Arithmetic intersection

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1 Definition of the height

Let X be a regular projective flat scheme over $\mathbb{Z}$, and $\overline{L}$ an hermitian line bundle over X. For every integral closed subset $Y \subset X$ we shall define a real number $h_{\overline{L}}(Y)$, called the (Faltings) *height* of Y ([4]). For this we need a few preliminaries.

1.1 Algebraic preliminaries

1.1.1 Length

Let A be a nœtherian (commutative and unitary) ring, and M an A-module of finite type.

There exists a filtration

(1) $$M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_r = 0$$

such that $M_{i+1} \neq M_i$ and $M_i/M_{i+1} = A/\wp_i$, where $\wp_i$ is a prime ideal, $0 \leq i \leq r-1$.

Definition. M has *finite length* when there exists a filtration as (1) above where, for all i, $\wp_i$ a maximal ideal in A.

Lemma 1 (Jordan-Hölder). If M has finite length, r does not depend on the choice of the filtration (1) with $\wp_i$ maximal. We call this number the *length* of M and denote it $\ell(M) \in \mathbb{N}$.

Lemma 2. Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of A-modules of finite type. Then

$$\ell(M) = \ell(M') + \ell(M'') .$$

The proofs of Lemma 1 and Lemma 2 are left to the reader.

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1.1.2 Order

Let A be as above. The *dimension* of A is

$$\dim(A) = \max\{n \backslash \exists \text{ a chain of prime ideals } \wp_0 \subset \wp_1 \subset \wp_2 \ldots \subset \wp_n \subset A,$$

$$\text{with } \wp_i \neq \wp_{i+1}\}.$$

Let A be an integral ring of dimension 1, and $a \in A$, $a \neq 0$.

Lemma 3. A/aA has finite length.

Proof. Let

$$\overline{\wp_0} \subset \ldots \subset \overline{\wp_n}$$

a maximal chain in A/aA, with $\overline{\wp_i} \neq \overline{\wp_{i+1}}$, and $\varphi : A \to A/aA$ the projection. Let $\wp_i = \varphi^{-1}(\overline{\wp_i})$. We get a chain

$$\wp_0 \subset \ldots \subset \wp_n$$

with $\wp_i \neq \wp_{i+1}$. Since A is integral, (0) is a prime ideal. And $\wp_0 \neq (0)$ since $\wp_0$ contains a. We conclude that

$$\dim(A/aA) \leq \dim(A) - 1\,.$$

Since $\dim(A) = 1$ this implies that every prime ideal of A/aA is maximal. Therefore A/aA has finite length.

q.e.d.

Let A be as in Lemma 3 and $K = \mathrm{frac}(A)$ the field of fractions of A. If $x \in K$ we define, if $x = a/b$,

$$\mathrm{ord}_A(x) = \ell(A/aA) - \ell(A/bA) \in \mathbb{Z}\,.$$

Lemma 4. i) $\mathrm{ord}_A(x)$ does not depend on the choice of a and b.
ii) $\mathrm{ord}_A(xy) = \mathrm{ord}_A(x) + \mathrm{ord}_A(y)$.

The proof of Lemma 4 is left to the reader.

Example. Assume A is *local* (i.e. A has only one maximal ideal $\mathcal{M}$) and *regular* (i.e. $\dim A = \dim(\mathcal{M}/\mathcal{M}^2)$). When $\dim(A) = 1$, K has a *discrete valuation*

$$v : K \to \mathbb{Z} \cup \{\infty\}\,,$$

$$A = \{x \in K \text{ such that } v(a) \geq 0\}$$

and $\mathrm{ord}_A(x) = n$ *iff* $x \in \mathcal{M}^n$ and $x \notin \mathcal{M}^{n+1}$. Therefore

$$\mathrm{ord}_A(x) = v(x)\,.$$

1.1.3 Divisors

Let X be a nœtherian scheme and O_X the sheaf of functions on X.

Definition. A *line bundle* on X is a locally free O_X-module L of rank one.

In other words L is a sheaf of abelian groups on X with a map

$$\mu : O_X \times L \to L$$

such that there exists an open cover

$$X = \bigcup_\alpha U_\alpha$$

such that

- $L(U_\alpha) \simeq O(U_\alpha)$
- μ on $L(U_\alpha)$ is the multiplication.

Assume now that X is integral (for every open subset $U \subset X$, $O(U)$ is integral). Let $\eta \in X$ be the generic point.

Definition. A *rational section* of L is an element $s \in L_\eta$.

Let $Z^1(X)$ be the free abelian group spanned by the closed irreducible subsets $Y \subset X$ of codimension one. We call $Z^1(X)$ the group of *divisors of* X.

If $s \in L_\eta$ is a rational section, its *divisor* is defined as

$$\mathrm{div}(s) = \sum_Y n_Y [Y] \in Z^1(X)\,,$$

where n_Y is computed as follows. If $Y \subset X$ has codimension 1 and $Y = \overline{\{y\}}$ is integral, the ring $A = O_{X,y}$ is local, integral, of dimension 1. Its fraction field is

$$K = O_{X,\eta}\,.$$

Choose an isomorphism $L_y \simeq A$, hence $L_\eta \simeq K$. If $s \in L_\eta - \{0\} = K^*$, we let

$$n_Y = \mathrm{ord}_A(s)$$

(we shall also write $n_Y = \mathrm{ord}_Y(s)$).

One can prove that n_Y does not depend on choices, and $n_Y = 0$ for almost all Y.

Example. Let K be a number field and $X = \mathrm{Spec}\,(O_K)$. Giving L amounts to give

$$\Lambda = L(X)\,,$$

a projective O_K-module of rank one. If $s \in \Lambda$, $s \neq 0$, we have a decomposition

$$\Lambda / O_K s \simeq \prod_{\wp \text{ prime}} (O_K / \wp^{n_\wp})$$

where $n_\wp = \mathrm{ord}_{O_\wp}(s)$, hence

$$\mathrm{div}(s) = \sum_{\wp} n_\wp \, [\wp] \, .$$

1.2 Analytic preliminaries

Let X be an analytic smooth manifold over $\mathbb{C}$, and $O_{X,\mathrm{an}}$ the sheaf of holomorphic functions on X.

Definitions. i) An *holomorphic line bundle* on X is a locally free $O_{X,\mathrm{an}}$-module of rank one.
ii) A *metric* $\| \cdot \|$ on L consists of maps

$$L(x) \xrightarrow{\| \cdot \|} \mathbb{R}_+$$

for any x, where $L(x) = L_x / \mathcal{M}_x$ is the fiber at x. We ask that

- $\|\lambda s\| = |\lambda| \, \|s\|$ if $\lambda \in \mathbb{C}$;
- $\|s\| = 0$ iff $s = 0$;
- Let $U \subset X$ be an open subset and s a section of L over U vanishing nowhere; then the map
$$x \longmapsto \|s(x)\|^2$$
is C^∞.

We write $\overline{L} = (L, \| \cdot \|)$.

Denote by $A^n(X)$ the $\mathbb{C}$-vector space of C^∞ differential forms of degree n on X. Recall that $A^n(X)$ decomposes as

$$A^n(X) = \bigoplus_{p+q=n} A^{pq}(X) \, ,$$

where $A^{pq}(X)$ consists of those differential forms which can be written locally as a sum of forms of type

$$u \, d\, z_{i_1} \wedge \ldots \wedge d\, z_{i_p} \wedge d\, \overline{z}_{j_1} \wedge \ldots \wedge d\, \overline{z}_{j_q}$$

where u is a C^∞ function, $d\, z_\alpha = d\, x_\alpha + i\, d\, y_\alpha$ and $d\, \overline{z}_\alpha = d\, x_\alpha - i\, d\, y_\alpha$.

The differential

$$d : A^n(X) \to A^{n+1}(X)$$

is a sum $d = \partial + \overline{\partial}$ where

$$\partial : A^{pq}(X) \to A^{p+1,q}(X)$$

and

$$\overline{\partial} : A^{pq}(X) \to A^{p,q+1}(X) .$$

We have $\partial^2 = \overline{\partial}^2 = d^2 = 0$ and we let

$$d^c = \frac{\partial - \overline{\partial}}{4\,\pi\, i} ,$$

so that

$$dd^c = \frac{\overline{\partial}\,\partial}{2\,\pi\, i} .$$

Lemma 4. Let $\overline{L} = (L, \|\cdot\|)$ be an analytic line bundle with metric. There exists a smooth form

$$c_1(\overline{L}) \in A^{1,1}(X)$$

such that, if $U \subset X$ is an open subset and $s \in \Gamma(U, L)$ is such that $s(x) \neq 0$ for every $x \in U$,

$$c_1(\overline{L})_{|U} = -\, dd^c \log \|s\|^2 .$$

Proof. Let $s' \in \Gamma(U, L)$ be another section such that $s(x) \neq 0$ when $x \in U$. We need to show that

$$(2) \qquad -\, dd^c \log \|s'\|^2 = -\, dd^c \log \|s\|^2 \quad \text{in} \quad A^{11}(U) .$$

There exists $f \in \Gamma(U, O_{X_{\rm an}})$ such that

$$s' = f s .$$

We get

$$-\, dd^c \log \|s'\|^2 = -\, dd^c \log \|s\|^2 - dd^c \log |f|^2 .$$

But

$$\partial\, \overline{\partial} \log |f|^2 = \partial \left[\frac{\overline{\partial}\, f}{f} + \frac{\overline{\partial}\, \overline{f}}{\overline{f}} \right] = -\, \overline{\partial}\, \frac{\partial\, (\overline{f})}{\overline{f}} = 0 ,$$

and (2) follows. q.e.d.

The form $c_1(\overline{L})$ is called the *first Chern form* of $\overline{L}$.

1.3 Heights

Let X be a regular, projective, flat scheme over $\mathbb{Z}$. We denote by $X(\mathbb{C})$ the set of complex points of X, an analytic manifold.

Definition. An *hermitian line bundle* on X is a pair $\overline{L} = (L, \|\cdot\|)$, where L is a line bundle on X and $\|\cdot\|$ is a metric on the holomorphic line bundle

$$L_{\mathbb{C}} = L_{|X(\mathbb{C})}\,.$$

We also assume that $\|\cdot\|$ is invariant by the complex conjugaison

$$F_\infty : X(\mathbb{C}) \to X(\mathbb{C})\,.$$

Let $\overline{L}$ be an hermitian line bundle on X. We let

$$c_1(\overline{L}) = c_1(\overline{L}_{\mathbb{C}}) \in A^{1,1}(X(\mathbb{C}))\,.$$

Theorem 1. There is a unique way to associate to every integral closed subset $Y \subset X$ a real number

$$h_{\overline{L}}(Y) \in \mathbb{R}$$

in such a way that:

i) If $\dim(Y) = 0$, i.e. when $Y = \{y\}$ where $y \in X$ is a closed point, we let $k(y) = O_{X,y}/\mathcal{M}_{X,y}$ be the residue field. Then $k(y)$ is finite and

$$h_{\overline{L}}(Y) = \log \# \left(k(y)\right).$$

ii) If $\dim(Y) > 0$, let s be a rational section of L over Y. If

$$\mathrm{div}_Y(s) = \sum_\alpha n_\alpha\, Y_\alpha\,,$$

then

$$h_{\overline{L}}(Y) = \sum_\alpha n_\alpha\, h_{\overline{L}}(Y_\alpha) - \int_{Y(\mathbb{C})} \log \|s\|\, c_1(\overline{L})^{\dim Y(\mathbb{C})}.$$

2 Existence of the height

2.1 Resolutions

To prove Theorem 1, we first need to make sense of the integral in ii). For that we use Hironaka's resolution theorem.

Theorem 2 (Hironaka). Let X be a scheme of finite type over $\mathbb{C}$, and $Z \subset X$ a proper closed subset of X such that $X - Z$ is smooth. Then there exists a proper map

$$\pi : \widetilde{X} \to X$$

such that:
i) $\widetilde{X}$ is smooth;
ii) $\widetilde{X} - \pi^{-1}(Z) \xrightarrow{\sim} X - Z$;
iii) $\pi^{-1}(Z)$ is a divisor with normal crossings.

In the situation of ii) in Theorem 1, we apply Theorem 2 to $X = Y(\mathbb{C})$, and to the union $Z = \mathrm{div}(s) \cup Y(\mathbb{C})^{\mathrm{sing}}$ of the support of $\mathrm{div}(s)$ and the singular locus of $Y(\mathbb{C})$. Let $\pi : \widetilde{Y} \to Y(\mathbb{C})$ be the resolution of $Y(\mathbb{C})$, d the dimension of $Y(\mathbb{C})$ and $\omega \in A^{dd}(Y(\mathbb{C}) - Z)$. Then we define

$$\int_{Y(\mathbb{C})} \log \|s\| \, \omega = \int_{\widetilde{Y}} \log \|\pi^*(s)\| \, \pi^*(\omega) \, .$$

To see that the integral converges choose local coordinates $z_1, \ldots, z_d$ of $\widetilde{Y}$ such that

$$\pi^*(s) = z_1^n \, u \, ,$$

with u invertible. Therefore

$$\log \|\pi^*(s)\| = n \log |z_1| + \alpha \, ,$$

with $\alpha \, C^\infty$, and

$$\pi^*(\omega) = \beta \prod_{i=1}^{d} d\, z_i \, d\, \overline{z}_i \, ,$$

with $\beta \, \mathbb{C}^\infty$. Since

$$\int_0^\varepsilon \log(z) \, d\, z \, d\, \overline{z} = \int_0^\varepsilon \log(r) \, r \, dr \, d\theta < +\infty \, ,$$

the integral converges.

2.2

By induction on $\dim(Y)$, the unicity of $h_{\overline{L}}(Y)$ is clear.

Example. Let K be a number field and $X = \mathrm{Spec}\,(O_K)$. If Σ is the set of complex embeddings of K we have

$$X(\mathbb{C}) = \coprod_{\sigma \in \Sigma} \mathrm{Spec}\,(\mathbb{C}) \, .$$

To give $\overline{L} = (L, \| \cdot \|)$ amounts to give a pair $\overline{\Lambda} = (\Lambda, \| \cdot \|_\sigma)$ where $\Lambda = L(X)$ is a projective O_K-module of rank one and, for any $\sigma \in \Sigma$, $\| \cdot \|_\sigma$ is a metric on $\Lambda \underset{\sigma}{\otimes} \mathbb{C} \simeq \mathbb{C}$ such that

$$\|F_\infty(x)\|_{F_\infty \circ \sigma} = \|x\|_\sigma \, .$$

If $s \in \Lambda$, $s \neq 0$, we have

$$\operatorname{div}(s) = \sum_{\wp} n_{\wp}\,[\wp]$$

and

$$h_{\overline{L}}(X) = \sum_{\wp} n_{\wp} \log(N\wp) - \sum_{\sigma \in \Sigma} \log \|\sigma(s)\|_{\sigma}\,,$$

where $N\wp = \#\,(O/\wp)$.

Since

$$\Lambda/Os = \prod_{\wp} (O_{\wp}/\wp^{n_{\wp}})$$

we get

$$\sum_{\wp} n_{\wp} \log(N\wp) = \log \#\,(\Lambda/Os)\,.$$

Lemma 6. $h_{\overline{L}}(X)$ does not depend on the choice of s.

Proof. Let

$$d(s) = \log \#\,(\Lambda/Os) - \sum_{\sigma \in \Sigma} \log \|\sigma(s)\|_{\sigma}\,.$$

If $s' \in \Lambda$, $s' \neq 0$, we have

$$s' = f\,s$$

with $f \in K^*$. Therefore

$$d(s') - d(s) = \sum_{\wp} v_{\wp}(f) \log(N\wp) - \sum_{\sigma \in \Sigma} \log \|\sigma(f)| = 0$$

by the product formula. q.e.d.

2.3

Let us prove Theorem 1 when Y has dimension one and Y is horizontal, i.e. Y maps surjectively onto $\operatorname{Spec}(\mathbb{Z})$. We have then

$$Y = \overline{\{y\}}\,,$$

where y is a closed point in $X \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. The residue field $K = k(y)$ is a number field and

$$Y = \operatorname{Spec}(R)$$

where R is an integral ring with fraction field K. Denote by $\widetilde{R}$ the integral closure of R in K (i.e. $\widetilde{R} = O_K$) and let

$$\pi : \widetilde{Y} = \operatorname{Spec}(\widetilde{R}) \to Y$$

be the projection. If

$$s \in \Gamma(Y, L) - \{0\}\,,$$

$$\pi^*(s) \in \Gamma(\widetilde{Y}, \pi^* L) - \{0\}\,.$$

We shall prove that

$$d(s) = d(\pi^*(s))\,. \tag{3}$$

By 2.2 this will imply that $d(s)$ is independent of the choice of s. To prove (3) we first notice that

$$Y(\mathbb{C}) = \widetilde{Y}(\mathbb{C}) = \coprod_{\sigma \in \Sigma} \mathrm{Spec}\,(\mathbb{C})\,,$$

hence

$$\sum_{\sigma \in \Sigma} \log \|s\|_\sigma = \sum_{\sigma \in \Sigma} \log \|\pi^*(s)\|_\sigma\,. \tag{4}$$

Next we consider the commutative diagram

$$\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & & & 0 & & 0 & & K & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & R & \xrightarrow{\;s\;} & L & \longrightarrow & L/R\,s & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \widetilde{R} & \xrightarrow{\;\widetilde{s}\;} & \widetilde{L} & \longrightarrow & \widetilde{L}/R\,\widetilde{s} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & \widetilde{R}/R & \longrightarrow & \widetilde{L}/L & \longrightarrow & K'' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}$$

where $\widetilde{s} = \pi^*(s) \in \widetilde{L} = \pi^*(L)$.

By diagram chase we get

$$\#\,K = \#\,K'\,.$$

On the other hand, for any prime ideal $\wp$ in O_K, we have

$$\#\left(\frac{\widetilde{L}}{L}\right)_{\wp} = \#\left(\frac{\widetilde{R}}{R}\right)_{\wp}$$

since $\widetilde{L}_\wp = L_\wp \underset{R_\wp}{\otimes} \widetilde{R}_\wp$ and L and $\widetilde{L}$ are locally trivial. This implies

$$\# K' = \# K''$$

and $\# K = \# K''$. Therefore

$$\# (L/R\,s) = \# (\widetilde{L}/\widetilde{R}\,s)\,. \tag{5}$$

The assertion (3) follows from (4) and (5).

When $\dim(Y) = 1$ and Y is vertical i.e. its image in $\mathrm{Spec}\,(\mathbb{Z})$ is a closed point of finite residue field k, Theorem 1 is proved by considering a resolution

$$\pi : \widetilde{Y} \to Y\,.$$

The proof is the same as in the case Y is horizontal, the product formula being replaced by the equality

$$\sum_{x \in \widetilde{Y}} v_x(f)\,[k(x):k] = 0$$

for any $f \in k(Y)^*$. Indeed,

$$\log \# k(x) = [k(x):k] \log \# k\,.$$

2.4

Assume from now on that $\dim(Y) \geq 2$, with $Y \subset X$ a closed integral subscheme, $Y = \overline{\{y\}}$. If $s \in Ly$, $s \neq 0$,

$$\mathrm{div}(s) = \sum_\alpha n_\alpha\, Y_\alpha\,.$$

Lemma 7. There exists $t \in Ly$ such that, for every α, the restriction of t to Y_α is not zero.

Proof. Let $Y_\alpha = \overline{\{y_\alpha\}}$. The ring

$$R = \varinjlim_{\exists\, \alpha\, \mathrm{s.t.}\, y_\alpha \in U} O(U)$$

is *semi-local*, i.e. it has finitely many maximal ideals $\mathcal{M}_\alpha$, $\alpha \in A$. Let

$$I = \bigcap_{\alpha \in \Lambda} \mathcal{M}_\alpha$$

be the radical of R, and

$$\Lambda = \varinjlim_{\exists\, \alpha\, \mathrm{s.t.}\, y_\alpha \in U} L(U)\,.$$

Note that, for every α,

$$R_{\mathcal{M}_\alpha} = O_{y_\alpha}$$

and, for every pair $\alpha \neq \beta$

$$\mathcal{M}_\alpha + \mathcal{M}_\beta = R\,.$$

Since L is locally trivial

$$\Lambda \otimes R/I = \prod_\alpha (\Lambda \otimes O_{y_\alpha})/\mathcal{M}_\alpha \simeq \prod_\alpha O_{y_\alpha}/\mathcal{M}_\alpha = R/I\,.$$

Denote by $t \in \Lambda$ an element such that its class in $\Lambda \otimes R/I$ maps to $1 \in R/I$ by the above isomorphism. The module

$$M = \Lambda / R\,t$$

is such that $M = IM$. Therefore, by Nakayama's lemma, $M = 0$. Since

$$\Lambda = R\,t\,,$$

for every $\alpha \in A$ the restriction of t to Y_α does not vanish. q.e.d.

2.5

Given s and t as above we write

$$\mathrm{div}(s) = \sum_\alpha n_\alpha\, Y_\alpha$$

and

$$\mathrm{div}(t) = \sum_\beta m_\beta\, Z_\beta\,,$$

with $Z_\beta \neq Y_\alpha$ for all β and α. Consider

$$\mathrm{div}(s) \cdot \mathrm{div}(t) = \sum_\alpha n_\alpha\, \mathrm{div}(t \mid Y_\alpha)$$

and

$$\mathrm{div}(t) \cdot \mathrm{div}(s) = \sum_\beta m_\beta\, \mathrm{div}(s \mid Z_\beta)\,.$$

These are cycles of codimension two in Y.

Proposition 1. We have

$$\mathrm{div}(s) \cdot \mathrm{div}(t) = \mathrm{div}(t) \cdot \mathrm{div}(s)\,.$$

The proof of Proposition 1 will be given later.

Assume $\dim Y(\mathbb{C}) = d$, and define

$$d(s) = h_{\overline{L}}(\mathrm{div}(s)) - \int_{Y(\mathbb{C})} \log \|s\| \, c_1(\overline{L})^d \, .$$

By induction hypothesis we have

$$\begin{aligned} d(s) &= \sum_\alpha n_\alpha \, h_{\overline{L}}(\mathrm{div}(t_{|Y_\alpha})) \\ &- \sum_\alpha n_\alpha \int_{Y_\alpha(\mathbb{C})} \log \|t\| \, c_1(\overline{L})^{d-1} - \int_{Y(\mathbb{C})} \log \|s\| \, c_1(\overline{L})^d \\ &= h_{\overline{L}}(\mathrm{div}(s) \cdot \mathrm{div}(t)) - I(s,t) \end{aligned}$$

where

$$I(s,t) = \sum_\alpha n_\alpha \int_{Y_\alpha(\mathbb{C})} \log \|t\| \, c_1(\overline{L})^{d-1} + \int_{Y(\mathbb{C})} \log \|s\| \, c_1(\overline{L})^d \, .$$

Proposition 2. $I(s,t) = I(t,s)$.

From Proposition 1 and Proposition 2 we deduce that $d(s) = d(t)$ when $\mathrm{div}(s)$ and $\mathrm{div}(t)$ are transverse. When s and s' are two sections of L there exists a section t such that $\mathrm{div}(s)$ and $\mathrm{div}(t)$ (resp. $\mathrm{div}(s')$ and $\mathrm{div}(t)$) are transverse. Therefore

$$d(s) = d(t) = d(s')$$

and Theorem 1 follows.

2.6

To prove Proposition 1 we write

$$\mathrm{div}(s) \cdot \mathrm{div}(t) = \sum_W n_W \, [W]$$

with $\mathrm{codim}_Y(W) = 2$. Let $W = \overline{\{w\}}$ and

$$R = O_{Y,w} \, .$$

Since $L_w \simeq O_{Y,w}$ one can assume that t (resp. s) corresponds to $a \in R$ (resp. $b \in R$). Since R is integral and $a \neq 0$, we know from the proof of Lemma 3 that, if $A = R/a\,R$,

$$\dim(A) \leq \dim(R) - 1 = 1 \, .$$

Let $\overline{b} \in A$ be the image of b and let $\overline{\wp} \subset A$ be a minimal prime ideal of A. The inverse image $\wp \subset R$ of $\overline{\wp}$ is a minimal nontrivial prime ideal. Since $a \in \wp$ we have $a\,R = \wp$ and $b \notin \wp$, otherwise $\mathrm{div}(t)$ is not prime to $\mathrm{div}(s)$. Hence $\overline{b} \notin \overline{\wp}$.

Furthermore $\bar{b}$ does not divide zero in A. Otherwise there exists a minimal prime ideal $\wp \supset a\,R$ such that $\bar{b}$ divides zero in $R/\wp$. Since $\bar{b} \neq 0$ and R is integral, we get a contradiction.

Since $\bar{b}$ does not divide zero we have $\dim(A/\bar{b}) \leq \dim(A) - 1$. But $\dim(A) \leq 1$. Therefore $\dim(A) = 1$ and $\dim(A/\bar{b}) = 0$. It follows that $A/\bar{b}$ has finite length. If $\langle a, b\rangle \subset R$ is the ideal spanned by a and b, $A/\bar{b} = R/\langle a, b\rangle$. We shall prove that

$$n_W = \ell(R/\langle a, b\rangle)\,.$$

2.7

Let A be as above and let M be an A-module of finite type. If $x \in A$ we have an exact sequence

$$0 \to M[x] \to M \xrightarrow{\times x} M \to M/x\,M \to 0\,. \tag{6}$$

If $M[x]$ and $M/x\,M$ have finite length we define

$$e(x, M) = \ell(M/x\,M) - \ell(M[x]) \in \mathbb{Z}\,.$$

Lemma 7. i) $M[\bar{b}]$ and $M/\bar{b}\,M$ have finite length.

ii)

$$e(\bar{b}, M) = \sum_{\substack{\wp \subset A \\ \wp \text{ minimal}}} \ell_{A_\wp}(M_\wp)\, e\,(\bar{b}, A/\wp)\,.$$

iii)

$$e(\bar{b}, A/\wp) = \ell(A/(\wp + b\,A))\,.$$

Proof of i) and ii). Note that both sides in ii) are additive in M. Therefore we can assume that $M = A/q$ where q is a prime ideal. We distinguish two cases:

a) If q is maximal, for any minimal prime ideal $\wp$ we have $M_\wp = 0$. Therefore $\ell(M)$ is finite. From Lemma 2 and (6) we conclude that

$$e(\bar{b}, M) = 0\,.$$

b) Assume $q = \wp$ is minimal. If $\wp' \neq \wp$ is any prime ideal different from $\wp$ we have

$$M_{\wp'} = 0\,.$$

Therefore the right hand side reduces to one summand and i) holds. Furthermore

$$\ell_{A_\wp}(M_\wp) = 1$$

and

$$e(\bar{b}, M) = e(\bar{b}, A/\wp)$$

so ii) is true.

To prove iii) it is also enough to check the case $M = A/\wp$. We saw that $b \notin \wp$ and $A/\wp$ is integral, therefore $M[\bar{b}] = 0$.

On the other hand

$$\dim(A/(\wp + bA)) \leq \dim(A/\wp) - 1 = 0\,.$$

Therefore

$$e(\bar{b}, A/\wp) = \ell(A/(\wp + bA))\,.$$

q.e.d.

2.8

We shall apply Lemma 7 to

$$M = A = R/a\,R\,.$$

Let $\wp$ be a minimal prime in A and $Y \subset |\mathrm{div}(s)|$ the corresponding component of the support of $\mathrm{div}(s)$. We have

$$\ell_{A_\wp}(A_\wp) = \mathrm{ord}_{A_\wp}(a) = \mathrm{ord}_Y(s)$$

and

$$\ell(A/(\wp + bA)) = \mathrm{ord}_W(t_{|Y})\,.$$

Lemma 7 iii) says that

$$e(\bar{b}, A) = n_W\,.$$

But $\bar{b}$ does not divide zero, so

$$e(\bar{b}, A) = \ell(R/\langle a, b\rangle)\,.$$

Therefore $n_W = \ell(R/\langle a, b\rangle)$. Since $\langle a, b\rangle = \langle b, a\rangle$ we conclude that

$$\mathrm{div}(s) \cdot \mathrm{div}(t) = \sum_W n_W[W] = \mathrm{div}(t) \cdot \mathrm{div}(s)\,.$$

This ends the proof of Proposition 1.

2.9

We shall now prove Proposition 2. For this we need some more analytic preliminaries. Let X be a smooth complex compact manifold of dimension d.

Definition. A *current* $T \in D^{pq}(X)$ is a $\mathbb{C}$-linear form

$$T : A^{d-p,d-q}(X) \to \mathbb{C}$$

which is continuous for the Schwartz' topology.

Examples. 1) If $\eta \in L^1(X) \underset{C^\infty(X)}{\otimes} A^{pq}(X)$ is an integrable differential, η defines a current by the formula

$$\eta(\omega) = \int_X \eta \wedge \omega \, .$$

2) If $Z = \sum_\alpha n_\alpha \, Z_\alpha$ is a cycle of codimension p on X, it defines a Dirac current $\delta_Z \in D^{pp}(X)$ by the formula

$$\delta_Z(\omega) = \sum_\alpha n_\alpha \int_{Z_\alpha} \omega \, ,$$

where the integrals converge by Hironaka's theorem.

We can derivate a current $T \in D^{pq}(X)$ by the formulae

$$\partial \, T(\omega) = (-1)^{p+q+1} \, T(\partial \, \omega)$$

and

$$\overline{\partial} \, T(\omega) = (-1)^{p+q+1} \, T(\overline{\partial} \, \omega) \, .$$

By the Stokes formula we get a commutative diagram

$$\begin{array}{ccc} D^{pq}(X) & \xrightarrow{\partial} & D^{p+1,q}(X) \\ \cup & & \cup \\ A^{pq}(X) & \xrightarrow{\partial} & A^{p+1,q}(X) \end{array}$$

and idem for $\overline{\partial}$ and $d = \partial + \overline{\partial}$.

Proposition 3 (Poincaré-Lelong). Let $\overline{L}$ be an hermitian line bundle on X and s a meromorphic section of L. Then we have the following formula in $D^{1,1}(X)$

$$dd^c(-\log \|s\|^2) + \delta_{\mathrm{div}(s)} = c_1(\overline{L}) \, . \tag{7}$$

2.10

To prove Proposition 3 let $Z = |\mathrm{div}(s)|$ be the support of the divisor of s. By Theorem 2, there exists a birational resolution

$$\pi : \widetilde{X} \to X$$

where $\pi^{-1}(Z)$ has local equation $z_1 \ldots z_k = 0$. Therefore

$$\pi^*(s) = z_1^{n_1} \ldots z_k^{n_k}$$

locally. If Proposition 3 holds for $\pi^*(\overline{L})$ and $\pi^*(s)$, by applying π_* we get (7). So we can assume that $X = \widetilde{X}$. By additivity we can assume that

a) $\|s\| = |z_1|$

or

b) $\log \|s\| = \rho \in C^\infty(X)$.

In case b) $\mathrm{div}(s) = 0$ and (7) is true by definition of $c_1(\overline{L})$ (Lemma 4). In case a) we have to show that, for every differerntial form ω, with compact support in U,

$$-\int_U \log |z_1|^2 \, dd^c(\omega) = \int_{|z_1|=\varepsilon} \omega \, .$$

But, by Stokes' theorem, we have

$$\begin{aligned} & -\lim_{\varepsilon\to 0} \int_{|z_1|\geq\varepsilon} \log |z_1|^2 \, dd^c(\omega) \\ = \; & \lim_{\varepsilon\to 0} \int_{|z_1|=\varepsilon} \log |z_1| \, d^c \omega + \lim_{\varepsilon\to 0} \int_{|z_1|\geq\varepsilon} d \log |z_1|^2 \, d^c \omega \, . \end{aligned}$$

The first summand vanishes and, applying Stokes' theorem again,

$$\begin{aligned} & \lim_{\varepsilon\to 0} \int_{|z_1|\geq\varepsilon} d \log |z_1|^2 \, d^c \omega = -\lim_{\varepsilon\to 0} \int_{|z_1|\geq\varepsilon} d^c \log |z_1|^2 \, d\omega \\ = \; & \lim_{\varepsilon\to 0} \int_{|z_1|=\varepsilon} d^c \log |z_1|^2 \, \omega - \lim_{\varepsilon\to 0} \int_{|z_1|\geq\varepsilon} dd^c \log |z_1|^2 \, \omega \, . \end{aligned}$$

The second summand vanishes and, taking polar coordinates $z_1 = r_1 \, e^{i\theta_1}$, we get

$$d^c \log |z_1|^2 = \frac{d\,\theta_1}{2\,\pi}$$

and

$$\lim_{\varepsilon\to 0} \int_{|z_1|=\varepsilon} \frac{d\,\theta_1}{2\,\pi} \, \omega = \int_{z_1=0} \omega \, .$$

q.e.d.

2.11

Coming back to Proposition 2 we consider the current

$$T_{s,t} = \delta_{\mathrm{div}(s)} \log \|t\|^2 + \log \|s\|^2 \, c_1(\overline{L}) \, .$$

Then

$$I(s,t) = T_{s,t}(c_1(\overline{L})^{d-1})/2 \, .$$

Proposition 3 implies

$$T_{s,t} = (c_1(\overline{L}) + dd^c \log \|s\|^2) \log \|t\|^2 + \log \|s\|^2 \, c_1(\overline{L})$$

at least formally: we have to make sense of the product of currents $(dd^c \log \|s\|^2) \log \|t\|^2$. By Stokes' theorem we have (at least formally)

$$\begin{aligned} dd^c(T_1) \, T_2 &= d(d^c(T_1) \, T_2) + d^c(T_1) \, d(T_2) \\ &= d(d^c(T_1) \, T_2) + d^c(T_1 \, d\, T_2) - T_1 \, d^c d(T_2) \, . \end{aligned}$$

Since $d^c d = -dd^c$ and $d(c_1(\overline{L})^{d-1}) = d^c(c_1(\overline{L})^{d-1}) = 0$ we get

$$2I(s,t) = T_{s,t}(c_1(\overline{L})^{d-1}) = T_{t,s}(c_1(\overline{L})^{d-1}) = 2I(t,s)\,.$$

q.e.d.

2.12 The height of the projective space

Let $N \geq 1$ be an integer and $\mathbb{P}^N$ the N-dimensional projective space over $\mathbb{Z}$. The tautological line bundle $O(1)$ on $\mathbb{P}^N$ is a quotient of the trivial vector bundle of rank $N+1$

$$O_{\mathbb{P}^N}^{N+1} \to O(1) \to 0\,.$$

We equip $O_{\mathbb{P}^N}^{N+1}$ with the trivial metric and $O(1)$ with the quotient metric.

Proposition 4. The height of $\mathbb{P}^N$ is

$$h_{\overline{O(1)}}(\mathbb{P}^N) = \frac{1}{2} \sum_{k=1}^{N} \sum_{m=1}^{k} \frac{1}{m}\,.$$

Proof of Proposition 4. Let s be the section of $O(1)$ defined by the homogeneous coordinate X_0. Then $\mathrm{div}(s) = \mathbb{P}^{N+1}$ and we get, from Theorem 1 ii),

$$h(\mathbb{P}^N) = h(\mathbb{P}^{N-1}) - \int_{\mathbb{P}^N(\mathbb{C})} \log \|s\| \, d\mu$$

where $d\mu$ is the probability measure on $\mathbb{P}^N(\mathbb{C})$ invariant under rotation by $U(N+1)$. If dv is the probability measure on the sphere S^{2N+1} invariant under $U(N+1)$ we have

$$\int_{\mathbb{P}^N(\mathbb{C})} \log \|s\| \, d\mu = \int_{S^{2N+1}} \log |X_0| \, dv$$

and Proposition 4 follows from

Lemma 8.

$$\int_{S^{2N+1}} \log |X_0| \, dv = \frac{1}{2} \sum_{m=1}^{N} \frac{1}{m}\,.$$

3 Arithmetic Chow groups

3.1 Definition

Let X be a regular projective flat scheme over $\mathbb{Z}$ and $p \geq 0$ an integer. Let $Z^p(X)$ be the group of codimension p cycles on X.

Definition. A *Green current* for $Z \in Z^p(X)$ is a real current $g \in D^{p-1,p-1}(X(\mathbb{C}))$ such that $F_\infty^*(g) = (-1)^{p-1}\, g$ and

$$dd^c\, g + \delta_Z = \omega$$

is a smooth form $\omega \in A^{pp}(X(\mathbb{C}))$.

We let $\widehat{Z}^p(X)$ be the group generated by pairs (Z, g), $Z \in Z^p(X), g$ Green current for Z, with $(Z_1, g_1) + (Z_2, g_2) = (Z_1 + Z_2, g_1 + g_2)$.

Examples. i) Let $Y \subset X$ be a closed irreducible subset with $\operatorname{codim}_X(Y) = p-1$, and $f \in k(Y)$ a rational function on Y. Define $\log|f|^2 \in D^{p-1,p-1}(X(\mathbb{C}))$ by the formula

$$(\log|f|^2)(\omega) = \int_{Y(\mathbb{C})} \log|f|^2\, \omega$$

(which makes sense by Theorem 2). We may think of f as a rational section of the trivial line bundle on Y. Therefore Poincaré-Lelong formula (Proposition 3) reads

$$dd^c(-\log|f|^2) + \delta_{\operatorname{div}(f)} = 0\,.$$

Hence the pair

$$\widehat{\operatorname{div}}(f) = (\operatorname{div}(f), -\log|f|^2)$$

is an element of $\widehat{Z}^p(X)$.
ii) Given $u \in D^{p-2,p-1}(X(\mathbb{C}))$ and $v \in D^{p-1,p-2}(X(\mathbb{C}))$ we have

$$dd^c(\partial\, u + \overline{\partial}\, v) = 0\,,$$

so $(0, \partial\, u + \overline{\partial}\, v) \in \widehat{Z}^p(X)$.

We let $\widehat{R}^p(X) \subset \widehat{Z}^p(X)$ be the subgroup generated by all elements $\widehat{\operatorname{div}}(f)$ and $(0, \partial\, u + \overline{\partial}\, v)$.

Definition. The *arithmetic Chow group* of codimension p of X is the quotient

$$\widehat{\operatorname{CH}}^p(X) = \widehat{Z}^p(X)/\widehat{R}^p(X)\,.$$

3.2 Example

Let $\widehat{\operatorname{Pic}}(X)$ be the group of isometric isomorphism classes of hermitian line bundles on X, equipped with the tensor product.

If $\overline{L} = (L, \|\cdot\|) \in \widehat{\operatorname{Pic}}(X)$ and if $s \neq 0$ is a rational section of L we let

$$\widehat{\operatorname{div}}(s) = (\operatorname{div}(s), -\log\|s\|^2) \in \widehat{Z}^1(X)$$

(Proposition 3), and we define

$$\widehat{c}_1(\overline{L}) \in \widehat{\operatorname{CH}}^1(X)$$

to be the class of $\widehat{\mathrm{div}}(s)$. It does not depend on the choice of s: if s' is another section of L we have

$$s' = f\, s$$

with $f \in k(X)$. Therefore

$$\widehat{\mathrm{div}}(s') - \widehat{\mathrm{div}}(s) = \widehat{\mathrm{div}}(f) \in \widehat{R}^1(X)\,.$$

Proposition 5. The map $\widehat{c}_1$ induces a group isomorphism

$$\widehat{c}_1 : \widehat{\mathrm{Pic}}(X) \to \widehat{\mathrm{CH}}^1(X)\,.$$

To prove Proposition 5 we consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^\infty(X(\mathbb{C})) & \xrightarrow{a} & \widehat{\mathrm{Pic}}(X) & \xrightarrow{\zeta} & \mathrm{Pic}(X) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \widehat{c}_1} & & \downarrow{\scriptstyle c_1} & & \\
0 & \longrightarrow & C^\infty(X(\mathbb{C})) & \xrightarrow{a'} & \widehat{\mathrm{CH}}^1(X) & \xrightarrow{\zeta'} & \mathrm{CH}^1(X) & \longrightarrow & 0
\end{array}$$

where $a(\varphi)$ is the trivial line bundle on X equipped with the norm such that $\|1\| = \exp(\varphi)$, $\zeta(\overline{L}) = L$, $a'(\varphi) = (0, -\log|\varphi|^2)$ and $\zeta(Z,g) = Z$. Since c_1 is an isomorphism the same is true for $\widehat{c}_1$.

3.3 Products

3.3.1

Denote by $\widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}}$ the tensor product $\widehat{\mathrm{CH}}^p(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

Theorem 3. When $p \geq 0$ and $q \geq 0$ there is an intersection pairing

$$\begin{array}{rcl}
\widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q(X) & \longrightarrow & \mathrm{CH}^{p+q}(X)_{\mathbb{Q}} \\
x \otimes y & \longmapsto & x \cdot y
\end{array}$$

It makes $\underset{p\geq 0}{\oplus} \widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}}$ a commutative graded $\mathbb{Q}$-algebra.

Let $\zeta : \widehat{\mathrm{CH}}^p(X) \to \mathrm{CH}^p(X)$ be the map sending the class of (Z,g) to the class of Z, and let $\omega : \widehat{\mathrm{CH}}^p(X) \to A^{pp}(X)$ be the map sending (Z,g) to $dd^c g + \delta_Z$. Then

$$z(x \cdot y) = z(x)\, z(y)$$

and

$$\omega(x \cdot y) = \omega(x)\, \omega(y)\,.$$

3.3.2

To prove Theorem 3, let $y = (Y, g_Y) \in \widehat{Z}^p(X)$ and $z = (Z, g_Z) \in \widehat{Z}^q(X)$.

We first define a cycle $Y \cap Z$. For this we assume that the restrictions $Y_{\mathbb{Q}}$ and $Z_{\mathbb{Q}}$ of Y and Z to the generic fiber $X_{\mathbb{Q}}$ meet properly, i.e. the components of $|Y_{\mathbb{Q}}| \cap |Z_{\mathbb{Q}}|$ have codimension $p+q$ (the moving lemma allows one to make this hypothesis). It follows that there exists a well defined intersection cycle $Y_{\mathbb{Q}} \cdot Z_{\mathbb{Q}} \in Z^{p+q}(X_{\mathbb{Q}})$, supported on the closed set $|Y_{\mathbb{Q}}| \cap |Z_{\mathbb{Q}}|$. Let

$$\mathrm{CH}^p_Y(X) = \ker(\mathrm{CH}^p(X) - \mathrm{CH}^p(X - Y))$$

be the Chow group with supports in Y, and $\mathrm{CH}^p_{\mathrm{fin}}(X)$ the Chow group with supports in finite fibers of X. There is a canonical map

$$\mathrm{CH}^p_Y(X) = \mathrm{CH}^p_{\mathrm{fin}}(X) \oplus Z^p(X_{\mathbb{Q}}) \,.$$

One can define an intersection paring

$$\mathrm{CH}^p_Y(X) \otimes \mathrm{CH}^q_Z(X) \to \mathrm{CH}^{p+q}_{Y \cap Z}(X)_{\mathbb{Q}} \,.$$

One method to do so ([1], [2], [5]) is to interpret $\mathrm{CH}^p_Y(X)_{\mathbb{Q}}$ as the subspace of $K_0^Y(X)_{\mathbb{Q}}$ where the Adams operations ψ^k act by multiplication by k^p $(k \geq 1)$, and to use the tensor product

$$K_0^Y(X) \otimes K_0^Z(X) \to K_0^{Y \cap Z}(X) \,.$$

We let $Y \cap Z \in \mathrm{CH}^{p+q}_{\mathrm{fin}}(X)_{\mathbb{Q}} \oplus Z^{p+q}(X_{\mathbb{Q}})_{\mathbb{Q}}$ be the image of

$$[Y] \otimes [Z] \in \mathrm{CH}^p_Y(X) \otimes \mathrm{CH}^p_Z(X)$$

by the maps

$$\mathrm{CH}^p_Y(X) \otimes \mathrm{CH}^q_Z(X) \to \mathrm{CH}^{p+q}_{Y \cap Z}(X)_{\mathbb{Q}} \to \mathrm{CH}^{p+q}_{\mathrm{fin}}(X)_{\mathbb{Q}} \oplus Z^{p+q}(X)_{\mathbb{Q}} \,.$$

Next we define a Green current for $Y \cap Z$. For this we write

$$dd^c\, g_Y + \delta_Y = \omega_Y$$

and

$$dd^c\, g_Z + \delta_Z = \omega_Z \,,$$

and we let

$$g_Y * g_Z = \delta_Y\, g_Z + g_Y\, \omega_Z \,.$$

However $g_Y\, \delta_Z$, being a product of currents, is not well defined a priori. But g_Y is defined up to the addition of a term $\partial(u) + \overline{\partial}(v)$ and one shows that g_Y can be chosen to be an L^1-form on $X(\mathbb{C}) - Y(\mathbb{C})$, with restriction an L^1-form η on $Z(\mathbb{C}) - Z(\mathbb{C}) \cap Y(\mathbb{C})$. We let $g_Y\, \delta_Z$ be the current defined by η on $Z(\mathbb{C})$ (see above Example 1) in §2.9):

$$g_Y\, \delta_Z(\omega) = \int_{Z(\mathbb{C}) - (Z(\mathbb{C}) \cap Y(\mathbb{C}))} \eta\, \omega \,.$$

To prove that $g_Y * g_Z$ is a Green current for $Y \cap Z$ we proceed formally:

$$\begin{aligned} dd^c(g_Y * g_Z) &= dd^c(\delta_Y\, g_Z) + dd^c(g_Y\, \omega_Z) \\ &= \delta_Y\, dd^c(g_Z) + dd^c(g_Y)\, \omega_Z \\ &= \delta_Y (\omega_Z - \delta_Z) + \omega_Y - \delta_Y)\, \omega_Z \\ &= \omega_Y\, \omega_Z - \delta_Y\, \delta_Z \\ &= \omega_Y\, \omega_Z - \delta_{Y\cap Z}\,. \end{aligned}$$

We refer to [2] for the justification of this series of equalities.

3.4 Functoriality

Let $f : X \to Y$ be a morphism.

Theorem 5. For every $p \geq 0$ there is a morphism

$$f^* : \widehat{\mathrm{CH}}^p(Y) \to \widehat{\mathrm{CH}}^p(X)\,.$$

If the restriction of f to $X(\mathbb{C})$ is a smooth map of complex manifolds, there are morphisms

$$f_* : \widehat{\mathrm{CH}}^p(X) \to \widehat{\mathrm{CH}}^{p+\dim(Y)-\dim(X)}(Y)\,.$$

Both f^* and f_* are compatible to ζ and ω. Furthermore

$$f^*(x \cdot y) = f^*(x) \cdot f^*(y)$$

and

$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y\,.$$

3.5 Heights and intersection numbers

3.5.1

Let X be a projective regular flat scheme over $\mathbb{Z}$ and $Y \subset X$ a closed integral subscheme. We assume that X is equidimensional of dimension d and $\mathrm{codim}_X(Y) = p$. One can then define as follows a morphism

$$\int_Y : \widehat{\mathrm{CH}}^{d-p}(Y) \to \mathbb{R}\,.$$

First, assume that $X = Y$ and that $x \in \widehat{\mathrm{CH}}^d(X)$ is the class of (Z, g_Z) where Z is a zero-cycle and $g_Z \in D^{d-1,d-1}(X(\mathbb{C}))$. The cycle Z is then a finite sum

$$Z = \sum_\alpha n_\alpha\, y_\alpha$$

where y_α is a closed point with finite residue field $k(y_\alpha)$, and there exist currents u and v such that $\eta_Z = g_Z + \partial(u) + \overline{\partial}(v)$ is smooth. By definition

$$\int_X x = \sum_\alpha n_\alpha \log \# (k(y_\alpha)) - \frac{1}{2} \int_{X(\mathbb{C})} \eta_Z \, .$$

In general we let g_Y be a Green current for Y in $X(\mathbb{C})$, and $y = (Y, g_Y)$. If $x \in \widehat{\mathrm{CH}}^{d-p}(Y)$ we have $x \cdot y \in \widehat{\mathrm{CH}}^{d}(X)$ and we define

$$\int_Y x = \int_X x \cdot y - \frac{1}{2} \int_{X(\mathbb{C})} \omega(x) \, g_Y \, .$$

One checks that this number is independent on the choice of g_Y.

Theorem 5. The height of Y is

$$h_{\overline{L}}(Y) = \int_Y \widehat{c}_1(\overline{L})^{d-p} \, .$$

3.5.2

To prove Theorem 5 we shall check that the two properties in Theorem 1 hold true for the number $\int_Y \widehat{c}_1(\overline{L})^{d-p}$.

When $p = d$, Y is a closed point y and, if x is the class of $(y, 0)$ in $\widehat{\mathrm{CH}}^{d}(X)$, we have

$$\int_X x = \log \# k(y) = h_{\overline{L}}(Y) \, .$$

Assume $\dim(Y) > 0$. Let g_Y be a Green current for Y and $y = (Y, g_Y)$. Close a rational section s of L on Y, and an extension $\widetilde{s}$ of s to X. Then

$$\widehat{c}_1(\overline{L}) = (\mathrm{div}(\widetilde{s}), -\log \|\widetilde{s}\|^2) \, .$$

If $x = \widehat{c}_1(\overline{L})^{d-p-1}$ we get, from the definition of $\int_Y$,

$$\int_Y x \, \widehat{c}_1(\overline{L}) = \int_X x \, \widehat{c}_1(\overline{L}) \, y - \frac{1}{2} \int_{X(\mathbb{C})} \omega(x \, \widehat{c}_1(\overline{L})) \, g_Y \, . \tag{8}$$

But

$$\begin{aligned} x \cdot \widehat{c}_1(\overline{L}) \cdot y &= x \cdot (\mathrm{div}(\widetilde{s} \mid Y), -\log \|\widetilde{s}\|^2 * g_Y) \\ &= x \cdot (\mathrm{div}(s), -\log \|\widetilde{s}\|^2 \, \delta_Y + c_1(\overline{L}) \, g_Y) \, . \end{aligned}$$

If $x =: \widehat{c}_1(\overline{L})^{d-p-1}$ is the class of (Z, g_Z), we get

$$x \cdot \widehat{c}_1(\overline{L}) \cdot y = (Z \cdot \mathrm{div}(s), \omega(x)(-\log \|\widetilde{s}\|^2 \, \delta_Y + c_1(\overline{L}) \, g_Y) + g_Z \, \delta_{\mathrm{div}(s)}) \, . \tag{9}$$

Since

$$\int_X (Z \cdot \mathrm{div}(s), g_Z \, \delta_{\mathrm{div}(s)}) = \int_{\mathrm{div}(s)} x$$

we deduce from (9) that

$$(10)\quad \int_X x\cdot \widehat{c}_1(\overline{L})\cdot y = \int_{\mathrm{div}(s)} x - \frac{1}{2}\int_{Y(\mathbb{C})} \omega(x)\log\|s\|^2 + \frac{1}{2}\int_{X(\mathbb{C})} \omega(x)\, c_1(\overline{L})\, g_Y\,.$$

Since $\omega(x\,\widehat{c}_1(\overline{L})) = \omega(x)\, c_1(\overline{L}_{\mathbb{C}})$, (8) and (10) imply that

$$\int_Y \widehat{c}_1(\overline{L})^{d-p} = \int_{\mathrm{div}(s)} \widehat{c}_1(\overline{L})^{d-p-1} - \frac{1}{2}\int_{Y(\mathbb{C})} c_1(\overline{L})^{d-p-1}\log\|s\|\,.$$

q.e.d.

References

[1] Gillet, H., Soulé, C.: Intersection theory using Adams operations, *Inventiones Math.* **90**,1987,243-277.

[2] Gillet, H., Soulé, C.: Arithmetic intersection theory. *Inst. Hautes études Sci. Publ. Math.* **72** (1990), 93–174.

[3] Bost, J.-B.; Gillet, H.; Soulé, C.: Heights of projective varieties and positive Green forms. J. Am. Math. Soc. 7, No.4, 903-1027 (1994)

[4] Faltings, G.: Diophantine approximation on abelian varieties, Ann. of Math. (2) **133** (1991), no. 3, 549–576.

[5] Soulé, C., Abramovich, D., Burnol, J.-F. and Kramer, J.: Lectures on Arakelov Geometry. Cambridge University Press 1991.