcal{O}_{X_\Omega})$ the associated Ω-affinoid space.

Definition 1.2.13. Let $\mathcal{A}$ and $\mathcal{B}$ be two k-affinoid algebras. Let $\mathcal{A} \to \mathcal{B}$ be a morphism of k-affinoid algebras, such that $\mathcal{B}$ is a finite A-module. We will call *finite morphism of k-affinoid spaces* any morphism of k-affinoid spaces $Y \to X$ induced by such a morphism of k-affinoid algebras.

A boundary in $\mathcal{M}(\mathcal{A})$, where A is a k-Banach algebra, is a closed subset Γ of $\mathcal{M}(\mathcal{A})$, such that every element of A has its maximum in Γ. Minimal boundaries (with respect to the inclusion) exist. If there is a smallest boundary, it called the *Shilov boundary* of A, and it is denoted by $\Gamma(A)$.

Proposition 1.2.14. *Let $\mathcal{A}$ be a k-affinoid algebra. The Shilov boundary $\Gamma(\mathcal{A})$ of $\mathcal{A}$ exists and it is finite.*

1.2.2 Good analytic spaces

Quasiaffinoid spaces

A *k-quasiaffinoid space* is the data of a pair $(\mathcal{U}, \psi)$, where $\mathcal{U}$ is a locally ringed space and ψ is an open immersion of $\mathcal{U}$ in a k-affinoid space $X := \mathcal{M}(\mathcal{A})$. A closed subset $V \subset \mathcal{U}$ is called an affinoid domain if $\psi(V)$ is an affinoid domain in X. The corresponding k-affinoid algebra is denoted by $\mathcal{A}_V$. Let $(\mathcal{U}, \psi)$ and $(\mathcal{V}, \psi')$ be two k-quasiaffinoid spaces and let $\mathcal{A}$ and $\mathcal{B}$ their respective associated k-affinoid algebras. A morphism of k-quasiaffinoid spaces $\tau : (\mathcal{U}, \psi) \to (\mathcal{V}, \psi')$ is a morphism of locally ringed spaces that satisfies: for each affinoid domains $U \subset \mathcal{U}$ and $V \subset \mathcal{V}$ with $\tau(U) \subset Int(V/\mathcal{V})$ (the topological interior of V in $\mathcal{V}$), the induced morphism $\mathcal{B}_V \to \mathcal{A}_U$ is bounded.

Definition 1.2.15. Let X be a locally ringed space. A *local k-analytic chart* of X is a k-quasiaffinoid space $(\mathcal{U}, \psi)$, where $\mathcal{U}$ is a sub-locally ringed open space of X.

Definition 1.2.16. A *k-analytic atlas* $\mathbf{A}$ on X is the data of a collection of charts $\{(\mathcal{U}_i, \psi_i)\}_{i\in I}$, that satisfies the following conditions:

- $\bigcup_{i\in I} \mathcal{U}_i = X$.
- [The compatibility of the charts] The morphisms of locally ringed spaces $\psi_j\psi_i^{-1}$: $\psi_i(\mathcal{U}_i \cap \mathcal{U}_j) \to \psi_j(\mathcal{U}_i \cap \mathcal{U}_j)$ are isomorphisms of k-quasiaffinoid spaces.

We define an equivalence relation over the set of k-analytic atlas over X as follows: $\mathbf{A} \sim \mathbf{A}'$ if and only if $\mathbf{A} \cup \mathbf{A}'$ is an atlas on X (i.e. the charts of $\mathbf{A}$ are compatible with the charts of $\mathbf{A}'$). The equivalence class associated to an atlas $\mathbf{A}$ will be denoted by $\widetilde{\mathbf{A}}$.

Definition 1.2.17. A *good k-analytic space* is a locally ringed space equipped with an equivalence class of k-analytic atlases on X. A *good analytic space over k* is a good Ω-analytic space for some $\Omega \in E(k)$.

A morphism between two good k-analytic spaces X and Y is a morphism of locally ringed spaces $\Phi : X \to Y$ such that there exists an atlas $\{(\mathcal{U}_i, \psi_i)\}_i \in \widetilde{\mathbf{A}}$ on X and an atlas $\{(\mathcal{V}_i, \psi'_i)\}_i \in \widetilde{\mathbf{A}'}$ on Y, such that $\psi'_j \Phi \psi_i^{-1} : \psi_i(\mathcal{U}_i) \to \psi'_j(\mathcal{V}_j)$ is a morphism of k-quasiaffinoid spaces. We define in an analogous way morphisms between two analytic good spaces over k. This define the *category of good k-analytic spaces* and the *category of good analytic spaces over k.*

Remark 1.2.18. Note that, for $\Omega \in E(k)$ there exists an natural functor from the category of good k-analytic spaces to the category of good Ω-analytic spaces. For a good k-analytic space $(X, \mathcal{O}_X)$ we denote by $(X_\Omega, \mathcal{O}_{X_\Omega})$ the associated good Ω-analytic space.

Remark 1.2.19. We have a canonical surjective morphism, that we will call a projection:

$$\pi_{\Omega/k} : (X_\Omega, \mathcal{O}_{X_\Omega}) \to (X, \mathcal{O}_X) \tag{1.8}$$

For $x \in X$, the set theoretic fiber $\pi_{\Omega/k}^{-1}(x)$ of x is homeomorphic to $\mathcal{M}(\mathscr{H}(x)\hat{\otimes}_k\Omega)$.

The category of good k-analytic spaces admits *fiber products*. However, the category of good analytic spaces over k does not admit fiber products in general except in special cases. We give the following important examples. Let $\Phi : Y \to X$ be a morphism of good k-analytic spaces, and let $x \in X$. Then the fiber product $Y_x := Y \times_X \mathcal{M}(\mathscr{H}(x))$ exists. Indeed there exists an isomorphism $Y_x \xrightarrow{\sim} Y_{\mathscr{H}(x)} \times_{X_{\mathscr{H}(x)}} \mathcal{M}(\mathscr{H}(x))$. This is a good $\mathscr{H}(x)$-analytic space called the *fiber of Φ at x.*

Definition 1.2.20. A morphism of k-analytic spaces $\Phi : Y \to X$ is called an *analytic domain* if Φ induces a homeomorphism of Y with its image in X, and if, for any morphism $\Psi : Z \to X$ with $\Psi(Z) \subset \Phi(Y)$, there exists a unique morphism $\sigma : Z \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[rr, "\sigma"] \arrow[dr, "\Psi"'] & & Y \arrow[dl, "\Phi"] \\ & X & \end{tikzcd}.$$

If in addition $\Phi(Y)$ is a k-affinoid (resp. k-quasiaffinoid) space, it is called *affinoid (resp. quasiaffinoid) domain*.

Properties 1.2.21. *Let X be a good k-analytic space and $x \in X$. The affinoid neighbourhoods of x form a basis of neighbourhoods of x.*

Note that, if V an affinoid neighbourhood of x, the completed residue field $\mathscr{H}(x)$ does not depend on the choice of this affinoid domain. Indeed, on the one hand we have $\mathcal{O}_{X,x} \simeq \mathcal{O}_{V,x}$. On the other hand the residue field of the local ring $\mathcal{O}_{V,x}$ is naturally valued and $\mathscr{H}(x)$ is its completion.

Definition 1.2.22. A morphism of good k-analytic spaces $\varphi : Y \to X$ is said to be *finite at a point* $y \in Y$ if there exist affinoid neighbourhoods V of y and U of $\varphi(y)$ such that φ induces a finite morphism $V \to U$. It is said to be *quasifinite* if it is finite at any point $y \in Y$.

Definition 1.2.23. A quasifinite morphism of good k-analytic spaces $\varphi : Y \to X$ is said to be *étale* at a point y if $\mathcal{O}_{Y,y}/m_x\mathcal{O}_{Y,y}$ is a unramified finite separable extension of the field $\kappa(x) := \mathcal{O}_{X,x}/m_x\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{X,x}$, where $x := \varphi(y)$.

Definition 1.2.24. An analytic map $\varphi : Y \to X$ is said to be *étale*, if it for all y there exists an affinoid neighbourhood V (resp. U) of y (resp. $\varphi(x)$) such that $\varphi|_V : V \to U$ is étale at y.

Affine analytic space

The *affine analytic space* of dimension n over k is the set $\mathbb{A}_k^{n,\mathrm{an}}$ of multiplicative semi-norms on $k[T_1, \cdots, T_n]$ that whose restrictions to k coincide with the absolute value of k. As for the analytic spectrum (cf. 1.1.2), we endow $\mathbb{A}_k^{n,\mathrm{an}}$ with the initial topology of the map $\mathbb{A}_k^{n,\mathrm{an}} \to \mathbb{R}_+^{k[T_1,\cdots,T_n]}$. The closed disk centred at $\underline{c} = (c_1, \cdots, c_n)$ with radius $\underline{r} = (r_1, \cdots, r_n) \in (\mathbb{R}_+^*)^n$ is the set:

$$D_k^+(\underline{c}, \underline{r}) := \{x \in \mathbb{A}_k^{n,\mathrm{an}} |\ |T_i(x) - c_i| \leq r_i,\ 1 \leq i \leq n\}.$$

The open disk centred at $\underline{c} = (c_1, \cdots, c_n)$ with radius $\underline{r} = (r_1, \cdots, r_n) \in (\mathbb{R}_+^*)^n$ is the set:

$$D_k^-(\underline{c}, \underline{r}) := \{x \in \mathbb{A}_k^{n,\mathrm{an}} |\ |T_i(x) - c_i| < r_i,\ 1 \leq i \leq n\}.$$

We may delete the index k when it is obvious from the context.

We have $\mathbb{A}_k^{n,\mathrm{an}} = \bigcup_{\underline{r} \in (\mathbb{R}_+^*)^n} D^+(\underline{0}, \underline{r})$. The topological space $D^+(\underline{0}, \underline{r})$ is homeomorphic to $\mathcal{M}(k\{\underline{r}^{-1}\underline{T}\})$. This induces a structure of good k-analytic space on $\mathbb{A}_k^{n,\mathrm{an}}$.

Remark 1.2.25. Note that, for $\Omega \in E(k)$ the good Ω-analytic space $(\mathbb{A}_k^{n,\mathrm{an}})_\Omega$ coincides with $\mathbb{A}_\Omega^{n,\mathrm{an}}$.

1.2.3 Analytic line in the sens of Berkovich

The analytic line

The *affine analytic line* is the analytic affine space $\mathbb{A}_k^{1,\mathrm{an}}$ of dimension one. The *projective analytic line* $\mathbb{P}_k^{1,an}$is obtained by gluing $\mathcal{M}(k\{T\})$ and $\mathcal{M}(k\{T^{-1}\})$ through $\mathcal{M}\{T, T^{-1}\}$ by the maps:

$$\begin{tikzcd}
k\{T\} \arrow[dr] & \\
& k\{T, T^{-1}\}. \\
k\{T^{-1}\} \arrow[ur] &
\end{tikzcd}$$

We can describe the analytic affine line $\mathbb{A}_k^{1,\mathrm{an}}$. For that purpose we can assume that k is algebraically closed. Indeed, the Galois group Gal(k^{alg}/k) acts over k^{alg} and the natural map:

$$\mathbb{A}^{1,\mathrm{an}}_{\widehat{k^{alg}}}/\mathrm{Gal}(k^{alg}/k) \xrightarrow{\sim} \mathbb{A}_k^{1,\mathrm{an}}$$

is a homeomorphism (cf. [Ber90, Corollary 1.3.6]).

Affinoid domains of $\mathbb{A}_k^{1,\mathrm{an}}$

In the case where k is algebraically closed, the affinoid domains of $\mathbb{A}_k^{1,\mathrm{an}}$ are *finite unions* of connected affinoid domains of the form:

$$X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$$

where $c_0, \cdots, c_\mu \in k$ with $|c_i - c_0| \leq r_0$ for each i and $0 < r_0, \cdots, r_\mu \leq r_0$ (the case where $\mu = 0$ is included). The associated affinoid algebra is

$$\mathcal{O}_X(X) = \bigoplus_{i=1}^{\mu}\{\sum_{j\in\mathbb{N}^*} \frac{a_{ij}}{(T-c_i)^j} |\ a_{ij} \in k;\ \lim_{j\to+\infty} |a_{ij}|r_i^{-j} = 0\} \bigoplus k\{r_0^{-1}(T-c_0)\}$$

where $\| \sum_{j\in\mathbb{N}^*} \frac{a_{ij}}{(T-c_i)^j}\| = \max_j |a_{ij}|r_i^{-j}$ and the direct sum above is equipped with the maximum norm (cf. [FV04, Proposition 2.2.6]).

For r_1, $r_2 \in \mathbb{R}_+$, such that $0 < r_1 \leq r_2$, we will call a *closed annulus* the affinoid domain

$$C_k^+(c, r_1, r_2) := D_k^+(c, r_2) \setminus D_k^-(c, r_1)$$

and for $r_1 < r_2$, we will call *open annulus* the open set

$$C_k^-(c, r_1, r_2) := D_k^-(c, r_2) \setminus D_k^+(c, r_1).$$

We may delete the index k when it is obvious from the context.

Type of points of $\mathbb{A}_k^{1,\mathrm{an}}$

Let $\Omega \in E(k)$, we set $E_{\Omega/k} := \dim_{\mathbb{Q}}(|\Omega^*|/|k^*| \otimes_{\mathbb{Z}} \mathbb{Q})$ (dimension as $\mathbb{Q}$-vector space) and $F_{\Omega/k} := \mathrm{tr.deg}(\widetilde{\Omega}/\tilde{k})$(transcendence degree of $\widetilde{\Omega}$ over $\tilde{k}$). For a point $x \in \mathbb{A}_k^{1,\mathrm{an}}$ we have *Abhyankar's inequality*:

$$E_{\mathscr{H}(x)/k} + F_{\mathscr{H}(x)/k} \leq 1. \tag{1.9}$$

Therefore, there are four types of points $\mathbb{A}_k^{1,\mathrm{an}}$:

- *x is a point of type (1):* if $\mathscr{H}(x) \subset \widehat{k^{alg}}$.
- *x is a point of type (2):* if $F_{\mathscr{H}(x)/k} = 1$.
- *x is a point of type (3):* if $E_{\mathscr{H}(x)/k} = 1$.
- *x is a point of type (4):* if $E_{\mathscr{H}(x)/k} = F_{\mathscr{H}(x)/k} = 0$ and x is not of type (1).

In the case where k is algebraically closed we can discribe the nature of this points more explicitly:

- *Type (1):* if x is a point of type (1), then there exists $c \in k$ such that:

$$\begin{array}{rccc} |.|_x : & k[T] & \longrightarrow & \mathbb{R}_+ \\ & P(T) & \mapsto & |P(c)| \end{array}.$$

- *Type (2):* if x is a point of type (2), then there exists $c \in k$ and $r \in |k^*|$ such that:

$$\begin{array}{rccc} |.|_x : & k[T] & \longrightarrow & \mathbb{R}_+ \\ & \sum\limits_{i=0}^{m} a_i(T-c)^i & \mapsto & \max\limits_{0\leq i\leq m} |a_i|r^i \end{array}.$$

  We will denote such point by $x_{c,r}$.

- *Type (3):* if x is a point of type (3), then its multiplicative semi-norm is defined as for the points of type (2) but with $r \in \mathbb{R}_+ \setminus |k|$ and we will denote it also by $x_{c,r}$.

- *Type (4):* if x is a point of type (4), then there exists a family of *nested* closed disks $\mathscr{E}$ (i.e the closed disks of $\mathscr{E}$ are indexed by a totally ordred set $(I, \leq)$ and for each $i \leq j$ we have $D^+(c_j, r_j) \subset D^+(c_i, r_i)$) with $(\bigcap_{i\in I} D^+(c_i, r_i)) \cap k = \varnothing$ such that:

$$\begin{array}{rccc} |.|_x : & k[T] & \longrightarrow & \mathbb{R}_+ \\ & P(T) & \mapsto & \inf\limits_{i\in I} |P(T)|_{x_{c_i,r_i}} \end{array}$$

  Note that here, we have $\bigcap_{i\in I} D^+(c_i, r_i) = \{x\}$.

Remark 1.2.26. Note that a point $x_{c,r}$ of type (2) or (3) correspond to the unique point of the Shilov boundary of the closed disk $D^+(c, r)$.

Definition 1.2.27. Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ and $y \in \pi^{-1}_{\widehat{k^{alg}}/k}(x)$. We defined the radius of x to be the value:

$$r_k(x) = \inf_{c\in k^{alg}} |T(y) - c|.$$

It does not depend on the choice of y. We may delete k if it is obvious from the context.

Remark 1.2.28. Assume that k is algebraically closed. For a point $x_{c,r}$ of type (1), (2) or (3), we have $r_k(x_{c,r}) = r$; and for a point x of type (4), we have $r_k(x) = \inf_{i\in I} r_i$ where the r_i are the radii of the disks of $\mathscr{E}$ (introduced above).

Remark 1.2.29. Let $\Omega \in E(k)$ and $\pi_{\Omega/k} : \mathbb{A}^{1,\mathrm{an}}_{\Omega} \to \mathbb{A}^{1,\mathrm{an}}_k$ be the canonical projection. Let $x \in \mathbb{A}^{1,\mathrm{an}}_k$ and $y \in \pi^{-1}_{\Omega/k}(x)$. In general, we have

$$r_k(x) \neq r_\Omega(y).$$

We will show further when the equality holds.

Assume that k is algebraically closed. Let $c \in k$. The following map

$$\begin{array}{ccc} [0,+\infty) & \longrightarrow & \mathbb{A}^{1,\mathrm{an}}_k \\ r & \mapsto & x_{c,r} \end{array}$$

induces a homeomorphism between $[0,+\infty)$ and its image.

Notation 1.2.30. We will denote by $[x_{c,r},\infty)$ (resp. $(x_{c,r},\infty)$) the image of $[r,+\infty)$ (resp. (r,∞)), by $[x_{c,r},x_{c,r}]$ (resp. $(x_{c,r},x_{c,r'}]$, $[x_{c,r},x_{c,r'})$, $(x_{c,r},x_{c,r'})$) the image of $[r,r']$ (resp. $(r,r']$, $[r,r')$, (r,r')).

Completed residue fields of the points of $\mathbb{A}^{1,\mathrm{an}}_k$

In the case where k is algebraically closed, we can describe the field $\mathscr{H}(x)$, where x is a point of $\mathbb{A}^{1,\mathrm{an}}_k$, in a more explicit way. In the case where x is of type (1), we have $\mathscr{H}(x) = k$. If x is of type (3) of the form $x = x_{c,r}$ where $c \in k$ and $r \notin |k|$, then it is easy to see that $\mathscr{H}(x) = \mathcal{O}(C^+(c,r,r))$. But for the points of type (2) and (4), a description is not obvious, we have the following Propositions.

Convention 1.2.31. In this section we assume that k is algebraically closed.

Proposition 1.2.32 (Mittag-Leffler Decomposition)**.** *Let $x = x_{c,r}$ be a point of type (2) of $\mathbb{A}^{1,\mathrm{an}}_k$ ($c \in k$ and $r \in |k^*|$). We have the decomposition:*

$$\mathscr{H}(x) = E \oplus \mathcal{O}(D^+(c,r))$$

where E is the closure in $\mathscr{H}(x)$ of the ring of rational fractions in $k(T-c)$ whose poles are in $D^+(c,r)$. i.e. for $\gamma \in k$ with $|\gamma| = r$:

$$E := \widehat{\bigoplus}_{\tilde{\alpha}\in\tilde{k}} \{ \sum_{i\in\mathbb{N}^*} \frac{a_{\alpha i}}{(T-c+\gamma\alpha)^i} |\ a_{\alpha i} \in k,\ \lim_{i\to+\infty} |a_{\alpha i}| r^{-i} = 0\},$$

where α is an element of k that corresponds to the class $\tilde{\alpha}$.

Proof. In the case where k is not trivially valued we refer to [Chr83, Theorem 2.1.6]. Otherwise, the only point of type (2) of $\mathbb{A}^{1,\mathrm{an}}_k$ is $x_{0,1}$, which corresponds to the trivial norm on $k[T]$. Therefore, we have $\mathscr{H}(x) = k(T)$. □

Lemma 1.2.33. *Let $x \in \mathbb{A}^{1,\mathrm{an}}_k$ be a point of type (4). The field $\mathscr{H}(x)$ is the completion of $k[T]$ with respect to the norm $|.|_x$.*

Proof. Recall that for a point $x \in \mathbb{A}_k^{1,\mathrm{an}}$ of type (4), the field $\mathscr{H}(x)$ is the completion of $k(T)$ with respect to $|.|_x$. To prove that $\mathscr{H}(x)$ is the completion of $k[T]$, it is enough to show that $k[T]$ is dense in $k(T)$ with respect to $|.|_x$. For all $a \in k$ it is then enough to show that there exists a sequence $(P_i)_{i\in\mathbb{N}} \subset k[T]$ which converges to $\frac{1}{T-a}$. Let $a \in k$. Since x is of type (4), there exists $c \in k$ such that $|T-c|_x < |T-a|_x$. Therefore we have $|c-a|_x = |T-a|_x$ and we obtain:

$$\frac{1}{T-a} = \frac{1}{(T-c)+(c-a)} = \frac{1}{c-a}\sum_{i\in\mathbb{N}}\frac{(T-c)^i}{(a-c)^i}.$$

So we conclude. $\square$

Proposition 1.2.34. *Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (4). Then there exists an isometric isomorphism $\psi : \mathcal{O}(D^+_{\mathscr{H}(x)}(T(x), r_k(x))) \to \mathscr{H}(x)\hat{\otimes}_k\mathscr{H}(x)$ of $\mathscr{H}(x)$-Banach algebras.*

Proof. Note that, for any element $f \in \mathscr{H}(x)\hat{\otimes}_k\mathscr{H}(x)$ with $|f| \leq r_k(x)$, we can define a morphism of $\mathscr{H}(x)$-Banach algebras $\psi : \mathcal{O}(D^+_{\mathscr{H}(x)}(T(x), r_k(x))) \to \mathscr{H}(x)\hat{\otimes}_k\mathscr{H}(x)$, that associates f to $T-T(x)$. To prove the statement we choose $f = T(x)\otimes 1 - 1\otimes T(x)$. This is possible since, for all $a \in k$ we have $T(x)\otimes 1-1\otimes T(x) = (T(x)-a)\otimes 1 - 1\otimes(T(x)-a)$, hence,

$$|T(x)\otimes 1 - 1\otimes T(x)| \leq \inf_{a\in k}(\max(|(T(x)-a)|, |(T(x)-a)|) = \inf_{a\in k}|T(x)-a| = r_k(x)$$

By construction ψ is a contracting $\mathscr{H}(x)$-linear map. In order to prove that it is an isometric isomorphism, we need to construct its inverse map and show that it is also a contracting map. For all $a \in k$, $|T(x)-a| > r_k(x)$, hence $T-a$ is invertible in $\mathcal{O}(D^+_{\mathscr{H}(x)}(T(x), r_k(x)))$. This means that $k(T) \subset \mathcal{O}(D^+_{\mathscr{H}(x)}(T(x), r_k(x)))$ as k-vector space. As for all $a \in k$ we have:

$$|T-a| = \max(r_k(x), |T(x)-a|) = |T(x)-a|,$$

the restriction of the norm of $\mathcal{O}(D^+_{\mathscr{H}(x)}(T(x), r_k(x)))$ to $k(T)$ coincides with $|.|_x$. Consequently, the closure of $k(T)$ in $\mathcal{O}(D^+_{\mathscr{H}(x)}(T(x), r_k(x)))$ is exactly $\mathscr{H}(x)$, which means that we have an isometric embedding $\mathscr{H}(x) \hookrightarrow \mathcal{O}(D^+_{\mathscr{H}(x)}(T(x), r_k(x)))$ of k-algebras which associates $T(x)$ to T. This map extends uniquely to a contracting morphism of $\mathscr{H}(x)$-algebras:

$$\begin{tikzcd}
\mathscr{H}(x) \arrow[r, hook] \arrow[d] & \mathcal{O}(D^+_{\mathscr{H}(x)}(T(x), r_k(x))) \\
\mathscr{H}(x)\hat{\otimes}_k\mathscr{H}(x) \arrow[ur, "\exists!\varphi"'] &
\end{tikzcd}$$

Then we have $\varphi(T(x)\otimes 1 - 1\otimes T(x)) = T - T(x)$. Since $(T-T(x))$ (resp. $T(x)\otimes 1 - 1\otimes T(x)$) is a topological generator of the $\mathscr{H}(x)$-algebra $D^+_{\mathscr{H}(x)}(T(x), r(x))$ (resp. $\mathscr{H}(x)\otimes\mathscr{H}(x)$) and both of $\varphi\circ\psi$ and $\psi\circ\varphi$ are bounded morphisms of k-Banach algebras, we have $\varphi\circ\psi = \mathrm{Id}_{\mathcal{O}(D^+_{\mathscr{H}(x)}(T(x), r_k(x)))}$ and $\psi\circ\varphi = \mathrm{Id}_{\mathscr{H}(x)\hat{\otimes}_k\mathscr{H}(x)}$. Hence, we obtain the result. $\square$

Universal points and fiber of a point under extension of scalars

Definition 1.2.35. A point $x \in \mathbb{A}_k^{1,\mathrm{an}}$ is said to be *universal* if, for any $\Omega \in E(k)$, the tensor norm on the algebra $\mathscr{H}(x)\hat{\otimes}_k\Omega$ is multiplicative. In this case, it defines a point of $\pi_{\Omega/k}^{-1}(x)$ in $\mathbb{A}_\Omega^{1,\mathrm{an}}$ that we denote by $\sigma_{\Omega/k}(x)$.

Proposition 1.2.36. *In the case where k is algebraically closed, any point $x \in \mathbb{A}_k^{1,\mathrm{an}}$ is universal.*

Proof. See [Poi13, Corollary 3.14.]. □

Theorem 1.2.37. *Suppose that k is algebraically closed. Let $\Omega \in E(k)$ algebraically closed.*

- *If x has type (i), where $i \in \{1,2\}$, then $\sigma_{\Omega/k}(x)$ has type (i). If x has type (j), where $j \in \{3,4\}$, then $\sigma_{\Omega/k}(x)$ has type (j) or (2).*
- *The fiber $\pi_{\Omega/k}^{-1}(x)$ is connected and the connected components of $\pi_{\Omega/k}^{-1}(x) \setminus \{\sigma_{\Omega/k}(x)\}$ are open disks with boundary $\{\sigma_{\Omega/k}(x)\}$. Moreover they are open in $\mathbb{A}_\Omega^{1,\mathrm{an}}$.*

Proof. See [PP15, Theorem 2.2.9]. □

Remark 1.2.38. In the case where x is a point of type (4), we have

$$\pi_{\mathscr{H}(x)/k}^{-1}(x) = D^+_{\mathscr{H}(x)}(T(x), r_k(x))$$

(cf. Remark 1.2.19 and Proposition 1.2.34) and $\sigma_{\mathscr{H}(x)/k}(x) = x_{T(x),r_k(x)}$.

Corollary 1.2.39. *Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (i), where $i \in \{2,3,4\}$. Let $\Omega \in E(k)$ algebraically closed such that there is no isometric k-embedding $\mathscr{H}(x) \hookrightarrow \Omega$. Then*

$$\pi_{\Omega/k}^{-1}(x) = \{\sigma_{\Omega/k}(x)\}.$$

Proof. Recall that $\pi_{\Omega/k}^{-1}(x)\setminus\{\sigma_{\Omega/k}(x)\}$ is a disjoint union of open disks (cf. Theorem 1.2.37). Therefore, if it is not empty, it contains points of type (1) which gives rise to isometric k-embeddings $\mathscr{H}(x) \hookrightarrow \Omega$, which contradict the hypothesis. □

Lemma 1.2.40. *Let $\Omega \in E(k)$ such that $k^{alg} \subset \Omega$. Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$. Then for any $y \in \pi_{\widehat{k^{alg}}/k}^{-1}(x)$ we have:*

$$r_\Omega(\sigma_{\Omega/\widehat{k^{alg}}}(y)) = r_k(x).$$

Proof. If x is of type (1), then for any $y \in \pi_{\widehat{k^{alg}}/k}^{-1}(x)$, the point $\sigma_{\Omega/\widehat{k^{alg}}}(y)$ is of type (1). Hence, we obtain $r_\Omega(\sigma_{\Omega/\widehat{k^{alg}}}(y)) = r_k(x) = 0$.

If x is of type (2) or (3), then any $y \in \pi^{-1}_{\widehat{k^{alg}}/k}(x)$ is of the form $x_{c,r_k(x)}$, where $c \in k^{alg}$. Since the morphism $\mathcal{O}(D^+_{\widehat{k^{alg}}}(c, r_k(x))) \to \mathscr{H}(y)$ is isometric, then so is

$$\mathcal{O}(D^+_{\Omega}(c, r_k(x))) \to \mathscr{H}(y)\hat{\otimes}_{\widehat{k^{alg}}}\Omega.$$

Therefore, we have $\sigma_{\Omega/\widehat{k^{alg}}}(y) = x_{c,r_k(x)}$ in $\mathbb{A}^{1,\mathrm{an}}_{\Omega}$. Hence $r_{\Omega}(\sigma_{\Omega/\widehat{k^{alg}}}(y)) = r_k(x)$.

Now suppose that x is a point of type (4), then for any $y \in \pi^{-1}_{\widehat{k^{alg}}/k}(x)$ there exists a family of nested disks $\mathscr{E}$ indexed by $(I, \leq)$ such that $\bigcap_{i\in I} D^+_{\widehat{k^{alg}}}(c_i, r_i) = \{y\}$. Note that we have $r_k(x) = r_k(y) = \inf_{i\in I} r_i$. Then we have:

$$\pi^{-1}_{\Omega/\widehat{k^{alg}}}(y) = \bigcap_{i\in I} D^+_{\Omega}(c_i, r_i).$$

We distinguish two cases: the first is $\bigcap_{i\in I} D^+_{\Omega}(c_i, r_i) = \{\sigma_{\widehat{k^{alg}}/\Omega}(y)\}$. Then, we have:

$$r(\sigma_{\Omega/\widehat{k^{alg}}}(y)) = \inf_{i\in I} r_i = r_{\widehat{k^{alg}}}(y) = r_k(x).$$

The second is $\bigcap_{i\in I} D^+_{\Omega}(c_i, r_i) = D^+_{\Omega}(c, r_{\widehat{k^{alg}}}(y))$, where $c \in \Omega \setminus \widehat{k^{alg}}$. Here, $\sigma_{\Omega/\widehat{k^{alg}}}(y)$ coincides with the Shilov boundary of $D^+_{\Omega}(c, r_{\widehat{k^{alg}}}(y))$. Therefore we have

$$r(\sigma_{\Omega/\widehat{k^{alg}}}(y)) = r_{\widehat{k^{alg}}}(y) = r_k(x).$$

□

Sheaf of differential forms and étale morphisms

Here we do not give the general definition of sheaf of differential forms given in [Ber93, §1.4.], but only how it looks like in the case of an analytic domain of $\mathbb{A}^{1,\mathrm{an}}_k$. Let X be an analytic domain of $\mathbb{A}^{1,\mathrm{an}}_k$. Let T be the global coordinate function on $\mathbb{A}^{1,\mathrm{an}}_k$ fixed above. It induces a global coordinate function T on X. The sheaf of differential forms $\Omega_{X/k}$ of X is free with dT as a basis.

Let $\frac{\mathrm{d}}{\mathrm{dT}} : \mathcal{O}_X \to \mathcal{O}_X$ be the formal derivation with respect to T. In this setting the canonical derivation $d_{X/k}$ satisfies:

$$\begin{array}{rcl} d_{X/k} : \mathcal{O}_X(U) & \longrightarrow & \Omega_{X/k}(U) \\ f & \mapsto & \frac{\mathrm{d}}{\mathrm{dT}}(f)\cdot \mathrm{dT} \end{array}, \tag{1.10}$$

where U is an open subset of X.

Lemma 1.2.41. *Let X and Y be two connected open analytic domain of $\mathbb{A}^{1,\mathrm{an}}_k$ and let T (resp. S) be a coordinate function defined on X (resp. Y). Let $\varphi : Y \to X$ be a finite morphism of k-affinoid spaces and let $\varphi^{\#} : \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ be the induced morphism of k-affinoid algebras. Then φ is étale if and only if the morphism*

$$\begin{array}{rcl} \Omega_{X/k}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_Y(Y) & \longrightarrow & \Omega_{Y/k}(Y) \\ h.dT \otimes g & \mapsto & \frac{\mathrm{d}}{\mathrm{dS}}(\varphi^{\#}(h))g.dS \end{array} \tag{1.11}$$

is an isomorphism of $\mathcal{O}_X(X)$-Banach modules.

Proof. See [Ber93, Proposition 3.5.3]. □

Remark 1.2.42. Note that, any morphism $\varphi : Y \to X$ between two connected open analytic domains of $\mathbb{A}_k^{1,\mathrm{an}}$ is obtained by a convenient choice of an element f of $\mathcal{O}_Y(Y)$, which is the image of T by $\varphi^{\#}$. In this setting, assume that φ is finite, then φ is étale if and only if $\frac{\mathrm{d}}{\mathrm{dS}}(f)$ is invertible in $\mathcal{O}_Y(Y)$.

Corollary 1.2.43. *Let $\varphi : Y \to X$ be a finite morphism between connected open analytic domains of $\mathbb{A}_k^{1,\mathrm{an}}$. If char(k)$= 0$, then for each $x \in X$ of type (2), (3) or (4) there exists an affinoid neighbourhood U of x in X such that $\varphi|_{\varphi^{-1}(U)} : \varphi^{-1}(U) \to U$ is an étale morphism.*

Proof. Let $f := \varphi^{\#}(T)$. Since $\mathrm{char}(\tilde{k}) = 0$, for each $x \in X$ not of type (1) we have: $\frac{\mathrm{d}}{\mathrm{dS}}(f)(x) = 0$ if and only if $f \in k$. Since φ is finite, $f \notin k$. Hence, there exists an affinoid neighbourhood U of x such that $\frac{\mathrm{d}}{\mathrm{dS}}(f)$ is invertible in $\mathcal{O}(U)$. The result follows by Remark 1.2.42. □

Lemma 1.2.44. *Let $\varphi : Y \to X$ be a finite morphism between affinoid domains of $\mathbb{A}_k^{1,\mathrm{an}}$ and let T (resp. S) be a coordinate function defined on X (resp. Y). Let y be a point of type (2), (3) or (4). Then the induced extension $\mathscr{H}(\varphi(y)) \hookrightarrow \mathscr{H}(y)$ is finite and we have*

$$\mathscr{H}(y) = \mathscr{H}(\varphi(y))(S(y)) \simeq \bigoplus_{i=0}^{n-1} \mathscr{H}(\varphi(y)) \cdot S(y)^i.$$

Proof. Since $\varphi : Y \to X$ is finite, we have $[\mathscr{H}(y) : \mathscr{H}(\varphi(y))] = n$ for some $n \in \mathbb{N}$ (cf. [PP15, Lemma 2.24]). Therefore, $S(y)$ is algebraic over $\mathscr{H}(\varphi(y))$. Hence, $\mathscr{H}(\varphi(y))(S(y))$ is a complete intermediate finite extension , that contains $k(S(y))$. As $k(S(y))$ is dense in $\mathscr{H}(y)$, we obtain

$$\mathscr{H}(y) = \mathscr{H}(\varphi(y))(S(y)) \simeq \bigoplus_{i=0}^{n-1} \mathscr{H}(\varphi(y)) \cdot S(y)^i.$$

□

Proposition 1.2.45. *Assume that k is algebraically closed. Let $\varphi : Y \to X$ be a finite étale cover between two analytic domains of $\mathbb{A}_k^{1,\mathrm{an}}$. Let $y \in Y$ and $x := \varphi(y)$. Let U be an neighbourhood of y such that U is a connected component of $\varphi^{-1}(\varphi(U))$. Then we have*

$$[\mathscr{H}(y) : \mathscr{H}(x)] = \#U \cap \varphi^{-1}(b)$$

for each $b \in \varphi(U) \cap k$.

Proof. See [BR10, Corollary 9.17] and [Duc, (3.5.4.3)]. □

II

Spectral theory in the sense of Berkovich

Contents

This chapter is organized as follows. The first section is divided into two parts, the first is to recall the definition and first properties of the spectrum given by Berkovich. In the second part, we prove some additional properties of the spectrum. In the second section, we will expose the map that plays the role of the Cauchy integral. The last section is devoted to proving continuity results for the variation of the spectra.

2.1 Definition and first properties

2.1.1 Definition and first properties

Recall the definition of the sheaf of analytic functions with values in a k-Banach space over an analytic space and the definition of the spectrum given by V. Berkovich in [Ber90, Chapter 7].

Definition 2.1.1. Let X be a k-affinoid space and B be a k-Banach space. We define the sheaf of analytic functions with values in B over X to be the sheaf:

$$U \mapsto \mathcal{O}_X(B)(U) = \varprojlim_{V \subset U} B\hat{\otimes}_k \mathcal{A}_V$$

where U is an open subset of X, V a special subset and $\mathcal{A}_V$ the k-Banach algebra associated to V (cf. Definition 1.2.10).

As each k-analytic space is obtained by gluing k-affinoid spaces (see [Ber90], [Ber93]), we can extend the definition to k-analytic spaces.

Let U be an open subset of X. Every element $f \in \mathcal{O}_X(B)(U)$ induces a function:

$$\begin{array}{rcl} f: U & \longrightarrow & \coprod_{x\in U} B\hat{\otimes}_k \mathscr{H}(x) \\ x & \mapsto & f(x) \end{array},$$

where $f(x)$ is the image of f by the map $\mathcal{O}_X(B)(U) \to B\hat{\otimes}_k \mathscr{H}(x)$.

Definition 2.1.2. Assume that X is reduced. We will call *analytic function over U with value in B* any function $\psi : U \mapsto \coprod_{x\in U} B\hat{\otimes}_k \mathscr{H}(x)$ induced by an element $f \in \mathcal{O}_X(B)(U)$.

Convention 2.1.3. Until the end, we will assume that all the Banach algebras are with unit element and morphisms preserve the unit elements.

Definition 2.1.4. Let E be a k-Banach algebra and $f \in E$. The spectrum of f is the set $\Sigma_{f,k}(E)$ of points $x \in \mathbb{A}_k^{1,\mathrm{an}}$ such that the element $f \otimes 1 - 1 \otimes T(x)$ is not invertible in the k-Banach algebra $E\hat{\otimes}_k \mathscr{H}(x)$. The resolvent of f is the function:

$$\begin{array}{rcl} R_f : \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_{f,k}(E) & \longrightarrow & \coprod_{x\in \mathbb{A}_k^{1,\mathrm{an}}\setminus \Sigma_{f,k}(A)} E\hat{\otimes}_k \mathscr{H}(x) \\ x & \longmapsto & (f\otimes 1 - 1 \otimes T(x))^{-1} \end{array}$$

Remark 2.1.5. If there is no confusion we denote the spectrum of f, as an element of E, just by Σ_f.

Theorem 2.1.6. *Let E be a Banach k-algebra and $f \in E$. Then:*

1. *The spectrum Σ_f is a compact subset of $\mathbb{A}_k^{1,\mathrm{an}}$ and non-empty if $E \neq 0$.*
2. *The radius of the smallest (with respect to inclusion) closed disk with center at zero which contains Σ_f is equal to $\|f\|_{\mathrm{Sp}}$.*
3. *The resolvent R_f extends to an analytic function on $\mathbb{P}_k^{1,an} \setminus \Sigma_f$ which is equal to zero at infinity.*

Proof. See [Ber90, Theorem 7.1.2]. □

Remark 2.1.7. Note that the set $\Sigma_f \cap k$ coincides with the classical spectrum, i.e.

$$\Sigma_f \cap k = \{a \in k |\ f - a \text{ is not invertible in } E\}.$$

Remark 2.1.8. Let E be a k-Banach algebra and $r \in \mathbb{R}_+^*$. Note that we have

$$\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{an}}}(E)(D^+(a,r)) = E\hat{\otimes}_k \mathcal{O}(D^+(a,r)),$$

i.e any element of $\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{an}}}(E)(D^+(a,r))$ has the form $\sum_{i\in\mathbb{N}} f_i \otimes (T-a)^i$ with $f_i \in E$ (which follows from Tate's acyclicity Theorem and the good behaviours of complete tensor product under exacte sequences (cf.[Gru66, Theorem 3.2.1])). Let $\varphi = \sum_{i\in\mathbb{N}} f_i \otimes (T-a)^i$ be an element of $\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{an}}}(E)(D^+(a,r))$. Since $\|f_i \otimes (T-a)^i\| = \|f_i\|\|(T-a)\|^i$ in $\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{an}}}(E)(D^+(a,r))$ (cf. Proposition 1.1.39), the radius of convergence of φ with respect to $T-a$ is equal to $\liminf\limits_{i\to+\infty} \|f_i\|^{-\frac{1}{i}}$.

Lemma 2.1.9. *Let E be a non-zero Banach k-algebra and $f \in E$. If $a \in (\mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_f) \cap k$, then the biggest open disk centered in a contained in $\mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_f$ has radius $R = \|(f-a)^{-1}\|_{\mathrm{Sp}}^{-1}$.*

Proof. Since Σ_f is compact and not empty, the biggest disk $D^-(a,R) \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_f$ has finite positive radius R. Since R_f is analytic on $D^-(a,R)$, $R_f = \sum_{i\in\mathbb{N}} f_i \otimes (T-a)^i$ in $\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{an}}}(E)(D^+(a,r))$ for all $0 < r < R$ and has a radius convergence with respect to $(T-a)$ equal to R. In the neighbourhood of the point a, for formal reasons we have:

$$R_f = (f\otimes 1 - 1\otimes T)^{-1} = ((f-a)\otimes 1 - 1\otimes(T-a))^{-1} = ((f-a)^{-1}\otimes 1)\sum_{i\in\mathbb{N}} \frac{1}{(f-a)^i}\otimes(T-a)^i.$$

The radius of convergence of the latter series with respect to $(T-a)$ is equal to $\|(f-a)^{-1}\|_{\mathrm{Sp}}^{-1}$ (cf. Remark 2.1.8). Therefore, we have $\|(f-a)^{-1}\|_{\mathrm{Sp}}^{-1} \geq R$. If $\|(f-a)^{-1}\|_{\mathrm{Sp}}^{-1} > R$, then $f\otimes 1 - 1\otimes T(x)$ for all $x \in D^-(a, \|(f-a)^{-1}\|_{\mathrm{Sp}}^{-1})$, this contradicts the assumption. Hence, we have $R = \|(f-a)^{-1}\|_{\mathrm{Sp}}^{-1}$. □

Proposition 2.1.10. *Let E be a commutative k-Banach algebra element, and $f \in E$.*

- *The spectrum Σ_f of f coincides with the image of the analytic spectrum $\mathcal{M}(E)$ by the map ${}^*f : \mathcal{M}(E) \to \mathbb{A}_k^{1,\mathrm{an}}$ induced by the ring morphism $k[T] \to E$, $T \mapsto f$.*

- *The canonical map*
$$\begin{array}{ccc} \mathcal{M}(E) & \longrightarrow & \prod_{f\in E} \Sigma_f \\ x & \mapsto & {}^*f(x) \end{array}$$
induces a homeomorphism of $\mathcal{M}(E)$ with its image.

Proof. See [Ber90, Proposition 7.1.4]. □

Definition 2.1.11. Let E be a k-Banach algebra and B a commutative k-subalgebra of E. We say that B is a maximal commutative subalgebra of E, if for any commutative subalgebra B' of E we have the following property:

$$(B \subset B' \subset E) \Leftrightarrow (B' = B).$$

Remark 2.1.12. A maximal subalgebra B is necessarily closed in E, hence a k-Banach algebra.

Proposition 2.1.13. *Let E be a k-Banach algebra. For any maximal commutative subalgebra B of E, we have:*

$$\forall f \in B; \qquad \Sigma_f(B) = \Sigma_f(E).$$

Proof. See [Ber90, Proposition 7.2.4]. □

Definition 2.1.14. Let E be a k-algebra and let B be a k-subalgebra of E. If any element of B invertible in E is invertible in B, we say that B is a *saturated* subalgebra of E.

Proposition 2.1.15. *Assume that k is not trivially valued. Let E be a k-Banach algebra and let B be a saturated k-Banach subalgebra of E. Then we have:*

$$\forall f \in B; \qquad \Sigma_f(B) = \Sigma_f(E).$$

Proof. See [Ber90, Proposition 7.2.4]. □

Let $P(T) \in k[T]$, let E be a Banach k-algebra and let $f \in E$. We set $P(f)$ to be the image of $P(T)$ by the morphism $k[T] \to E$, $T \mapsto f$, and $P : \mathbb{A}_k^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ to be the analytic map associated to $k[T] \to k[T]$, $T \mapsto P(T)$.

Lemma 2.1.16. *Let $P(T) \in k[T]$, let E be a Banach k-algebra and let $f \in E$. We have the equality of sets:*

$$\Sigma_{P(f)} = P(\Sigma_f)$$

Proof. Let B a maximal commutative k-subalgebra of E containing f (which exists by Zorn's Lemma). Then B contains also $P(f)$. By Proposition 2.1.13 we have $\Sigma_f(E) = \Sigma_f(B)$ and $\Sigma_{P(f)}(E) = \Sigma_{P(f)}(B)$. Let ${}^*f : \mathcal{M}(B) \to \mathbb{A}_k^{1,\mathrm{an}}$ (resp. ${}^*P(f) : \mathcal{M}(B) \to \mathbb{A}_k^{1,\mathrm{an}}$) be the map induced by $k[T] \to E$, $T \mapsto f$ (resp. $T \mapsto P(f)$). By Proposition 2.1.10 we have $\Sigma_f(B) =^* f(\mathcal{M}(B))$ and $\Sigma_{P(f)}(B) =^* P(f)(\mathcal{M}(B))$. Since ${}^*P(f) = P \circ^* f$, we obtain the equality. □

Remark 2.1.17. Note that we can imitate the proof provided in [Bou07, p.2] to prove the statement of Lemma 2.1.16.

Lemma 2.1.18. *Let E and E' be two Banach k-algebras and $\varphi : E \to E'$ be a bounded morphism of k-algebras. If $f \in E$ then we have:*

$$\Sigma_{\varphi(f)}(E') \subset \Sigma_f(E).$$

If moreover φ is a bi-bounded isomorphism then we have the equality.

Proof. Consequence of the definition. □

Proposition 2.1.19. *Let E be a k-Banach algebra, and let $f \in E$. Let $\Omega \in E(k)$, and let $\pi_{\Omega/k} : \mathbb{A}^{1,\mathrm{an}}_\Omega \to \mathbb{A}^{1,\mathrm{an}}_k$ be the canonical projection (cf. (6.3)). Then we have*

$$\Sigma_{f\otimes 1,\Omega}(E\hat{\otimes}_k\Omega) = \pi^{-1}_{\Omega/k}(\Sigma_{f,k}(E)).$$

Proof. See [Ber90, Proposition 7.1.6]. □

2.1.2 Additional properties

Lemma 2.1.20. *Let M_1 and M_2 be k-Banach spaces and let $M = M_1 \oplus M_2$ endowed with the max norm. Let p_1, p_2 be the respective projections associated to M_1 and M_2 and i_1, i_2 be the respective inclusions. Let $\varphi \in \mathcal{L}_k(M)$and set $\varphi_1 = p_1\varphi i_1 \in \mathcal{L}_k(M_1)$ and $\varphi_2 = p_2\varphi i_2 \in \mathcal{L}_k(M_2)$. If $\varphi(M_1) \subset M_1$, then we have:*

i) $\Sigma_{\varphi_i}(\mathcal{L}_k(M_i)) \subset \Sigma_\varphi(\mathcal{L}_k(M)) \cup \Sigma_{\varphi_j}(\mathcal{L}_k(M_j))$*, where i, $j \in \{1, 2\}$ and $i \neq j$.*

ii) $\Sigma_\varphi(\mathcal{L}_k(M)) \subset \Sigma_{\varphi_1}(\mathcal{L}_k(M_1)) \cup \Sigma_{\varphi_2}(\mathcal{L}_k(M_2))$*. Furthermore, if $\varphi(M_2) \subset M_2$, then we have the equality.*

iii) If $\Sigma_{\varphi_1}(\mathcal{L}_k(M_1))\cap\Sigma_{\varphi_2}(\mathcal{L}_k(M_2)) = \varnothing$*, then* $\Sigma_\varphi(\mathcal{L}_k(M)) = \Sigma_{\varphi_1}(\mathcal{L}_k(M_1))\cup\Sigma_{\varphi_2}(\mathcal{L}_k(M_2))$*.*

Proof. By Lemma 1.1.42, we can represent the elements of $\mathcal{L}_k(M)$ as follows:

$$\mathcal{L}_k(M) = \left\{\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} | L_1 \in \mathcal{L}_k(M_1), L_2 \in \mathcal{L}_k(M_2, M_1), L_3 \in \mathcal{L}_k(M_1, M_2), L_4 \in \mathcal{L}_k(M_2)\right\}$$

and φ has the form $\begin{pmatrix} \varphi_1 & L \\ 0 & \varphi_2 \end{pmatrix}$, where $L \in \mathcal{L}_k(M_2, M_1)$.
Let $x \in \mathbb{A}^{1,\mathrm{an}}_k$. We have an isomorphism of $k-$Banach algebras:

$$\mathcal{L}_k(M)\hat{\otimes}_k\mathscr{H}(x) = \left\{\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} | \begin{array}{c} L_1 \in \mathcal{L}_k(M_1)\hat{\otimes}_k\mathscr{H}(x), L_2 \in \mathcal{L}_k(M_2, M_1)\hat{\otimes}_k\mathscr{H}(x), \\ L_3 \in \mathcal{L}_k(M_1, M_2)\hat{\otimes}_k\mathscr{H}(x), L_4 \in \mathcal{L}_k(M_2)\hat{\otimes}_k\mathscr{H}(x) \end{array}\right\}$$

Consequently,

$$\varphi \otimes 1 - 1 \otimes T(x) = \begin{pmatrix} \varphi_1 \otimes 1 - 1 \otimes T(x) & L \otimes 1 \\ 0 & \varphi_2 \otimes 1 - 1 \otimes T(x) \end{pmatrix}.$$

We first prove i). Let $\begin{pmatrix} L_1 & C \\ 0 & L_2 \end{pmatrix}$ be an invertible element of $\mathcal{L}_k(M)\hat{\otimes}_k\mathscr{H}(x)$. We claim that if, for $i \in \{1,2\}$, L_i is invertible in $\mathcal{L}_k(M_i)\hat{\otimes}_k\mathscr{H}(x)$, then so is L_j, where $j \neq i$. Indeed, let $\begin{pmatrix} L_1' & C' \\ B & L_2' \end{pmatrix}$ such that we have:

$$\begin{pmatrix} L_1 & C \\ 0 & L_2 \end{pmatrix}\begin{pmatrix} L_1' & C' \\ B & L_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} L_1' & C' \\ B & L_2' \end{pmatrix}\begin{pmatrix} L_1 & C \\ 0 & L_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we obtain:

$$\begin{cases} L_1L_1' + CB = 1 \\ L_1C' + CL_2 = 0 \\ L_2B = 0 \\ L_2L_2' = 1 \end{cases}; \quad \begin{cases} L_1'L_1 = 1 \\ L_1'C + C'L_2 = 0 \\ BL_1 = 0 \\ BC + L_2'L_2 = 1 \end{cases}.$$

On the one hand, we deduce that L_1 is left invertible and L_2 is right invertible. If L_1 is invertible, then $B = 0$ which implies that L_2 is left invertible, hence invertible. If L_2 is invertible, then $B = 0$ which implies that L_1 is right invertible, hence invertible. On the other hand, if $\begin{pmatrix} L_1 & C \\ 0 & L_2 \end{pmatrix}$ is an element of $\mathcal{L}_k(M)\hat{\otimes}_k\mathscr{H}(x)$ with L_i is invertible in $\mathcal{L}_k(M_i)\hat{\otimes}_k\mathscr{H}(x)$ for $i \in \{1,2\}$, then $\begin{pmatrix} L_1 & C \\ 0 & L_2 \end{pmatrix}$ is invertible in $\mathcal{L}_k(M)\hat{\otimes}_k\mathscr{H}(x)$.

Therefore, if $\varphi \otimes 1 - 1 \otimes T(x)$ and $\varphi_i \otimes 1 - 1 \otimes T(x)$ are invertible where $i \in \{1,2\}$, then $\varphi_j \otimes 1 - 1 \otimes T(x)$ is invertible for $j \in \{1,2\} \setminus \{i\}$. We conclude that $\Sigma_{\varphi_j} \subset \Sigma_\varphi \cup \Sigma_{\varphi_i}$ where $i,\ j \in \{1,2\}$ and $i \neq j$.

We now prove ii). If $\varphi_1 \otimes 1 - 1 \otimes T(x)$ and $\varphi_2 \otimes 1 - 1 \otimes T(x)$ are invertible, then $\varphi \otimes 1 - 1 \otimes T(x)$ is invertible. This proves that $\Sigma_\varphi \subset \Sigma_{\varphi_1} \cup \Sigma_{\varphi_2}$. If $\varphi(M_2) \subset M_2$, then $C = 0$ which implies that: if $\varphi \otimes 1 - 1 \otimes T(x)$ is invertible, then $\varphi_1 \otimes 1 - 1 \otimes T(x)$ and $\varphi_2 \otimes 1 - 1 \otimes T(x)$ are invertible. Hence we have the equality.

We now prove iii). If $\Sigma_{\varphi_1} \cap \Sigma_{\varphi_2} = \varnothing$, then by the above we have $\Sigma_{\varphi_1} \subset \Sigma_\varphi$ and $\Sigma_{\varphi_2} \subset \Sigma_\varphi$. Therefore, $\Sigma_{\varphi_1} \cup \Sigma_{\varphi_2} \subset \Sigma_\varphi$.

□

Remark 2.1.21. We keep the notation of Lemma 2.1.20. In the proof above, we showed also: if $\varphi \otimes 1 - 1 \otimes T(x)$ is invertible, then $\varphi_1 \otimes 1 - 1 \otimes T(x)$ is left invertible and $\varphi_2 \otimes 1 - 1 \otimes T(x)$ is right invertible.

Lemma 2.1.22. *Let $\Omega \in E(k)$, let E be an Ω-Banach algebra and $f \in E$. Then we have*

$$\Sigma_{f,k}(E) = \pi_{\Omega/k}(\Sigma_{f,\Omega}(E)),$$

where $\pi_{\Omega/k} : \mathbb{A}_{\Omega}^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ is the canonical projection.

Proof. Let B be a maximal commutative Ω-subalgebra of E containing f. Let B' be a commutative k-subalgebra of E such that $B \subset B'$. Then B' is also an Ω-subalgebra of E. Therefore B is also maximal as a commutative k-subalgebra of E. Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$, we have an isometric isomorphism $B\hat{\otimes}_k\mathscr{H}(x) \simeq B\hat{\otimes}_\Omega(\Omega\hat{\otimes}_k\mathscr{H}(x))$ (cf. [BGR84, Section 2.1, Proposition 7]). For each $y \in \pi_{\Omega/k}^{-1}(x)$, we have a contracting map $\Omega\hat{\otimes}_k\mathscr{H}(x) \to \mathscr{H}(y)$. Therefore, the induced map $B\hat{\otimes}_k\mathscr{H}(x) \to B\hat{\otimes}_\Omega\mathscr{H}(y)$ is contracting too. Hence, if $f\otimes 1 - 1\otimes T(y)$ is not invertible in $B\hat{\otimes}_\Omega\mathscr{H}(y)$, then $f\otimes 1 - 1\otimes T(x)$ is not invertible too in $B\hat{\otimes}_k\mathscr{H}(x)$. Therefore, $\pi_{\Omega/k}(\Sigma_{f,\Omega}(B)) \subset \Sigma_{f,k}(B)$. Let now $x \in \Sigma_{f,k}(B)$. Since $f\otimes 1 - 1\otimes T(y)$ is not invertible in $B\hat{\otimes}_k\mathscr{H}(x) \simeq B\hat{\otimes}_\Omega(\Omega\hat{\otimes}_k\mathscr{H}(x))$, according to Lemma 1.1.30, there exists $y \in \mathcal{M}(\Omega\hat{\otimes}_k\mathscr{H}(x)) = \pi_{\Omega/k}^{-1}(x)$ such that $f\otimes 1 - 1\otimes T(y)$ is not invertible in $B\hat{\otimes}_\Omega\mathscr{H}(y)$. Therefore, $\Sigma_{f,k}(B) \subset \pi_{\Omega/k}(\Sigma_{f,\Omega}(B))$. Hence, by Proposition 2.1.13 we obtain

$$\Sigma_{f,k}(E) = \pi_{\Omega/k}(\Sigma_{f,\Omega}(E)).$$

□

Definition 2.1.23. Let $\Omega \in E(k)$ and $f \in \mathcal{M}_n(\Omega)$. Let $\{a_1, \cdots a_N\}$ be the set of eigenvalues of f in $\widehat{\Omega^{alg}}$. We will call the set $\pi_{\widehat{\Omega^{alg}}/\Omega}(\{a_1, \cdots, a_N\})$ the set of eigenvalues of f in $\mathbb{A}_{\Omega}^{1,\mathrm{an}}$.

Corollary 2.1.24. *Let $\Omega \in E(k)$ and $f \in \mathcal{M}_n(\Omega)$. Let $\{a_1, \cdots, a_N\}$ be the set of eigenvalues of f in $\mathbb{A}_{\Omega}^{1,\mathrm{an}}$. Then we have*

$$\Sigma_{f,k}(\mathcal{M}_n(\Omega)) = \pi_{\Omega/k}(\{a_1, \cdots, a_N\}).$$

Corollary 2.1.25. *Suppose that k is not trivially valued. Let $\Omega \in E(k)$ and $f \in \mathcal{M}_n(\Omega)$. Let $\{a_1, \cdots, a_N\}$ be the set of rigid points of $\mathbb{A}_{\Omega}^{1,\mathrm{an}}$ that correspond to the eigenvalues of f in some finite extension of Ω. Then we have*

$$\Sigma_{f,k}(\mathcal{L}_k(\Omega^n)) = \pi_{\Omega/k}(\{a_1, \cdots, a_N\}).$$

Proof. Since $\mathcal{M}_n(\Omega)$ is a saturated subalgebra of $\mathcal{L}_k(\Omega^n)$, according to Proposition 2.1.15, we have $\Sigma_{f,k}(\mathcal{M}_n(\Omega)) = \Sigma_{f,k}(\mathcal{L}_k(\Omega^n))$. By Corollary 2.1.24, we obtain the result. □

2.2 The Cauchy integral

This section is devoted to the presentation of the map that plays the role of the Cauchy integral along a contour around a compact set introduced by Berkovich. In the first part, we will give the necessary notations and definitions. In the second part, we will explain the construction of the map and refer the reader to [Ber90, Chapter 8] for the proofs.

Convention 2.2.1. For simplicity we will assume that k is algebraically closed.

2.2.1 Notations and definitions

Definition 2.2.2. Let X be an analytic domain of $\mathbb{P}_k^{1,an}$. Let E be a k-Banach algebra. Let Σ be a compact subset of X. The k-algebra of analytic functions defined in a neighbourhood of Σ with value in E is the following:

$$\mathcal{O}_X(E)(\Sigma) := \varinjlim_{\Sigma \subset U} \mathcal{O}_X(E)(U)$$

where U is an open subset of X. We endow $\mathcal{O}_X(E)$ with the semi-norm

$$\|f\|_\Sigma := \max_{x\in\Sigma} |f(x)|.$$

We set $\mathcal{O}_X(\Sigma) := \mathcal{O}_X(k)(\Sigma)$.

Notation 2.2.3. Let V be an affinoid domain of $\mathbb{P}_k^{1,an}$. If $\infty \in V$, we set

$$(\mathcal{A}_V)_0 := \{f \in \mathcal{A}_V |\ f(\infty) = 0\}.$$

Otherwise, we set $(\mathcal{A}_V)_0 := \mathcal{A}_V$.

Notation 2.2.4. Let Σ be a compact (resp. open) subset of $\mathbb{P}_k^{1,an}$. If $\infty \in \Sigma$, we set

$$(\mathcal{O}_{\mathbb{P}_k^{1,an}}(E)(\Sigma))_0 := \{f \in \mathcal{O}_{\mathbb{P}_k^{1,an}}(E)(\Sigma) |\ f(\infty) = 0\}.$$

Otherwise, we set $(\mathcal{O}_{\mathbb{P}_k^{1,an}}(E)(\Sigma))_0 := \mathcal{O}_{\mathbb{P}_k^{1,an}}(E)(\Sigma)$.

Theorem 2.2.5 (Holomorphic functional calculus,[Ber90, Corollary 7.3.4])**.** *Let E be a k-Banach algebra and let $f \in E$. The morphism of k-algebras $K[T] \to E$ which assigns f to T extends to a bounded morphism*

$$\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{an}}}(\Sigma_{f,k}(E)) \to E.$$

The image of an element $g \in \mathcal{O}_{\mathbb{A}_k^{1,\mathrm{an}}}(\Sigma_{f,k}(E))$ will be denoted by $g(f)$.

Theorem 2.2.6 (Shilov Idempotent Theorem, [Ber90, Theorem 7.4.1])**.** *Let E be a commutative k-Banach algebra, and let $\mathcal{U}$ be an open and closed subset of $\mathcal{M}(E)$. Then there exists a unique idempotent $e \in E$ which is equal to 1 on $\mathcal{U}$ and 0 outside $\mathcal{U}$.*

A closed disk D of $\mathbb{P}_k^{1,an}$ is a set of the form $D^+(a,r)$ or $\mathbb{P}_k^{1,an} \setminus D^-(a,r)$ with $a \in k$. Its radius $r(D)$ is equal to r. Any affinoid domain of $\mathbb{P}_k^{1,an}$ is a finite disjoint union of connected affinoid domains. Note that any connected affinoid domain $V \subset \mathbb{P}_k^{1,an}$ is a finite intersection of closed disks of $\mathbb{P}_k^{1,an}$. Moreover, we can represent V uniquely as $V = \bigcap_{i=0}^{\mu} D_i$, where D_i are closed disks in $\mathbb{P}_k^{1,an}$ with $D_i \not\subset D_j$ for any $i \neq j$.

Notation 2.2.7. Let V be a connected affinoid domain of $\mathbb{P}_k^{1,an}$. Let $V = \bigcap_{i=0}^{\mu} D_i$ be the unique representation introduced above. We set

$$\mathscr{E}(V) := \{D_0, \cdots, D_\mu\}.$$

Let V be an affinoid domain of $\mathbb{P}_k^{1,an}$ and let $V_1, \cdots, V_\nu$ be its connected components. We set

$$\mathscr{E}(V) := \bigcup_{i=1}^{\nu} \mathscr{E}(V_i).$$

Remark 2.2.8. In the case where V is not connected, we can find D and D' in $\mathscr{E}(V)$ such that $D \subset D'$. For example $(\mathbb{P}_k^{1,an} \setminus D^-(0,1)) \cup D^+(0,\frac{1}{2})$.

Notation 2.2.9. We denote by A the set of affinoid domain of $\mathbb{P}_k^{1,an}$ and by A_c the set of connected affinoid domain of $\mathbb{P}_k^{1,an}$. The set of finite unions of closed disks which are not equal to $\mathbb{P}_k^{1,an}$ will be denoted by A_d.

Example 2.2.10. The affinoid domain $(\mathbb{P}_k^{1,an} \setminus D^-(0,1)) \cup D^+(0,\frac{1}{2})$ is an element of A_d.

Let $V := \coprod_{i=1}^{m} V_i \subset \mathbb{P}_k^{1,an}$ be an affinoid domain. Using the Mittag-Leffler decomposition for connected affinoid domains [FV04, Proposition 2.2.6], we obtain an isometric isomorphism of Banach spaces:

$$\begin{array}{ccc} \bigoplus\limits_{D\in\mathscr{E}(V)} (\mathcal{A}_D)_0 & \longrightarrow & (\mathcal{A}_V)_0 \\ (f_D)_{D\in\mathscr{E}(V)} & \mapsto & f := \sum\limits_{i=1}^{m} (\sum\limits_{D\in\mathscr{E}(V_i)} f_D|_{V_i}).1_{V_i} \end{array} \tag{2.1}$$

where 1_{V_i} is the characteristic function of V_i.

Let D and D' be two disks. We say that D and D' are of the same type if $\{\infty\} \cap D = \{\infty\} \cap D'$.

Let V and V' be two connected affinoid domains such that $V \subset V'$. We assume that $\infty \in V$ or $\infty \notin V'$. Then each disk $D' \in \mathscr{E}(V')$ contains exactly one disk $D \in \mathscr{E}(V)$. We thus get a map:

$$\begin{array}{ccc} \mathscr{E}(V') & \longrightarrow & \mathscr{E}(V) \\ D' & \mapsto & D \end{array}.$$

Notation 2.2.11. Let V and V' be two connected affinoid domains. We write $V \prec V'$, if the map $\mathscr{E}(V') \to \mathscr{E}(V)$: $D' \mapsto D$ is a bijection and $D \neq D'$ for all $D' \in \mathscr{E}(V')$. More generally, for V, $V' \in \mathsf{A}$ where $V = \bigcup_{i=1}^{\nu} V_i$ (resp. $V' = \bigcup_{i=1}^{\nu'} V_i'$) is the decomposition into connected affinoids, we write $V \prec V'$, if for each $i \in \{1, \cdots, \nu\}$ there exists $j \in \{1, \cdots, \nu\}$ such that $V_i \prec V_j'$.

Definition 2.2.12. Let D and D' be two disks. We say that D and D' are *complementary* if they are of different types, $D \cup D' = \mathbb{P}^{1,an}_k$, and $r(D) \neq r(D')$. Let $V, V' \in \mathsf{A}$. We say that V and V' are *complementary* if $V \cup V' = \mathbb{P}^{1,an}_k$ and there exists a bijection $\mathscr{E}(V) \to \mathscr{E}(V')$ sending each disk $D \in \mathscr{E}(V)$ to a complementary disk $D' \in \mathscr{E}(V')$.

Remark 2.2.13. If such bijection exists then it is unique.

Example 2.2.14. Let $V = C^+(0, \frac{1}{2}, \frac{3}{2}) \cup D^+(0, \frac{1}{4})$. The affinoid $V' = (\mathbb{P}^{1,an}_k \setminus D^-(0,1)) \cup C^+(0, \frac{1}{8}, \frac{3}{4})$ is a complementary set of V.

Remark 2.2.15. Note that, there exist affinoid domains that do not admit complementary sets. Indeed, for example the closed annulus $C^+(a,r,r)$ does not admit complementary sets. Indeed, we have $\mathscr{E}(C^+(a,r,r)) = \{D^+(a,r),\ \mathbb{P}^{1,an}_k \setminus D^-(a,r)\}$. If V' is a complementary affinoid set then we must have $\mathscr{E}(V') = \{D^+(a,r_1),\ \mathbb{P}^{1,an}_k \setminus D^-(a,r_2)\}$ with $r_1 > r$ and $r_2 < r$. Then the unique possibility is $V' = C^+(a, r_2, r_1)$ which does not verify $V' \cup C^+(a,r,r) = \mathbb{P}^{1,an}_k$.

Notation 2.2.16. We denote by A' the set of affinoid domains which admits complementary sets. We set $\mathsf{A}'_c := \mathsf{A}_c \cap \mathsf{A}'$.

Lemma 2.2.17. *Let $V \in \mathsf{A}$. Then $V \in \mathsf{A}'$ if and only if there exists $V' \in \mathsf{A}$ with $V' \prec V$.*

Lemma 2.2.18. *Let $V := \coprod_{i=1}^n V_i \in \mathsf{A}'$, are the V_i are connected affinoid domains. Let U be a complementary set of V. Then we have:*

$$U = \bigcap_{i=1}^n \bigcup_{D \in \mathscr{E}(V_i)} D' \tag{2.2}$$

where $D' \in \mathscr{E}(U)$ is the complementary disk of D. In particular, If $V_1, V_2 \in \mathsf{A}'$, then $V_1 \cup V_2 \in \mathsf{A}'$.

Proof. We set $X := \bigcap_{i=1}^n \bigcup_{D\in\mathscr{E}(V_i)} D' = \bigcap_{i=1}^n \bigcup_{j=1}^{n_i} D'_{ij}$. Since the bijection $D \mapsto D'$ is unique, it is enough to show that $\mathscr{E}(X) = \mathscr{E}(U)$ and $X \cup V = \mathbb{P}^{1,an}_k$. We set $I := \prod_{i=1}^n \{1, \cdots, n_i\}$, the we have $X := \bigcup_{\ell \in I} \bigcap_{i=1}^n D'_{i\ell_i}$. Since for each i the D'_{ij}'s does not have the same radius, then for each i and j there exits $\ell \in I$ such that $D'_{ij} \in \mathscr{E}(\bigcap_{i=1}^n D_{i\ell_i})$. Hence, $\mathscr{E}(X) = \mathscr{E}(U)$. We now prove that $V \cup X = \mathbb{P}^{1,an}_k$. By construction we have $\mathbb{P}^{1,an}_k \setminus V \subset X$, hence we obtain the result.

□

Lemma 2.2.19. *Let (U, V) be a pair of complementary subsets. Then*

$$U \cap V = \bigcup_{D \in \mathscr{E}(V)} (D \cap D'),$$

where D' is the disk in $\mathscr{E}(U)$ complementary to D.

Proof. It follows from the decomposition (2.2). □

Lemma 2.2.20. *Let $\Sigma \subset \mathbb{P}_k^{1,an}$ be a closed subset with $\Sigma \neq \varnothing, \mathbb{P}_k^{1,an}$. Then the intersection of the affinoid neighbourhoods of Σ that belong to A' coincides with Σ.*

Proof. Since A' is stable under union and Σ is compact, the $V \in \mathsf{A}'$ containing Σ form a basis of neighbourhood of Σ. Hence, the result holds. □

Let $V \in \mathsf{A}$. We can represent V as intersection $\bigcap_{i=1}^n V_i$ of elements in A_d with $V_i \cup V_j = \mathbb{P}_k^{1,an}$ for $i \neq j$ as follows. Let $V' \in \mathsf{A}_d$ such that $V \prec V'$ and let U be a complementary set of V'. Then $\mathscr{E}(V) \xrightarrow{\sim} \mathscr{E}(U)$, $D \mapsto D'$. Let $U_1, \cdots, U_n$ be the connected components of U. We set $V_i = \bigcup_{D' \in \mathscr{E}(U_i)} D$. Then we have

$$V = \bigcap_{i=1}^{n} V_i \qquad \text{and} \qquad \forall i \neq j \qquad V_j \cup V_i = \mathbb{P}_k^{1,an}. \tag{2.3}$$

This representation does not depend on the choice of U.

Example 2.2.21. Let $V = C^+(0, \frac{1}{2}, \frac{3}{2}) \cup D^+(0, \frac{1}{4})$. Then $V = V_1 \cap V_2$ with

$$V_1 = D^+(0, \frac{3}{2}) \qquad \text{and} \qquad V_2 = (\mathbb{P}_k^{1,an} \setminus D^-(0, \frac{1}{2})) \cup D^+(0, \frac{1}{4}).$$

Using Mayer-Vietoris sequence we obtain an isomorphism $(\mathcal{A}_V)_0 \xrightarrow{\sim} \bigoplus_{i=1}^n (\mathcal{A}_{V_i})_0$ of Banach space. By combining this isomorphism with (2.1), we obtain the isomorphism:

$$\begin{array}{ccc} \bigoplus\limits_{D \in \mathscr{E}(V)} (\mathcal{A}_D)_0 & \longrightarrow & (\mathcal{A}_V)_0 \\ (f^D)_{D \in \mathscr{E}(V)} & \mapsto & f := \sum\limits_{i=1}^{n} \Big(\sum\limits_{D \in \mathscr{E}(V_i)} (f^D.1_D) \Big)|_V \end{array} \tag{2.4}$$

where 1_D is the characteristic function of D.

Remark 2.2.22. Note that if V is belong A_d or A_c, then the isomorphisms (2.1) and (2.4) coincide. Indeed, in the case where $V \in \mathsf{A}_d$, any complementary set U is a connected affinoid domain, hence $V = \bigcup_{D' \in \mathscr{E}(U)} D$. In the case where $V \in \mathsf{A}_c$, then an complementary set U is an element of A_d, therefore the disks $D \in \mathscr{E}(V)$ coincide with the V_i's of (2.3).

Remark 2.2.23. The isomorphisms (2.1) and (2.4) are connected as follows: $f_D = \sum f^{D'}$ where the sum is taken over all $D' \in \mathscr{E}(V)$ with $D \subset D'$.

Example 2.2.24. Let $V = C^+(0, \frac{1}{2}, \frac{3}{2}) \sqcup D^+(0, \frac{1}{4})$. We set $X_1 := C^+(0, \frac{1}{2}, \frac{3}{2})$, $X_2 := D^+(0, \frac{1}{4})$, $D_1 := D^+(0, \frac{3}{2})$, $D_2 := \mathbb{P}_k^{1,an} \setminus D^+(0, \frac{1}{2})$ and $D_3 := D^+(0, \frac{1}{4})$. Then the V_i's of (2.3) are: $V_1 = D_1$ and $V_2 = D_2 \cup D_3$. The image of $(f^{D_1}, f^{D_2}, f^{D_3})$ by the isomorphism (2.4) is $f = f^{D_1}|_V + (f^{D_2}.1_{D_2} + f^{D_3}.1_{D_3})|_V$. We have $f^{D_1}|_V = f^{D_1}.1_{X_1} + f^{D_1}.1_{X_2}$, $(f^{D_2}.1_{D_2})|_V = f^{D_2}.1_{X_1}$ and $(f^{D_3}.1_{D_3})|_V = f^{D_3}.1_{X_2}$. Then,

$$f = (f^{D_1} + f^{D_2}).1_{X_1} + (f^{D_1} + f^{D_3}).1_{X_2}.$$

Consequently, we have $f_{D_1} = f^{D_1}$, $f_{D_2} = f^{D_2}$ and $f_{D_3} = f^{D_1} + f^{D_3}$.

Example 2.2.25. Let $V = C^+(0, \frac{1}{4}, \frac{1}{2}) \sqcup C^+(1, \frac{4}{,}\frac{1}{2})$. We set $X_1 := C^+(0, \frac{1}{4}, \frac{1}{2})$, $X_2 := C^+(1, \frac{4}{,}\frac{1}{2})$, $D_1 := D^+(0, \frac{1}{2})$, $D_2 := D^+(1, \frac{1}{2})$, $D_3 := \mathbb{P}_k^{1,an} \setminus D^-(0, \frac{1}{4})$ and $D_4 := \mathbb{P}_k^{1,an} \setminus D^-(1, \frac{1}{4})$. Then the $V_i's$ of (2.3) are: $V_1 = D_1 \cup D_2$, $V_2 = D_3$ and $V_3 = D_4$. The image of $(f^{D_1}, f^{D_2}, f^{D_3}, f^{D_4})$ by the isomorphism (2.4) is $f = (f^{D_1}.1_{D_1} + f^{D_2}.1_{D_2})|_V + f^{D_3}|_V + f^{D_4}|_V$. We have $(f^{D_1}.1_{D_1})|_V = f^{D_1}.1_{X_1}$, $(f^{D_2}.1_{D_2})|_V = f^{D_2}.1_{X_2}$, $(f^{D_3})|_V = f^{D_3}.1_{X_1} + f^{D_3}.1_{X_2}$ and $(f^{D_4})|_V = f^{D_4}.1_{X_1} + f^{D_4}.1_{X_2}$. Here, $f^{D_3}.1_{X_2} \in \mathcal{A}_{D_2}$ and $f^{D_4}.1_{X_1} \in \mathcal{A}_{D_1}$. Then,

$$f = (f^{D_1} + f^{D_3} + f^{D_4}).1_{X_1} + (f^{D_2} + f^{D_3} + f^{D_4}).1_{X_2}.$$

Consequently, we have $f_{D_1} = f^{D_1} + f^{D_4}$, $f_{D_2} = f^{D_2} + f^{D_3}$, $f_{D_3} = f^{D_3}$ and $f_{D_4} = f^{D_4}$.

Definition 2.2.26. Let D be a closed disk. The *residue operator* $\mathrm{Res}_D : \mathcal{A}_D \to k$ is defined as follows. If $D \subset \mathbb{A}_k^{1,\mathrm{an}}$, then $\mathrm{Res}_D \equiv 0$. If $D = \mathbb{P}_k^{1,an} \setminus D^-(a, r)$ with $a \in k$ then

$$\begin{array}{rcl} \mathrm{Res}_D : \mathcal{A}_D & \longrightarrow & k \\ \sum_{i=0}^{\infty} f_i (T-a)^{-i} & \mapsto & f_1 \end{array}.$$

It does not depends on the choice of a.

Remark 2.2.27. It easy to see that, if $D = \mathbb{P}_k^{1,an} \setminus D^-(a, r)$ then Res_D is bounded and $\|\mathrm{Res}_D\| = r$. Otherwise, $\|\mathrm{Res}_D\| = 0$.

Let D and D' be complementary closed disks. Since we have the isomorphism (2.4):

$$\begin{array}{rcl} (\mathcal{A}_D)_0 \oplus (\mathcal{A}_{D'})_0 & \longrightarrow & (\mathcal{A}_{D\cap D'})_0 \\ (f, f') & \mapsto & f|_{D\cap D'} + f'|_{D\cap D'} \end{array},$$

we can define:

$$\begin{array}{rcl} \mathrm{Res}_{D\cap D'} : (\mathcal{A}_{D\cap D'})_0 & \longrightarrow & k \\ f & \mapsto & \mathrm{Res}_D(f_D) + \mathrm{Res}_{D'}(f_{D'}), \end{array} \tag{2.5}$$

Moreover, if $D \subset \mathbb{A}_k^{1,\mathrm{an}}$ (resp. $D' \subset \mathbb{A}_k^{1,\mathrm{an}}$) then $\mathrm{Res}_{D\cap D'}(f) = \mathrm{Res}_{D'}(f_{D'})$ (resp. $\mathrm{Res}_{D\cap D'}(f) = \mathrm{Res}_D(f_D)$).

Remark 2.2.28. Let B be a k-Banach space. The operators Res_D and $\mathrm{Res}_{D\cap D'}$ extend naturally to bounded operators $\mathrm{Res}_D : B\hat{\otimes}_k \mathcal{A}_D \to B$ and $\mathrm{Res}_{D\cap D'} : B\hat{\otimes}_k (\mathcal{A}_{D\cap D'})_0 \to B$.

Definition 2.2.29. Let Σ be a closed subset of $\mathbb{P}_k^{1,an}$ different from $\varnothing$ and $\mathbb{P}_k^{1,an}$. A *contour around* Σ is a pair (U, V) of complementary subsets such that U is a neighbourhood of Σ and $V \subset \mathbb{P}_k^{1,an} \setminus \Sigma$.

Example 2.2.30. Let $\Sigma = \{a, b\}$ be a subset of k. Let $U := D^+(a, r_a) \sqcup D^+(b, r_b)$ and $V := \mathbb{P}_k^{1,an} \setminus (D^-(a, r'_a) \sqcup D^-(b, r'_b))$ with $r'_a < r_a$ and $r'_b < r_b$. Then (U, V) is a contour of Σ and $U \cap V = C^+(a, r'_a, r_a) \cup C^+(b, r'_b, r_b)$.

2.2.2 The Cauchy integral

We now expose the Cauchy integral. Let E be a k-Banach algebra and let Σ be a compact subset of $\mathbb{P}_k^{1,an}$. Let $\gamma = (U, V)$ be a contour of Σ. Then the *Cauchy integral* around γ is the bounded k-linear map $\mathcal{E} : E\hat{\otimes}_k(\mathcal{A}_V)_0 \to \mathcal{L}_k((\mathcal{A}_U)_0, E))$ defined as follows:

$$\begin{array}{rcl} \mathcal{E} : E\hat{\otimes}_k(\mathcal{A}_V)_0 & \longrightarrow & \mathcal{L}_k((\mathcal{A}_U)_0, E) \\ \varphi & \mapsto & \mathcal{E}\varphi : f \to \sum\limits_{D\in\mathscr{E}(V)} (Res_{D\cap D'})(f^{D'}|_{D\cap D'} \cdot \varphi_D|_{D\cap D'}) \end{array}, \tag{2.6}$$

where D' is the disk in $\mathscr{E}(U)$ complementary to D.

Proposition 2.2.31. *Let E be a k-Banach algebra and let $g \in E$. Let (U, V) be a contour of $\Sigma_{g,k}(E)$. Then we have*

$$\mathcal{E}R_g(f) = f(g),$$

where $f(g)$ is the image of g by the map of Theorem 2.2.5.

Proof. See [Ber90, Theorem 8.1.1, Remark 8.1.2] □

2.3 Variation of the spectrum and continuity results

In this section we will discuss the behaviour of the spectrum in in families, and especially its continuity properties. For that we need to define a topology on the set $K(\mathbb{A}_k^{1,\text{an}})$ of nonempty compact subsets of $\mathbb{A}_k^{1,\text{an}}$. Note that in the case where $\mathbb{A}_k^{1,\text{an}}$ is metrizable, we can endow $K(\mathbb{A}_k^{1,\text{an}})$ with a metric called Hausdorff metric. However, in general $\mathbb{A}_k^{1,\text{an}}$ is not metrizable. Indeed, it is metrizable if and only if $\tilde{k}$ is countable. In the first part of the section, we will introduce the topology on $K(\mathcal{T})$ (the set of nonempty compact subsets of a Hausdorff topological space $\mathcal{T}$), that coincides in the metrizable case with the topology induced by the Hausdorff metric. In the second part, we will prove continuity results of the spectrum, analogous to the complex case. For the continuity results for the spectrum in the complex case, we refer the reader to [Aup79].

2.3.1 The topology on $K(\mathcal{T})$

Let $\mathcal{T}$ be a Hausdorff topological space. We will denote by $K(\mathcal{T})$ the set of nonempty compact subsets of $\mathcal{T}$. Recall that in the case where $\mathcal{T}$ is metrizable, for an associated metric $\mathscr{M}$, the respective Hausdorff metric $\mathscr{M}_H$ defined on $K(\mathcal{T})$ is given as follows. Let Σ, $\Sigma' \in K(\mathcal{T})$

$$\mathscr{M}_H(\Sigma,\Sigma') = \max\{\sup_{\beta\in\Sigma'}\inf_{\alpha\in\Sigma}\mathscr{M}(\alpha,\beta), \sup_{\alpha\in\Sigma}\inf_{\beta\in\Sigma'}\mathscr{M}(\alpha,\beta)\}. \tag{2.7}$$

We introduce below a topology on $K(\mathcal{T})$ for an arbitrary Hausdorff topological space $\mathcal{T}$, that coincides with the topology induced by the Hausdorff metric in the metrizable case.

The topology on $K(\mathcal{T})$**:** Let U be an open of $\mathcal{T}$ and $\{U_i\}_{i\in I}$ be a finite family of open subsets of U. We set:

$$(U,\{U_i\}_{i\in I}) = \{\Sigma \in K(\mathcal{T})|\ \Sigma\subset U,\ \Sigma\cap U_i \neq \varnothing\ \forall i\}. \tag{2.8}$$

The family of sets of this form is stable under finite intersection. Indeed, we have:

$$(U,\{U_i\}_{i\in I})\cap(V,\{V_j\}_{j\in J}) = (U\cap V, \{U_i\cap V\}_{i\in I}\cup\{V_j\cap U\}_{j\in J}).$$

We endow $K(\mathcal{T})$ with the topology generated by this family of sets.

Lemma 2.3.1. *The topological space $K(\mathcal{T})$ is Hausdorff.*

Proof. Let Σ and Σ' be two compact subsets of $\mathcal{T}$ such that $\Sigma \neq \Sigma'$. We may assume that $\Sigma' \not\subset \Sigma$. Let $x\in\Sigma\setminus\Sigma'$. Since $\mathcal{T}$ is a Hausdorff space, there exists an open neighbourhood U_x of x and an open neighbourhood U' of Σ', such that $U_x\cap U' = \varnothing$. Let U be an open neighbourhood of Σ such that $U_x\subset U$. Then the open set $(U,\{U_x\})$ (resp. $(U',\{U'\})$) is a neighbourhood of Σ (resp. Σ') in $K(\mathcal{T})$ such that $(U,\{U_x\})\cap(U',\{U'\}) = \varnothing$.

□

Lemma 2.3.2. *Assume that $\mathcal{T}$ is metrizable. The topology on $K(\mathcal{T})$ coincides with the topology induced by the Hausdorff metric.*

Proof. Let $\mathscr{M}$ be a metric associated to $\mathcal{T}$. For $x\in\mathcal{T}$ and $r\in\mathbb{R}_+^*$ we set

$$B_{\mathscr{M}}(x,r) := \{y\in\mathcal{T}|\ \mathscr{M}(x,y) < r\}.$$

For $\Sigma\in K(\mathcal{T})$ and $r\in\mathbb{R}_+^*$, we set $B_{\mathscr{M}_H}(\Sigma,r) := \{\Sigma'\in K(\mathcal{T})|\ \mathscr{M}_H(\Sigma,\Sigma') < r\}$. Let $\Sigma\in K(\mathcal{T})$. To prove the statement, we need to show that for all $r\in\mathbb{R}_+^*$ there exists an open neighbourhood $(U,\{U_i\}_{i\in I})$ of Σ such that $(U,\{U_i\}_{i\in I})\subset B_{\mathscr{M}_H}(\Sigma,r)$, and for each open neighbourhood $(U,U_i)_{i\in I}$ of Σ, there exists $r\in\mathbb{R}_+^*$ such that $B_{\mathscr{M}_H}(\Sigma,r)\subset(U,\{U_i\}_{i\in I})$.

Let $r\in\mathbb{R}_+^*$. Let $\{c_1,\cdots,c_m\}\subset\Sigma$ such that $\Sigma\subset\bigcup_{i=1}^m B_{\mathscr{M}}(c_i,\frac{r}{3})$, we set $U = \bigcup_{i=1}^m B_{\mathscr{M}}(c_i,\frac{r}{3})$. We claim that $(U,\{B_{\mathscr{M}}(c_i,\frac{r}{3})\}_{i=1}^m)\subset B_{\mathscr{M}_H}(\Sigma,r)$. Indeed, let $\Sigma'\in(U,\{B_{\mathscr{M}}(c_i,\frac{r}{3})\}_{i=1}^m)$. As $\Sigma'\subset U$, for all $y\in\Sigma'$ we have $\min_{1\leq i\leq m}(\mathscr{M}(y,c_i)) < \frac{r}{3}$. Therefore,

$$\sup_{y\in\Sigma'}\inf_{x\in\Sigma}\mathscr{M}(y,x)\leq\frac{r}{3} < r.$$

Since for each c_i there exists $y \in \Sigma'$ such that $\mathscr{M}(c_i, y) < \frac{r}{3}$, for each $x \in \Sigma$ there exists $y \in \Sigma'$ such that $\mathscr{M}(x, y) < \frac{2r}{3}$. Indeed, there exists c_i such that $x \in B(c_i, \frac{r}{3})$, therefore we have

$$\mathscr{M}(x, y) \leq \mathscr{M}(c_i, y) + \mathscr{M}(c_i, x) < \frac{2r}{3}.$$

This implies

$$\sup_{x \in \Sigma} \inf_{y \in \Sigma'} \mathscr{M}(x, y) \leq \frac{2r}{3} < r.$$

Consequently, $\mathscr{M}_H(\Sigma, \Sigma') < r$.

Now let $(U, \{U_i\}_{i \in I})$ be an open neighbourhood of Σ. Let $\alpha = \inf_{y \in \mathcal{T} \setminus U} \inf_{x \in \Sigma} \mathscr{M}(x, y)$, since $\Sigma \subsetneq U$ we have $\alpha \neq 0$. For each $1 \leq i \leq m$, let $c_i \in \Sigma \cap U_i$. There exists $0 < \beta < \alpha$ such that for all $0 < r < \beta$ we have $B_{\mathscr{M}}(c_i, r) \subset U_i$ for each $1 \leq i \leq m$.

We claim that $B_{\mathscr{M}_H}(\Sigma, r) \subset (U, \{U_i\}_{i=1}^m)$. Indeed, let $\Sigma' \in B_{\mathscr{M}_H}(\Sigma, r)$ this means that:

$$\sup_{y \in \Sigma'} \inf_{x \in \Sigma} \mathscr{M}(x, y) < r; \qquad \sup_{x \in \Sigma} \inf_{y \in \Sigma'} \mathscr{M}(x, y) < r.$$

The first inequality implies $\Sigma' \subset U$. The second implies that for each c_i there exists $y \in \Sigma'$ such that $\mathscr{M}(c_i, y) < r$. Hence, $\Sigma' \cap U_i = \varnothing$ for each i. □

Lemma 2.3.3. *The following function is continuous*

$$\begin{array}{rcl} \Upsilon : K(\mathcal{T}) \times K(\mathcal{T}) & \longrightarrow & K(\mathcal{T}) \\ (\Sigma, \Sigma') & \mapsto & \Sigma \cup \Sigma' \end{array}$$

Proof. Let Σ and Σ' be two non-empty compact subset of $\mathcal{T}$. Let $(U, \{U_i\}_{i \in I})$ be an open neighbourhood of $\Sigma \cup \Sigma'$. We set

$$J := \{i \in I |\ \Sigma \cap U_i \neq \varnothing\} \text{ and } J' := \{i \in I |\ \Sigma' \cap U_i \neq \varnothing\}.$$

Then $(U, \{U_i\}_{i \in J})$ (resp. $(U, \{U_i\}_{i \in J'})$) is an open neighbourhood of Σ (resp. Σ') and we have

$$(U, \{U_i\}_{i \in J}) \times (U, \{U_i\}_{i \in J'}) \subset \Upsilon^{-1}((U, \{U_i\}_{i \in I})).$$

Hence we obtain the result. □

2.3.2 Continuity results

Continuity of spectrum function

Notation 2.3.4. For a non-zero k-Banach algebra E, we set

$$\begin{array}{rccl} \Sigma_{.,k}(E) : & E & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ & f & \mapsto & \Sigma_{f,k}(E), \end{array} \tag{2.9}$$

where $\Sigma_{f,k}(E)$ is the spectrum of f as an element of E.

Since the spectral semi-norm $\|.\|_{\mathrm{Sp}} : E \to \mathbb{R}_+$ may not be continuous, the map $\Sigma_{,k}(E)$ may fail to be continuous. For that we provide the following example:

Example 2.3.5. This example is inspired by the example given in [Aup79, p. 34]. Assume that $\mathrm{char}(\tilde{k}) = p > 0$. Let $E := \mathcal{L}_k(k\{T\})$. Let $\mathrm{Sh} \in E$ be the operator defined by

$$\forall n \in \mathbb{N}, \quad \mathrm{Sh}(T^n) = \alpha_n T^{n+1},$$

where

$$\alpha_n = \begin{cases} p^\ell & \text{if } n = 2^\ell(2m+1) - 1 \text{ with } \ell \in \mathbb{N} \setminus \{0\},\ m \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}.$$

For $l \in \mathbb{N} \setminus \{0\}$, let $\mathrm{S}_\ell \in E$ be the operator defined by

$$\forall n \in \mathbb{N}, \quad \mathrm{S}_\ell(T^n) = \begin{cases} 0 & \text{if } n = 2^\ell(2m+1) - 1 \text{ with } m \in \mathbb{N} \\ \mathrm{Sh}(T^n) & \text{otherwise} \end{cases}.$$

Then we have $(\mathrm{Sh} - \mathrm{S}_\ell)(\sum_{i\in\mathbb{N}} a_i T^i) = \sum_{m\in\mathbb{N}} p^\ell a_{2^\ell(2m+1)-1} T^{2^\ell(2m+1)}$. Therefore, $\|\mathrm{Sh} - \mathrm{S}_\ell\| \leq p^\ell$ and $\mathrm{S}_\ell \overset{\ell\to\infty}{\to} \mathrm{Sh}$. On the one hand we have $\mathrm{S}_\ell^{2^\ell} = 0$. Hence, $\|\mathrm{S}_\ell\|_{\mathrm{Sp}} = 0$ and $\Sigma_{\mathrm{S}_\ell,k}(E) = \{0\}$. On the other hand, since $\mathrm{Sh}^{n'}(T^n) = \alpha_n\alpha_{n+1}\cdots\alpha_{n+n'-1}T^{n+n'}$, we have

$$\|\mathrm{Sh}^{n'}\| = \sup_{n\in\mathbb{N}} |\alpha_n\alpha_{n+1}\cdots\alpha_{n+n'-1}|.$$

By construction we have:

$$\alpha_1\alpha_2\cdots\alpha_{2^\ell-1} = \prod_{j=1}^{\ell-1} p^{j2^{\ell-j-1}} = (p^{\sum_{j=1}^{\ell-1} j2^{-j-1}})^{2^\ell},$$

thus,

$$|\alpha_0\alpha_2\cdots\alpha_{2^\ell-2}|^{\frac{1}{2^\ell-1}} > (p^{-\sum_{j=1}^{\ell-1} j2^{-j-1}})^2.$$

We set $\sigma = \sum_{j=1}^{\infty} j2^{-j-1}$. Then we obtain

$$0 < p^{-2\sigma} \leq \lim_{n'\to\infty} \|\mathrm{Sh}^{n'}\|^{\frac{1}{n'}} = \|\mathrm{Sh}\|_{\mathrm{Sp}}.$$

Consequently, the sequence $(\Sigma_{\mathrm{S}_\ell,k}(E))_{\ell\in\mathbb{N}}$ does not converge to $\Sigma_{\mathrm{Sh},k}(E)$ and $\Sigma_{.,k}(E)$ is not continuous at Sh.

In the following, we will prove that under some assumption the map $\Sigma_{.k}(E) : E \to K(\mathbb{A}_k^{1,\mathrm{an}})$ is continuous.

Lemma 2.3.6. *Let E be a k-Banach algebra and let $f \in E$. For any open neighbourhood $U \subset \mathbb{A}_k^{1,\mathrm{an}}$ of $\Sigma_{f,k}(E)$, there exists a positive real number δ such that*

$$\|f - g\| < \delta \Rightarrow \Sigma_{g,k}(E) \subset U.$$

Proof. We set $\Sigma_f := \Sigma_{f,k}(E)$ and $\Sigma_g := \Sigma_{g,k}(E)$. Let U be an open neighbourhood of Σ_f. Let $\varepsilon > 0$, we have:

$$\mathbb{A}_k^{1,\mathrm{an}} \setminus U = [(\mathbb{A}_k^{1,\mathrm{an}} \setminus U) \cap D^+(0, \|f\|_{\mathrm{Sp}} + \varepsilon)] \bigcup [\mathbb{A}_k^{1,\mathrm{an}} \setminus (U \cup D^+(0, \|f\|_{\mathrm{Sp}} + \varepsilon))].$$

Since the spectral semi-norm is upper semi-continuous, there exists $\eta_0 > 0$ such that

$$\|f - g\| < \eta_0 \Rightarrow \|g\|_{\mathrm{Sp}} < \|f\|_{\mathrm{Sp}} + \varepsilon.$$

Therefore, we obtain

$$\Sigma_g \subset D^+(0, \|g\|_{\mathrm{Sp}}) \subset D^+(0, \|f\|_{\mathrm{Sp}} + \varepsilon)$$

(cf. Theorem 2.1.6). Hence, we have

$$\|f - g\| < \eta_0 \Rightarrow \mathbb{A}_k^{1,\mathrm{an}} \setminus (U \cup D^+(0, \|f\|_{\mathrm{Sp}} + \varepsilon)) \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_g.$$

We now prove that there exists $\eta_1 > 0$ such that

$$\|f - g\| < \eta_1 \Rightarrow (\mathbb{A}_k^{1,\mathrm{an}} \setminus U) \cap D^+(0, \|f\|_{\mathrm{Sp}} + \varepsilon) \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_g.$$

Let $x \in \mathbb{A}_k^{1,\mathrm{an}} \setminus U$, and let $V_x \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_f$ be an affinoid neighbourhood of x. Then $f \otimes 1 - 1 \otimes T$ is invertible in $E\hat{\otimes}_k \mathcal{A}_{V_x}$. We set $\eta_x = \|(f \otimes 1 - 1 \otimes T)^{-1}\|^{-1}$. Since for all $g \in E$ we have $\|f - g\| = \|f \otimes 1 - g \otimes 1\|$ (cf. Proposition 1.1.39), if $\|f - g\| < \eta_x$ then $g \otimes 1 - 1 \otimes T$ is invertible in $B\hat{\otimes}_k \mathcal{A}_{V_x}$ (cf. [Bou07, §2.4 Proposition 3]). Therefore, we obtain

$$\|f - g\| < \eta_x \Rightarrow V_x \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_g.$$

As $(\mathbb{A}_k^{1,\mathrm{an}} \setminus U) \cap D^+(0, \|f\|_{\mathrm{Sp}} + \varepsilon)$ is compact, there exists a finite subset $\{x_1, \cdots, x_m\} \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus U$ such that $(\mathbb{A}_k^{1,\mathrm{an}} \setminus U) \cap D^+(0, \|f\|_{\mathrm{Sp}} + \varepsilon) \subset \bigcup_{i=1}^m V_{x_i}$. We set $\eta_1 := \min_{1 \leq i \leq m} \eta_{x_i}$. Hence, we obtain

$$\|f - g\| < \eta_1 \Rightarrow (\mathbb{A}_k^{1,\mathrm{an}} \setminus U) \cap D^+(0, \|f\|_{\mathrm{Sp}} + \varepsilon) \subset \bigcup_{i=1}^m V_{x_i} \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_g.$$

We set $\delta := \min(\eta_0, \eta_1)$. Consequently, we obtain:

$$\|f - g\| < \delta \Rightarrow \mathbb{A}_k^{1,\mathrm{an}} \setminus U \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_g.$$

□

Theorem 2.3.7. *Let E be a* commutative *k-Banach algebra. The spectrum map*

$$\Sigma_{.,k} : E \to K(\mathbb{A}_k^{1,\mathrm{an}})$$

is continuous.

Proof. We will prove the statement by contradiction. Suppose that there exists an element $f \in E$ and a sequence $(f_n)_{n\in\mathbb{N}}$ that converges to f, such that $(\Sigma_{f_n,k}(E))_{n\in\mathbb{N}}$ does not converges to $\Sigma_{f,k}(E)$. We set $\Sigma_f := \Sigma_{f,k}(E)$ and $\Sigma_{f_n} := \Sigma_{f_n,k}(E)$. Note that, by Lemma 2.3.6 for any open neighbourhood $U \subset \mathbb{A}_k^{1,\mathrm{an}}$ of Σ_f, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $\Sigma_{f_n} \subset U$ for all $n > N$. Therefore, we can assume that there exists $x \in \Sigma_f$ and an open neighbourhood U_x of x, such that $\Sigma_{f_n} \cap U_x = \varnothing$. We know that $\pi^{-1}_{\mathscr{H}(x)/k}(\Sigma_{f_n}) = \Sigma_{f_n\otimes 1,\mathscr{H}(x)}(E\hat{\otimes}_k\mathscr{H}(x))$(cf. Proposition 2.1.19). Then we have

$$\Sigma_{f_n\otimes 1,\mathscr{H}(x)}(E\hat{\otimes}_k\mathscr{H}(x)) \cap \pi^{-1}_{\mathscr{H}(x)/k}(U_x) = \varnothing$$

Since $\pi^{-1}_{\mathscr{H}(x)/k}(U_x)$ is a neighbourhood of $T(x)$, there exists $\varepsilon > 0$ such that $D^-_{\mathscr{H}(x)}(T(x),\varepsilon) \subset \pi^{-1}_{\mathscr{H}(x)/k}(U_x)$. Hence, we have $D^-_{\mathscr{H}(x)}(T(x),\varepsilon) \subset \mathbb{A}^{1,\mathrm{an}}_{\mathscr{H}(x)} \setminus \Sigma_{f_n\otimes 1,\mathscr{H}(x)}(E\hat{\otimes}_k\mathscr{H}(x))$. Therefore, we obtain

$$\|(f_n\otimes 1 - 1\otimes T(x))^{-1}\|_{\mathrm{Sp}} < \frac{1}{\varepsilon}$$

(cf. Lemma 2.1.9). Since $E\hat{\otimes}_k\mathscr{H}(x)$ is commutative, the spectral semi-norm is submultiplicative (cf. Corollary 1.1.29). We set $u := f\otimes 1 - 1\otimes T(x)$ and $u_n := f_n\otimes 1 - 1\otimes T(x)$. Therefore, we obtain

$$\|1 - uu_n^{-1}\|_{\mathrm{Sp}} = \|u_n^{-1}(u_n - u)\|_{\mathrm{Sp}} \leq \|u_n^{-1}\|_{\mathrm{Sp}}\|u_n - u\|_{\mathrm{Sp}} \leq \frac{1}{\varepsilon}\|u_n - u\|.$$

Since f_n converges to f, u_n converges to u. This implies that uu_n^{-1} is invertible and u is right invertible. By analogous arguments, $u_n^{-1}u$ is invertible and u is left invertible. Hence, u is invertible which contradicts the hypothesis.

□

Lemma 2.3.8. *Let E be a k-Banach algebra and $f \in E$. Let $\mathcal{S}$ be a closed and open subset of $\Sigma_{f,k}(E)$. For any open subset $\mathcal{U} \subset \mathbb{A}_k^{1,\mathrm{an}}$ containing $\mathcal{S}$, there exists a positive real number δ such that*

$$\|f - g\| < \delta \Rightarrow \Sigma_{g,k}(E) \cap \mathcal{U} \neq \varnothing.$$

Proof. We set $\Sigma_f := \Sigma_{f,k}(E)$. We suppose that there exists an open neighbourhood $\mathcal{U}$ of $\mathcal{S}$ and a sequence $(f_n)_{n\in\mathbb{N}}$ that converges to f with $\Sigma_{f_n,k}(E) \cap \mathcal{U} = \varnothing$ for all $n \in \mathbb{N}$. Let (U,V) be a contour of Σ_f (which exists by Proposition 2.2.20) such that $U = U_1 \cup U_2$ where U_1 (resp. U_2) is an affinoid neighbourhood of $\mathcal{S}$ (resp. $\Sigma_f \setminus \mathcal{S}$) and $U_1 \cap U_2 = \varnothing$. We can assume that $\Sigma_{f_n,k}(E) \subset U$ (cf. Lemma 2.3.6) and $V \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_{f_n,k}(E)$ for all $n \in \mathbb{N}$, hence (U,V) is a contour of $\Sigma_{f_n,k}(E)$ for all $n \in \mathbb{N}$. We can assume that $U_1 \subset \mathcal{U}$. Let $1_{U_1} \in \mathcal{A}_U$ be the characteristic function of U_1. By Proposition 2.2.31, we have $\mathcal{E}R_f(1_{U_1}) = 1_{U_1}(f)$, which is an idempotent element of E (cf. Propositions 2.1.10, 2.1.13 and Theorem 2.2.6), since U_1 is non-empty $1_{U_1}(f)$ is not equal to 0. Since $\Sigma_{f_n} \cap U_1 = \varnothing$, by Theorem 2.2.5 we have $f_n(1_{U_1}) = 0$. Hence, by Proposition 2.2.31 we obtain $\mathcal{E}R_{f_n}(1_{U_1}) = f_n(1_{U_1}) = 0$. On the one hand we have $(R_{f_n})_{n\in\mathbb{N}}$ converges to R_f in $E\hat{\otimes}_k\mathcal{A}_V$. On the other hand, the map $\mathcal{E} : E\hat{\otimes}_k(\mathcal{A}_V)_0 \to \mathcal{L}_k((\mathcal{A}_U)_0, E)$ is continuous. Hence we obtain a contradiction since $1_{U_1}(f) \neq 0$. □

Theorem 2.3.9. *Let E be a k-Banach algebra and $f \in E$. If $\Sigma_{f,k}(E)$ is totally disconnected, then the map $\Sigma_{.,k} : E \to K(\mathbb{A}_k^{1,\mathrm{an}})$ is continuous at f.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset E$ be a sequence that converges to f. Let $(U, \{U_i\}_{i\in I}) \subset K(\mathbb{A}_k^{1,\mathrm{an}})$ be an open neighbourhood of $\Sigma_{f,k}(E)$. Let $x \in \Sigma_{f,k}(E)$, since $\Sigma_{f,k}(E)$ is totally disconnected, the closed-open subsets of $\Sigma_{f,k}(E)$ form a fondamental system of neighbourhoods of x. Hence, for each i there exists a closed-open subset $\mathcal{S} \subset \Sigma_{f,k}(E)$ such that $\mathcal{S} \subset U_i$. By Lemma 2.3.6, there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$ we have $\Sigma_{f_n,k}(E) \subset U$. By Lemma 2.3.8, we have for all $i \in I$ there exists $N_i \in \mathbb{N}$ such that for all $n > N_i$, $\Sigma_{f_n,k}(E) \cap U_i \neq \varnothing$. Therefore, for all $n > \max(N_0, \max_{i\in I}(N_i))$ we have $\Sigma_{f_n,k}(E) \in (U, \{U_i\}_{i\in I})$. Hence, we obtain the result. □

Corollary 2.3.10. *Let $\Omega \in E(k)$. The map $\Sigma_{.,k}(\mathcal{M}_n(\Omega)) : \mathcal{M}_n(\Omega) \to K(\mathbb{A}_k^{1,\mathrm{an}})$ is continuous.*

Other continuity result

Assume that k is algebraically closed. Let $c \in k$, we set

$$\begin{array}{rcl} D_c^+ : [0,\infty) & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ r & \mapsto & D^+(c,r) \end{array} \tag{2.10}$$

Proposition 2.3.11. *The function D_c^+ satisfies the following continuity properties:*

1. *it is right continuous;*
2. *it is continuous at r if and only if $r \notin |k|$.*

Proof. Let $r \geq 0$. Let $(U, \{U_i\}_{i\in I})$ be an open neighbourhood of $D_c^+(r)$. We may assume that of U is an open disk. Let $R > 0$ be the radius of U.

1. For all $r \leq r' < R$, $D_c^+(r) \subset D_c^+(r')$. Therefore, for all $r \leq r' < R$ and all $i \in I$, $D_c^+(r') \cap U_i \neq \varnothing$. Then we obtain $[r, R) \subset (D_c^+)^{-1}((U, \{U_i\}_{i\in I})$.
2. (i) Assume that $r \in |k|$. Then there exists $b \in (D^+(c,r) \setminus D^-(c,r)) \cap k$. As for all $0 \leq r' < r$, $D_c^+(r') \cap D^-(b,r)$, we have $D_c^+(r) \notin (U, \{U_i\} \cup \{D^-(b,r)\})$ for all $0 \leq r' < r$. Therefore D_c^+ is not left continuous at r.

   (ii) Assume that $r \notin k$. Then $D_c^+(r) = D^-(c,r) \cup \{x_{c,r}\}$. In order to prove the continuity at r it is enough to prove that D_c^+ is left continuous (by the above results, which ensures right continuity at r). Let $0 \leq r' < r$. We have $D_c^+(r') \subset D_c^+(r) \subset U$. In order to obtain the result, it is enough to prove that there exists $R' < r$ such that for all $R' < r' < r$, $D_c^+(r') \cap U_i \neq \varnothing$.

Let $\alpha_1, \cdots \alpha_m \in D_c^+(r)$ such that $\alpha_i \in D_c^+(r) \cap U_i$ for each i. Since $D_c^+(r)$ is a disk we can assume that α_i is either of type (1) or (3) [1] for each i.
If α_i is a point of type (1), then $\alpha_i \in D^-(c, r)$. Since the open disks form a basis of neighbourhoods for the point of type (1), there exists $D^-(\alpha_i, L_i) \subset D^-(c, r) \cap U_i$. We set $R_i := \max(|c - \alpha_i|, L_i)$. Then for all $R_i < r' < r$, $D^-(\alpha_i, R_i) \subset D^-(c, r') \subset D_c^+(r')$.
Suppose now that $\alpha_i = x_{b_i, L_i}$ is a point of type (3). If $\alpha_i \neq x_{c,r}$, we set $R_i := \max(|c - b_i|, L_i)$. For all $R_i < r' < r$, we have $\alpha_i \in D^-(c, r') \subset D_c^+(r')$. Assume now that $\alpha_i = x_{c,r}$. Open annulus $C^-(c, L_i^1, L_i^2)$ form a basis of neighbourhoods of α_i. Therefore, there exists $C^-(c, L_i^1, L_i^2) \subset U_i$ containing α_i. This implies that there exists $C^+(c, R_i, r) \subset D_c^+(r) \cap U_i$. Then for all $R_i < r' < r$ we have $C^+(c, R_i, r) \cap D_c^+(r') = C^+(c, R_i, r')$. Then we obtain $D_c^+(r') \cap U_i \neq \varnothing$. Consequently, for all $\max_i R_i < r' < r$, $D_c^+(r') \cap U_i = \varnothing$.

□

Corollary 2.3.12. *The function*

$$\begin{array}{rcl} \bigcup_{i=1}^{n} D_{c_i}^+ : [0, \infty) & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ r & \mapsto & \bigcup_{i=1}^{n} D^+(c_i, r) \end{array}$$

satisfies the following continuity properties:

1. *it is right continuous;*

2. *it is continuous at r if and only if $r \notin |k|$.*

Proof. The result follows form Proposition 2.3.11 and Lemma 2.3.3. □

1. Note that in the case where k is not trivially valued, we may assume that the α_i are of type (1).

Part II

Differential equations and their spectra

III

Differential modules and spectra

Contents

This chapter is organized as follows. In the first section, we recall the definition of differential modules and their main properties. In the second section, we explain in which way we associate a spectrum to a differential module, and show how it behaves under exact sequences. In the third one, we state and prove the analogue of Young's theorem in rank one (cf. Theorem 3.3.1, [You92]). In the last one, we will explain how the spectrum behaves under push-forward.

3.1 Preliminaries

Recall that a differential k-algebra, denoted by (A, d), is a commutative k-algebra A endowed with a non zero k-linear derivation $d : A \to A$. A differential module (M, ∇) over (A, d) is a finite free A-module M equipped with a k-linear map $\nabla : M \to M$, called connection of M, satisfying $\nabla(fm) = df.m + f.\nabla(m)$ for all $f \in A$ and $m \in M$. If we fix a basis of M, then we get an isomorphism of A-modules $M \tilde{\to} A^n$, and the operator ∇ is given in the induced basis $\{e_1, \cdots, e_n\}$ by the formula:

$$\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} + G \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \tag{3.1}$$

where $G \in \mathcal{M}_n(A)$ is the matrix that its i th column is the vector $\nabla(e_i)$. Conversely the data of such a matrix defines a differential module structure on A^n by the formula (3.1).

A morphism between differential modules is a k-linear map $M \to N$ commuting with connections.

Remark 3.1.1. In particular k endowed with the zero derivation is a differential k-algebra. The differential modules over $(k, 0)$ are vector spaces of finite dimension endowed with an endomorphism.

We set $\mathscr{D}_A = \bigoplus_{i\in\mathbb{N}} A.D^i$ to be the ring of differential polynomials equipped with the non-commutative multiplication defined by the rule: $D.f = df + f.D$ for all $f \in A$. Let $P(D) = g_0 + \cdots + g_{\nu-1}D^{\nu-1} + D^\nu$ be a monic differential polynomial. The quotient $\mathscr{D}_A/\mathscr{D}_A.P(D)$ is a finite free A-module of rank ν. Equipped with the multiplication by D, it is a differential module over (A, d). In the basis $\{1, D, \ldots, D^{\nu-1}\}$ the multiplication by D satisfies:

$$D \begin{pmatrix} f_1 \\ \vdots \\ f_\nu \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_\nu \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & -g_0 \\ 1 & 0 \cdots & 0 & \vdots \\ 0 & \ddots & 0 & \vdots \\ 0 \cdots & 0 & 1 & -g_{\nu-1} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_\nu \end{pmatrix}$$

Remark 3.1.2. Equip a free finite A-module M with a differential module over (A, d) structure is equivalent to equip M with a $\mathscr{D}_A$-left-module structure.

Theorem 3.1.3 (The cyclic vector theorem)**.** *Let (A, d) be a k-differential field (i.e A is a field), with $d \neq 0$, and let (M, ∇) be a differential module over (A, d) of rank n. Then there exists $m \in M$ such that $\{m, \nabla(m), \ldots \nabla^{n-1}(m)\}$ is a basis of M. In this case we say that m is cyclic vector.*

Proof. See [Ked10, Theorem 5.7.2.]. □

Remark 3.1.4. The last theorem means that there exists an isomorphism of differential modules between (M, ∇) and $(\mathscr{D}_A/\mathscr{D}_A.P(D), D)$ for some monic differential polynomial $P(D)$ of degree n.

Lemma 3.1.5. *Let L, P and Q be differential polynomials, such that $L = QP$. Then we have an exact sequence of differential modules:*

$$0 \longrightarrow \mathscr{D}_A/\mathscr{D}_A.Q \xrightarrow{i} \mathscr{D}_A/\mathscr{D}_A.L \xrightarrow{p} \mathscr{D}_A/\mathscr{D}_A.P \longrightarrow 0$$

where the maps i and p are defined as follows: for a differential polynomial R, $i(\bar{R}) = \overline{RP}$ and $p(\bar{R}) = \bar{R}$.

Proof. See [Chr83, Section 3.5.6]. $\square$

Notation 3.1.6. Let (A, d) be a differential k-algebra. We denote by $d - \mathbf{Mod}(A)$ the category whose objects are differential modules over (A, d) and arrows are morphisms of differential modules.

Let (M_1, ∇_1) and (M_2, ∇_2) be two differential modules over (A, d), respectively of rank n_1 and n_2. The tensor product $M_1 \otimes_A M_2$ equipped with the map:

$$\nabla_{M_1 \otimes_A M_2} := \nabla_1 \otimes 1 + 1 \otimes \nabla_2 \tag{3.2}$$

is a differential module over (A, d).

Let (A, d) and (A', d') be two differential k-algebras. We assume that A' is an A-algebra and the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "d"'] & A' \arrow[d, "d'"] \\ A \arrow[r] & A' \end{tikzcd}$$

commutes. Then we have a functor:

$$\begin{array}{rcl} d - \mathbf{Mod}(A) & \longrightarrow & d' - \mathbf{Mod}(A) \\ (M, \nabla) & \mapsto & (M \otimes_A A', \nabla_{M \otimes_A A'}) \end{array} \tag{3.3}$$

Note that, if (M, ∇) has rank equal to n, then so has $(M \otimes_A A', \nabla_{M \otimes_A A'})$.

If moreover A' is a finite free A-algebra with rank equal to n_A, then we have another functor

$$\begin{array}{rcl} d' - \mathbf{Mod}(A') & \longrightarrow & d - \mathbf{Mod}(A) \\ (M, \nabla) & \mapsto & (M_A, \nabla_A) \end{array} \tag{3.4}$$

where M_A is the restriction of scalars of M via $A \to A'$, and $\nabla = \nabla_A$ are equal as k-linear maps. If (M, ∇) has rank equal to n, then the rank of (M_A, ∇_A) is equal to $n.n_a$.

3.2 Spectrum associated to a differential module

Convention 3.2.1. From now on A will be either a k-affinoid algebra associated to an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$ or a non-algebraic complete extension $\Omega \in E(k)$ of k. Let d be a bounded derivation on A.

Let (M, ∇) be a differential module over (A, d). In order to associate to this differential module a spectrum we need to endow it with a structure of k-Banach space.

Definition 3.2.2. A Banach A-module M is said to be finite if there exists an admissible [1] epimorphism $\pi : A^n \to M$.

Proposition 3.2.3. *The Forgetful functor induces an equivalence of category between the category of finite Banach A-modules with bounded A-linear maps as morphisms and the category of finite A-modules with A-linear maps as morphisms.*

Proof. See [Ber90, Proposition 2.1.9]. □

These Propositions mean that we can endow M with a structure of finite Banach A-module isomorphic to A^n equipped with the maximum norm, and any other structure of finite Banach A-module on M is equivalent to the previous one. This induces a structure of Banach k-space on M. As ∇ satisfies the rule (3.1) and $d \in \mathcal{L}_k(A)$, we have $\nabla \in \mathcal{L}_k(M)$. The spectrum associated to (M, ∇) is denoted by $\Sigma_{\nabla,k}(\mathcal{L}_k(M))$ [2] (or just by Σ_∇ if the dependence is obvious from the context).

Let $\varphi : (M, \nabla) \to (N, \nabla')$ be a morphism of differential modules. If we endow M and N with structures of k-Banach spaces (as above) then φ is automatically an admissible bounded k-linear map (see [Ber90, Proposition 2.1.10]). In the case φ is an isomorphism, then it induces a bi-bounded k-linear isomorphism and according to Lemma 2.1.18 we have:

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{\nabla',k}(\mathcal{L}_k(N)). \tag{3.5}$$

This prove the following proposition:

Proposition 3.2.4. *The spectrum of a connection is an invariant by bi-bounded isomorphisms of differential modules.*

Proposition 3.2.5. *Let $0 \to (M_1, \nabla_1) \to (M, \nabla) \to (M_2, \nabla_2) \to 0$ be an exact sequence of differential modules over (A, d).*
Then we have: $\Sigma_\nabla(\mathcal{L}_k(M)) \subset \Sigma_{\nabla_1}(\mathcal{L}_k(M_1)) \cup \Sigma_{\nabla_2}(\mathcal{L}_k(M_2))$, with equality if $\Sigma_{\nabla_1}(\mathcal{L}_k(M_1)) \cap \Sigma_{\nabla_2}(\mathcal{L}_k(M_2)) = \varnothing$.

Proof. As M_1, M_2 and M are free A-modules, the sequence:

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

splits. Hence, we can identify M with $M_1 \oplus M_2$ so that f corresponds to the inclusion of M_1 into M and g to projection of M onto M_2. Let p_1 be the projection of M onto M_1

1. Which means that: $A^n / \operatorname{Ker} \pi$ endowed with the quotient norm is isomorphic as Banach A-module to $\operatorname{Im} \pi$.
2. Note that, since all the structures of finite Banach A-module on M are equivalent, the spectrum does not depend on the choice of such a structure.

and i_2 be the inclusion of M_2 into M. As both f and g are morphisms of differential modules, we have $\nabla(M_1) \subset M_1$, $\nabla_1 = p_1 \nabla f$ and $\nabla_2 = g\nabla i_2$. By Lemma 2.1.20 and Remark 2.1.21 we obtain the result. $\square$

Lemma 3.2.6. *We keep the assumptions of Proposition 3.2.5. If in addition we have an other exact sequence of the form:*

$$0 \to (M_2, \nabla_2) \to (M, \nabla) \to (M_1, \nabla_1) \to 0,$$

then we have $\Sigma_\nabla(\mathcal{L}_k(M)) = \Sigma_{\nabla_1}(\mathcal{L}_k(M_1)) \cup \Sigma_{\nabla_2}(\mathcal{L}_k(M_2))$.

Proof. This is a consequence of Remark 2.1.21. Indeed, If $\nabla \otimes 1 - 1 \otimes T(x)$ is invertible, then the first exact sequence shows that $\nabla_1 \otimes 1 - 1 \otimes T(x)$ is left invertible and $\nabla_2 \otimes 1 - 1 \otimes T(x)$ is right invertible, the second exact sequence to $\nabla_2 \otimes 1 - 1 \otimes T(x)$ is left invertible and $\nabla_1 \otimes 1 - 1 \otimes T(x)$ is right invertible. Therefore, both of $\nabla_1 \otimes 1 - 1 \otimes T(x)$ and $\nabla_2 \otimes 1 - 1 \otimes T(x)$ are invertible. Hence, we obtain $\Sigma_{\nabla_1} \cup \Sigma_{\nabla_2} \subset \Sigma_\nabla$. $\square$

Remark 3.2.7. In particular, if we have $(M, \nabla) = (M_1, \nabla_1) \oplus (M_2, \nabla_2)$ as differential modules then we have $\Sigma_\nabla = \Sigma_{\nabla_1} \cup \Sigma_{\nabla_2}$.

Proposition 3.2.8. *Let $P \in \mathscr{D}_A$ be a differential polynomial. Let $(M, \nabla) := (\mathscr{D}_A/\mathscr{D}_A.P^m, D)$ and $(N, \nabla') := (\mathscr{D}_A/\mathscr{D}_A.P, D)$. Then we have*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{\nabla',k}(\mathcal{L}_k(N)).$$

Proof. By Lemma 3.1.5, we have the exact sequences:

$$0 \to (\mathscr{D}_A/\mathscr{D}_A.P^{m-1}, D) \to (\mathscr{D}_A/\mathscr{D}_A.P^m, D) \to (\mathscr{D}_A/\mathscr{D}_A.P, D) \to 0$$

and

$$0 \to (\mathscr{D}_A/\mathscr{D}_A.P, D) \to (\mathscr{D}_A/\mathscr{D}_A.P^m, D) \to (\mathscr{D}_A/\mathscr{D}_A.P^{m-1}, D) \to 0$$

By induction and Lemma 3.2.6, we obtain $\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{\nabla',k}(\mathcal{L}_k(N))$. $\square$

Remark 3.2.9. From the last proposition we observe clearly that the spectrum does not take multiplicity into account.

Remark 3.2.10. Set notation as in Proposition 3.2.5. We suppose that $A = \mathscr{H}(x)$ for some point $x \in \mathbb{A}_k^{1,\mathrm{an}}$ not of type (1). It is then known (see [Ked10, Lemma 6.2.8]) that we have:

$$\|\nabla\|_{\mathrm{Sp}} = \max\{\|\nabla_1\|_{\mathrm{Sp}}, \|\nabla_2\|_{\mathrm{Sp}}\}.$$

Definition 3.2.11. We say that a differential module (M, ∇) over (A, d) of rank n is trivial if it isomorphic to (A^n, d) as a differential module.

Remark 3.2.12. From now on, for any differential module (M, ∇) over (A, d), we consider $\mathrm{Ker}\, \nabla$ as a differential module over $(k, 0)$ with connection equal to 0.

Lemma 3.2.13. *Let (M, ∇) be a differential module of rank n over a differential field (K, d). If $(\mathrm{Ker}\,\nabla \otimes_k A, d_{\mathrm{Ker}\,\nabla\otimes A}) \simeq (M, \nabla)$ as differential module, then (M, ∇) is a trivial differential module.*

Proof. See [Chr83, Proposition 3.5.3]. □

Corollary 3.2.14. *We suppose that $A = \mathscr{H}(x)$ for some $x \in \mathbb{A}_k^{1,\mathrm{an}}$ not of type (1). Let (M, ∇) be a differential module over (A, d). If $(\mathrm{Ker}\,\nabla \otimes_k A, d_{\mathrm{Ker}\,\nabla\otimes A}) \simeq (M, \nabla)$ as differential module, then $\Sigma_\nabla(\mathcal{L}_k(M)) = \Sigma_d(\mathcal{L}_k(A))$.*

Proof. Direct consequence of Proposition 3.2.4 and Remark 3.2.7. □

Lemma 3.2.15. *We suppose that k is algebraically closed. Let (M, ∇) be a differential module over (A, d) such that $G \in \mathcal{M}_n(k)$(cf. (3.1)) and $\{a_1, \cdots, a_N\}$ is the set of the eigenvalues of G. Then we have an isomorphisme of differential modules:*

$$(M, \nabla) \simeq \Big(\bigoplus_{1\leq i\leq N}\bigoplus_{1\leq j\leq N_i} \mathscr{D}_A/\mathscr{D}_A.(D - a_i)^{n_{i,j}}, D\Big)$$

where the $n_{i,j}$ are positive integers such that $\sum_{j=1}^{N_i} n_{i,j}$ is the multiplicity of a_i for each i.

Proof. Let $P \in \mathrm{GL}_n(k)$ such that $J := P^{-1}GP$ has form of Jordan. Since $d(P) = 0$ (derivation component by component), in some basis we have :

$$\nabla\begin{pmatrix} f_1\\ \vdots\\ f_n\end{pmatrix} = \begin{pmatrix} df_1\\ \vdots\\ df_n\end{pmatrix} + J\begin{pmatrix} f_1\\ \vdots\\ f_n\end{pmatrix}.$$

Hence, we obtain the result. □

Lemma 3.2.16. *Let (M, ∇) be the differential module over (A, d) associated to the differential polynomial $(D - a)^n$, where $a \in k$. The spectrum of ∇ is $\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = a + \Sigma_{d,k}(\mathcal{L}_k(A))$ (the image of $\Sigma_{d,k}(\mathcal{L}_k(A))$ by the polynomial $T + a$).*

Proof. From Proposition 3.2.7, we have $\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{d+a,k}(\mathcal{L}_k(A))$. By Lemma 2.1.16, we obtain $\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = a + \Sigma_{d,k}(\mathcal{L}_k(A))$.

□

Proposition 3.2.17. *We suppose that k is algebraically closed. Let (M, ∇) be a differential module over (A, d) such that:*

$$\nabla\begin{pmatrix} f_1\\ \vdots\\ f_n\end{pmatrix} = \begin{pmatrix} df_1\\ \vdots\\ df_n\end{pmatrix} + G\begin{pmatrix} f_1\\ \vdots\\ f_n\end{pmatrix},$$

with $G \in \mathcal{M}_n(k)$. The spectrum of ∇ is $\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \bigcup_{i=1}^N (a_i + \Sigma_{d,k}(\mathcal{L}_k(A)))$, where $\{a_1, \ldots, a_N\}$ are the eigenvalues of G.

Proof. Using the decomposition of Lemma 3.2.15, Lemma 3.2.16 and Remark 3.2.7, we obtain the result.

□

Remark 3.2.18. This claim shows that the spectrum of a connection depends highly on the choice of the derivation d.

Remark 3.2.19. In the next chapters (cf. 4 and 5), in order to obtain the spectra of important classes of differential equations, we will use Proposition 3.2.17 to reduce to the computation of the spectrum of a suitable derivation d.

3.3 Spectral version of Young's theorem in rank one

In this section we will give a spectral version in rank one of Young's theorem [You92], [Ked10, Theorem 6.5.3], [CM02, Theorem 6.2], which states the following :

Theorem 3.3.1 (Young). *Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2), (3) or (4). Let $\mathcal{L} = \sum_{i=0}^{n} g_{n-i} \frac{\mathrm{d}}{\mathrm{dS}}^i$ with $g_0 = 1$ and $g_i \in \mathscr{H}(x)$, and let (M, ∇) be the associated differential module over $(\mathscr{H}(x), \frac{\mathrm{d}}{\mathrm{dS}})$. We set $|\mathcal{L}|_{Sp} = \max_{0 \leq i \leq n} |g_i|^{\frac{1}{i}}$. If $|\mathcal{L}|_{Sp} > \| \frac{\mathrm{d}}{\mathrm{dS}} \|$ then $\|\nabla\|_{\mathrm{Sp}} = |\mathcal{L}|_{Sp}$.*

In this section, for the definitions and notation we refer the reader to Section 1.2.3. In order to state and prove the main statement of the section, we will need the following additional notations and results.

Convention 3.3.2. We suppose in this section that k is algebraically closed.

Notation 3.3.3. Let X be an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$ and let $f \in \mathcal{O}_X(X)$. We can see f as an analytic morphism $X \to \mathbb{A}_k^{1,\mathrm{an}}$ that we still denote f.

Lemma 3.3.4. *Let $\Omega \in E(k)$. Consider the isometric embedding of k-algebras*

$$\begin{array}{ccc} \Omega & \longrightarrow & \mathcal{L}_k(\Omega) \\ a & \mapsto & b \mapsto a.b \end{array}.$$

With respect to this embedding, Ω is a maximal commutative subalgebra of $\mathcal{L}_k(\Omega)$.

Proof. Let A be a commutative subalgebra of $\mathcal{L}_k(\Omega)$ such that $\Omega \subset A$. Then each element of A is an endomorphism of Ω that commutes with the elements of Ω. Therefore, $A \subset \mathcal{L}_\Omega(\Omega) = \Omega$. Hence, we have $A = \Omega$. □

Lemma 3.3.5. *Let $\Omega \in E(k)$ and $\pi_{\Omega/k} : \mathbb{A}_\Omega^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ be the canonical projection. Let $\alpha \in \Omega$. The spectrum of α as an element of $\mathcal{L}_k(\Omega)$ is $\Sigma_\alpha(\mathcal{L}_k(\Omega)) = \{\pi_{\Omega/k}(\alpha)\}$.*

Proof. By Proposition 2.1.10, the spectrum of α as an element of Ω is the point which corresponds to the character $k[T] \to \Omega$, $T \mapsto \alpha$. By Lemma 3.3.4 and Proposition 2.1.15 we conclude. □

We now state the spectral version in rank one of Young's theorem:

Theorem 3.3.6. *Assume that k is algebraically closed. Let $\Omega \in E(k)$ and $d : \Omega \to \Omega$ be a bounded k-linear derivation. Let (Ω, ∇) be the differential module over (Ω, d) with $\nabla := d + f$ for some $f \in \Omega$. If $r_k(\pi_{\Omega/k}(f)) > \|d\|$ (cf. Definition 1.2.27), then we have*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(\Omega)) = \{\pi_{\Omega/k}(f)\}.$$

Proof. By Lemma 3.3.5, we have $\Sigma_{f,k}(\mathcal{L}_k(\Omega)) = \{\pi_{\Omega/k}(f)\}$. Let us prove now that $\Sigma_{\nabla,k}(\mathcal{L}_k(\Omega)) = \{\pi_{\Omega/k}(f)\}$. Let $y \in \mathbb{A}_k^{1,\mathrm{an}} \setminus \{\pi_{\Omega/k}(f)\}$. We know that $f \otimes 1 - 1 \otimes T(y)$ is invertible in $\Omega\hat{\otimes}_k\mathscr{H}(y)$, hence invertible in $\mathcal{L}_k(\Omega)\hat{\otimes}_k\mathscr{H}(y)$. Since $d \otimes 1 = (\nabla \otimes 1 - 1 \otimes T(y)) - (f \otimes 1 - 1 \otimes T(y))$, in order to prove that $\nabla \otimes 1 - 1 \otimes T(y)$ is invertible, it is enough to show that

$$\|d \otimes 1\| < \|(f \otimes 1 - 1 \otimes T(y))^{-1}\|^{-1}.$$

In order to do so, since $\|d\| = \|d \otimes 1\|$ (cf. Lemma 1.1.38) and $r_k(\pi_{\Omega/k}(f)) > \|d\|$, it is enough to show that $r_k(\pi_{\Omega/k}(f)) \leq \|(f \otimes 1 - 1 \otimes T(y))^{-1}\|^{-1}$. On the one hand, since $\Omega \hookrightarrow \mathcal{L}_k(\Omega)$ is an isometric embedding, then so is $\Omega\hat{\otimes}_k\mathscr{H}(y) \to \mathcal{L}_k(\Omega)\hat{\otimes}_k\mathscr{H}(y)$ (cf. [Poi13, Lemme 3.1]). On the other hand, the norm on $\Omega\hat{\otimes}_k\mathscr{H}(y)$ is multiplicative (cf. Proposition 1.2.36). Therefore, we have

$$\|(f \otimes 1 - 1 \otimes T(y))^{-1}\|^{-1} = \|f \otimes 1 - 1 \otimes T(y)\|.$$

To avoid confusion, let S be an other coordinate function on $\mathbb{A}_k^{1,\mathrm{an}}$. Note that we have an isometric embedding $\mathscr{H}(\pi_{\Omega/k}(f)) \hookrightarrow \Omega$, that assigns to $S(\pi_{\Omega/k}(f))$ the element f. By the same argument as above, we have an isometric embedding of k-algebra $\mathscr{H}(\pi_{\Omega/k}(f))\hat{\otimes}_k\mathscr{H}(y) \hookrightarrow \Omega\hat{\otimes}_k\mathscr{H}(y)$. Therefore, it enough to show that

$$r_k(\pi_{\Omega/k}(f)) \leq \|(S(\pi_{\Omega/k}(f)) \otimes 1 - 1 \otimes T(y))\|.$$

The natural map $\mathscr{H}(\pi_{\Omega/k}(f))\hat{\otimes}_k\mathscr{H}(y) \to \mathscr{H}(\sigma_{\mathscr{H}(y)/k}(\pi_{\Omega/k}(f)))$ is an isometric map (cf. Lemma 1.1.12). The image of $S(\pi_{\Omega/k}(f)) \otimes 1 - 1 \otimes T(y)$ by this map is

$$S(\sigma_{\mathscr{H}(y)/k}(\pi_{\Omega/k}(f))) - T(y).$$

By Lemma 1.2.40, we have $r_{\mathscr{H}(y)}(\sigma_{\mathscr{H}(y)/k}(\pi_{\Omega/k}(f))) = r_k(\pi_{\Omega/k}(f))$. Therefore, we obtain

$$r_k(\pi_{\Omega/k}(f)) \leq |S(\sigma_{\mathscr{H}(y)/k}(\pi_{\Omega/k}(f))) - T(y)| \leq \|S(\pi_{\Omega/k}(f)) \otimes 1 - 1 \otimes T(y)\|.$$

Since $\Sigma_{\nabla,k}(\mathcal{L}_k(\Omega))$ is not empty, we conclude that $\Sigma_{\nabla,k}(\mathcal{L}_k(\pi_{\Omega/k}(f))) = \{\pi_{\Omega/k}(f)\}$. □

Note that for higher rank equations, in order to compute the spectra of the differential modules that are extensions of differential module of rank one, we can combine Proposition 3.2.5 and Theorem 3.3.6. More generally, we conjecture the following:

Conjecture 3.3.7. *Let $\mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2), (3) or (4). Let d be a bounded derivation on $\mathscr{H}(x)$ and let (M,∇) be a differential module over $(\mathscr{H}(x),d)$. Let $\{m,\nabla(m),\cdots,\nabla^{n-1}(m)\}$ be a cyclic basis and let G be the associated matrix in this basis. Let $\Sigma_{G,k}(\mathcal{M}_n(\mathscr{H}(x))):=\{y_1,\cdots,y_N\}$. Then there exists $C_G:=(C_{y_1},\cdots,C_{y_N})\in\mathbb{R}_+^N$ that depends only on G, such that*

$$r_k(y_i)>C_{y_i}\|d\|\Rightarrow y_i\in\Sigma_{\nabla,k}(\mathcal{L}_k(M)).$$

If moreover, for all $y_i\in\Sigma_{G,k}(\mathcal{M}_n(\mathscr{H}(x)))$ we have $r_k(y_i)>C_{y_i}\|d\|$, then

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M))=\Sigma_{G,k}(\mathcal{M}_n(\mathscr{H}(x))).$$

3.4 Push-forward and spectrum

In this section, we refer the reader to the last part of Section 1.2.3 for the definitions and notation.

Notation 3.4.1. From now on we will fix S to be the coordinate function of the analytic domain (of the affine line) where the linear differential equation is defined and T to be the coordinate function on $\mathbb{A}_k^{1,\mathrm{an}}$ (for the computation of the spectrum).

Let Y and X be two connected affinoid domains of $\mathbb{A}_k^{1,\mathrm{an}}$ and Z (resp. S) a coordinate function on X (resp. Y). Let $\varphi:Y\to X$ be a finite étale morphism and $\varphi^{\#}:\mathcal{O}_X\to\varphi_*\mathcal{O}_Y$ the induced sheave morphism. Let $f:=\varphi^{\#}(Z)$ and $f':=\frac{\mathrm{d}}{\mathrm{dS}}(f)$. Since φ is étale, f' is invertible in $\mathcal{O}_Y(Y)$ (cf. Lemma 1.2.41). To any bounded derivation $d=g\frac{\mathrm{d}}{\mathrm{dZ}}$ on X we assign the bounded derivation

$$\varphi^*d:=\frac{\varphi^{\#}(g)}{f'}\frac{\mathrm{d}}{\mathrm{dS}} \tag{3.6}$$

on Y, so-called the *pull-back* of d by φ.

Let $y\in Y$ and $x=\varphi(y)$. Since the derivation $\frac{\mathrm{d}}{\mathrm{dS}}$ (resp. $\frac{\mathrm{d}}{\mathrm{dZ}}$) extends to a bounded derivation on Y (resp. X), the derivation d (resp. φ^*d) extends to a bounded derivation on Y (resp. X). We have the commutative diagram:

$$\begin{tikzcd} \mathscr{H}(x) \arrow[r, hook] \arrow[d, "d"'] & \mathscr{H}(y) \arrow[d, "\varphi^*d"] \\ \mathscr{H}(x) \arrow[r, hook] & \mathscr{H}(y) \end{tikzcd}$$

Since $\varphi^{\#}$ induce a finite extension $\mathscr{H}(x) \hookrightarrow \mathscr{H}(y)$, we have the *push-forward* functor by φ defined as in (3.4):

$$\begin{array}{rcl} \varphi_* : \varphi^* d - \mathbf{Mod}(\mathscr{H}(y)) & \longrightarrow & d - \mathbf{Mod}(\mathscr{H}(x)) \\ (M, \nabla) & \mapsto & (\varphi_* M, \varphi_* \nabla) \end{array} \tag{3.7}$$

and the *pull-back* functor by φ defined as in (3.3):

$$\begin{array}{rcl} \varphi^* : d - \mathbf{Mod}(\mathscr{H}(x)) & \longrightarrow & \varphi^* d - \mathbf{Mod}(\mathscr{H}(y)) \\ (M, \nabla) & \mapsto & (\varphi^* M, \varphi^* \nabla) \end{array} \tag{3.8}$$

Remark 3.4.2. In chapters 5 and 6, we will discribe the above functors more explicitly.

Proposition 3.4.3. *We have the set-theoretic equality:*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \Sigma_{\varphi_*\nabla,k}(\mathcal{L}_k(\varphi_* M)).$$

Proof. Since M and $\varphi_* M$ are the same as $\mathscr{H}(x)$-Banach spaces, then they are isomorphic as k-Banach spaces. As ∇ and $\varphi_*\nabla$ coincide as k-linear maps, the equality of spectra holds. $\square$

IV

Spectrum of a differential equation with constant coefficients

Contents

This chapter is devoted to the computation of the spectrum of a differential equation with constant coefficients. It is divided into three parts. The first one we compute the spectrum of $\frac{\mathrm{d}}{\mathrm{dS}}$, the second one we state and prove the main result which is the computation of the spectrum associated to a linear differential equation with constant coefficients and in the last we describe the variation of the spectrum of such equations.

Convention 4.0.1. In this chapter we will suppose that k is algebraically closed.

Notation 4.0.2.

$$\omega = \begin{cases} |p|^{\frac{1}{p-1}} & \text{if } \mathrm{char}(\tilde{k}) = p \\ 1 & \text{if } \mathrm{char}(\tilde{k}) = 0 \end{cases}. \tag{4.1}$$

Lemma 4.0.3. • *Let $X := D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$ be a connected affinoid domain and set $r = \min\limits_{0 \leq i \leq \mu} r_i$. The operator norm of $(\frac{\mathrm{d}}{\mathrm{dS}})^n$ as an element of $\mathcal{L}_k(\mathcal{O}(X))$ satisfies:*

$$\|(\frac{\mathrm{d}}{\mathrm{dS}})^n\|_{\mathcal{L}_k(\mathcal{O}(X))} = \frac{|n!|}{r^n}, \qquad \|\frac{\mathrm{d}}{\mathrm{dS}}\|_{Sp,\mathcal{L}_k(\mathcal{O}(X))} = \frac{\omega}{r}.$$

- *Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2), (3) or (4). The operator norm of $(\frac{\mathrm{d}}{\mathrm{dS}})^n$ as an element of $\mathcal{L}_k(\mathscr{H}(x))$ satisfies:*

$$\|(\frac{\mathrm{d}}{\mathrm{dS}})^n\|_{\mathcal{L}_k(\mathscr{H}(x))} = \frac{|n!|}{r(x)^n}, \qquad \|\frac{\mathrm{d}}{\mathrm{dS}}\|_{Sp,\mathcal{L}_k(\mathscr{H}(x))} = \frac{\omega}{r(x)},$$

where $r(x)$ is the value defined in Definition 1.2.27.

Proof. See [Pul15, Lemma 4.4.1]. □

Remark 4.0.4. Let X be an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$. Let (X_i) be the finite family of connected components of X. We have $\mathcal{O}(X) = \bigoplus_{i=1}^{\mu} \mathcal{O}(X_i)$. As $\frac{\mathrm{d}}{\mathrm{dS}}$ stabilises each Banach space of the direct sum, we have:

$$\|\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp},\mathcal{L}_k(\mathcal{O}(X))} = \max_{0\leq i\leq \mu}\|\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp},\mathcal{L}_k(\mathcal{O}(X_i))}.$$

Let $\Omega \in E(k)$ and let X be an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$. Let $d = f(S)\frac{\mathrm{d}}{\mathrm{dS}}$ be a derivation defined on $\mathcal{O}(X)$. We can extend it to a derivation $d_\Omega = f(S)\frac{\mathrm{d}}{\mathrm{dS}}$ defined on $\mathcal{O}(X_\Omega)$. The derivation d_Ω is the image of $d \otimes 1$ by the morphism $\mathcal{L}_k(\mathcal{O}(X))\hat{\otimes}_k\Omega \to \mathcal{L}_\Omega(\mathcal{O}(X_\Omega))$ defined in Lemma 1.1.40.

Lemma 4.0.5. *Let $\pi_{\Omega/k} : X_\Omega \to X$ be the canonical projection. We have:*

$$\pi_{\Omega/k}(\Sigma_{d_\Omega,\Omega}(\mathcal{L}_\Omega(\mathcal{O}(X_\Omega))) \subset \Sigma_{d,k}(\mathcal{L}_k(\mathcal{O}(X))).$$

Proof. By Lemma 1.1.40 and 2.1.18 we have $\Sigma_{d_\Omega,\Omega}(\mathcal{L}_\Omega(\mathcal{O}(X_\Omega))) \subset \Sigma_{d\otimes 1,\Omega}(\mathcal{L}_k(\mathcal{O}(X))\hat{\otimes}_k\Omega)$. Since $\Sigma_{d\otimes 1,\Omega}(\mathcal{L}_k(\mathcal{O}(X)\hat{\otimes}_k\Omega)) = \pi_{\Omega/k}^{-1}(\Sigma_{d,k}(\mathcal{L}_k(\mathcal{O}(X))))$ (see [Ber90, Proposition 7.1.6]), we obtain the result. □

4.1 The spectrum of $\frac{\mathrm{d}}{\mathrm{dS}}$ on several domains

Let X be an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$ and $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2), (3) or (4). In this part we compute the spectrum of $\frac{\mathrm{d}}{\mathrm{dS}}$ as a derivation of $A = \mathcal{O}(X)$ or $\mathscr{H}(x)$ as an element of $\mathcal{L}_k(A)$. We treat the case of positive residue characteristic separately. We will also distinguish the case where X is a closed disk from the case where it is a connected affinoid subdomain, and the case where x is point of type (4) from the others ones.

4.1.1 The case of positive residue characteristic

In this section we assume that $\mathrm{char}(\tilde{k}) = p > 0$. In this case $\omega = |p|^{\frac{1}{p-1}}$.

Proposition 4.1.1. *Let* $X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$ *be a connected affinoid domain of* $\mathbb{A}_k^{1,\mathrm{an}}$. *The spectrum of* $\frac{\mathrm{d}}{\mathrm{dS}}$ *as an element of* $\mathcal{L}_k(\mathcal{O}(X))$ *is:*

$$\Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}(\mathcal{L}_k(\mathcal{O}(X))) = D^+(0, \frac{\omega}{\min\limits_{0\leq i\leq\mu} r_i}).$$

Proof. We distinguish the case of the disk from the other one.

- **Case** $\mu = 0$: We set $A = \mathcal{O}(D^+(c, r))$ and $d = \frac{\mathrm{d}}{\mathrm{dS}}$. We prove firstly this claim for a field k that is spherically complete and satisfies $|k| = \mathbb{R}_+$. By Lemma 4.0.3 the spectral norm of d is equal to $\|d\|_{\mathrm{Sp}} = \frac{\omega}{r}$. By Theorem 2.1.6 we have $\Sigma_d \subset D^+(0, \frac{\omega}{r})$. We prove now that $D^+(0, \frac{\omega}{r}) \subset \Sigma_d$.

  Let $x \in D^+(0, \frac{\omega}{r}) \cap k$. Then

  $$d \otimes 1 - 1 \otimes T(x) = (d - a) \otimes 1$$

  where $T(x) = a \in k$. The element $d \otimes 1 - 1 \otimes T(x)$ is invertible in $\mathcal{L}_k(A)\hat{\otimes}\mathscr{H}(x)$ if and only if $d - a$ is invertible in $\mathcal{L}_k(A)$ [Ber90, Lemma 7.1.7].

  If $|a| < \frac{\omega}{r}$, then $\exp(a(S - c)) = \sum_{n\in\mathbb{N}}(\frac{a^n}{n!})(S - c)^n$ exists and it is an element of A. Hence, $\exp(a(S - c)) \in \ker(d - a)$, in particular $d - a$ is not invertible. Consequently, $D^-(0, \frac{\omega}{r}) \cap k \subset \Sigma_d$.

  Now we suppose that $|a| = \frac{\omega}{r}$. We prove that $d - a$ is not surjective.
  Let $g(S) = \sum_{n\in\mathbb{N}} b_n(S - c)^n \in A$. If there exists $f(S) = \sum_{n\in\mathbb{N}} a_n(S - c)^n \in A$ such that $(d - a)f = g$, then for each $n \in \mathbb{N}$ we have:

  $$a_n = \frac{(\sum_{i=0}^{n-1} i!b_i a^{n-1-i}) + a^n a_0}{n!}. \tag{4.2}$$

  We now construct a series $g \in A$ such that its pre-image f does not converge on the closed disk $D^+(c, r)$. Let $\alpha, \beta \in k$, such that $|\alpha| = r$ and $|\beta| = |p|^{1/2}$. For $n \in \mathbb{N}$ we set:

  $$b_n = \begin{cases} \frac{\beta^\ell}{\alpha^{p^\ell - 1}} & \text{if } n = p^\ell - 1 \text{ with } \ell \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

  Then, $|b_n|r^n$ is either 0 or $|p|^{\log_p(n+1)}$ and $g \in A$. If we suppose that there exists $f \in A$ such that $(d - a)f = g$ then we have:

  $$\forall \ell \in \mathbb{N}; \qquad a_{p^l} = \frac{a^{p^\ell - 1}}{(p^\ell)!}[\sum_{j=0}^{\ell} \frac{(p^j - 1)!\beta^j}{a^{p^j-1}\alpha^{p^j-1}} + aa_0].$$

As $|p^\ell!| = \omega^{p^\ell-1}$ (cf. [DGS94, p. 51]) we have:

$$|a_{p^\ell}| = \frac{1}{r^{p^\ell-1}}|\sum_{j=0}^{\ell}\frac{(p^j-1)!\beta^j}{a^{p^j-1}\alpha^{p^j-1}} + aa_0|.$$

Since $|\frac{(p^j-1)!\beta^j}{a^{p^j-1}\alpha^{p^j-1}}| = |p|^{-j/2}$, we have:

$$|\sum_{j=0}^{\ell}\frac{(p^j-1)!\beta^j}{a^{p^j-1}\alpha^{p^j-1}}| = \max_{0\leq j\leq \ell}|p|^{-j/2} = |p|^{-\ell/2},$$

therefore $|a_{p^\ell}|r^{p^\ell} \xrightarrow{\ell\to+\infty} +\infty$, which proves that the power series f is not in A and this is a contradiction. Hence, $D^+(0,\frac{\omega}{r})\cap k \subset \Sigma_d$. As the points of type (1) are dense in $D^+(0,\frac{\omega}{r})$ and Σ_d is compact, we deduce that $D^+(0,\frac{\omega}{r})\subset\Sigma_d$.

Let us now consider an arbitrary field k. Let $\Omega\in E(k)$ algebraically closed, spherically complete such that $|\Omega| = \mathbb{R}_+$. We set $A_\Omega = \mathcal{O}(X_\Omega)$ and $d_\Omega = \frac{\mathrm{d}}{\mathrm{dS}}$ the derivation on A_Ω. From above, we have $\Sigma_{d_\Omega} = D^+_\Omega(0,\frac{\omega}{r})$, then $\pi_{\Omega/k}(\Sigma_{d_\Omega}) = D^+(0,\frac{\omega}{r})$. By Lemma 4.0.5 we have $D^+(0,\frac{\omega}{r}) = \pi_{\Omega/k}(\Sigma_{d_\Omega})\subset\Sigma_d$. As $\|d\|_{\mathrm{Sp}} = \frac{\omega}{r}$, we obtain $\Sigma_d = D^+(0,\frac{\omega}{r})$.

- **Case** $\mu\geq 1$: We set $A = \mathcal{O}(X)$ and $d = \frac{\mathrm{d}}{\mathrm{dS}}$. The spectral norm of d is equal to $\|d\|_{\mathrm{Sp}} = \frac{\omega}{\min\limits_{0\leq i\leq\mu} r_i}$ (cf. Lemma 4.0.3), which implies $\Sigma_d\subset D^+(0,\frac{\omega}{\min\limits_{0\leq i\leq\mu} r_i})$. It is easy to see that $0\in\Sigma_d$. Now, let $x\in D^+(0,\frac{\omega}{\min\limits_{0\leq i\leq\mu} r_i})\setminus\{0\}$. We set $A_{\mathscr{H}(x)} = \mathcal{O}(X_{\mathscr{H}(x)})$ and $d_{\mathscr{H}(x)} = \frac{\mathrm{d}}{\mathrm{dS}} : A_{\mathscr{H}(x)}\to A_{\mathscr{H}(x)}$. From Lemma 1.1.40 we have the bounded morphism:

$$\mathcal{L}_k(A)\hat{\otimes}_k\mathscr{H}(x)\to\mathcal{L}_{\mathscr{H}(x)}(A_{\mathscr{H}(x)}).$$

The image of $d\otimes 1$ by this morphism is the derivation $d_{\mathscr{H}(x)}$. By the Mittag-Leffler decomposition, we have:

$$\mathcal{O}(X_{\mathscr{H}(x)}) = \bigoplus_{i=1}^{n}\{\sum_{j\in\mathbb{N}^*}\frac{a_{ij}}{(S-c_i)^j}|\ a_{ij}\in\mathscr{H}(x),\ \lim_{j\to+\infty}|a_{ij}|r_i^{-j} = 0\}\oplus\mathcal{O}(D^+_{\mathscr{H}(x)}(c_0,r_0)).$$

Each Banach space of the direct sum above is stable under $d_{\mathscr{H}(x)}$.
We set $F_i = \{\sum_{j\in\mathbb{N}^*}\dfrac{a_{ij}}{(S-c_i)^j}|\ a_{ij}\in\mathscr{H}(x),\ \lim\limits_{j\to+\infty}|a_{ij}|r_i^{-j} = 0\}$, and $d_i = d_{\mathscr{H}(x)|_{F_i}}$.
By Lemma 2.1.20, $\Sigma_{d_{\mathscr{H}(x)}} = \bigcup\Sigma_{d_i}$. Let $i_0 > 0$ be the index such that $r_{i_0} = \min\limits_{0\leq i\leq\mu} r_i$.
We will prove that $d_{i_0} - T(x)$ is not surjective. Indeed, let $g(S) = \sum_{n\in\mathbb{N}^*}\frac{b_n}{(S-c_{i_0})^n}\in F_{i_0}$, if there exists $f(S) = \sum_{n\in\mathbb{N}^*}\frac{a_n}{(S-c_{i_0})^n}\in F_{i_0}$ such that $(d_{i_0}-T(x))f(S) = g(S)$, then for each $n\in\mathbb{N}^*$ we have:

$$a_n = \frac{(n-1)!}{(-T(x))^n}\sum_{i=1}^{n}\frac{(-T(x))^{i-1}}{(i-1)!}b_i.$$

We choose $g(S) = \frac{1}{S-c_{i_0}}$, in this case $a_n = \frac{(n-1)!}{(-T(x))^n}$ and $|a_n| = \frac{|(n-1)!|}{|T(x)|^n}$. As $|T(x)| \leq \frac{\omega}{r_{i_0}}$, the sequence $|a_n|r_{i_0}^{-n}$ diverges. We obtain contradiction since $f \in F_{i_0}$. Hence, $d_{i_0} - T(x)$ is not invertible, and so neither is $d_{\mathscr{H}(x)} - T(x)$ is not invertible too. Therefore, its pre-image $d \otimes 1 - 1 \otimes T(x)$ in $\mathcal{L}_k(A)\hat{\otimes}\mathscr{H}(x)$ cannot be invertible. Hence $x \in \Sigma_d$.

□

Remark 4.1.2. The statement holds even if the field k is not algebraically closed. Indeed, we did not use this assumption.

Corollary 4.1.3. *Let X be an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$. The spectrum of $\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathcal{O}(X))$ is:*

$$\Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}(\mathcal{L}_k(\mathcal{O}(X))) = D^+(0, \|\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}}).$$

Proof. In this case we may write $X = \bigcup_{i=1}^{u} X_i$, where the X_i are connected affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$ such that $X_i \cap X_j = \varnothing$ for $i \neq j$. We have:

$$\mathcal{O}(X) = \bigoplus_{i=1}^{u} \mathcal{O}(X_i).$$

Each Banach space of the direct sum above is stable under d, we denote by d_i the restriction of d to $\mathcal{O}(X_i)$. We have $\|d\|_{\mathrm{Sp}} = \max_{1\leq i\leq u} d_i$ (cf. Remark 4.0.4). By Lemma 2.1.20 and Proposition 4.1.1 we have $\Sigma_d = \bigcup_{i=1}^{u} D^+(0, \|d_i\|_{\mathrm{Sp}}) = D^+(0, \max_i \|d_i\|_{\mathrm{Sp}})$. Hence, we obtain the result. □

Proposition 4.1.4. *Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2), (3) or (4). The spectrum of $\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathscr{H}(x))$ is:*

$$\Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}(\mathcal{L}_k(\mathscr{H}(x))) = D^+(0, \frac{\omega}{r(x)}),$$

where $r(x)$ is the value defined in Definition 1.2.27.

Proof. We set $d = \frac{\mathrm{d}}{\mathrm{dS}}$. We distinguish three cases:

- x **is point a of type (2):** Let $c \in k$ such that $x = x_{c,r(x)}$. By Proposition 1.2.32, we have:

$$\mathscr{H}(x) = E \oplus \mathcal{O}(D^+(c, r(x))).$$

$$E := \widehat{\bigoplus}_{\tilde{\alpha}\in\tilde{k}}\{\sum_{i\in\mathbb{N}^*} \frac{a_{\alpha i}}{(T-c+\gamma\alpha)^i} |\ a_{\alpha i} \in k,\ \lim_{i\to+\infty} |a_{\alpha i}|r^{-i} = 0\},$$

where α is an element of k that corresponds to the class $\tilde{\alpha}$.

As both E and $\mathcal{O}(D^+(c, r(x)))$ are stable under d, by Lemma 2.1.20 we have $\Sigma_d = \Sigma_{d_{|_E}} \cup \Sigma_{d_{|_{\mathcal{O}(D^+(c,r(x)))}}}$. By Proposition 4.1.1 $\Sigma_{d_{|_{\mathcal{O}(D^+(c,r(x)))}}} = D^+(0, \frac{\omega}{r(x)})$. Since $\|d\|_{\mathrm{Sp}} = \frac{\omega}{r(x)}$ (cf. Lemma 4.0.3), then $\Sigma_d = D^+(0, \frac{\omega}{r(x)})$.

- x **is point a of type (3):** Let $c \in k$ such that $x = x_{c,r(x)}$. In this case $\mathscr{H}(x) = \mathcal{O}(C^+(c, r(x), r(x)))$. By Proposition 4.1.1 we obtain the result.

- x **is point a of type (4):** By Proposition 1.2.34 we have
$$\mathscr{H}(x)\hat{\otimes}_k\mathscr{H}(x) \simeq \mathcal{O}(D^+_{\mathscr{H}(x)}(S(x), r(x))).$$
From Lemma 1.1.40 we have the bounded morphism:
$$\mathcal{L}_k(\mathscr{H}(x))\hat{\otimes}_k\mathscr{H}(x) \to \mathcal{L}_{\mathscr{H}(x)}(\mathcal{O}(D^+_{\mathscr{H}(x)}(S(x), r(x)))).$$
The image of $d \otimes 1$ by this morphism is the derivation
$$d_{\mathscr{H}(x)} = \frac{\mathrm{d}}{\mathrm{dS}} : \mathcal{O}(D^+_{\mathscr{H}(x)}(S(x), r(x))) \to \mathcal{O}(D^+_{\mathscr{H}(x)}(S(x), r(x))).$$
From Proposition 4.1.1 we have $\Sigma_{d_{\mathscr{H}(x)}} = D^+_{\mathscr{H}(x)}(0, \frac{\omega}{r(x)})$, then $\pi_{\mathscr{H}(x)/k}(\Sigma_{d_{\mathscr{H}(x)}}) = D^+(0, \frac{\omega}{r(x)})$. By Lemma 4.0.5 we have $D^+(0, \frac{\omega}{r(x)}) \subset \Sigma_d$. Since $\|d\|_{\mathrm{Sp}} = \frac{\omega}{r(x)}$ (cf. Lemma 4.0.3), we obtain $\Sigma_d = D^+(0, \frac{\omega}{r(x)})$ (cf. Theorem 2.1.6).

□

4.1.2 The case of residue characteristic zero

In this section we assume that $\mathrm{char}(\tilde{k}) = 0$.

Proposition 4.1.5. *The spectrum of $\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathcal{O}(D^+(c, r)))$ is:*
$$\Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}(\mathcal{L}_k(\mathcal{O}(D^+(c, r)))) = \overline{D^-(0, \frac{\omega}{r})} = D^-(0, \frac{\omega}{r}) \cup \{x_{0,\frac{\omega}{r}}\}$$
(the topological closure of $D^-(0, \frac{1}{r})$).

Proof. Recall here that $\omega = 1$. We set $A = \mathcal{O}(D^+(c, r))$ and $d = \frac{\mathrm{d}}{\mathrm{dS}}$. The spectral norm of d is equal to $\|d\|_{\mathrm{Sp}} = \frac{1}{r}$ (cf. Lemma 4.0.3), which implies that $\Sigma_d \subset D^+(0, \frac{1}{r})$ (cf. Theorem 2.1.6). Let $x \in D^-(0, \frac{1}{r})$. We set $A_{\mathscr{H}(x)} = A\hat{\otimes}_k\mathscr{H}(x) = \mathcal{O}(D^+_{\mathscr{H}(x)}(c, r))$ and $d_{\mathscr{H}(x)} = \frac{\mathrm{d}}{\mathrm{dS}} : A_{\mathscr{H}(x)} \to A_{\mathscr{H}(x)}$. From Lemma 1.1.40 we have the bounded morphism:
$$\mathcal{L}_k(A)\hat{\otimes}_k\mathscr{H}(x) \to \mathcal{L}_{\mathscr{H}(x)}(A_{\mathscr{H}(x)})$$
The derivation $d_{\mathscr{H}(x)}$ is the image of $d \otimes 1$ by this morphism. As $|T(x)(S - c)| < 1$, $f = \exp(T(x)(S - c))$ exists and it is an element of $A_{\mathscr{H}(x)}$. As $f \in \mathrm{Ker}(d_{\mathscr{H}(x)} - T(x))$, $d_{\mathscr{H}(x)} - T(x)$ is not invertible. Therefore $d \otimes 1 - 1 \otimes T(x)$ is not invertible which is equivalent to saying that $x \in \Sigma_d$. By compactness of the spectrum we have $\overline{D^-(0, \frac{1}{r})} \subset \Sigma_d$. In order to conclude the proof, we need to first prove the statement for the case where k is trivially valued.

- **Trivially valued case:** We need to distinguish two cases:
  - $\mathbf{r \neq 1}$: In this case we have $D^+(0, \frac{1}{r}) = \overline{D^-(0, \frac{1}{r})}$, hence $\Sigma_d = \overline{D^-(0, \frac{1}{r})}$.
  - $\mathbf{r = 1}$: In this case we have $\mathcal{O}(D^+(c,1)) = k[S-c]$ equipped with the trivial valuation, and $\mathcal{L}_k(k[S-c])$ is the k-algebra of all k-linear maps equipped with the trivial norm (i.e. $\|\varphi\| = 1$ for all $\varphi \in \mathcal{L}_k(k[S-c]) \setminus \{0\}$). Let $a \in k \setminus \{0\}$. Since the power series $\exp(a(S-c)) = \sum_{n\in\mathbb{N}} \frac{a^n}{n!}(S-c)^n$ does not converge in $\mathcal{O}(D^+(c,1))$, the operator $d - a : k[S-c] \to k[S-c]$ is injective. It is also surjective. Indeed, let $g(S) = \sum_{n=0}^m b_n(S-c)^n \in \mathcal{O}(D^+(c,1))$. The polynomial $f(S) = \sum_{n=0}^m a_n(S-c)^n \in \mathcal{O}(D^+(c,1))$ whose coefficients are given by the formula
  $$a_n = \frac{-a^{n-1}}{n!} \sum_{i=n}^{m} i! b_i a^{-i},$$
  for all $0 \leq n \leq m$, satisfies $(d-a)f = g$. Hence, $d - a$ is invertible in $\mathcal{L}_k(k[S-c])$. Since the norm is trivial on $\mathcal{L}_k(k[S-c])$, we have $\|(d-a)^{-1}\|_{\mathrm{Sp}}^{-1} = 1$. Therefore, by Lemma 2.1.9 , for all $x \in D^-(a,1)$ the element $d \otimes 1 - 1 \otimes T(x)$ is invertible. Consequently, for all $a \in k \setminus \{0\}$ the disk $D^-(a,1)$ is not meeting the spectrum Σ_d. This means that Σ_d is contained in $D^+(0,1) \setminus \bigcup_{a\in k\setminus\{0\}} D^-(a,1) = [0, x_{0,1}]$. Since $[0, x_{0,1}] = \overline{D^-(0,1)}$ we have $\Sigma_d = \overline{D^-(0,1)}$.

- **Non-trivially valued case:** We need to distinguish two cases:
  - $\mathbf{r \notin |k^*|}$: In this case we have $D^+(0, \frac{1}{r}) = \overline{D^-(0, \frac{1}{r})}$, hence $\Sigma_d = \overline{D^-(0, \frac{1}{r})}$.
  - $\mathbf{r \in |k^*|}$: We can reduce our case to $r = 1$. Indeed, since k is algebraically closed, there exists an isomorphism of k-Banach algebras
  $$\mathcal{O}(D^+(c,r)) \to \mathcal{O}(D^+(c,1)),$$
  that associates to $S - c$ the element $\alpha(S-c)$, with $\alpha \in k$ and $|\alpha| = r$. This induces an isomorphism of k-Banach algebras
  $$\mathcal{L}_k(\mathcal{O}(D^+(c,r))) \to \mathcal{L}_k(\mathcal{O}(D^+(c,1))),$$
  which associates to $d : \mathcal{O}(D^+(c,r)) \to \mathcal{O}(D^+(c,r))$ the derivation $\frac{1}{\alpha} \cdot \frac{\mathrm{d}}{\mathrm{dS}} : \mathcal{O}(D^+(c,1)) \to \mathcal{O}(D^+(c,1))$. By Lemmas 2.1.18 and 2.1.16 we obtain
  $$\Sigma_d(\mathcal{L}_k(\mathcal{O}(D^+(c,r)))) = \frac{1}{\alpha} \Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}(\mathcal{L}_k(\mathcal{O}(D^+(c,1)))).$$
  We now suppose that $r = 1$. Let k' be a maximal (for the partial order given by the inclusion) trivially valued field included in k (which exists by Zorn's Lemma). As k is algebraically closed then so is k'. The complete residue field of $x_{0,1} \in \mathbb{A}^{1,\mathrm{an}}_{k'}$ is $\mathscr{H}(x_{0,1}) = k'(S)$ endowed with the trivial valuation, so the maximality of k' implies that $\mathscr{H}(x_{0,1})$ cannot be included in k, therefore $k \notin E(\mathscr{H}(x_{0,1}))$. By Proposition 1.2.39, we obtain $\pi_{k/k'}^{-1}(x_{0,1}) = \{x_{0,1}\}$. We set $d' =$

$\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_{k'}(\mathcal{O}(D^+_{k'}(c,1)))$. We know by [Ber90, Proposition 7.1.6] that the spectrum of $d'\otimes 1$, as an element of $\mathcal{L}_{k'}(\mathcal{O}(D^+_{k'}(c,1)))\hat{\otimes}_{k'}k$, satisfies $\Sigma_{d'\otimes 1}=\pi^{-1}_{k/k'}(\Sigma_{d'})$, by the previous result we have

$$\pi^{-1}_{k/k'}(\Sigma_{d'})=D^-(0,1)\cup\pi^{-1}_{k/k'}(x_{0,1})=D^-(0,1)\cup\{x_{0,r}\}.$$

From Lemma 1.1.40 we have a bounded morphism $\mathcal{L}_{k'}(\mathcal{O}(D^+_{k'}(c,1)))\hat{\otimes}_{k'}k\to\mathcal{L}_k(\mathcal{O}(D^+(c,1)))$, the image of $d'\otimes 1$ by this morphism is d. Therefore, $\Sigma_d\subset\pi^{-1}_{k/k'}(\Sigma_{d'})=\overline{D^-(0,1)}$. Then we obtain the result.

□

Proposition 4.1.6. *Let $X=D^+(c_0,r_0)\setminus\bigcup_{i=1}^{\mu}D^-(c_i,r_i)$ be a connected affinoid domain of $\mathbb{A}^{1,\mathrm{an}}_k$ with $\mu\geq 1$. The spectrum of $\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathcal{O}(X))$ is:*

$$\Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}(\mathcal{L}_k(\mathcal{O}(X)))=D^+(0,\frac{1}{\min\limits_{0\leq i\leq\mu}r_i}).$$

Proof. The proof is similar to that of the case $\mu\geq 1$ of Proposition 4.1.1. □

Corollary 4.1.7. *Let X be an affinoid domain of $\mathbb{A}^{1,\mathrm{an}}_k$ which does not contain a closed disk as a connected component. The spectrum of $\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathcal{O}(X))$ is:*

$$\Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}(\mathcal{L}_k(\mathcal{O}(X)))=D^+(0,\|\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}}).$$

Remark 4.1.8. Let X be an affinoid domain of $\mathbb{A}^{1,\mathrm{an}}_k$. Then $X=YD$, where Y is an affinoid domain as in the corollary above and D is a disjoint union of disks. We set $d_Y=\frac{\mathrm{d}}{\mathrm{dS}}_{|\mathcal{O}(Y)}$ and $d_D=\frac{\mathrm{d}}{\mathrm{dS}}_{|\mathcal{O}(D)}$. If $\|d_Y\|_{\mathrm{Sp}}\geq\|d_D\|_{\mathrm{Sp}}$ then $\Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}=D^+(0,\|d_Y\|_{\mathrm{Sp}})$. Otherwise, $\Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}=\overline{D^-(0,\|d_D\|_{\mathrm{Sp}})}$.

Proposition 4.1.9. *Let $x\in\mathbb{A}^{1,\mathrm{an}}_k$ be a point of type (2), (3) . The spectrum of $\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathscr{H}(x))$ is:*

$$\Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}(\mathcal{L}_k(\mathscr{H}(x)))=D^+(0,\frac{1}{r(x)}),$$

where $r(x)$ is the value defined in Definition 1.2.27.

Proof. We set $d=\frac{\mathrm{d}}{\mathrm{dS}}$. We distinguish two cases:

- x **is point of type (2):** Let $c\in k$ such that $x=x_{c,r(x)}$. By Proposition 1.2.32 we have
$$\mathscr{H}(x)=F\oplus\mathcal{O}(C^+(c,r(x),r(x)))$$

where,

$$F := \widehat{\bigoplus}_{\tilde{\alpha}\in\tilde{k}\setminus\{0\}}\{\sum_{i\in\mathbb{N}^*}\frac{a_{\alpha i}}{(T-c+\gamma\alpha)^i}|\ a_{\alpha i}\in k,\ \lim_{i\to+\infty}|a_{\alpha i}|r^{-i}=0\},$$

where α is an element of k that corresponds to the class $\tilde{\alpha}$. We use the same arguments as in Proposition 4.1.4.

- **x is point of type (3):** Let $c\in k$ such that $x=x_{c,r(x)}$. In this case $\mathscr{H}(x)=\mathcal{O}(C^+(c,r(x),r(x)))$, by Proposition 4.1.6 we conclude.

□

Proposition 4.1.10. *Let $x\in\mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (4). The spectrum of $\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathscr{H}(x))$ is:*

$$\Sigma_{\frac{\mathrm{d}}{\mathrm{dS}}}(\mathcal{L}_k(\mathscr{H}(x)))=\overline{D^-(0,\frac{1}{r(x)})},$$

where $r(x)$ is the value defined in Definition 1.2.27

Proof. We set $d=\frac{\mathrm{d}}{\mathrm{dS}}$. By Proposition 1.2.34 we have

$$\mathscr{H}(x)\hat{\otimes}_k\mathscr{H}(x)\simeq\mathcal{O}(D^+_{\mathscr{H}(x)}(S(x),r(x))).$$

From Lemma 1.1.40 we have the bounded morphism:

$$\mathcal{L}_k(\mathscr{H}(x))\hat{\otimes}_k\mathscr{H}(x)\to\mathcal{L}_{\mathscr{H}(x)}(\mathcal{O}(D^+_{\mathscr{H}(x)}(S(x),r(x))))$$

which associates to $d\otimes 1$ the derivation

$$d_{\mathscr{H}(x)}=\frac{\mathrm{d}}{\mathrm{dS}}:\mathcal{O}(D^+_{\mathscr{H}(x)}(S(x),r(x)))\to\mathcal{O}(D^+_{\mathscr{H}(x)}(S(x),r(x))).$$

From Proposition 4.1.5 we have $\Sigma_{d_{\mathscr{H}(x)}}=\overline{D^-_{\mathscr{H}(x)}(0,\frac{1}{r(x)})}$, hence

$$\pi_{\mathscr{H}(x)/k}(\Sigma_{d_{\mathscr{H}(x)}})=\overline{D^-(0,\frac{1}{r(x)})}.$$

By Lemma 4.0.5 we have $\overline{D^-(0,\frac{1}{r(x)})}\subset\Sigma_d$. From now on we set $r=r(x)$. Since $\|d\|_{\mathrm{Sp}}=\frac{1}{r}$ (cf. Lemma 4.0.4) and $\Sigma_d\subset D^+(0,\|d\|_{\mathrm{Sp}})$ (cf. Theorem 2.1.6), in order to prove the statement it is enough to show that for all $a\in k$ such that $|a|=\frac{1}{r}$, we have $D^-(a,\frac{1}{r})\subset\mathbb{A}_k^{1,\mathrm{an}}\setminus\Sigma_d$. Let $a\in k$ such that $|a|=\frac{1}{r}$. The restriction of $d-a$ to the normed k-algebra $k[S]$ is a bijective bounded map $d-a:k[S]\to k[S]$ with respect to the restriction of $|.|_x$. We set $\varphi=(d-a)_{|_{k[S]}}$. As $\mathscr{H}(x)$ is the completion of $k[S]$ with respect to $|.|_x$ (cf. Lemma 1.2.33). It suffices to prove that $\varphi^{-1}:k[S]\to k[S]$ is a bounded k-linear map. Indeed, by density of $k[S]$ inside $\mathscr{H}(x)$, this will ensure that φ^{-1} extends to a bounded endomorphism $\psi : \mathscr{H}(x) \to \mathscr{H}(x)$, and we will then necessarily have $\psi \circ (d-a) = (d-a)\circ\psi = \mathrm{Id}$, because this holds on the dense subset $k[S]$ of $\mathscr{H}(x)$. A family of closed disks $\{D^+(c_\ell, r_\ell)\}_{\ell\in I}$ is called nested if the set of index I is endowed with total order $\leq$ and for $i \leq j$ we have $D^+(c_i, r_i) \subset D^+(c_j, r_j)$. Since x is a point of type (4), then there exists a family of nested disks $\{D^+(c_\ell, r_\ell)\}_{\ell\in I}$ such that $\bigcap_{\ell\in I} D^+(c_\ell, r_\ell) = \{x\}$ and $r_l > r$ for all $\ell \in I$. If we consider $d-a$ as an element of $\mathcal{L}_k(\mathcal{O}(D^+(c_\ell, r_\ell)))$, then it is invertible (cf. Proposition 4.1.5) and its restriction to $k[S]$ coincides with φ as k-linear map. Let $f(S) = \sum_{i\in\mathbb{N}} a_i(S-c_\ell)^i$ and $g(S) = \sum_{i\in\mathbb{N}} b_i(S-c_\ell)^i$ be two elements of $\mathcal{O}(D^+(c_\ell, r_\ell))$ such that $(d-a)f = g$. Using the same induction to obtain the equation (6.6) we obtain: for all $n \in \mathbb{N}$

$$a_n = \frac{-a^{n-1}}{n!}\sum_{i\geq n} i!b_i a^{-i}. \tag{4.3}$$

Hence,

$$|a_n| = \frac{1}{r^{n-1}}|\sum_{i\geq n} i!b_i a^{-i}| \leq \frac{1}{r^{n-1}}\max_{i\geq n}|b_i|r^i \leq r\max_{i\geq n}|b_i|r^{i-n} \leq r\max_{i\geq n}|b_i|r_\ell^{i-n} \leq \frac{r}{r_\ell^n}\max_{i\geq n}|b_i|r_\ell^i \tag{4.4}$$

Therefore,

$$|a_n|r_\ell^n \leq r\max_{i\geq n}|b_i|r_\ell^i. \tag{4.5}$$

Consequently,

$$|f|_{x_{c_\ell,r_\ell}} \leq r|g|_{x_{c_\ell,r_\ell}}$$

In the special case where f and g are in $k[S]$, then $f = \varphi^{-1}(g)$ and we have for all $l \in \mathbb{N}$:

$$|\varphi^{-1}(g)|_{x_{c_\ell,r_\ell}} \leq r|g|_{x_{c_\ell,r_\ell}}.$$

Hence,

$$|\varphi^{-1}(g)|_x = \inf_{l\in\mathbb{N}}|\varphi^{-1}(g)|_{x_{c_\ell,r_\ell}} \leq \inf_{l\in\mathbb{N}} r|g|_{x_{c_\ell,r_\ell}} = r|g|_x.$$

This means that φ^{-1} is bounded, hence $d-a$ is invertible in $\mathcal{L}_k(\mathscr{H}(x))$ and $\|(d-a)^{-1}\| \leq r$, hence $\|(d-a)^{-1}\|_{\mathrm{Sp}} \leq r$. Since $\|(d-a)^{-1}\|_{\mathrm{Sp}}^{-1}$ is the radius of the biggest disk centred in a contained in $\mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_d$ (cf. Lemma 2.1.9), we obtain $D^-(a, \frac{1}{r}) \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_d$.

□

4.2 Spectrum of a linear differential equation with constant coefficients

Let X be an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$ and $x \in X$ a point of type (2), (3) or (4). We set here $A = \mathcal{O}(X)$ or $\mathscr{H}(x)$ and $d = \frac{\mathrm{d}}{\mathrm{dS}}$. Recall that a linear differential equation with constant coefficients is a differential module (M, ∇) over (A, d) associated to a differential polynomial $P(D) = g_0 + g_1 D + \cdots + g_{\nu-1}D^{\nu-1} + D^\nu$ with $g_i \in k$, or in an equivalent way there exists a basis for which the matrix G of the formula (3.1) has constant coefficients (i.e $G \in \mathcal{M}_\nu(k)$). Here we compute the spectrum of ∇ as an element of $\mathcal{L}_k(M)$ (cf. Section 3.2).

Theorem 4.2.1. *Let X be a connected affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$. We set here $A = \mathcal{O}(X)$. Let (M, ∇) be a defferential module over (A, d) such that the matrix G of the formula (3.1) has constant entries (i.e. $G \in \mathcal{M}_\nu(k)$), and let $\{a_1, \cdots, a_N\}$ be the set of eigenvalues of G. Then we have:*

- *If $X = D^+(c_0, r_0)$,*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \begin{cases} \bigcup_{i=1}^{N} D^+(a_i, \frac{\omega}{r_0}) & \text{If } \mathrm{char}(\tilde{k}) > 0 \\ \\ \bigcup_{i=1}^{N} \overline{D^-(a_i, \frac{1}{r_0})} & \text{If } \mathrm{char}(\tilde{k}) = 0 \end{cases}.$$

- *If $X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$ with $\mu \geq 1$,*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \bigcup_{i=1}^{N} D^+(a_i, \frac{\omega}{\min\limits_{0\leq i\leq \mu} r_i}).$$

Where ω is the positive real number introduced in (4.1).

Proof. By Propositions 4.1.1, 4.1.1, 4.1.5, 4.1.6 and 3.2.17 we obtain the result. □

Theorem 4.2.2. *Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2), (3) or (4). We set here $A = \mathscr{H}(x)$. Let (M, ∇) be a defferential module over (A, d) such taht the matrix G of the formula (3.1) has constant entries (i.e. $G \in \mathcal{M}_\nu(k)$), and let $\{a_1, \cdots, a_N\}$ be the set of eigenvalues of G. Then we have:*

- *If x is a point of type (2) or (3),*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \bigcup_{i=1}^{N} D^+(a_i, \frac{\omega}{r(x)}).$$

- *If x is a point of type (4),*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \begin{cases} \bigcup_{i=1}^{N} D^+(a_i, \frac{\omega}{r(x)}) & \text{If } \mathrm{char}(\tilde{k}) > 0 \\ \\ \bigcup_{i=1}^{N} \overline{D^-(a_i, \frac{1}{r(x)})} & \text{If } \mathrm{char}(\tilde{k}) = 0 \end{cases},$$

where ω is the positive real number introduced in (4.1).

Proof. By Propositions 4.1.4, 4.1.9, 4.1.10 and 3.2.17 we obtain the result. □

Remark 4.2.3. Notice that since the spectrum of ∇ is independant of the choice of the basis, if G' is another associated matrix to the differential module (M,∇) with constant entries, then the set of eigenvalues $\{a'_1,\cdots,a'_{N'}\}$ of G' can not be arbitrary, namely it must satisfy: for each a'_i there exists a_j such that a'_i belongs to the connected componente of the spectrum containing a_j.

Remark 4.2.4. If we consider the differential polynomial $P(d)$ as an element of $\mathcal{L}_k(A)$, then its spectrum is $\Sigma_{P(d)}=P(\Sigma_d)$ (cf. Lemma 2.1.16) which is in general diffrent from the spectrum of the associated connexion.

4.3 Variation of the spectrum

In this section, we will discuss the behaviour of the spectrum of (M,∇) over $(\mathscr{H}(x),d)$, when we make x vary inside $[x_1,x_2]\subset\mathbb{A}_k^{1,\mathrm{an}}$, where x_1 and x_2 are points of type (2), (3) or (4).

Let X be an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$. Let (M,∇) be a differential module over $(\mathcal{O}(X),\frac{\mathrm{d}}{\mathrm{dS}})$ such that there exists a basis for which the associated matrix G has constant entries. For a point $x\in X$ not of type (1), the differential module (M,∇) extends to a differential module (M_x,∇_x) over $(\mathscr{H}(x),\frac{\mathrm{d}}{\mathrm{dS}})$. In the corresponding basis of (M_x,∇_x) the associated matrix is G.

Theorem 4.3.1. *Let $X=D^+(c_0,r_0)\setminus\bigcup_{i=1}^{\mu}D^-(c_i,r_i)$ be a connected affinoid domain and $x\in X$ be a point of type (2), (3) or (4). Let (M,∇) be a differential module over $(\mathcal{O}(X),\frac{\mathrm{d}}{\mathrm{dS}})$ such that there exists a basis for which the corresponding matrix G has constant entries. We set:*

$$\begin{array}{rcl}\Psi:[x,x_{c_0,r_0}] & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}})\\ y & \mapsto & \Sigma_{\nabla_y}(\mathcal{L}_k(M_y))\end{array}.$$

Then we have:

1. *for each $y\in[x,x_{c_0,r_0}]$, the restriction of Ψ to $[x,y]$ is continuous at y.*
2. *for $y\in(x,x_{c_0,r_0}]$, Ψ is continuous at y if and only if y is a point of type (3), .*
3. *If $\mathrm{char}(\tilde{k})=0$ and x is a point of type (4), then Ψ is continuous at x.*

Proof. Let $\{a_1,\cdots,a_N\}\subset k$ be the set of eigenvalues of G. We identifie $[x,x_{c_0,r_0}]$ with the interval $[r(x),r_0]$ by the map $y\mapsto r(y)$ (cf. Definition 1.2.27). Let $y\in[x,x_{c_0,r_0}]$. We set $\Sigma_y=\Sigma_{\nabla_y}(\mathcal{L}_k(M_y))$. By Theorem 4.2.2, for all $y'\in(x,x_{c_0,r_0}]$ (y' can not be a point of type (4)) we have $\Sigma_{y'}=\bigcup_{i=1}^N D^+(a_i,\frac{\omega}{r(y')})$.

- For the claims 1. and 2. it follows from Corollary 2.3.12.
- We assume that $\mathrm{char}(\tilde{k}) = 0$. Let $y \in [x, x_{c_0,r_0}]$ be a point of type (4). Since $\Sigma_y = \bigcup_{i=1}^{N}(D^-(a_i, \frac{\omega}{r(y)}) \cup \{x_{a_i,\frac{\omega}{r(y)}}\})$ (cf. Theorem 4.2.2), using Lemma 2.3.3 the proof is similar to the second case of Proposition 2.3.11.

$\square$

V

Spectrum of a regular singular differential module

Contents

This chapter is devoted to the computation of the spectrum of a regular singular module (M,∇) over $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$, with $x\in\mathbb{A}_k^{1,\mathrm{an}}$ a point of type (2), (3) or (4). We will proceed as follows. Firstly we compute the spectrum of the derivation $S\frac{\mathrm{d}}{\mathrm{dS}}$ by using the push-forward functor (cf. (3.7)). We treat the case of positive residual characteristic separately. As in the previous chapters, we have different spectra in the positive and zero residual characteristics cases. In the end, we will use Proposition 3.2.15 in order to prove the main result, which is the computation of the spectrum of a regular singular differential module. We will discuss at the end the variation of the spectrum.

Convention 5.0.1. In this chapter, we assume that k is algebraically closed.

Definition 5.0.2. A differential module (M,∇) over $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$ is said to be regular singular, if there exists a basis for which the associated matrix G (cf. (3.1)) has constant entries (i.e $G\in\mathcal{M}_n(k)$).

5.1 Spectrum of the derivation $S\frac{\mathrm{d}}{\mathrm{dS}}$

Lemma 5.1.1. *Let $x := x_{0,r}$, with $r > 0$. The norm and spectral semi-norm of $S\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathscr{H}(x))$ satisfy:*

$$\|S\frac{\mathrm{d}}{\mathrm{dS}}\| = 1, \quad \|S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}} = 1.$$

Proof. Since $\|S\| = |S| = r$ and $\|\frac{\mathrm{d}}{\mathrm{dS}}\| = \frac{1}{r}$ (cf. Lemma 4.0.3), we have $\|S\frac{\mathrm{d}}{\mathrm{dS}}\| \leq 1$. Hence also, $\|S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}} \leq 1$. The map

$$\begin{array}{ccc} \mathcal{L}_k(\mathscr{H}(x)) & \longrightarrow & \mathcal{L}_k(\mathscr{H}(x)) \\ \varphi & \mapsto & S^{-1}\circ\varphi\circ S \end{array}$$

is bi-bounded and induces change of basis. Therefore, as $S^{-1}\circ(S\frac{\mathrm{d}}{\mathrm{dS}})\circ S = S\frac{\mathrm{d}}{\mathrm{dS}} + 1$, we have $\|S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}} = \|S\frac{\mathrm{d}}{\mathrm{dS}} + 1\|_{\mathrm{Sp}}$. Since 1 commutes with $S\frac{\mathrm{d}}{\mathrm{dS}}$, we have:

$$1 = \|1\|_{\mathrm{Sp}} = \|S\frac{\mathrm{d}}{\mathrm{dS}} + 1 - S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}} \leq \max(\|S\frac{\mathrm{d}}{\mathrm{dS}} + 1\|_{\mathrm{Sp}}, \|S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}})$$

(cf. Corollary 1.1.29). Consequently, we obtain

$$\|S\frac{\mathrm{d}}{\mathrm{dS}}\| = \|S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}} = 1.$$

□

5.1.1 The case of positive residual characteristic

We assume here that $\mathrm{char}(\tilde{k}) = p > 0$. We start with the case where $x = x_{0,r}$. In order to compute the spectrum of $S\frac{\mathrm{d}}{\mathrm{dS}}$, we will use the push-forward by the Frobenius map. We refer the reader to Section A.2.1 for the definition of Frobenius map $\mathbf{Frob}_p : \mathbb{A}_k^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ and its properties. Since it induces an étale map $(\mathbf{Frob}_p)^n : \mathbb{A}_k^{1,\mathrm{an}}\backslash\{0\} \to \mathbb{A}_k^{1,\mathrm{an}}\backslash\{0\}$, the push-forward functor (cf. (3.7)) is well defined for any $x_{0,r} \in \mathbb{A}_k^{1,\mathrm{an}}$ with $r > 0$. Recall that $(\mathbf{Frob}_p)^n(x_{0,r}) = x_{0,r^{p^n}}$ and $[\mathscr{H}(x_{0,r}) : \mathscr{H}(x_{0,r^{p^n}})] = p^n$ (cf. Properties A.2.1).

Let $x := x_{0,r}$ and $y := (\mathbf{Frob}_p)^n(x)$. According to formula (3.6) the pull-back of the derivation $p^nS\frac{\mathrm{d}}{\mathrm{dS}} : \mathscr{H}(y) \to \mathscr{H}(y)$ is the derivation $S\frac{\mathrm{d}}{\mathrm{dS}} : \mathscr{H}(x) \to \mathscr{H}(x)$. To avoid confusion in the following we set $p^nS(y)\frac{\mathrm{d}}{\mathrm{dS(y)}} := p^nS\frac{\mathrm{d}}{\mathrm{dS}}$ and $S(x)\frac{\mathrm{d}}{\mathrm{dS(x)}} := S\frac{\mathrm{d}}{\mathrm{dS}}$.

Now let (M_{p^n}, ∇_{p^n}) be the push-forward of the differential module $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$ by $(\mathbf{Frob}_p)^n$. Since $M_{p^n} \simeq \mathscr{H}(x)$ as an $\mathscr{H}(y)$-Banach space and according to Lemma 1.2.44, we can take $\{1, S(x), \cdots, S(x)^{p^n-1}\}$ as a basis of (M_{p^n}, ∇_{p^n}). Since

$$\nabla_{p^n}(S(x)^i) = S(x)\frac{\mathrm{d}}{\mathrm{dS(x)}}(S(x)^i) = iS(x)^i,$$

in this basis we have:

$$\nabla_{p^n}\begin{pmatrix} f_1 \\ \vdots \\ f_{p^n}\end{pmatrix} = \begin{pmatrix} p^nS(y)\frac{\mathrm{d}}{\mathrm{dS(y)}}f_1 \\ \vdots \\ p^nS(y)\frac{\mathrm{d}}{\mathrm{dS(y)}}f_{p^n}\end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p^n-1 \end{pmatrix}\begin{pmatrix} f_1 \\ \vdots \\ f_{p^n}\end{pmatrix}. \tag{5.1}$$

In other terms, we have an isomorphism of differential modules

$$(M_{p^n}, \nabla_{p^n}) \simeq \bigoplus_{i=0}^{p^n-1}(\mathscr{H}(y), p^nS(y)\frac{\mathrm{d}}{\mathrm{dS(y)}} + i). \tag{5.2}$$

Notation 5.1.2. For the simplecity, we still denote here $S(x)\frac{\mathrm{d}}{\mathrm{dS(x)}}$ by $S\frac{\mathrm{d}}{\mathrm{dS}}$.

Proposition 5.1.3. *Let* $r > 0$*. We set* $x := x_{0,r}$*. The spectrum of* $S\frac{\mathrm{d}}{\mathrm{dS}}$ *as an element of* $\mathcal{L}_k(\mathscr{H}(x))$ *is*

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = \mathbb{Z}_p.$$

Proof. Since for all $l \in \mathbb{N}$ we have $S\frac{\mathrm{d}}{\mathrm{dS}}(S(x)^l) - l(S(x)^l) = 0$, we obtain

$$\mathbb{N} \subset \Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))).$$

By compactness of the spectrum, we obtain

$$\mathbb{Z}_p \subset \Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))).$$

Let $n \in \mathbb{N}\setminus\{0\}$. We set $y := (\mathbf{Frob}_p)^n(x) = x_{0,p^n}$. Let (M_{p^n}, ∇_{p^n}) be the push-froward of $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$ by $(\mathbf{Frob}_p)^n$. On the one hand, according to Proposition 3.4.3 we have

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = \Sigma_{\nabla_{p^n},k}(\mathcal{L}_k(M_{p^n})).$$

On the other hand, since we have the isomorphism (5.2), we have

$$\Sigma_{\nabla_{p^n},k}(\mathcal{L}_k(M_{p^n})) = \bigcup_{i=0}^{p^n-1}\Sigma_{p^nS(y)\frac{\mathrm{d}}{\mathrm{dS(y)}}+i,k}(\mathcal{L}_k(\mathscr{H}(y))) = \bigcup_{i=0}^{p^n-1}p^n\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(y)))+i$$

(cf. Remark 3.2.7 and Lemma 2.1.16). By Lemma 5.1.1, we know that $\|S\frac{\mathrm{d}}{\mathrm{dS}}\|_{\mathrm{Sp}} = 1$ in $\mathcal{L}_k(\mathscr{H}(y))$. Therefore, we have $\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(y))) \subset D^+(0,1)$ (cf. Theorem 2.1.6). Consequently, $\Sigma_{\nabla_{p^n},k}(\mathcal{L}_k(M_{p^n})) \subset \bigcup_{i=0}^{p^n-1}D^+(i,|p|^n)$. Applying this process for all n, we obtain

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) \subset \bigcap_{n\in\mathbb{N}\setminus\{0\}}\bigcup_{i=0}^{p^n-1}D^+(i,|p|^n) = \mathbb{Z}_p,$$

which ends the proof. □

We now assume that $x \in \mathbb{A}_k^{1,\mathrm{an}}$ is a point of type (2), (3) or (4) not of the form $x_{0,r}$. Then there exists $c \in k\backslash\{0\}$ such that $x \in D^-(c, |c|)$. The logarithm map (cf. Section A.1)

$$\mathrm{Log}_c : D^-(c, |c|) \to \mathbb{A}_k^{1,\mathrm{an}}$$

is well defined and induces an infinite étale cover. We set $y := \mathrm{Log}_c(x)$. Let $r_k : \mathbb{A}_k^{1,\mathrm{an}} \to \mathbb{R}_+$ be the radius map (cf. Definition 1.2.27) and $\omega = |p|^{\frac{1}{p-1}}$. We have:

- if $0 < r_k(x) < |c|\omega$, then $0 < r_k(y) < \omega$ and $[\mathscr{H}(x) : \mathscr{H}(y)] = 1$.
- Let $n \in \mathbb{N} \setminus \{0\}$, if $|c|\omega^{\frac{1}{p^{n-1}}} \leq r_k(x) < |c|\omega^{\frac{1}{p^n}}$, then $\frac{\omega}{|p|^{n-1}} \leq r_k(y) < \frac{\omega}{p^n}$ and $[\mathscr{H}(x) : \mathscr{H}(y)] = p^n$

(cf. Properties A.1.4). Note that, since $|T(x)| = |c|$ the inequalities above do not depend on the choice of c. According to formula (3.6) the pull-back of the derivation $\frac{\mathrm{d}}{\mathrm{dS}} : \mathscr{H}(y) \to \mathscr{H}(y)$ is the derivation $S\frac{\mathrm{d}}{\mathrm{dS}} : \mathscr{H}(x) \to \mathscr{H}(x)$.

Let (M, ∇) be the push-forward of the differential module $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$ by Log_c. Assume that $|c|\omega^{\frac{1}{p^{n-1}}} \leq r_k(x) < |c|\omega^{\frac{1}{p^n}}$. By Lemma 1.2.44, we can take $\{1, S(x), \cdots, S(x)^{p^n-1}\}$ as a basis of (M, ∇). Since

$$\nabla(S(x)^i) = S\frac{\mathrm{d}}{\mathrm{dS}}(S(x)^i) = iS(x)^i,$$

in this basis we have:

$$\nabla\begin{pmatrix} f_1 \\ \vdots \\ f_{p^n} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{dS}} f_1 \\ \vdots \\ \frac{\mathrm{d}}{\mathrm{dS}} f_{p^n} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p^n - 1 \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_{p^n} \end{pmatrix}. \tag{5.3}$$

Proposition 5.1.4. *Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2), (3) or (4) not of the form $x_{0,r}$. Let $c \in k \setminus \{0\}$ such that $x \in D^-(c, |c|)$. We set $y := \mathrm{Log}_c(x)$. In the case where $r_k(x) \leq |c|\omega$, the spectrum of $S\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathscr{H}(x))$ is*

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = D^+(0, \frac{\omega}{r_k(y)}).$$

In the case where $|c|\omega^{\frac{1}{p^{n-1}}} < r_k(x) \leq |c|\omega^{\frac{1}{p^n}}$ with $n \in \mathbb{N}\setminus\{0\}$, the spectrum is a disjoint union of p^n closed disks

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = \bigcup_{i=0}^{p^n-1} D^+(i, \frac{\omega}{r_k(y)}).$$

Proof. Assume that $r_k(x) < |c|\omega$. Since $[\mathscr{H}(x) : \mathscr{H}(y)] = 1$, the push-forward of $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$ by Log_c is isomorphic to $(\mathscr{H}(y), \frac{\mathrm{d}}{\mathrm{dS}})$. Therefore, by Propositions 4.1.4 and 3.4.3 we obtain

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = \Sigma_{\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(y))) = D^+(0, \frac{\omega}{r_k(y)}).$$

We now assume that $|c|\omega^{\frac{1}{p^{n-1}}} \leq r_k(x) < |c|\omega^{\frac{1}{p^n}}$ with $n \in \mathbb{N}\setminus\{0\}$. Let (M, ∇) be the push-forward of $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$ by Log_c. Since we have the formula (5.3) and according to Propositions 3.4.3 and 4.2.2, we have

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = \Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \bigcup_{i=0}^{p^n-1} D^+(i, \frac{\omega}{r_k(y)}).$$

If moreover $|c|\omega^{\frac{1}{p^{n-1}}} < r_k(x)$, then $|p|^n < \frac{\omega}{r_k(y)} < |p|^{n-1}$. Consequently, the spectrum $\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x)))$ is a disjoint union of p^n disks. For the case where $r_k(x) = |c|\omega^{\frac{1}{p^n}}$ with $n \in \mathbb{N}$, we have $r_k(y) = \frac{\omega}{|p|^n}$ (cf. Properties A.1.4). For all $0 \leq i \leq p^n - 1$ and $1 \leq l \leq p-1$, we have $D^+(i, |p|^n) = D^+(i + lp^n, |p|^n)$. Hence, we obtain

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = \bigcup_{i=0}^{p^{n+1}-1} D^+(i, |p|^n) = \bigcup_{i=0}^{p^n-1} D^+(i, |p|^n),$$

which is obviously a disjoint union. □

Corollary 5.1.5. *Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2), (3) or (4) not of the form $x_{0,r}$. Let $c \in k\setminus\{0\}$ such that $x \in D^-(c, |c|)$. We set $y := \mathrm{Log}_c(x)$. The spectrum of $S\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathscr{H}(x))$ is*

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = \bigcup_{i\in\mathbb{N}} D^+(i, \frac{\omega}{r_k(y)}).$$

Proof. By Proposition 5.1.4, we have $\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = \bigcup_{i=0}^{p^n-1} D^+(i, \frac{\omega}{r_k(y)})$ for some $n \in \mathbb{N}$. Since for all $l \in \mathbb{N}$ we have $S\frac{\mathrm{d}}{\mathrm{dS}}(S(x)^l) - l(S(x)^l) = 0$, we obtain

$$\mathbb{N} \subset \Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))).$$

Therefore, for each $l \in \mathbb{N}$ there exists $0 \leq i_l \leq p^n - 1$ such that $D^+(l, \frac{\omega}{r_k(y)}) = D^+(i_l, \frac{\omega}{r_k(y)})$. Consequently, we obtain $\bigcup_{i\in\mathbb{N}} D^+(i, \frac{\omega}{r_k(y)}) \subset \Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x)))$ which ends the proof. □

5.1.2 The case of residue characteristic zero

We assume here that $\mathrm{char}(\tilde{k}) = 0$.

Proposition 5.1.6. *Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2) of the form $x_{0,r}$. The spectrum of $S\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathscr{H}(x))$ is*

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = D^+(0,1).$$

Proof. We set $d := S\frac{\mathrm{d}}{\mathrm{dS}}$ and $\Sigma_d = \Sigma_{d,k}(\mathcal{L}_k(\mathscr{H}(x)))$. Since $\|d\|_{\mathrm{Sp}} = 1$, we have $\Sigma_d \subset D^+(0,1)$. Let $y \in D^+(0,1)$. We set $A_{\mathscr{H}(y)} = \mathscr{H}(x)\hat{\otimes}_k\mathscr{H}(y)$ and $d_{\mathscr{H}(y)} = S\frac{\mathrm{d}}{\mathrm{dS}} : A_{\mathscr{H}(y)} \to A_{\mathscr{H}(y)}$. From Lemma 1.1.40 we have a bounded morphism:

$$\mathcal{L}_k(\mathscr{H}(x))\hat{\otimes}_k\mathscr{H}(y) \to \mathcal{L}_{\mathscr{H}(y)}(A_{\mathscr{H}(y)}).$$

The image of $d \otimes 1$ by this morphism is the derivation $d_{\mathscr{H}(y)}$. We now show that the image of $d \otimes 1 - 1 \otimes T(y)$ is not invertible in $\mathcal{L}_{\mathscr{H}(y)}(A_{\mathscr{H}(y)})$. Let α an element of k that corresponds to the class $\tilde{\alpha}$ in $\tilde{k}$. We have the following decomposition

$$\mathscr{H}(x) = \widehat{\bigoplus}_{\tilde{\alpha}\in\tilde{k}}\{\sum_{i\in\mathbb{N}^*}\frac{a_{\alpha i}}{(S+\gamma\alpha)^i}|\ a_{\alpha i} \in k,\ \lim_{i\to+\infty}|a_{\alpha i}|r^{-i} = 0\} \oplus \mathcal{O}(D^+(0,r))$$

with $\gamma \in k$ and $|\gamma| = r$ (cf. Proposition 1.2.32). Therefore, we obtain the isometric isomorphism

$$\mathscr{H}(x)\hat{\otimes}_k\mathscr{H}(y) \simeq \widehat{\bigoplus}_{\tilde{\alpha}\in\tilde{k}}\{\sum_{i\in\mathbb{N}^*}\frac{a_{\alpha i}}{(S+\gamma\alpha)^i}|\ a_{\alpha i} \in \mathscr{H}(y),\ \lim_{i\to+\infty}|a_{\alpha i}|r^{-i} = 0\}\oplus\mathcal{O}(D^+_{\mathscr{H}(y)}(0,r)).$$

Each Banach space of the completed direct sum is stable under $d_{\mathscr{H}(y)} - T(y)$. The operator $d_{\mathscr{H}(y)} - T(y)$ is not surjective. Indeed, let $c := \gamma\alpha$ with $\tilde{\alpha} \in \tilde{k}\setminus\{0\}$ and let $g = \frac{c}{S-c}$. If there exists $f \in \mathscr{H}(x)\hat{\otimes}_k\mathscr{H}(y)$ such that $(d_{\mathscr{H}(y)} - T(y))(f) = g$, then we can choose f of the form $f = \sum_{i\in\mathbb{N}\setminus\{0\}}\frac{a_i}{(S-c)^i}$, such that for each $i \in \mathbb{N}\setminus\{0\}$ we have

$$a_i = -\frac{(-c)^i(i-1)!}{\prod_{j=1}^i(T(y)+j)}.$$

We observe that $|a_i|r^{-i} \geq 1$ for each $i \in \mathbb{N}\setminus\{0\}$. This means that such f does not exist in $\mathscr{H}(x)\hat{\otimes}_k\mathscr{H}(y)$. Hence, $d\otimes 1 - 1\otimes T(y)$ is not invertible in $\mathcal{L}_k(\mathscr{H}(x))$ and we conclude that $D^+(0,1) \subset \Sigma_d$. □

Proposition 5.1.7. *Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (3) of the form $x_{0,r}$. The spectrum of $S\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathscr{H}(x))$ is*

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = \mathbb{Z}\cup\{x_{0,1}\}.$$

Proof. We set $d := S\frac{\mathrm{d}}{\mathrm{dS}}$ and $\Sigma_{d-n} := \Sigma_{d-n,k}(\mathcal{L}_k(\mathscr{H}(x)))$. As $\|d\|_{\mathrm{Sp}} = 1$ (cf. Lemma 5.1.1), we have $\Sigma_d \subset D^+(0,1)$. Recall that

$$\mathscr{H}(x) = \mathcal{O}(C^+(0,r,r)) = \{\sum_{i\in\mathbb{Z}}a_iS^i|\ \lim_{|i|\to\infty}|a_i|r^i = 0\}.$$

Let $a \in k \cap D^+(0,1)$. If $a \in \mathbb{Z}$, then we have $(d-a)(S^a) = 0$. Hence, $d-a$ is not injective and $\mathbb{Z} \subset \Sigma_d$. As the spectrum is compact, we have $\mathbb{Z} \cup \{x_{0,1}\} \subset \Sigma_d$. If $a \notin \mathbb{Z}$, then $d-a$ is invertible in $\mathcal{L}_k(\mathscr{H}(x))$. Indeed, let $g(S) = \sum_{i\in\mathbb{Z}} b_i S^i \in \mathscr{H}(x)$, if there exists $f = \sum_{i\in\mathbb{Z}} a_i S^i \in \mathscr{H}(x)$ such that $(d-a)f = g$, then for each $i \in \mathbb{Z}$ we have

$$a_i = \frac{b_i}{(i-a)}.$$

If there exists $i_0 \in \mathbb{Z}$ such that $a \in D^-(i_0,1)$, then for each $i \neq i_0$ we have $|a_i| = |b_i|$ and $|a_{i_0}| = \frac{|b_{i_0}|}{|i_0-a|}$. Otherwise, for each $i \in \mathbb{Z}$ we have $|a_i| = |b_i|$. This means that f it is unique and converges in $\mathscr{H}(x)$. We obtain also $|f| \leq \frac{|g|}{|i_0-a|}$ or $|f| = |g|$. Consequently, the set theoretical inverse $(d-a)^{-1}$ is bounded. We claim that if $a \in D^-(i_0,1)$ then $\|(d-a)^{-1}\|_{\mathrm{Sp}} = \frac{1}{|i_0-a|}$, otherwise $\|(d-a)^{-1}\|_{\mathrm{Sp}} = 1$. Indeed, in the first case, similar computations show that $\|(d-a)^{-n}\| \leq \frac{1}{|i_0-a|^n}$. Since $(d-a)^{-n}(S^{i_0}) = \frac{S^{i_0}}{(i_0-a)^n}$, the equality holds and we obtain $\|(d-a)^{-1}\|_{\mathrm{Sp}} = \frac{1}{(i_0-a)}$. In the second case, by the above computations $(d-a)^{-1}$ is an isometry. Therefore, we have $\|(d-a)^{-1}\|_{\mathrm{Sp}} = 1$. Setting $R_a := \inf_{i\in\mathbb{Z}} |i-a|$, we have $\|(d-a)^{-1}\|_{\mathrm{Sp}} = \frac{1}{R_a}$. According to Lemma 2.1.9, we have $D^-(a,R_a) \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_d$.

In order to end the proof, since $D^+(0,1) = \bigcup_{a\in k\setminus\mathbb{Z}} D^-(a,R_a) \cup \bigcup_{n\in\mathbb{Z}} [n, x_{0,1}]$, it is enough to show that $(n, x_{0,1}) \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_d$ for all $n \in \mathbb{Z}$. Let $n \in \mathbb{Z}$. Then we have an equality of spaces in $\mathbf{Ban}_k^{\leq 1}$ (cf. Notations 1.1.34 and Lemma 1.1.21)

$$\mathscr{H}(x) = k.S^n \oplus \widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i.$$

The operator $(d-n)$ stabilises both $k.S^n$ and $\widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i$. We set $(d-n)|_{k.S^n} = \nabla_1$ and $(d-n)|_{\widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i} = \nabla_2$. We set $\Sigma_{\nabla_1} := \Sigma_{\nabla_1,k}(\mathcal{L}_k(k.S^n))$ and $\Sigma_{\nabla_2} := \Sigma_{\nabla_2,k}(\mathcal{L}_k(\widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i))$. We have $\nabla_1 = 0$. By Lemma 2.1.20, we have:

$$\Sigma_{d-n} = \Sigma_{\nabla_1} \cup \Sigma_{\nabla_2} = \{0\} \cup \Sigma_{\nabla_2}.$$

We now prove that

$$D^-(0,1) \cap \Sigma_{\nabla_2} = \varnothing.$$

The operator ∇_2 is invertible in $\mathcal{L}_k(\widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i)$. Indeed, let $g(S) = \sum_{i\in\mathbb{Z}\setminus\{n\}} b_i S^i \in \widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i$. If there exists $f = \sum_{i\in\mathbb{Z}\setminus\{n\}} a_i S^i \in \widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i$ such that $\nabla_2(f) = g$, then for each $i \in \mathbb{Z} \setminus \{n\}$ we have

$$a_i = \frac{b_i}{(i-n)}.$$

Since $|a_i| = |b_i|$, the element f exists and it is unique, moreover $|f| = |g|$. Hence, ∇_2 is invertible in $\mathcal{L}_k(\widehat{\bigoplus}_{i\in\mathbb{Z}\setminus\{n\}} k.S^i)$ and as a k-linear map it is isometric. Therefore, we have $\|\nabla_2^{-1}\|_{\mathrm{Sp}} = 1$. Hence, by Lemma 2.1.9 $D^-(0,1) \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_{\nabla_2}$. Consequently, $D^-(0,1) \cap \Sigma_{d-n} = \{0\}$. As $\Sigma_d = \Sigma_{d-n} + n$ (cf. Lemma 2.1.16), we have $D^-(n,1) \cap \Sigma_d = \{n\}$. Therefore, for all $n \in \mathbb{Z}$ we have $(n, x_{0,1}) \subset \mathbb{A}_k^{1,\mathrm{an}} \setminus \Sigma_d$ and the claim follows. □

Proposition 5.1.8. *Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2), (3) or (4) not of the form $x_{0,r}$. Let $c \in k \setminus \{0\}$ such that $x \in D^-(c, |c|)$. The spectrum of $S\frac{\mathrm{d}}{\mathrm{dS}}$ as an element of $\mathcal{L}_k(\mathscr{H}(x))$ is*

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = \begin{cases} \overline{D^-(0, \frac{|c|}{r_k(x)})} & \text{if } x \text{ is of type (4)} \\ \\ D^+(0, \frac{|c|}{r_k(x)}) & \text{otherwise} \end{cases}.$$

Proof. Let $\mathrm{Log}_c : D^-(c, |c|) \to D^-(0,1)$ be the logarithm and setting $y := \mathrm{Log}_c(x)$. Since $\mathrm{char}(\tilde{k}) = 0$, Log_c is an isomorphism analytic map and $[\mathscr{H}(x) : \mathscr{H}(y)] = 1$. Therefore, the push-forward of $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$ by Log_c is isomorphic to $(\mathscr{H}(y), \frac{\mathrm{d}}{\mathrm{dS}})$. Therefore, by Propositions 4.2.2 and 3.4.3 we obtain

$$\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(x))) = \Sigma_{\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(y))) = \begin{cases} \overline{D^-(0, \frac{1}{r_k(y)})} & \text{if } x \text{ is of type (4)} \\ \\ D^+(0, \frac{1}{r_k(y)}) & \text{otherwise} \end{cases}.$$

Since $r_k(y) = \frac{r_k(x)}{|c|}$, the result follows. □

5.2 Spectrum of a regular singular differential module

As we have mentioned at the beginning of the chapter, the computation of the spectrum of a regular differential module follows directly from the computation of the spectrum done above and Proposition 3.2.17. In this section, we will summarize all the different case discussed in the previous section. We will also discuss the variation of the spectrum.

5.2.1 Spectrum of a regular singular differential module

Notation 5.2.1. We denote by $\overline{\mathbb{Z}}$ the topological closure of $\mathbb{Z}$ in $\mathbb{A}_k^{1,\mathrm{an}}$.

Theorem 5.2.2. *Assume that $\mathrm{char}(\tilde{k}) = p > 0$. Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a point of type (2), (3) or (4). Let (M, ∇) be a regular singular differential module over $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$. Let G be the matrix associated to ∇ with constant entries (i.e. $G \in \mathcal{M}_\nu(k)$), and let $\{a_1, \cdots, a_N\}$ be the set of eigenvalues of G.*

- *If x is a point of the form $x_{0,r}$, then we have*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \bigcup_{i=1}^{N} a_i + \overline{\mathbb{Z}}.$$

- *Otherwise, let* $c \in k \setminus \{0\}$ *such that* $x \in D^-(c, |c|)$ *and* $y := \mathrm{Log}_c(x)$. *Then we have*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \begin{cases} \bigcup\limits_{j=1}^{N} D^+(a_j, \frac{\omega}{r_k(y)}) & \text{if } r_k(x) \in (0, |c|\omega\ ] \\ \\ \bigcup\limits_{j=1}^{N} \bigcup\limits_{i=0}^{p^n} D^+(a_j + i, \frac{\omega}{r_k(y)}) & \text{if } r_k(x) \in (|c|\omega^{\frac{1}{p^{n-1}}}, |c|\omega^{\frac{1}{p^n}}\ ] \\ & \text{with } n \in \mathbb{N} \setminus \{0\}. \end{cases}$$

Theorem 5.2.3. *Assume that* $\mathrm{char}(\tilde{k}) = 0$. *Let* $x \in \mathbb{A}_k^{1,\mathrm{an}}$ *be a point of type (2), (3) or (4). Let* (M, ∇) *be a regular singular differential module over* $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$. *Let* G *be the matrix associated to* ∇ *with constant entries (i.e.* $G \in \mathcal{M}_\nu(k)$*), and let* $\{a_1, \cdots, a_N\}$ *be the set of eigenvalues of* G.

- *If* x *is a point of type (2) of the form* $x_{0,r}$, *then we have*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \bigcup_{j=1}^{N} D^+(a_j, 1).$$

- *If* x *is a point of type (3) of the form* $x_{0,r}$, *then we have*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \bigcup_{j=1}^{N} a_j + \overline{\mathbb{Z}}.$$

- *Otherwise, let* $c \in k \setminus \{0\}$ *such that* $x \in D^-(c, |c|)$. *Then we have*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \begin{cases} \bigcup\limits_{j=1}^{N} \overline{D^-(a_j, \frac{|c|}{r_k(x)})} & \text{if } x \text{ is of type (4)} \\ \\ \bigcup\limits_{j=1}^{N} D^+(a_j, \frac{|c|}{r_k(x)}) & \text{otherwise} \end{cases}.$$

5.2.2 Variation of the spectrum

Let X be an affinoid domain of $\mathbb{A}_k^{1,\mathrm{an}}$. Let (M, ∇) be a differential module over $(\mathcal{O}(X), S\frac{\mathrm{d}}{\mathrm{dS}})$ such that there exists a basis for which the associated matrix G has constant entries. For a point $x \in X$ not of type (1), the differential module (M, ∇) extends to a differential module (M_x, ∇_x) over $(\mathscr{H}(x), S\frac{\mathrm{d}}{\mathrm{dS}})$. In the corresponding basis of (M_x, ∇_x) the associated matrix is G.

The case of positive residue characteristic

Assume that $\mathrm{char}(\tilde{k}) = p > 0$. We observe from Theorem 5.2.2 that: although the spectrum is roughly different from the constant case studied in Chapter 4, it satisfies analogous continuity properties.

Theorem 5.2.4. *Let $X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$ be a connected affinoid domain and $x \in X$ be a point of type (2), (3) or (4). Let (M, ∇) be a differential module over $(\mathcal{O}_X(X), S\frac{\mathrm{d}}{\mathrm{dS}})$ such that there exists a basis for which the corresponding matrix G has constant entries. We set:*

$$\begin{array}{rcl} \Psi : [x, x_{c_0,r_0}] & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ y & \mapsto & \Sigma_{\nabla_y,k}(\mathcal{L}_k(M_y)) \end{array}.$$

Then we have:

- *for each $y \in [x, x_{c_0,r_0}]$, the restriction of Ψ to $[x, y]$ is continuous at y.*
- *Let $y \in [x, x_{c_0,r_0}]$. The map Ψ is continuous at y if and only if y is of type (3) or of the form $x_{0,R}$.*

Proof. We identify $[x, x_{c_0,r_0}]$ with the interval $[r(x), r_0]$ by the map $y \mapsto r(y)$ (cf. Definition 1.2.27). Let $y \in [x, x_{c_0,r_0}]$. Assume that there exists $x' \in [x, x_{c_0,r_0})$ such that $[x, x'] \cap (0, \infty) = \varnothing$ and $[x, y] \subset [x, x']$. Let $y' \in [x, x']$. By Theorem 5.2.2 and Corollary 5.1.5, we have $\Psi(y) = \bigcup_{i=1}^{N} D^+(a_i, \varphi(y))$ and $\Psi(y') = \bigcup_{i=1}^{N} D^+(a_i, \varphi(y'))$, where $\varphi : [x, y] \to \mathbb{R}_+$ is a decreasing continuous function and $\varphi(y) \notin |k|$ if y is of type (3). Therefore, the claims:

- Ψ is continuous at y if and only if y is of type (3),
- the restriction of Ψ to $[x, y]$ is continuous at y,

holds by Corollary 2.3.12.

Now assume that $y \in [x_{0,R}, x_{c_0,r_0}]$, this means that $x_{c_0,r_0} = x_{0,r_0}$. In the case where $y \neq x_{0,R}$, the restriction of Ψ to $[x_{0,R}, x_{c_0,r_0}]$ is constant (cf. Theorem 5.2.2). Hence the restriction of Ψ to $[x_{0,R}, x_{c_0,r_0}]$ is continuous. Otherwise, on the one hand the restriction of Ψ to $[y, x_{c_0,r_0}]$ is continuous at y. On the other hand, since for all $y' \in [x, y]$ we have $\Psi(y') = \bigcup_{i=1}^{N} \alpha_i + \Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}},k}(\mathcal{L}_k(\mathscr{H}(y)))$, it is enough to show the result for the differential module $(\mathcal{O}_X(X), S\frac{\mathrm{d}}{\mathrm{dS}})$ (cf. Lemma 2.3.3). Hence, we reduce to the case where $\Psi(y) = \mathbb{Z}_p$ and $\Psi(y') = \bigcup_{i \in \mathbb{N}} D^+(i, \varphi(y'))$, with $\varphi : [x, y) \to \mathbb{R}_+$ a decreasing continuous function and $\lim\limits_{y' \to y} \varphi(y') = 0$ (cf. Theorem 5.2.2 and Corollary 5.1.5). Let $(U, \{U_i\}_{i \in I})$ be an open neighbourhood of $\Psi(y)$. Since $\Psi(y)$ is a set of points of type (1), we can assume that the U_i are open disks. Since $\mathbb{Z}_p$ is the topological closure of $\mathbb{N}$ in $\mathbb{A}_k^{1,\mathrm{an}}$, for each $i \in I$ we have $\mathbb{N} \cap U_i \neq \varnothing$. Therefore, for each $i \in I$ we have $\Psi(y') \cap U_i \neq \varnothing$. We now prove that there exists $x' \in [x, y)$ such that for all $y' \in (x', y)$ we have $\Psi(y') \subset U$. Let L be the smallest radius of the disks U_i. Since φ is a decreasing continuous function, there exists y_L such that for all $y' \in (y_L, y)$ we have $\varphi(y') < L$. Therefore, since $\mathbb{N} \subset U = \bigcup_{i\in I} U_i$, for all $j \in \mathbb{N}$ there exists $i \in I$ such that $D^+(j, \varphi(y')) \subset U_i$. Consequently, we have $\Psi(y') \subset U$ and $\Psi(y') \in (U, \{U_i\}_{i\in I})$.

$\square$

The case of residue characteristic zero

Assume that $\mathrm{char}(\tilde{k}) = 0$. We observe from Theorem 5.2.3 that, the spectrum behaves differently from the case where $\mathrm{char}(\tilde{k}) = p > 0$. In the special case where k is not trivially valued and $|k| \neq \mathbb{R}_+$, the map

$$\begin{array}{rccc} \Psi : & (0,\infty) & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ & y & \mapsto & \Sigma_{\nabla_y,k}(\mathcal{L}_k(M_y)) \end{array}$$

is not continuous at all. Indeed, let $y \in (0,\infty)$ be a point of type (2). Assume that $(M, \nabla) = (\mathcal{O}_X(X), S\frac{\mathrm{d}}{\mathrm{dS}})$. Then we have $\Psi(y) = D^+(0,1)$. Let U be an open neighbourhood of $\Psi(y)$ in $\mathbb{A}_k^{1,\mathrm{an}}$. Let $a \in (D^+(0,1) \cap k) \setminus \mathbb{Z}$ and let $0 < r < 1$ such that $D^-(a,r) \cap \mathbb{Z} = \varnothing$. For any $y' \in (0,\infty)$ of type (3) we have $\Psi(y') = \mathbb{Z} \cup \{x_{0,1}\}$, hence $\Psi(y') \cap D^-(a,r) = \varnothing$. Therefore, $\Psi(y') \notin (U, \{U, D^+(a,r)\})$.

In the case where k is trivially valued the only point where there is no continuity is $x_{0,1}$. For the other points of $(0,\infty)$, since Ψ is constant it is continuous on $(0,\infty)\setminus\{x_{0,1}\}$.

For branches $(c, x_{0,|c|}\,]$ with $c \in k \setminus \{0\}$, the map

$$\begin{array}{rccc} \Psi : & (c, x_{0,|c|}\,] & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ & y & \mapsto & \Sigma_{\nabla_y,k}(\mathcal{L}_k(M_y)) \end{array}$$

satisfies the same continuity properties as those of Theorem 4.3.1. Indeed, for any $y \in (c, x_{c,|c|}]$ we have $\Psi(y) = \bigcup_{i=1}^N D^+(a_i, \varphi(y))$ with $\varphi : (c, x_{0,|c|}] \to \mathbb{R}_+$ a decreasing continuous function and $\varphi(y) \notin |k|$ if y is of type (3).

We have the following results:

Theorem 5.2.5. *Assume that $|k| = \mathbb{R}_+$. Let $X = D^+(c_0, r_0) \setminus \bigcup_{i=1}^{\mu} D^-(c_i, r_i)$ be a connected affinoid domain and $x \in X$ be a point of type (2), (3) or (4). Let (M, ∇) be a differential module over $(\mathcal{O}_X(X), S\frac{\mathrm{d}}{\mathrm{dS}})$ such that there exists a basis for which the corresponding matrix G has constant entries. We set:*

$$\begin{array}{rccc} \Psi : & [x, x_{c_0,r_0}] & \longrightarrow & K(\mathbb{A}_k^{1,\mathrm{an}}) \\ & y & \mapsto & \Sigma_{\nabla_y,k}(\mathcal{L}_k(M_y)) \end{array}.$$

Then we have:

- *for each* $y \in [x, x_{c_0, r_0}]$, *the restriction of* Ψ *to* $[x, y]$ *is continuous at* y.
- *Let* $y \in [x, x_{c_0, r_0}]$. *The map* Ψ *is continuous at* y *if and only if* y *is of type (3), (4) or of the form* $x_{0,R}$.

Proof. The proof is analogous to the proof of Theorem 4.3.1 and 5.2.4. □

VI

Spectrum of a linear differential equation over a field of formal power series

Contents

Let $k((S))$ be the field of Laurent power series. This chapter is devoted to the computation of the spectrum of a differential module (M, ∇) over $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$, which is an application of the results already proved in the previous chapters.

Convention 6.0.1. In this chapter, we assume that k is trivially valued and algebraically closed.

Convention 6.0.2. We fix $r \in (0, 1)$ and endow $k((S))$ with the S-adic absolute value given by

$$|\sum_{i\geq N} a_i S^i| := r^N,$$

if $a_N \neq 0$. In this setting $k((S))$ coincides with $\mathscr{H}(x_{0,r})$, where $x_{0,r} \in \mathbb{A}_k^{1,\mathrm{an}}$.

6.1 Spectrum of a differential module after ramified ground field extension

Recall that if F is a finite extension of $k((S))$ of degree m, then we have $F \simeq k((S^{\frac{1}{m}}))$ [VS12, Proposition 3.3]. The absolute value $|.|$ on $k((S))$ extends uniquely to an absolute value on F. The pair $(F, |.|)$ is an element of $E(k)$ and can be identified with $\mathscr{H}(x_{0,r^{\frac{1}{m}}})$. The derivation $S\frac{\mathrm{d}}{\mathrm{dS}}$ extends uniquely to a derivation d on F, where $d(S^{\frac{1}{m}}) = \frac{1}{m}S^{\frac{1}{m}}$. Then (F, d) is a finite differential extension of $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$. Now, if we set $Z = S^{\frac{1}{m}}$, then we have $F = k((Z))$ and $d = \frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}}$. Since $F \simeq \mathscr{H}(x_{0,r^{\frac{1}{m}}})$ and $k((S)) \simeq \mathscr{H}(x_{0,r})$, we can see $k((S)) \hookrightarrow F$ as the inclusion induced by the power function $(.)^m : \mathbb{A}_k^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ (cf. Section A.2), we set $I_F := (.)^m$. The derivation d is nothing but then the pull-back of d. Note that we can see $(F, \frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}})$ as a differential module over $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$. In the basis $\{1, Z, \cdots, Z^{m-1}\}$ we have:

$$\frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}}\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} S\frac{\mathrm{d}}{\mathrm{dS}}f_1 \\ \vdots \\ S\frac{\mathrm{d}}{\mathrm{dS}}f_m \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{m} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{m-1}{m} \end{pmatrix}\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \tag{6.1}$$

We have a functor (cf. (3.8)):

$$\begin{array}{rcl} I_F{}^* : S\frac{\mathrm{d}}{\mathrm{dS}} - \mathbf{Mod}(k((S))) & \longrightarrow & \frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}} - \mathbf{Mod}(F) \\ (M, \nabla) & \mapsto & (I_F{}^*M, I_F{}^*\nabla) \end{array} \tag{6.2}$$

where $I_F{}^*M = M \otimes_{k((S))} F$ and the connection $I_F{}^*\nabla$ is defined as follows:

$$I_F{}^*\nabla = \nabla \otimes 1 + 1 \otimes \frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}}$$

Let (M, ∇) be an object of $S\frac{\mathrm{d}}{\mathrm{dS}} - \mathbf{Mod}(k((S)))$ of rank n. If $\{e_1, \dots, e_n\}$ is a basis of M such that we have:

$$\nabla\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} S\frac{\mathrm{d}}{\mathrm{dS}}f_1 \\ \vdots \\ S\frac{\mathrm{d}}{\mathrm{dS}}f_n \end{pmatrix} + G\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},$$

with $G \in \mathcal{M}(k((S)))$, then $(I_F{}^*M, I_F{}^*\nabla)$ is of rank n and in the basis $\{e_1 \otimes 1, \dots, e_n \otimes 1\}$ we have:

$$I_F{}^*\nabla\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} \frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}}f_1 \\ \vdots \\ \frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}}f_n \end{pmatrix} + G\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}. \tag{6.3}$$

We have also the push-forward functor (cf. (3.7)):

$$\begin{array}{rcl} I_{F*} : \frac{Z}{m}\frac{\mathrm{d}}{\mathrm{dZ}} - \mathbf{Mod}(F) & \longrightarrow & S\frac{\mathrm{d}}{\mathrm{dS}} - \mathbf{Mod}(k((S))) \\ (M, \nabla) & \mapsto & (I_{F*}M, I_{F*}\nabla) \end{array} \tag{6.4}$$

Let (M, ∇) be an object of $S\frac{\mathrm{d}}{\mathrm{dS}} - \mathbf{Mod}(k(\!(S)\!))$ of rank n. The differential module $(I_{F*}I_F{}^*M, I_{F*}I_F{}^*\nabla)$ has rank nm. Let $\{e_1, \cdots, e_n\}$ be a basis of (M, ∇) and let G be the associated matrix in this basis. Then the associated matrix of $(I_{F*}I_F{}^*M, I_{F*}I_F{}^*\nabla)$ in the basis $\{e_1 \otimes 1, \cdots, e_n \otimes 1, e_1 \otimes Z, \cdots, e_n \otimes Z, \cdots, e_1 \otimes Z^{m-1}, \cdots, e_n \otimes Z^{m-1}\}$ is:

$$\begin{pmatrix} G & 0 & \cdots & 0 \\ 0 & G + \frac{1}{m} \cdot I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & G + \frac{m-1}{m} \cdot I_n \end{pmatrix}.$$

Therefore we have the following isomorphism:

$$(I_{F*}I_F{}^*M, I_{F*}I_F{}^*\nabla) \simeq \bigoplus_{i=0}^{m-1} (M, \nabla + \frac{i}{m}). \tag{6.5}$$

As k-Banach spaces $I_{F*}I_F{}^*M$ and $I_F{}^*M$ are the same, and $I_{F*}I_F{}^*\nabla$ as a k-linear map coincides with $I_F{}^*\nabla$. Therefore, by Remark 3.2.7 and Lemma 2.1.16 we have:

$$\Sigma_{I_F{}^*\nabla,k}(\mathcal{L}_k(I_F{}^*M)) = \bigcup_{i=0}^{m-1} \frac{i}{m} + \Sigma_{\nabla,k}(\mathcal{L}_k(M)). \tag{6.6}$$

6.2 Newton polygon and the decomposition according to the slopes

Let $v : k(\!(S^{\frac{1}{m}})\!) \to \mathbb{Z} \cup \{\infty\}$ be the valuation map associated to the absolute value of $k(\!(S^{\frac{1}{m}})\!)$, that satisfies $v(S^{\frac{1}{m}}) = \frac{1}{m}$. Let $P = \sum_{i=0}^{n} g_i D^i$ be an element of $\mathscr{D}_{k(\!(S)\!)}$. Let L_P to be the convex hull in $\mathbb{R}^2$ of the set of points

$$\{(i, v(g_i)) |\ 0 \leq i \leq n\} \cup \{(0, \min_{0 \leq i \leq n} v(g_i))\}.$$

Definition 6.2.1 ([VS12, Definition 3.44])**.** The Newton polygon $\mathrm{NP}(P)$ of P is the boundary of L_P. The finite slopes γ_i of P are called the slopes of $\mathrm{NP}(P)$. The horizontal width of the segment of $NP(P)$ of slope γ_i is called the multiplicity of γ_i.

Proposition 6.2.2. *Let (M, ∇) be a differential module over $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$. The following properties are equivalent:*

- *(M, ∇) is regular singular;*
- *There exists differential polynomial $P(D)$ with only one slope equal to 0 such that $(M, \nabla) \simeq \mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P(D)$;*

- *There exists* $P(D) = g_0 + g_1 D + \cdots + g_{n-1}D^{n-1} + D^n$ *with* $g_i \in k[\![S]\!]$, *such that* $(M,\nabla) \simeq (\mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P(D), D)$;
- *There exists* $P(D) = g_0 + g_1 D + \cdots + g_{n-1}D^{n-1} + D^n$ *with* $g_i \in k$, *such that* $(M,\nabla) \simeq (\mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P(D), D)$;

Proof. See [Ked10, Corollary 7.1.3] and [Chr, Proposition 10.1]. □

Proposition 6.2.3 ([Ked10, Proposition 7.3.6])**.** *Let* $P(D) = g_0+g_1D+\cdots+g_{n-1}D^{n-1}+D^n$ *such that* $g_i \in k[\![S]\!]$. *Then we have the isomorphism in* $S\frac{\mathrm{d}}{\mathrm{dS}} - \mathbf{Mod}(k(\!(S)\!))$:

$$(\mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P(D), D) \simeq (\mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P_0(D)),$$

where $P_0(D) = g_0(0) + g_1(0)D + \cdots g_{n-1}(0)D^{n-1} + D^n$.

Remark 6.2.4. This Proposition means in particular that for all $f \in k[\![S]\!]$ there exists $g \in k(\!(S)\!) \setminus \{0\}$ such that $f - \frac{S\frac{\mathrm{d}}{\mathrm{dS}}(g)}{g} = f(0)$. Indeed, by Proposition 6.2.3 we have $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}} + f) \simeq (k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}} + f(0))$. This is equivalente to say that there exists $g \in k(\!(S)\!) \setminus \{0\}$ such that

$$g^{-1} \circ S\frac{\mathrm{d}}{\mathrm{dS}} \circ g + f = S\frac{\mathrm{d}}{\mathrm{dS}} + f(0).$$

Definition 6.2.5. Let (M,∇) be a differential module of $S\frac{\mathrm{d}}{\mathrm{dS}} - \mathbf{Mod}(k(\!(S)\!))$, and let $P(D) \in \mathscr{D}_{k(\!(S)\!)}$ such that $(M,\nabla) \simeq (\mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P(D), D)$. If all the slopes of $P(D)$ are different from 0, then we say that (M,∇) is without regular part.

Proposition 6.2.6. *Let* $P \in \mathscr{D}_{k(\!(S)\!)}$ *and* γ *a slope of* P. *Let* ν *be the multiplicity of* γ. *Then there exist differential polynomials* R, R', Q *and* Q' *which satisfy the following properties:*

- $P = RQ = Q'R'$.
- *The degree of* R *and* R' *is equal to* ν, *and their only slopes are* γ *with multiplicity equal to* ν.
- *All the slopes of* Q *and* Q' *are different from* γ.
- $\mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.P = \mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.R \oplus \mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.Q = \mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.Q' \oplus \mathscr{D}_{k(\!(S)\!)}/\mathscr{D}_{k(\!(S)\!)}.R'$.

Proof. See [Chr, Proposition 12.1] and [VS12, Theorem 3.48]. □

Corollary 6.2.7. *Let* (M,∇) *be a differential module over* $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$. *Then we have the decomposition in* $S\frac{\mathrm{d}}{\mathrm{dS}} - \mathbf{Mod}(k(\!(S)\!))$:

$$(M,\nabla) = (M_{reg}, \nabla_{reg}) \oplus (M_{irr}, \nabla_{irr})$$

where (M_{reg}, ∇_{reg}) *is a regular singular differential module and* (M_{irr}, ∇_{irr}) *is a differential module without regular part.*

6.3 Spectrum of a differential module

In this section we compute the spectrum of a differential module (M, ∇) over $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$, which is the aim of this chapter. According to Corollary 6.2.7 we have the decomposition:

$$(M, \nabla) = (M_{\text{reg}}, \nabla_{\text{reg}}) \oplus (M_{\text{irr}}, \nabla_{\text{irr}})$$

We know that $\Sigma_\nabla = \Sigma_{\nabla_{\text{reg}}} \cup \Sigma_{\nabla_{\text{irr}}}$ (cf. Remark 3.2.7). Therefore, in order to obtain the general statement, it is enough to know the spectrum of a regular singular differential module and the spectrum of a pure irregular singular differential module. The case of a regular singular is done in Chapter 5. It just remains to treat the case of differential module without regular part.

Recall the statement for the case of a regular singular module.

Theorem 6.3.1. *Let (M, ∇) be a regular singular differential module over $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$. Let G the matrix associated to ∇ with constant entries (i.e. $G \in \mathcal{M}_\nu(k)$), and let $\{a_1, \cdots, a_N\}$ be the set of eigenvalues of G. The spectrum of ∇ is*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \bigcup_{i=1}^{N} (a_i + \mathbb{Z}) \cup \{x_{0,1}\}.$$

Proof. Since $k((S)) \simeq \mathscr{H}(x)$ where x is a point of type (3), it follows directly from Theorem 5.2.3. □

Lemma 6.3.2 ([VS12, Proposition 3.12])**.** *We can assume that the set of the eigenvalues $\{a_1, \cdots, a_N\}$ satisfies $a_i - a_j \notin \mathbb{Z}$ for each $i \neq j$.*

Definition 6.3.3. We will call the a_i of Theorem 6.3.1 the exponents of (M, ∇).

Recall the following Theorem, which is the celebrated theorem of Turrittin. It ensures that any differential module becomes extension of rank one differential modules after pull-back by a convenient ramification ramified extension.

Theorem 6.3.4 ([Tur55])**.** *Let (M, ∇) be a differential module over $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$. There exists a finite extension $F = k((S^{\frac{1}{m}}))$ such that we have:*

$$({I_F}^*M, {I_F}^*\nabla) = \bigoplus_{i=1}^{N} (\mathscr{D}_F / \mathscr{D}_F.(D - f_i)^{\alpha_i}, D) \tag{6.7}$$

where $f_i \in k[\![S^{-\frac{1}{m}}]\!]$ and $\alpha_i \in \mathbb{N}$.

Proof. See [VS12, Theorem 3.1]. □

Now, in order to compute the spectrum, we need Theorem 3.3.1, that we recall here under the hypotheses of the chapter.

Lemma 6.3.5. *Let $f = \sum_{i\in\mathbb{Z}} a_i S^{\frac{i}{m}}$ an element of $k(\!(S^{\frac{1}{m}})\!)$ and let $(k(\!(S^{\frac{1}{m}})\!), \nabla)$ be the differential module of rank one such that $\nabla = S\frac{\mathrm{d}}{\mathrm{dS}} + f$. If $v(f) < 1$, then the spectrum of ∇ as an element of $\mathcal{L}_k(k(\!(S^{\frac{1}{m}})\!))$ is:*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(k(\!(S^{\frac{1}{m}})\!))) = \{x_{0,r^{v(f)}}\}.$$

Proof. We can assume that $f = \sum_{i\in\mathbb{N}} a_i S^{\frac{-i}{m}}$. Indeed, since $f = f_- + f_+$ with $f_- := \sum_{i<0} a_i S^{\frac{-i}{m}}$ and $f_+ := \sum_{i\geq 0} a_i S^{\frac{-i}{m}}$, according to Proposition 6.2.3 there exists $g \in k(\!(S^{\frac{1}{m}})\!)$ such that $f_+ - a_0 = \frac{S\frac{\mathrm{d}}{\mathrm{dS}}(g)}{g}$. Therefore, we have $(k(\!(S^{\frac{1}{m}})\!), \nabla) \simeq (k(\!(S^{\frac{1}{m}})\!), S\frac{\mathrm{d}}{\mathrm{dS}} + f_- + a_0)$. By Theorem 3.3.6, we know that $\Sigma_{\nabla,k}(\mathcal{L}_k(k(\!(S^{\frac{1}{m}})\!))) = \{f(x_{0,r^{\frac{1}{m}}})\}$ (cf. Notation 3.3.3). As $f(x_{0,r^{\frac{1}{m}}}) = x_{0,|f|} = x_{0,r^{v(f)}}$, the statement follows. □

Proposition 6.3.6. *Let (M, ∇) be a differential module over $(k(\!(S)\!), S\frac{\mathrm{d}}{\mathrm{dS}})$ without regular part. The spectrum of ∇ as an element of $\mathcal{L}_k(M)$ is:*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \{x_{0,r^{v(f_1)}}, \cdots, x_{0,r^{v(f_N)}}\}$$

where the f_i are as in the formula (6.7).

Proof. We set $\Sigma_\nabla := \Sigma_{\nabla,k}(\mathcal{L}_k(M))$. By Theorem 6.3.4, there exists $F = k(\!(S^{\frac{1}{m}})\!)$ such

$$({I_F}^*M, {I_F}^*\nabla) = \bigoplus_{i=1}^{N} \mathscr{D}_F/\mathscr{D}_F.(D - f_i)^{\alpha_i}$$

where $f_i \in k[\![S^{-\frac{1}{m}}]\!]$. We set $\Sigma_{{I_F}^*\nabla} := \Sigma_{{I_F}^*\nabla,k}(\mathcal{L}_k({I_F}^*M))$. Since (M, ∇) is purely irregular singular, we have $f_i \in k[\![S^{-\frac{1}{m}}]\!] \setminus k$. By Proposition 3.2.8 and Remark 3.2.7, we have:

$$\Sigma_{{I_F}^*\nabla} = \bigcup_{i=1}^{N} \Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}}+f_i}(\mathcal{L}_k(F)).$$

By Lemma 6.3.5, we have $\Sigma_{S\frac{\mathrm{d}}{\mathrm{dS}}+f_i}(\mathcal{L}_k(F)) = \{x_{0,r^{v(f_i)}}\}$. Hence,

$$\Sigma_{{I_F}^*\nabla} = \{x_{0,r^{v(f_1)}}, \cdots, x_{0,r^{v(f_N)}}\}.$$

By the formula (6.6), we have:

$$\Sigma_{{I_F}^*\nabla} = \bigcup_{i=0}^{m-1} \frac{i}{m} + \Sigma_\nabla.$$

Since $r^{v(f_i)} > 1$ for all $1 \leq i \leq N$, then each element of $\Sigma_{I_F{}^*\nabla}$ is invariant by translation by $\frac{j}{m}$ where $1 \leq j \leq m$. This means that $\Sigma_\nabla = \Sigma_\nabla + \frac{j}{m}$. Therefore, we have $\Sigma_{I_F{}^*\nabla} = \Sigma_\nabla$. □

Remark 6.3.7. Note that, it is not easy to compute the f_i of the formula (6.7). However, the values $-v(f_i)$ coincide with the slopes of the differential module (cf. [Kat87] and [VS12, Remarks 3.55]).

We now announce the main statement that summarizes all the previous result of the chapter:

Theorem 6.3.8. *Let (M, ∇) be a differential module over $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$. Let $\{\gamma_1, \cdots, \gamma_{\nu_1}\}$ be the set of the slopes of (M, ∇) and let $\{a_1, \cdots, a_{\nu_2}\}$ be the set of the exponents of the regular part of (M, ∇). Then the spectrum of ∇ as an element of $\mathcal{L}_k(M)$ is:*

$$\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \{x_{0,r^{-\gamma_1}}, \cdots, x_{0,r^{-\gamma_{\nu_1}}}\} \cup \bigcup_{i=1}^{\nu_2}(a_i + \mathbb{Z})$$

Proof. According to Theorem 6.3.1, Proposition 6.3.6 and Remark 6.3.7 we obtain the result. □

Remark 6.3.9. We observe that although differential modules over $(k((S)), S\frac{\mathrm{d}}{\mathrm{dS}})$ are algebraic objects, their spectra in the sense of Berkovich depends highly on the choice of the absolute value on $k((S))$.

Appendices

A

Description of some étale morphisms

This part is devoted to summarizing the properties of the analytic morphisms: logarithm and power map.

A.1 Logarithm

Let $a \in k \setminus \{0\}$. We define the logarithm function $\mathrm{Log}_a : D^-(a, |a|) \to \mathbb{A}_k^{1,\mathrm{an}}$ to be the analytic map associated to the ring morphism:

$$\begin{array}{rcl} k[T] & \longrightarrow & \mathcal{O}(D^-(a,|a|)) \\ T & \mapsto & \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{(-1)^{n-1}}{a^n n}(T-a)^n \end{array}.$$

We define the exponential function $\exp_a : D^-(0,\omega) \to D^-(a, |a|\omega)$ to be the analytic map associated to the ring morphism:

$$\begin{array}{rcl} \mathcal{O}(D^-(a,|a|\omega)) & \longrightarrow & \mathcal{O}(D^-(0,\omega)) \\ \frac{T-a}{a} & \mapsto & \sum_{n\in\mathbb{N}} \frac{T^n}{n!} \end{array},$$

where

$$\omega = \begin{cases} |p|^{\frac{1}{p-1}} & \text{if } \mathrm{char}(\tilde{k}) = p \\ 1 & \text{if } \mathrm{char}(\tilde{k}) = 0 \end{cases}.$$

Lemma A.1.1. *Let $b \in D^-(a,|a|) \cap k$. Then we have* $\mathrm{Log}_b = \mathrm{Log}_a - \mathrm{Log}_a(b)$.

Proof. We have $\mathrm{Log}_a(T) = \mathrm{Log}_1(\frac{T}{a})$, in particular $\mathrm{Log}_1(\frac{b}{a})$ is well defined. Therefore,

$$\mathrm{Log}_b(T) = \mathrm{Log}_1\left(\frac{T}{b}\right) = \mathrm{Log}_1\left(\frac{\frac{T}{a}}{\frac{b}{a}}\right) = \mathrm{Log}_1\left(\frac{T}{a}\right) - \mathrm{Log}_1\left(\frac{b}{a}\right) = \mathrm{Log}_a(T) - \mathrm{Log}_a(b).$$

□

Lemma A.1.2. *The logarithm function* Log_a *induces an analytic isomorphism* $D^-(a, |a|\omega) \to D^-(0, \omega)$*, whose reverse isomorphism is* $\exp_a$*.*

Proof. Since $\exp_a : D^-(0, \omega) \to D^-(a, |a|\omega)$ is surjective, we obtain the isomorphism. □

Lemma A.1.3. *Assume that* k *is algebraically closed and* $\mathrm{char}(\tilde{k}) = p > 0$*. Let* ζ_{p^n} *be a* p^n*th root of the unity. Then we have*

- $|\zeta_{p^n} - 1| = \omega^{\frac{1}{p^{n-1}}}$*;*
- *if* $x \in D^-(a, |a|) \cap k$*, then* $\mathrm{Log}_a(x) = 0 \iff x = a\zeta_{p^n}$*.*

Proof. It is easy to see that $\mathrm{Log}_a(a\zeta_{p^n}) = 0$. Indeed,

$$\mathrm{Log}_a(a\zeta_{p^n}) = \mathrm{Log}_1(\zeta_{p^n}) = \frac{1}{p^n}\mathrm{Log}_1(1) = 0.$$

Since $\mathrm{Log}_a(T) = \mathrm{Log}_1(\frac{T}{a})$, it is enough to show that $\mathrm{Log}_1(x) = 0 \iff x = \zeta_{p^n}$. By Lemma A.1.2, we have $\mathrm{Log}_1(x) = 0 \iff x = 1$ if $x \in D^-(1, \omega) \cap k$. Now let $x \in D^-(1, 1) \cap k$, we have

$$x^p - 1 = (x-1)^p + p(x-1)\sum_{i=1}^{p-1} p^{-1}\binom{p}{i}(x-1)^{i-1}.$$

Therefore,

$$|x^p - 1| \leq \max(|p||x-1|, |x-1|^p).$$

Hence, there exists $n \in \mathbb{N}$ such that $|x^{p^n} - 1| < \omega$. Since $\mathrm{Log}_1(x^{p^n}) = \mathrm{Log}_1(x)$, if $\mathrm{Log}_1(x) = 0$ then $x^{p^n} = 1$. Then we obtain the result.

□

Properties A.1.4. *Assume that* k *is algebraically closed and* $\mathrm{char}(\tilde{k}) = p > 0$*. Let* $D^+(b, r)$ *be a closed subdisk of* $D^-(a, |a|)$*. Then :*

- *The logarithm function induces an étale cover* $D^-(a, |a|) \to \mathbb{A}_k^{1,\mathrm{an}}$*.*

- $\mathrm{Log}_a(D^+(b,r)) = D^+(\mathrm{Log}_a(b), \varphi(r))$ *where*

$$\begin{array}{rcl} \varphi : (0,|a|) & \longrightarrow & \mathbb{R}_+ \\ r & \mapsto & |\mathrm{Log}_b(x_{b,r})| \end{array}.$$

- *The function φ depends only on the choice of the radius of the disk $D^+(b,r)$. In particular, it is an increasing continuous function and piecewise logarithmically affine on $(0,|a|)$ and*

- $\varphi(|a|\omega^{\frac{1}{p^n}}) = \frac{\omega}{|p|^n}$, *where $n \in \mathbb{N}$.*

- *If $|a|\omega^{\frac{1}{p^{n-1}}} \leq r < |a|\omega^{\frac{1}{p^n}}$, then $\mathrm{Log}_a^{-1}(\mathrm{Log}_a(b)) \cap D^+(b,r) = \{b\zeta_{p^n}^i | 0 \leq i \leq p^n - 1\}$ where ζ_{p^n} is a p^nth root of the unity.*

Proof.

- Since $\frac{\mathrm{d}}{\mathrm{dT}} \mathrm{Log}_a(T) = \frac{1}{T}$ is invertible in $\mathcal{O}(D^-(a,|a|))$, by Remark 1.2.42 Log_a is locally étale. Hence, it is an étale cover.

- We know that the image of the disk $D^+(b,r)$ by the analytic map Log_a is the disk $D^+(\mathrm{Log}_a(b), \varphi(r))$ with radius is equal to $\varphi(r) = |\mathrm{Log}_a(x_{b,r}) - \mathrm{Log}_a(b)|$. By Lemma A.1.1 we obtain $\varphi(r) = |\mathrm{Log}_b(x_{b,r})|$.

- Since $|b| = |a|$ and $\varphi(r) = |\mathrm{Log}_b(x_{b,r})|$, by construction it depends only on the value r. Since Log_b is an analytic map well defined on $(b, x_{a,|a|})$, the map φ is an increasing continuous function piecewise logarithmically affine on $(0,|a|)$.

- We have $\varphi(|a|\omega^{\frac{1}{p^n}}) = \max_{i\in\mathbb{N}\setminus\{0\}} |i|^{-1}\omega^{\frac{i}{p^n}} = \frac{\omega}{p^n}$.

- Since $\mathrm{Log}_b = \mathrm{Log}_a - \mathrm{Log}_a(b)$, we conclude by Lemma A.1.3.

□

Proposition A.1.5. *Assume that $\mathrm{char}(\tilde{k}) = p > 0$. Let $y \in D^-(a,|a|)$ and $x := \mathrm{Log}_a(y)$, then we have:*

- *If $0 < r_k(y) < |a|\omega$, then $[\mathscr{H}(y) : \mathscr{H}(x)] = 1$*

- *If $|a|\omega^{\frac{1}{p^{n-1}}} \leq r_k(y) < |a|\omega^{\frac{1}{p^n}}$ with $n \in \mathbb{N} \setminus \{0\}$, then $[\mathscr{H}(y) : \mathscr{H}(x)] = p^n$.*

Proof. Consequence of Propositions A.1.4 and 1.2.45. □

A.2 Power map

For the details of this part we refer the reader for example to [Pul15, Section 5] and [Ked10, Chapter 10]. We define the nth power map $\Delta_n : \mathbb{A}_k^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ to be the analytic map associated to the ring morphism:

$$\begin{array}{ccc} k[T] & \longrightarrow & k[T] \\ T & \mapsto & T^n \end{array}$$

A.2.1 Frobenius map

We assume here that $\mathrm{char}(\tilde{k}) = p$, with $p > 0$.
We define the Frobenius map to be the pth power map. We will denote it by $\mathbf{Frob}_p$.

Properties A.2.1. *Let $a \in k$ and $r \in \mathbb{R}_+^*$. The Frobenius map satisfies the following properties:*

- *It induces an finite étale morphism $\mathbb{A}_k^{1,\mathrm{an}} \setminus \{0\} \to \mathbb{A}_k^{1,\mathrm{an}} \setminus \{0\}$.*
- $\mathbf{Frob}_p(D^+(a,r)) = D^+(a^p, \varphi(a,r))$ *where* $\varphi(a,r) = \max(|p||a|^{p-1}r, r^p)$.
- $\mathbf{Frob}_p(x_{a,r}) = x_{a^p,\varphi(a,r)}$.

Proposition A.2.2. *Let $y := x_{a,r}$ with $r > 0$. We set $x := \mathbf{Frob}_p(y)$. Then we have:*

- *If $r < \omega|a|$, then $[\mathscr{H}(y) : \mathscr{H}(x)] = 1$.*
- *If $r \geq \omega|a|$, then $[\mathscr{H}(y) : \mathscr{H}(x)] = p$.*

Corollary A.2.3. *Let $y := x_{0,r}$ with $r > 0$. Let $n \in \mathbb{N} \setminus \{0\}$, we set $x := (\mathbf{Frob}_p)^n(y)$. Then we have $[\mathscr{H}(y) : \mathscr{H}(x)] = p^n$.*

A.2.2 Tame case

Let $n \in \mathbb{N} \setminus \{0\}$. We assume that n is prime to $\mathrm{char}(\tilde{k})$.

Properties A.2.4. *Let $a \in k$ and $r \in \mathbb{R}_+^*$. The nth power map satisfies the following propreties:*

- *It induces a finite étale morphism $\mathbb{A}_k^{1,\mathrm{an}} \setminus \{0\} \to \mathbb{A}_k^{1,\mathrm{an}} \setminus \{0\}$.*
- $\Delta_n(D^+(a,r)) = D^+(a^n, \varphi(a,r))$ *where* $\varphi(a,r) = \max(|a|^{n-1}r, r^n)$.
- $\Delta_n(x_{a,r}) = x_{a^n,\varphi(a,r)}$.

Proposition A.2.5. *Let $y := x_{a,r}$ with $r > 0$. We set $x := \Delta_n(y)$, then we have:*

- *If $r < |a|$, then $[\mathscr{H}(y) : \mathscr{H}(x)] = 1$.*
- *If $r \geq |a|$, then $[\mathscr{H}(y) : \mathscr{H}(x)] = n$.*

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