Cours

École d'été 2017

Slopes and distribution of points

Plan

I) Heights and distribution
  1) Adelic metrics
  2) Arakelov heights
  3) Equidistribution
  4) Accumulating subsets

II) Slopes, freeness
  1) Definition
  2) Properties
  3) First examples ($\mathbb{P}^n$, $\mathbb{P}^1 \times \mathbb{P}^1$)
  4) Accumulating sets (rational curves fibrations)
  5) Open questions

III) Local distribution
  1) Mc Kinnon et Roth approximation
  2) Locally accumulating subsets
  3) Using slopes.

My aim in these lectures is to explain why I think the slopes à la Bost, which were introduced in the lectures of Éric GAUDRON, might be a central tool to study the distribution of rational points of bounded height on Fano-like varieties

Slopes and distribution of points

I Heights and distributions

1) Adelic metric

In these talks, I am going to use heights given by an adelic metric, wich is the analog of a Riemannian metric in the adelic setting. Let me explain this notion and fix my notations.

Notation

$\mathbb{K}$ is a number field,
$Val(\mathbb{K})$ the set of places of $\mathbb{K}$
For $w \in Val(\mathbb{K})$, $\mathbb{K}_w$ is the completion and $v \in Val(\mathbb{Q})$ restriction of w

$$|\cdot|_w : \mathbb{K}_w \longrightarrow \mathbb{R}_{\geq 0}$$

given by

$$|x|_w = |N_{\mathbb{K}_w/\mathbb{Q}_v}(x)|_v$$

which is the multiplyer for the Haar measure
dx_w Haar measure on $\mathbb{K}_w$
normalyzed by

$$\begin{cases} \int_{\mathcal{O}_w} dx_w = 1 & \text{if } w \text{ ultrametric} \\ dx_w = \text{Lebesgue measure} & \text{if } \mathbb{K}_w \xrightarrow{\sim} \mathbb{R} \\ dx_w = 2\,dx\,dy & \text{if } \mathbb{K}_w \xrightarrow{\sim} \mathbb{C} \end{cases}$$

We have

$$d(\lambda x)_w = |\lambda|_w dx_w$$

and the product formula

$$\forall \lambda \in \mathbb{K}^*, \prod_{w \in Val(\mathbb{K})} |\lambda| = 1$$

V is a nice variety$/\mathbb{K}$ that is smooth, projective and geometrically integral.

Definition

Let $\pi : E \to V$ be a vector bundle on V.
In these talks a (classical) adelic norm on E is a family $(\|\cdot\|_w)_{w \in Val(\mathbb{K})}$ of continuous maps

$$\|\cdot\|_w : E(\mathbb{K}_w) \longrightarrow \mathbb{R}_{\geqslant 0}$$

such that

(i) if w is ultrametric, $\forall x \in V(\mathbb{K}_w)$ $\|\cdot\|_w|_{E_x}$ is an ultrametric norm with values in $\mathrm{im}(|\cdot|_w)$

(ii) if $\mathbb{K}_w \simeq \mathbb{R}$, $\forall x \in V(\mathbb{K}_w)$ $\|\cdot\|_w|_{E_x}$ is euclidean

(iii) if $\mathbb{K}_w \xrightarrow{\sim} \mathbb{C}$, $\forall x \in V(\mathbb{K}_w)$ $\exists\, h_x$ positive definite hermitian form on E_x such that

$$\|y\|_w = h_x(y) \text{ for } y \in E_x.$$

(iv) $\exists$ a model $\mathcal{E}/\mathcal{V}$ of E/V over $\mathcal{O}_S$ for some finite $S \subset Val(\mathbb{K})$ such that $\forall w \in Val(\mathbb{K}) - S$, $\forall x \in \mathcal{V}(\mathcal{O}_w)$,

$$\mathcal{E}_x = \{ y \in E_x \mid \|y\|_w \leqslant 1 \}$$

We call adelic metric on V an adelic norm on TV.

Examples

The point of using this type of metrics is that you may do the usual constructions

1) direct sums $E \oplus F$, tensor product $E \otimes F$ and exterior product $\Lambda^k E$
   (if v is archimedean and $(e_1, \ldots, e_n)$ an orthonormal basis on E_x for $\|\cdot\|_v$ then $(e_{i_1} \wedge \ldots \wedge e_{i_k})_{i_1 < \ldots < i_k}$ is an orthonormal basis of $\Lambda^k E$)
   on the dual $E^\vee$

   In particular an adelic metric on V defines an adelic norm on $\omega_V^{-1} = \Lambda^n TV$ where $n = \dim(V)$.

2) we can define pull-backs for morphisms
$$\phi : X \to Y$$
of nice varieties $/\mathbb{K}$.

3) If $V = \mathrm{Spec}(\mathbb{K})$ $\quad E = \mathbb{K}$ vector space equipped with an adelic norm $(\|\cdot\|_w)_{w \in Val(\mathbb{K})}$
$$\mathcal{E} = \{ y \in E \mid \forall \text{ ultrametric } v \quad \|y\|_v \leq 1 \}$$
is a projective $\mathcal{O}_{\mathbb{K}}$ module of rank $r = \dim(E)$

If $r = 1$, by the product formula
$$\prod_{w \in Val(\mathbb{K})} \|y\|_w$$
is constant for $y \in E - \{0\}$

So we can define

$$\widehat{\deg}(E) = -\sum_{w \in Val(\mathbb{K})} \log \|y\|_w$$

Let $\widehat{Pic}(Spec(\mathbb{K}))$ be the set of isomorphisms class of line bundles with an adelic norm on $spec(\mathbb{K})$ then we get an exact sequence

$$0 \to Pic(Spec(O_{\mathbb{K}})) \to \widehat{Pic}(Spec(\mathbb{K})) \xrightarrow{\widehat{\deg}} \mathbb{R} \to 0$$

For arbitrary rank r, we may define

$$\widehat{\deg}(E) = \widehat{\deg}(\Lambda^r E).$$

2) Arakelov heights

Definition

For any vector bundle E/V equipped with an adelic norm, the corresponding logarithmic height is defined by

$$h_E : V(\mathbb{K}) \to \mathbb{R}$$
$$x \mapsto \widehat{\deg}(E_x)$$

In fact E_x = pull-back of E by $x : Spec(\mathbb{K}) \to V$.

The exponential height is $H_E = \exp \circ h_E$

Remark

If $r = rk(E)$ $h_E = h_{\Lambda^r E} = h_{\det(E)}$.

So we do not get more than the heights define by line bundle.

Example

$$\forall w \in Val(\mathbb{K}) \quad \|\cdot\|_w : \mathbb{K}_w^{N+1} \longrightarrow \mathbb{R}_{\geq 0}$$
$$(x_0, \ldots, x_N) \mapsto \max_{0 \leq i \leq n} |x_i|_w$$

The tautological line bundle $\mathcal{O}_{\mathbb{P}^N}(-1) \to \mathbb{P}^N_{\mathbb{K}}$ may be described as follows:

If $x \in \mathbb{P}^N(\mathbb{K}_w)$, $\mathcal{O}(-1)_x$ is the line corresponding to x in $\mathbb{K}_w^{N+1}$; by restricting $\|\cdot\|_w$ to these lines we get an adelic norm $(\|\cdot\|_w)_{w\in Val(\mathbb{K})}$ on $\mathcal{O}_{\mathbb{P}^N}(-1)$ and by duality on $\mathcal{O}_{\mathbb{P}^N}(1)$.

If $(x_0, \dots, x_N) \in \mathbb{K}^{N+1}$ $\quad x = [x_0 : \dots : x_N]$

$y = (x_0, \dots, x_n) \in \mathcal{O}(-1)$

$$H_{\mathcal{O}(-1)}(x) = \prod_{w \in Val(\mathbb{K})} \|y\|_w^{-1}$$

So
$$H_{\mathcal{O}(1)}(x) = \prod_{w \in Val(\mathbb{K})} \|y\|_w$$

If $\mathbb{K} = \mathbb{Q}$ and $x_0, \dots, x_N$ are coprime integers, then $\|(x_0, \dots, x_N)\|_v = 1$ for $v \neq \infty$ and we get

$$H_{\mathcal{O}(1)}(x) = \max_{0 \leq i \leq N} |x_i|$$

which is the naïve height on the projective space

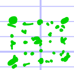
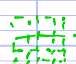
Notation

For any height $H : V(\mathbb{K}) \longrightarrow \mathbb{R}_{\geq 0}$, $W \subset V(\mathbb{K})$ and $B \geq 0$ we consider the set

$$W_{H \leq B} = \{P \in V(\mathbb{K}) \mid H(P) \leq B\}$$

which we want to study as B goes to ∞ty

Illustration

A few pictures $\mathbb{P}^2_{\mathbb{Q}}$  $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$  $S^2_{\mathbb{Q}}$ 

Diophantine statistics

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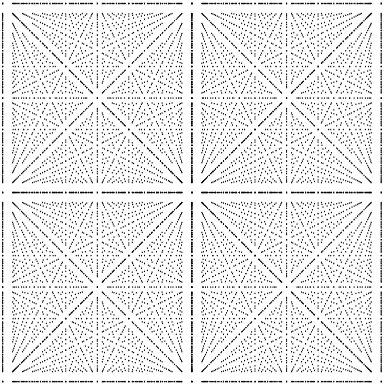
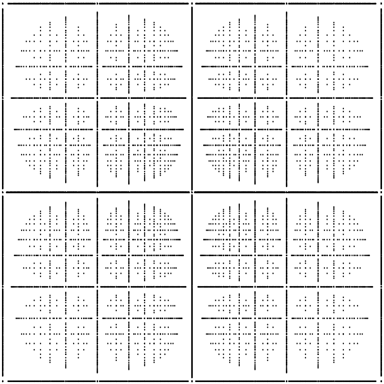
Projective plane



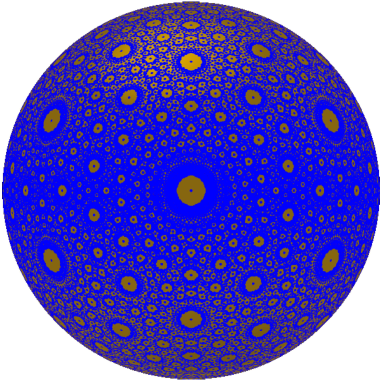
$$\{\,[x:y:1] \in \mathbf{P}^2(\mathbf{Q}) \mid H(x:y:1) < 40,\ |x| \leqslant 1 \text{ and } |y| \leqslant 1\,\}$$

The hyperboloid of one sheet



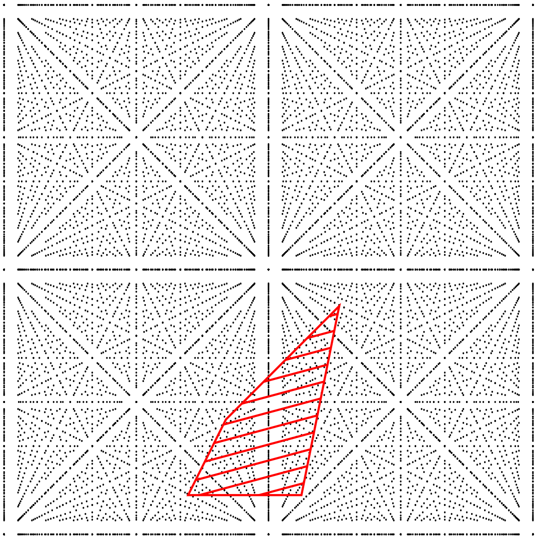
$$\{ P=[x:y:z:t]\in \mathbf{P}^3(\mathbf{Q}) \mid H(P)\leqslant 50,\ |x|\leqslant 1, \\ |y|\leqslant 1, z=1 \text{ and } xy=zt\}$$

The sphere



$$\{\, P = [x : y : z : t] \in \mathbf{P}^3(\mathbf{Q}) \mid H(P) \leqslant B \text{ and } x^2 + y^2 + z^2 = t^2 \}$$

$\mathbf{P}^2_{\mathbf{Q}}$



$Y^{+}Z^{2} = X^{3} - X$

Proposition

If L is big, then there exists $U \subset V$ dense open subset for Zariski topology such that

$$\forall B \in \mathbb{R}_{\geq 0}, \quad U(\mathbb{K})_{H \leq B} \text{ is finite}$$

Proof

Since L is big $\exists M > 0$ and $U \subset V$ such that the rational map

$$V \dashrightarrow \mathbb{P}(H^0(V, L^{\otimes M})^{\vee}) \simeq \mathbb{P}^N_{\mathbb{K}}$$

dual

is an immersion on U (write $L = F + M$, F effective and M ample and $U = V -$ base locus of F)

By Northcott's theorem

$$\# \mathbb{P}^N(\mathbb{K})_{H \leq B} \text{ is finite}. \quad \square$$

A natural question is the dependance of the height on the choices of the metric:

Prop

Let H and H' be defined by norms on a line bundle L then H/H' is bounded:

$$\exists \ 0 < c < c', \ \forall x \in V(K) \qquad c < \frac{H'(x)}{H(x)} < c'$$

Proof

$$\frac{\|\cdot\|'_w}{\|\cdot\|_w} : \underbrace{\mathbb{P}(L)(\mathbb{K}_w)}_{\text{compact}} \longrightarrow \mathbb{R}_{>0} \text{ continuous, thus bounded}$$

and $\equiv 1$ for almost all w.

3) Equidistribution

During these lectures, I will concentrate on the distribution of the points of bounded height:

a) Basic example: $\mathbb{P}^n_{\mathbb{K}}$

What do we mean by distribution

Question

Let $\mathbb{P}^n(\mathbb{A}_{\mathbb{K}}) = \prod_{w \in Val(\mathbb{K})} \mathbb{P}^n(\mathbb{K}_w)$ and $f : \mathbb{P}^n(\mathbb{A}_{\mathbb{K}}) \to \mathbb{R}$ be a continuous function

Does $S_B(f) = \dfrac{\sum_{x \in \mathbb{P}^n(\mathbb{K})_{H \leq B}} f(x)}{\# \, \mathbb{P}^n(\mathbb{K})_{H \leq B}}$ have a limit as $B \to +\infty$?

The answer is positive

Proposition

$$S_B(f) \xrightarrow[B \to +\infty]{} \int_{\mathbb{P}^n(\mathbb{A}_{\mathbb{K}})} f \, \mu_{\mathbb{P}^n}$$

where $\mu_{\mathbb{P}^n} = \prod_{w \in Val(\mathbb{K})} \mu_w$
for borelian measure ω_w defined by
- if w is non-archimedean:
For $\pi_k : \mathbb{P}^n(\mathbb{K}_w) \to \mathbb{P}^n(\mathcal{O}_w / m_w^k)$
$\mu_w(\pi_k^{-1}(X)) = \dfrac{\# X}{\# \, \mathbb{P}^n(\mathcal{O}_w/m_w^k)}$ natural counting measure
- if w is archimedean
for $\pi : \mathbb{K}_w^{n+1} - \{0\} \to \mathbb{P}^n(\mathbb{K}_w)$
$$\mu_w(U) = \frac{Vol(\pi^{-1}(U) \cap B_{\|\cdot\|_w}(1))}{Vol(B_{\|\cdot\|_w}(1))}.$$

Remark

If $W \subset \mathbb{P}^n(\mathbb{A}_{\mathbb{K}})$ open is such that $\omega_{\mathbb{P}^n}(\partial W) = 0$ then this implies that
$$\frac{\# (W \cap \mathbb{P}^n(\mathbb{K}))_{H \leq B}}{\# \, \mathbb{P}^n(\mathbb{K})_{H \leq B}} \xrightarrow[B \to +\infty]{} \omega_{\mathbb{P}^n}(W).$$

Sketch of the proof for $\mathbb{K} = \mathbb{Q}$

Take a cube $\mathcal{C}$ in $\mathbb{R}^n \hookrightarrow \mathbb{P}^n(\mathbb{R})$, $x_0 \in \mathbb{P}^n(\mathbb{Z}/M\mathbb{Z})$

We want to estimate

$$\# \{ x \in \mathbb{P}^n(\mathbb{Q}) \mid H(x) \leq B, \; x \in \mathcal{C}, \; x \equiv x_0 \pmod M \}$$

$$= \frac{1}{2} \# \{ y \in \mathbb{Z}^{n+1} \mid y \text{ primitive}, \|y\| \le B, y \in \pi^{-1}(\mathcal{C}), y \equiv \lambda y_0 (M) \} \qquad \pi(y_0) = x_0$$

$$= \frac{1}{2} \sum_{\substack{d > 0 \\ \lambda \in (\mathbb{Z}/M\mathbb{Z})^*}} \mu(d) \, \overbrace{\underbrace{\# \{ y \in d\mathbb{Z}^{n+1} - \{0\} \mid \|y\| \le B, y \in \pi^{-1}(\mathcal{C}), y \equiv \lambda y_0 (M) \}}_{\sim \frac{\mathrm{Vol}\{ y \in \pi^{-1}(\mathcal{C}) \mid \|y\| \le B \}}{(dM)^{n+1}} \text{ if } \gcd(d,M)=1 \\ = 0 \text{ otherwise}}}^{0 \text{ if } \gcd(d,M) \neq 1}$$

One gets

$$\frac{1}{2} \mathrm{Vol}(\pi^{-1}(\mathcal{C}) \cap B_{\|\cdot\|_\infty}(1)) \times \underbrace{\frac{\prod_{p \mid M} (1 - \frac{1}{p^{n+1}})^{-1}}{M^{n+1}} \varphi(M)}_{= \frac{1}{\# \mathbb{P}^n(\mathbb{Z}/M\mathbb{Z})}} \frac{1}{\zeta_{\mathbb{Q}}(n+1)} B^{n+1} + \text{error term}$$

□

b) Adelic measure

By choosing different norms, and thus different heights on $\mathbb{P}^n(\mathbb{Q})$, one realizes that the measures which gives the asymptotic distribution as $B \to +\infty$ may be directly defined from the adelic norm on ω_V^{-1}, exactly as a Riemannian metric defines a volume form. Moreover this construction aplies to any nice variety equipped with an adelic metric

Construction

V nice variety with $V(\mathbb{K}) \neq \emptyset$.

Let $(\|\cdot\|_w)_{w \in \mathrm{Val}(\mathbb{K})}$ be an adelic norm on ω_V^{-1}.

The formula for change of variables assures that the local measure

$$\left\| \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n} \right\|_w dx_{1,w} dx_{2,w} \dots dx_{n,w}$$

where $(x_1, \dots, x_n) : \underset{V(\mathbb{K}_w)}{\Omega \cap} \to \mathbb{K}_w^n$ is a local system of coordinates does not depend on the choice of coordinates; so, by glueing, it defines a measure ω_v on $V(\mathbb{K}_v)$, $\mathbb{N}_v = \frac{1}{\omega_v(V(\mathbb{K}_v))}\omega_v$ is a probability measure

$$\mathbb{N}_V = \prod_{v \in Val(\mathbb{K})} \mathbb{N}_v$$

is a probability measure on $V(\mathbb{A}_{\mathbb{K}})$.

For $\mathbb{P}^n$ it gives the right distribution.

Does it work in other cases? More precisely:

Definition

The counting measure on V with bound B is defined by

For $W \subset V(\mathbb{K})$ $W \neq \emptyset$ Dirac measure at x

$$\delta_{W_{H \leq B}} = \frac{1}{\# W_{H \leq B}} \sum_{x \in W_{H \leq B}} \delta_x$$

Over optimistic question

Does $\delta_{V(\mathbb{K})_{H \leq B}} \xrightarrow[B \to +\infty]{\text{weakly}} \mathbb{N}_V$? (NE) naïve

(that is $\int_{V(\mathbb{A}_{\mathbb{K}})} f \, \delta_{V(\mathbb{K})_{H \leq B}} \xrightarrow[B \to +\infty]{} \int_{V(\mathbb{A}_{\mathbb{K}})} f \, \mathbb{N}_V$ for any continuous $f : V(\mathbb{A}_{\mathbb{K}}) \to \mathbb{R}$)

Over-pessimistic question

Is there any case besides $\mathbb{P}^n$ where this holds?

Theorem

If $V = G/P$ where P is a parabolic subgroup of G then (NE) is true.

Idea of proof

Use harmonic analysis on $G/P(\mathbb{A}_{\mathbb{K}})$ and apply Langlands's work on Eisenstein series. $\square$

Example

Any quadric with $Q(\mathbb{A}_{\mathbb{K}})$ is such a homogeneous spaces. □

What about higher degrees?

Theorem [≤ Birch]

Let $V \subset \mathbb{P}^n$ be a smooth hypersurface of degree d with $V(\mathbb{A}_{\mathbb{K}}) \neq \emptyset$ and $n \geqslant 2^d$, then (NE) is true

Remark

It also applies to smooth complete intersection of sufficiently big dimension (with respect to the degrees)

First problem

$$\mathrm{Support}\,(S_{V(\mathbb{K})_{H \leq B}}) \subset \overset{\text{closure}}{\overline{V(\mathbb{K})}} \subset V(\mathbb{A}_{\mathbb{K}})$$

So

$$(NE) \Rightarrow \text{weak approximation} \Rightarrow V(\mathbb{K}) \text{ Zariski dense}$$

Let us assume that rational points are Zariski dense (We could reduce to that case by considering the Zariski closure of the rational points)

Assumption

$V(\mathbb{K})$ is Zariski dense

then there is a well known obstruction to weak approximation

Brauer-Manin obstruction to WA

We consider $Br(\mathbb{L}) = H^2(Gal(\overline{\mathbb{L}}/\mathbb{L}), \mathbb{G}_m)$

Then class field theory gives embeddings

$$inv_w : Br(\mathbb{K}_w) \hookrightarrow \mathbb{Q}/\mathbb{Z} \quad (\xrightarrow{\sim} \text{unless } w \text{ is archimedean})$$

so that

$$0 \to Br(\mathbb{K}) \xrightarrow{\text{functoriality of the Brauer group}} \bigoplus_{w \in Val(\mathbb{K})} Br(\mathbb{K}_w) \xrightarrow{\sum_w inv_w} \mathbb{Q}/\mathbb{Z} \to 0$$

For V we may consider the cohomological Brauer group

$$Br(V) = H^2_{ét}(V, \mathbb{G}_m)$$

Then we may define a coupling

$$Br(V) \times V(\mathbb{A}_{\mathbb{K}}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$\left(\alpha, (P_w)_{w \in Val(\mathbb{K})}\right) \longmapsto \sum_{w \in Val(\mathbb{K})} inv_w(\underbrace{\alpha(P_w)}_{\in Br(\mathbb{K}_w)})$$

(Specialization)

which may be seen as a map

$$V(\mathbb{A}_{\mathbb{K}}) \longrightarrow Br(V)^\vee = Hom(Br(V), \mathbb{Q}/\mathbb{Z})$$

$$P \longmapsto \eta_P$$

If $P \in V(\mathbb{K})$, the fact that the previous sequence is a complex implies that $\eta_P = 0$

In fact, by a continuity argument, one may prove that

$$\overline{V(\mathbb{K})} \subset V(\mathbb{A}_{\mathbb{K}})^{Br} = \{P \in V(\mathbb{A}_{\mathbb{K}}) \mid \eta_P = 0\}.$$

Assume that

$$Br(V)/im(Br(\mathbb{K})) \text{ is finite}$$

then

$$V(\mathbb{A}_{\mathbb{K}})^{Br} \subset V(\mathbb{A}_{\mathbb{K}}) \text{ is closed \& open}$$

<u>Definition</u>

$$\mathbb{N}_V^{Br}(W) = \frac{\mathbb{N}(W \cap V(\mathbb{A}_{\mathbb{K}}^{Br}))}{\mathbb{N}(V(\mathbb{A}_{\mathbb{K}}^{Br}))}$$

<u>Optimistic question</u>

Does

$$\delta_{V(\mathbb{K})_{H \leq B}} \xrightarrow[B \to +\infty]{\text{weakly}} \mathbb{N}_V^{Br} \quad ? \qquad (LNE)$$

Then we run into the problem of

<u>4) Accumulating subsets</u>

In general

Support $(\lim_{B \to +\infty} \delta_{V(\mathbb{K})_{H \leq B}})$ is much smaller than $\overline{V(\mathbb{K})}$!

Let me give you a few examples

A sandbox case is

a) The plane blown up in one point

$V \hookrightarrow \mathbb{P}^2_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}} \qquad xv = yu$

$[X:Y:Z] \quad [u:v]$

$\pi = pr_1, \quad P_0 = (0:0:1) \quad E = \pi^{-1}(P_0) \overset{pr_2}{\xrightarrow{\sim}} \mathbb{P}^1_{\mathbb{Q}}, \quad U = V - E \overset{\pi}{\xrightarrow{\sim}} \mathbb{P}^2_{\mathbb{Q}} - \{P_0\}$

As height we may take the map

$$H : V(\mathbb{Q}) \longrightarrow \mathbb{R}_{>0}$$
$$(P, Q) \longmapsto H_{\mathcal{O}_{\mathbb{P}^2}(1)}(P)^2 \, H_{\mathcal{O}_{\mathbb{P}^1}(1)}(Q)$$

it corresponds to a norm on ω_V^{-1}

Proposition (Serre, Batyrev & Manin, P.)

$$\# E(\mathbb{Q})_{H \leq B} \underset{B \to +\infty}{\sim} \frac{2}{\zeta_{\mathbb{Q}}(2)} B^2$$

$$\# U(\mathbb{Q})_{H \leq B} \sim \frac{8}{3\,\zeta_{\mathbb{Q}}(2)^2} B \log(B)$$

So There much more points on the exceptional line E

Remark

$\# U(\mathbb{Q})_{H \leq B} = o\left(E(\mathbb{Q})_{H \leq B}\right)$ and, in fact,

$$\delta_{V(\mathbb{Q})_{H \leq B}} \xrightarrow[B \to +\infty]{\text{weakly}} \mu_E \text{ with support on } E(\mathbb{A}_{\mathbb{Q}})$$

Prop

$$\delta_{U(\mathbb{Q})_{H \leq B}} \xrightarrow[B \to +\infty]{\text{weakly}} \mu_V$$

It may seem counter intuitive that by removing points, we get a measure with larger support. But this is precisely the reason for which we should remove the accumulating subset

Remark

If $\delta_W \longrightarrow \mu_V$ then for any subvariety $F \subset V$

$\#(W \cap F(\mathbb{Q}))_{H \leq B} = o\left(\# W_{H \leq B}\right)$ since $\mu_V(F(\mathbb{A}_{\mathbb{Q}})) = 0$

So any subscheme with a strictly positive contribution to the number of points has to be removed to get equidistribution.

Question

Can we find $U \underset{\text{open}}{\subset} V$ so that

$$\delta_{U(\mathbb{Q})_{H\leq B}} \xrightarrow[B\to+\infty]{} \mu_V^{Br} \quad ?$$

Again the answer is negative:

b) The counter-example of Batyrev and Tschinkel

We consider the hypersurface

$V \subset \mathbb{P}^3_{\mathbb{Q}} \times \mathbb{P}^3_{\mathbb{Q}}$ defined by the equation:

$$\sum_{i=0}^{3} X_i Y_i^3 = 0$$

$H(P,Q) = H_{\mathcal{O}_{\mathbb{P}^3}(1)}(P)^2 \, H_{\mathcal{O}_{\mathbb{P}^3}(1)}(Q)$ defines a height relative to ω_V^{-1}

$\pi = pr_1 : V \longrightarrow \mathbb{P}^3_{\mathbb{Q}}$ For $x = [x_0 : \dots : x_3] \in \mathbb{P}^3(\mathbb{Q})$, $\prod_{i=0}^{3} x_i \neq 0$,

$V_x = \pi^{-1}(x)$ is a smooth cubic surface $\supset$ 27 lines

$U_x = V_x - 27$ lines

So the expected result for this surface is

For $U \underset{\text{open}}{\subset} U_x \quad U \neq \emptyset$

$$\# U(\mathbb{Q})_{H\leq B} \underset{B\to+\infty}{\sim} C_x B \log(B)^{r_x - 1}$$

where $r_x = rk(Pic(U_x)) \in \{1,2,3,4\}$

In particular $rk(Pic(V_x)) = 4$ if x_i/x_j are cubes

But, by Lefschetz theorem

$$Pic(V) \cong Pic(\mathbb{P}^3_{\mathbb{Q}} \times \mathbb{P}^3_{\mathbb{Q}}) = \mathbb{Z}^2$$

The initial conjecture of Manin for V predicted:

$$\# V(\mathbb{Q})_{H\leq B} \underset{B\to+\infty}{\sim} C B \log(B)$$

So each fibre with a Picard group of rank bigger than the generic one contains too many points

But $\forall U$ open dense in V, $\{x \mid U\cap V_x \neq \emptyset$ and $rk(Pic(U_x)) > 1\}$

is infinite.
We are confronted with a thin accumulating subset.

Definition
Let V be a nice variety / $\mathbb{K}$
a subset $W \subset V(\mathbb{K})$ is said to be thin, if there exists
$\varphi : X \longrightarrow V$ morphism of varieties such that
(i) φ is generically finite
(ii) φ admits no rational section
(iii) $W \subset \mathrm{Im}(\varphi)$

Remarks
(i) E elliptic curve
$$\coprod_{E(\mathbb{Q})/2} E \longrightarrow E$$
$$P \longmapsto \underset{i}{P_i} + 2P$$
system of representants of $E(\mathbb{Q})/2$

So $E(\mathbb{Q})$ is thin!
(ii) Here
$$\bigcup_{\{x \mid rk(\mathrm{Pic}(V_x)) > 1\}} V_x \quad \text{is thin}$$
(iii) We may assume that φ is proper.
$$\varphi(X(\mathbb{A}_\mathbb{Q})) \subset V(\mathbb{A}_\mathbb{Q}) \text{ closed subset}$$
and under mild hypotheses
$$\mu_V(\varphi(X(\mathbb{A}_\mathbb{Q}))) = 0$$

c) The example of C. Le Rudulier

$V = \mathcal{H}ilb^2(\mathbb{P}^2_\mathbb{Q})$ Hilbert scheme of points of degree 2 in $\mathbb{P}^2_\mathbb{Q}$; $Y = \mathrm{Sym}^2(\mathbb{P}^2_\mathbb{Q}) = (\mathbb{P}^2_\mathbb{Q})^2/\mathfrak{S}_2 \supset \Delta$
$$\begin{tikzcd} P \arrow[r] \arrow[d] & (\mathbb{P}^2_\mathbb{Q})^2 \arrow[d, "g"] \\ V \arrow[r, "f"] & Y \end{tikzcd}$$
Blowing up of Δ

$$U_0 = V - f^{-1}(\Delta)$$

$M = f^{-1}(g(\mathbb{P}^2(\mathbb{Q})^2)) \cap U^0(\mathbb{Q})$ Zariski dense thin subset

Theorem (C. Le Rudulier)

$$\frac{\# M_{H \leq B}}{\# U_0(\mathbb{Q})_{H \leq B}} \xrightarrow[B \to +\infty]{} c > 0$$

But $\forall F \underset{\text{Zariski closed}}{\subset} V \quad \#(F(\mathbb{Q}) \cap M)_{H \leq B} = o\left(U_0(\mathbb{Q})_{H \leq B}\right)$

So M is a thin subset which is not the union of accumulating closed subsets but which is an obstruction to equidistribution nevertheless.

Conclusion (so far)

In all known cases, if ω_V^{-1} is big (enough)

$\exists\ W = V(\mathbb{Q}) - T$, T thin subset

$$\delta_{W_{H \leq B}} \longrightarrow \mu_V$$

Problem

How can you describe T?

II Slopes à la Bost

1) Definition

Let me give the description of slopes I am going to use:

a) Slopes of an adelic vector bundle / Spec($\mathbb{K}$)

Definition (Reminder)

Let E be a $\mathbb{K}$-vector space of dimension n equipped with

- $\Lambda \subset E$ projective $\mathcal{O}_{\mathbb{K}}$-module of constant rank n
- for $v \in \mathrm{Val}(\mathbb{K})$, complex

$$\|\cdot\|_v : E_v = E \otimes \mathbb{K}_v \longrightarrow \mathbb{R}_{\geq 0}$$

  given by a hermitian form
- for real v

$$\|\cdot\|_v : E_v \longrightarrow \mathbb{R}_{\geq 0} \quad \text{euclidean norm}$$

$$\widehat{\deg}(E) = \widehat{\deg}(\Lambda^n E)$$

Example

$\mathbb{K} = \mathbb{Q}$

Λ = subgroup of E generated by a basis of E

$\|\cdot\| : E_{\mathbb{R}} = E \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}_{\geq 0}$ euclidean

$$\widehat{\deg}(E) = -\log(\mathrm{Vol}(E/\Lambda))$$

↳ euclidean volume

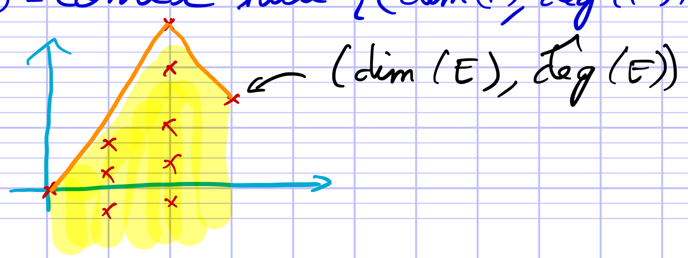
Definition

For $F \subset E$ subspace comes equipped with $\Lambda_F = \Lambda \cap F$ and the restriction of the norms.

Define the Newton polygon as

$$\mathcal{P}(E) = \text{Convex hull}\{(\dim(F), \widehat{\deg}(F)) \text{ for } F \text{ subspace of } E\}$$

Picture



$\mathcal{P}(E)$ is bounded from above so we can define

$$m_E : [0, n] \longrightarrow \mathbb{R}$$
$$m_E(x) = \max\{y \in \mathbb{R} \mid (x,y) \in \mathcal{P}(E)\}$$

This function is concave and affine in each segment $[i, i+1]$

The slopes of E are defined as

$$\mu_i(E) = m_E(i) - m_E(i-1)$$

for $i \in \{1, \dots, n\}$ these numbers are the slope of the affine pieces of the graph of m_E

Remark

(i) By construction,

$$\mu_n(E) \leq \mu_{n-1}(E) \leq \dots \leq \mu_1(E)$$

Note that the inequality is not strict in general!

(ii) $\widehat{\deg}(E) = \sum_{k=1}^{n} \mu_k(E)$.

b) Slopes on variety, freeness

Definition

- E vector bundle on nice $V/\mathbb{K}$, $n = \dim(V)$ equipped with an adelic norm $(\|\cdot\|_w)_{w \in Val(\mathbb{K})}$
  For $x \in V(\mathbb{K})$, $\mu_i^E(x) = \mu_i(E_x)$.
- If V is equipped with an adelic metric
  $$\mu_i(x) = \mu_i(T_x V)$$

Remark

(i) $\mu_n(x) \leq \mu_{n-1}(x) \leq \dots \leq \mu_1(x)$

(ii) $\widehat{\deg}(T_x V) = \sum_{k=1}^{n} \mu_i(x)$.

but $\widehat{\deg}(T_x V) = \widehat{\deg}((\omega_V^{-1})_x) = h(x) = \log(H(x))$

where H is a height relative to ω_V^{-1}

Thus these slopes give us information beyond the height of x

(iii) $\mu_n(x) \leq \frac{h(x)}{n} \leq \mu_1(x)$.

Definition

The **freeness** of a point x is defined by

$$\ell(x) = \begin{cases} n\,\dfrac{\mu_n(x)}{h(x)} & \text{if } \mu_n(x)>0 \\ 0 & \text{otherwise.} \end{cases}$$

Remarks

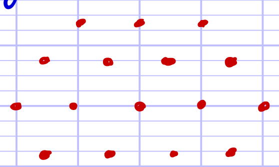
(i) $\ell(x) \in [0,1]$

(ii) $\ell(x)=0 \iff \mu_{min}(x)=\mu_n(x)\leq 0$

(iii) $\ell(x)=1 \iff \mu_1(x)=\mu_2(x)=\dots=\mu_n(x)$

$\iff T_xV$ is semi-stable.

e.g. $\mathbb{Z}^n \subset \mathbb{R}^n$ with usual euclidean structure



hexagonal lattice in $\mathbb{R}^2$

More generally for 2 dimensional lattices

Up to rescaling — fundamental domain

$\Lambda \simeq a(\mathbb{Z}+\mathbb{Z}\tau)$ with $\begin{cases} \mathrm{Re}(\tau)\in[-1,1], & |\tau|\geq 1 \\ \mathrm{Im}(\tau)>0 \end{cases}$

$\Lambda \subset E \simeq \mathbb{C}$

Λ semi-stable

$\iff \mathrm{Im}(\tau)\leq 1$

(iv) for a curve $\ell(x)\equiv 1$.

(v) For a surface $S/\mathbb{Q}$

$S(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$, $x \mapsto H(x)$ $\qquad$ $S(\mathbb{Q}) \to \mathcal{H}/PSL_2(\mathbb{Z})$, $x \mapsto [\tau_x]$ $\quad$ τ_x as above

$$\ell(x) = \begin{cases} 1 & \text{if } \operatorname{Im}(\tau_x) \leq 1 \\ 1 - \dfrac{\log \operatorname{Im}(\tau_x)}{\log H(x)} & \text{if in }]0,1[\\ 0 & \text{otherwise.} \end{cases}$$

2) Properties

First it is crucial to understand how this freeness of mine depends on the choice of metric

Lemma

Let $\varphi : E \to F$ be a morphism of vector bundles and let $(\|\cdot\|_w)_{w \in V_{al}(\mathbb{K})}$ (resp. $(\|\cdot\|'_w)_{w\in V_{al}(\mathbb{K})}$) be an adelic norm on E (resp F) then there exists a family $(\lambda_w)_{w \in V_{al}(\mathbb{K})}$ such that

(i) $\forall\ w \in V_{al}(\mathbb{K}),\ \forall\ x \in V(\mathbb{K}_w),\ \forall\ y \in E(x)$
$$\|\varphi(y)\|'_w \leq \lambda_w \ \|y\|_w$$
(ii) $\{w \mid \lambda_w \neq 1\}$ is finite.

Sketch of proof

- $\mathbb{P}(E)$: projective bundle of lines in E
  $\tilde{\varphi} : \mathbb{P}(E) \longrightarrow \mathbb{P}(F)$

  We may consider for $w \in V_{al}(\mathbb{K})$
  $$\begin{aligned} \mathbb{P}(E)(\mathbb{K}_w) &\longrightarrow \mathbb{R}_{\geq 0} \quad \text{continuous} \\ \mathbb{K}_w y &\longmapsto \frac{\|\varphi(y)\|'_w}{\|y\|_w} \end{aligned}$$

  thus bounded from above.
- for almost all w, any x in $V(\mathbb{K}_w)$
  $$\varphi(\{y \in E(x) \mid \|y\|_w \leq 1\}) \subset \{y \in F(x) \mid \|y\|'_w \leq 1\}$$
  because $\|\cdot\|_w$ and $\|\cdot\|'_w$ are defined by models of E and F for almost all w. $\square$

Remark

In particular, if $(\|\cdot\|_w)_{w\in Val(K)}$ and $(\|\cdot\|'_w)_{w\in Val(K)}$ are metrics, then

(i) $\dfrac{\|\cdot\|'_w}{\|\cdot\|_w}$ is bounded for any $w \in Val(K)$

(ii) $\|\cdot\|'_w = \|\cdot\|_w$ for almost all w.

Thus

$$\forall x \in V(K),\ \forall F \subset T_x V, \quad |\widehat{\deg}(F) - \widehat{\deg}'(F)| < C$$

Corollary 1

Let μ_i and μ'_i be the slopes defined by two metrics $/V$

(i) $|\mu_i - \mu'_i|$ is bounded;

(ii) $|\ell'(x) - \ell(x)| < \dfrac{C}{h(x)}$ when $h(x) > 0$.

Notation

The notion of slopes in the geometric setting over the field of rational fractions in one variable may be described as follows

$\varphi : \mathbb{P}^1 \longrightarrow V$ morphism of variety

By the decomposition of vector bundles on $\mathbb{P}^1$

$\varphi^*(TV) \simeq \bigoplus_{i=1}^{n} \mathcal{O}(a_i)$ with $a_1 \geq \cdots \geq a_n$

$\mu_i(\varphi) = a_i \qquad \deg_{w_i}(\varphi) = \sum_{i=1}^{n} \mu_i(\varphi)$

and

$$\ell(\varphi) = \begin{cases} n\, a_n / \deg_{a_v}(\varphi) & \text{if } a_n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Remarks

(i) $\ell(\varphi) \in [0,1] \cap \mathbb{Q}$

(ii) $\ell(\varphi) > 0 \Leftrightarrow \varphi$ is very free.

Prop

Let $\varphi : \mathbb{P}^1 \longrightarrow V$, V equipped with an adelic metric

$$\ell(\varphi(x)) \underset{\text{as } h(x)\to+\infty}{\longrightarrow} \ell(\varphi)$$

Proof

vector bundles

$$\underbrace{\varphi^*(TV)}_{\text{pull back of the norm}} \simeq \underbrace{\bigoplus_{i=1}^{n} \mathcal{O}(a_i)}_{\text{norm for direct sums}}$$

slopes $\mu_i(\varphi(x))$ slopes $\mu_i'(x) = a_i\, h_{\mathcal{O}(1)}(x)$

so if $a_n \geq 0$

$$\left| \ell(\varphi(x)) - \frac{a_n\, n}{\sum a_i} \right| < \frac{C_\varphi}{h_{\mathcal{O}(1)}(x)} .$$

and

$\ell(\varphi(x)) = 0$ if $a_n < 0$ except for a finite # of x. $\square$

3) Explicit computations

a) In the projective space

Proposition

Let $x \in \mathbb{P}^n(\mathbb{K})$

$$\ell(x) = \frac{n}{n+1} + \min_F \left(\frac{-n\, \widehat{\deg}(F)}{\operatorname{codim}_E(F)\, h(x)} \right)$$

where F goes over the subspaces $F \subsetneq E$ such that $x \in \mathbb{P}(F)$

Sketch of proof

Let $D \subset E$ be the line corresponding to x

$$T_x \mathbb{P}^n_{\mathbb{K}} \simeq D^\vee \otimes E / D^\vee \otimes D$$

giving a bijection

$$\{F \mid D \subset F \subsetneq E\} \longrightarrow \text{subspaces of } T_x \mathbb{P}^n_{\mathbb{K}}$$

$$F \longmapsto D^\vee \otimes F / D^\vee \otimes D$$

and $\widehat{\deg}(D^\vee \otimes F / \underbrace{D^\vee \otimes D}_{\simeq \mathbb{K}}) = \widehat{\deg}(D^\vee \otimes F) - \deg(\mathbb{K})$

$$= \widehat{\deg}(F) - \dim(F)\,\widehat{\deg}(D) - \widehat{\deg}(\mathbb{K})$$

$$h(x) = -(n+1)\,\widehat{\deg}(D) - \widehat{\deg}(\mathbb{K})$$

We get

$$\mu_n(x) = -\widehat{\deg}(D) + \min_F \left(\frac{-\widehat{\deg}(F)}{\operatorname{codim}(F)} \right)$$

$$\ell(x) = \frac{n}{n+1} + \min \left(\frac{-n\,\widehat{\deg}(F)}{\operatorname{codim}(F)\,h(x)} \right)$$

<u>Corollary</u>

$$\ell(x) \geq \frac{n}{n+1}$$

<u>Remark</u>

(i) Let us fix $F \subset E$

Then $\ell(x) \xrightarrow[h(x)\to+\infty]{} \frac{n}{n+1}$

(ii) $\exists C > 0$

$$\#\{x \in \mathbb{P}^n(\mathbb{K}) \mid H(x) \leq B \;\&\; \ell(x) \leq 1-\varepsilon\} < C B^{1-\varepsilon}$$

(and $\#\{x \in \mathbb{P}^n(\mathbb{K}) \mid H(x) \leq B\} \sim C(\mathbb{P}^n_{\mathbb{K}})\, B$)

which means that this number is negligible

But even on a homogeneous space the freeness can be small:

4) $(\mathbb{P}^1)^n$

<u>Proposition</u>

Let $\underline{x} = (x_1, \dots, x_n) \in \mathbb{P}^1(\mathbb{K})^n$

then

$$\ell(\underline{x}) = \frac{n \min_{1 \leq i \leq n} h(x_i)}{\sum_{1 \leq i \leq n} h(x_i)}$$

<u>Proof</u>

$T_{\underline{x}} (\mathbb{P}^1_{\mathbb{K}})^n = \bigoplus_{i=1}^{n} T_{x_i} \mathbb{P}^1_{\mathbb{K}}$ each is of dimension 1

So for $\sigma \in \mathfrak{S}_n$ such that

$$h(x_{\sigma(1)}) \geq h(x_{\sigma(2)}) \geq \dots \geq h(x_{\sigma(n)})$$

we get $\quad \mu_i(\underline{x}) = h(x_{\sigma(i)})$. $\square$

Corollary

For any $\varepsilon > 0$

$$\frac{\#\{\underline{x} \in \mathbb{P}^1(\mathbb{K})^n \mid H(x) \leq B \ \&\ \ell(x) \geq \varepsilon\}}{\#\{\underline{x} \in \mathbb{P}^1(\mathbb{K})^n \mid H(x) \leq B\}} \xrightarrow[B \to +\infty]{} C_\varepsilon > 0$$

with $C_\varepsilon = 1 - O(\varepsilon)$.

Sketch of the proof

We consider the maps

$$\underline{h} : \mathbb{P}^1(\mathbb{K})^n \longrightarrow \mathbb{R}^n_{\geq 0} \xrightarrow{s} \mathbb{R}_{\geq 0}$$

$$(x_1, \dots, x_n) \longmapsto (h(x_1), \dots, h(x_n))$$

$$(y_1, \dots, y_n) \longmapsto \sum_{i=1}^n y_i$$

$$H(\underline{x}) = s(\underline{h}(\underline{x}))$$

Then using the formula

$$\#\{x \in \mathbb{P}^1(\mathbb{K}) \mid H(x) \leq B\} = C(\mathbb{P}^1_{\mathbb{K}})\, B + O(B^{1/2} \log B)$$

and partial integration one gets

$$\#\{\underline{x} \in \mathbb{P}^1(\mathbb{K})^n \mid H(x) \leq B\} \sim C(\mathbb{P}^1_{\mathbb{K}})^n\, B \, \underbrace{\mathrm{Vol}\left\{(t_1, \dots, t_n) \in \mathbb{R}^n_{\geq 0} \,\middle|\, \sum_{i=1}^n t_i \leq \log(B)\right\}}_{1/n!}$$

and

$$\#\{x \in \mathbb{P}^1(\mathbb{K})^n \mid H(x) \leq B \ \&\ \ell(x) \geq \varepsilon\} \sim C(\mathbb{P}^1_{\mathbb{K}})^n\, B \, \underbrace{\mathrm{Vol}\left\{(t_1, \dots, t_n) \in \mathbb{R}^n_{\geq 0} \,\middle|\, \begin{cases} \sum_{i=1}^n t_i \leq \log(B) \\ \min_{1 \leq i \leq n} t_i \geq \varepsilon \frac{\sum_{i=1}^n t_i}{n} \end{cases}\right\}}_{V_\varepsilon}$$

So we get that the quotient converges to

$$V_\varepsilon \times n! \quad \text{and} \quad \left(V_\varepsilon - \frac{1}{n!}\right) = O(\varepsilon) \ \square$$

Remark

In particular the number of points with freeness $< \varepsilon$ is <u>not</u> negligible!

Let us now go back to the accumulating subvarieties
As we shall see rational points in accumulating subvarieties seem to have a small freeness which would mean that the freeness might be a criterion to distinguish between good points and bad points

4) Accumulating subsets

On surfaces accumulating subsets are given as rational curves of low degree

Proposition