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Minima and slopes of rigid adelic spaces

Warning: These notes are a preliminary version which may contain errors.

1) Introduction

Ref: [Ma 2003, Chaps 1 and 2], [Mi 1910]

We propose here a lecture on geometry of numbers for normed (adelic) vector spaces over an algebraic extension of $\mathbb{Q}$. We shall define several type of minima and slopes for these objects and we shall compare them.

First, let us recall some basic notions of the classical geometry of numbers

let Λ be a free $\mathbb{Z}$-module of rank $n \geq 1$ and let $\|.\|$ be an Euclidean norm on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. We shall say that the couple $(\Lambda, \|.\|)$ is an Euclidean lattice of rank n. To such a lattice are associated n positive real numbers, called the successive minima : $\forall i \in \{1, \dots, n\}$,

$$\lambda_i(\Lambda, \|.\|) = \min\{r > 0 \,;\, \dim \mathrm{Vect}_{\mathbb{R}}(x \in \Lambda \,;\, \|x\| \leq r) \geq i\}$$

$$= \min\{\max\{\|x_1\|, \dots, \|x_i\|\} \,;\, x_1, \dots, x_i \in \Lambda \text{ linearly independent}\}$$

We have $0 < \lambda_1(\Lambda, \|.\|) \leq \lambda_2(\Lambda, \|.\|) \leq \dots \leq \lambda_n(\Lambda, \|.\|)$

let $e_1, \dots, e_n$ be a $\mathbb{Z}$-basis of Λ. The (co-) volume of Λ is the positive real number $\mathrm{vol}(\Lambda) = \det(\langle e_i, e_j\rangle)_{1 \leq i,j \leq n}^{1/2}$ where $\langle , \rangle$ is the scalar product on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ associated to $\|.\|$.

Let us define

$$c_I(n,\mathbb{Q}) = \sup \frac{\lambda_1(\Lambda,\|.\|)}{\mathrm{vol}(\Lambda)^{1/n}} \quad \text{and} \quad c_{II}(n,\mathbb{Q}) = \sup \left(\frac{\lambda_1(\Lambda,\|.\|) \cdots \lambda_n(\Lambda,\|.\|)}{\mathrm{vol}(\Lambda)} \right)^{1/n}$$

where the suprema are taken over Euclidean lattices $(\Lambda,\|.\|)$ of rank n.

The square $\gamma_n = c_I(n,\mathbb{Q})^2$ is nothing but the famous Hermite constant. Its exact value is only known for $n \leq 8$ and $n = 24$. It can also be characterized as the smaller positive real number c such that, for all $(a_0,\dots,a_n) \in \mathbb{Z}^{n+1}\setminus\{0\}$, it exists $(x_0,\dots,x_n) \in \mathbb{Z}^{n+1}\setminus\{0\}$ satisfying

$$a_0x_0 + \dots + a_nx_n = 0 \quad \text{and} \quad x_0^2 + \dots + x_n^2 \leq c\,(a_0^2 + \dots + a_n^2)^{1/n}.$$

Minkowski proved (see Geometrie der Zahlen, 1896, §51)

<u>Theorem</u>: $c_I(n,\mathbb{Q}) = c_{II}(n,\mathbb{Q}) \leq \sqrt{n}$

In what follows, we generalize in the following way

$\mathbb{Q} \longrightarrow K$ algebraic extension of $\mathbb{Q}$

Euclidean lattice $(\Lambda,\|.\|) \longrightarrow E$ rigid adelic space over K

Minimum $\lambda_i(\Lambda,\|.\|) \longrightarrow$ Minimum $\Lambda_i(E)$

Volume $\mathrm{vol}(\Lambda) \longrightarrow$ Height $H(E)$

$\mathrm{vol}(\Lambda)^{1/n} \longrightarrow$ Slope $\mu(E)$

2) Rigid adelic spaces

a) Algebraic extensions of $\mathbb{Q}$

(Ref: [GR 2017, § 2])

Let $K|\mathbb{Q}$ be an algebraic extension.

let $V(K)$ = places of K (equivalence classes of non trivial absolute values over K)

= projective limit of $V(L)$ for $\mathbb{Q} \subset L \subset K$ finite subextension of K

The discrete topology on $V(L)$ induces a topology on $V(K)$ by projective limit.
It coincides with the topology generated by compact open subsets

$$V_v(K) = \{ w \in V(K) \; ; \; w_{|L} = v \} \quad \text{for } v \in V(L) \text{ and } L \text{ varies}$$

through number fields contained in K.

On $V(K)$ can be defined a Borel measure λ characterized by

$$\lambda(V_v(K)) = \frac{[L_v : \mathbb{Q}_v]}{[L : \mathbb{Q}]} \quad \text{for } v \in V(L)$$

($\mathbb{Q}_v = \mathbb{Q}_p$, $\mathbb{R}$ or $\mathbb{C}$ depending on $v | p$, v real or v complex non real).

We have $\lambda(V_p(K)) = 1$ for all $p \in V(\mathbb{Q})$.

We shall work with the adeles of K: $\mathbb{A}_K = K \otimes_{\mathbb{Z}} \mathbb{A}_{\mathbb{Q}}$ where

$$\mathbb{A}_{\mathbb{Q}} = \left\{ (x_p)_p \in \prod_{p \in V(\mathbb{Q}_p)} \mathbb{Q}_p \; ; \; \text{For all prime } p \text{, outside a finite subset, } |x_p|_p \leq 1 \right\}$$

If K is a number field $\mathbb{A}_K$ is the usual adele ring and we have

$$\mathbb{A}_K = \bigcup_{\substack{L \subset K \\ [L:\mathbb{Q}] < +\infty}} \mathbb{A}_L$$

For $v \in V(K)$, we denote by K_v the topological completion of K with respect to v and $|\cdot|_v$ is the unique absolute value on K_v such that $|p|_v \in \{1, p, p^{-1}\}$ for all prime number p.

b) <u>Rigid adelic spaces</u>

In the following, the letter K always denote an algebraic extension of $\mathbb{Q}$.

<u>Definition</u>: An adelic space E is a K-vector space of finite dimension endowed with norms $\|\cdot\|_{E,v}$ on $E \otimes_K K_v$ for every $v \in V(K)$.

The (adelic) standard space of dimension $n \geq 1$ is the vector space K^n endowed with the following norms: $\forall (x_1, \dots, x_n) \in K_v^n$,

$$|(x_1, \dots x_n)|_v = \begin{cases} \left(|x_1|_v^2 + \dots + |x_n|_v^2\right)^{1/2} & \text{if } v \mid \infty \\ \max(|x_1|_v, \dots, |x_n|_v) & \text{if } v \nmid \infty \end{cases}$$

<u>Definition</u>: A rigid adelic space is an adelic space E for which there exists an isomorphism $\varphi : E \to K^n$ and an adelic matrix $A = (A_v)_{v \in V(K)} \in GL_n(\mathbb{A}_K)$ such that $\|x\|_{E,v} = |A_v \varphi_v(x)|_v \quad \forall x \in E \otimes_K K_v$ where $\varphi_v = \varphi \otimes id_{K_v} : E \otimes K_v \to K_v^n$ is the natural extension of φ to $E \otimes_K K_v$.

In looser terms, a rigid adelic space is a compact deformation of a standard space.

<u>Remarks</u>: 1) Actually, if E is a rigid adelic space over K, for every isomorphism $\varphi : E \to K^n$, there exists $A \in GL_n(\mathbb{A}_K)$, upper triangular, such that (φ, A) defines the adelic structure on E.

2) If $x \in E \setminus \{0\}$ there exists a number field $K_0 \subset K$ such that $A \in GL_n(\mathbb{A}_{K_0})$ and $\varphi(x) \in K_0^n$. Thus, outside a compact subset of $V(K)$ (finite union of some $V_v(K)$ with $v \in V(K_0)$), we have $\|x\|_{E,v} = 1$ since $|\varphi(x)|_v = 1$ and A_v is an isometry.

Examples of rigid adelic spaces:

- K^n
- let $(\Lambda, \|.\|)$ be an Euclidean lattice, $\Lambda = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$

On $K = \mathbb{Q}$ we can consider $E_\Lambda = \Lambda \otimes \mathbb{Q}$ endowed with the norm $\|.\|$ at the infinite place and $\|x_1e_1 + \cdots + x_ne_n\|_{E_\Lambda, p} = \max\limits_{1 \leq i \leq n} |x_i|_p$ for prime p.

- When K is a number field, we have a one-to-one correspondence between rigid adelic spaces and Hermitian vector bundles over $\operatorname{Spec} \mathcal{O}_K$

rigid adelic space E $\longrightarrow$ $\overline{\mathcal{E}} = (\mathcal{E}, (\|.\|_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}})$

- $\mathcal{E} = \{x \in E \;;\; \|x\|_{E,v} \leq 1 \quad \forall v \in V(K) \setminus V_\infty(K)\}$ (projective $\mathcal{O}_K$-module of finite type)
- $\|.\|_\sigma = \|.\|_{E,v}$ $(v = \{\sigma, \bar{\sigma}\})$ Hermitian norms invariant by complex conjugation $\|x\|_\sigma = \|\bar{x}\|_{\bar{\sigma}}$ $\quad x \in E \otimes_\sigma \mathbb{C}$

When E and F are rigid adelic spaces over K, a linear map $f: E \to F$ is called isometric if $\|f_v(x)\|_{F,v} = \|x\|_{E,v}$ for all $v \in V(K)$.

Operations on adelic spaces

Let $E_{/K}$ be an adelic space and $F \subset E$ a vector subspace. One can consider the following adelic spaces:

1) Induced structure on F : F with norms on $E \otimes_K K_v$ restricted to $F \otimes_K K_v$

2) Quotient : E/F with quotient norms $\|\bar{x}\|_{E/F, v} = \inf\left\{ \|z\|_{E,v} \,;\, \begin{array}{l} z = x \bmod F \otimes_K K_v \\ z \in E \otimes_K K_v \end{array} \right\}$

3) Dual : $E^\vee = \mathrm{Hom}_K(E, K)$ (linear forms) with operator norms

$$\|\ell\|_{E^\vee, v} = \sup\left\{ \frac{|\ell(z)|_v}{\|z\|_{E,v}} \,;\, z \in E \otimes K_v \setminus \{0\} \right\} \qquad \ell \in E^\vee \otimes_K K_v$$

Consider another adelic space E' over K. We also have:

4) (Hermitian) direct sum : $E \oplus E'$ with

$$\begin{array}{l} x \in E \otimes K_v \\ y \in E' \otimes K_v \end{array}, \quad \|(x, y)\|_{E \oplus E', v} = \begin{cases} \left(\|x\|_{E,v}^2 + \|y\|_{E',v}^2 \right)^{1/2} & \text{if } v \mid \infty \\ \max\left(\|x\|_{E,v}, \|y\|_{E',v} \right) & \text{if } v \nmid \infty \end{cases}$$

5) Operator norm : $\mathrm{Hom}_K(E, E')$ (linear maps) with

$$\|\varphi\|_v = \sup\left\{ \frac{\|\varphi_v(x)\|_{E',v}}{\|x\|_{E,v}} \,;\, x \in E \otimes_K K_v \setminus \{0\} \right\}$$

Using the natural isomorphism $E \otimes E' \simeq \mathrm{Hom}(E^\vee, E')$, we get an adelic structure on $E \otimes E'$ that will be noted $E \otimes_\varepsilon E'$ in the rest of the text (the ε refers to the injective norm for tensor product of spaces)

6) Hilbert-Schmidt norm : When $E = (\varphi, A)$ and $E' = (\varphi', A')$ are rigid adelic spaces, the tensor product $E \otimes E'$ is endowed with the rigid structure given by $(\varphi \otimes \varphi', A \otimes A')$

7) Symmetric product : When $E = (\varphi, A)$ is a rigid space and $\ell \in \mathbb{N} \setminus \{0\}$, the symmetric product $S^\ell(E)$ is endowed with $(S^\ell(\varphi), S^\ell(A))$. It corresponds to the quotient structure by the natural surjection $E^{\otimes \ell} \to S^\ell(E)$.

We have $\|x^\ell\|_{S^\ell(E),v} = \|x\|_{E,v}^\ell$ for $x \in E \otimes_K K_v$. If $e_1, \dots, e_n$ is an orthonormal basis of $E \otimes K_v$ then $e_1^{i_1} \cdots e_n^{i_n}$ with $i_j \in \mathbb{N}$ and $i_1 + \dots + i_n = \ell$ is an orthogonal basis of $S^\ell(E)$ and $\|e_1^{i_1} \cdots e_n^{i_n}\|_{S^\ell(E),v} = \begin{cases} \left(\frac{i_1! \cdots i_n!}{\ell!}\right)^{1/2} & \text{if } v \mid \infty \\ 1 & \text{if } v \nmid \infty \end{cases}$.

8) Exterior product : When $E = (\varphi, A)$ is a rigid adelic space and $\ell \in \mathbb{N} \setminus \{0\}$, the exterior product $\Lambda^\ell(E)$ is endowed with $(\Lambda^\ell \varphi, \Lambda^\ell(A))$. It corresponds to the quotient structure $E^{\otimes \ell} \to \Lambda^\ell E$, $x_1 \otimes \dots \otimes x_\ell \mapsto x_1 \wedge \dots \wedge x_\ell$.

For $e_1, \dots, e_\ell \in E \otimes_K K_v$,

$$\|e_1 \wedge \dots \wedge e_\ell\|_{\Lambda^\ell E, v} = \left(\det \langle e_i, e_j \rangle_{E,v}\right)^{1/2} \quad \text{if } v \mid \infty$$
$$= \prod_{i=1}^{\ell} \|e_i\|_{E,v} \quad \text{if } v \nmid \infty.$$

(when $\ell = \dim E$, $\Lambda^\ell(E) = \det E$).

9) Scalar extension : let K'/K be an algebraic extension and $E = (\varphi, A)$ be a rigid adelic space. We endow $E \otimes_K K'$ with the rigid adelic structure given by $(\varphi \otimes \mathrm{id}_{K'}, A)$ where $\varphi \otimes \mathrm{id}_{K'} : E \otimes_K K' \to (K')^n$ is induced by φ and A is viewed in $GL_n(\mathbb{A}_{K'})$ by means of the diagonal embedding $\mathbb{A}_K \hookrightarrow \mathbb{A}_{K'}$.

These definitions do not depend on the chosen couple (φ, A).

Note that every rigid adelic space E/K can be written as the scalar extension $E_0 \otimes_{K_0} K$ of a rigid adelic space E_0 over a number field K_0 : take K_0 such that $A \in GL_n(\mathbb{A}_{K_0})$ and define $E_0 = \varphi^{-1}(K_0^n)$ with the structure given by $(\varphi_{|E_0}, A)$.

Theorem: When E and E' are rigid adelic spaces, all these adelic structures are rigid except (in general) the one on $E \otimes_{\xi} E'$.

Moreover the canonical isomorphisms $E \simeq E^{\vee\vee}$ and $E/F \simeq (F^{\perp})^{\vee}$ (where $F^{\perp} = \{\varphi \in E^{\vee} ; \varphi(F) = 0\}$) are isometries.

Height, degree and slope of rigid adelic spaces

Let E/K be a rigid adelic space defined by (φ, A).

Height of E: $H(E) = \exp \int_{V(K)} \operatorname{Log} |\det A_v|_v \, d\lambda(v)$

If $E = \{0\}$, one has $H(E) = 1$. This definition does not depend on the choice of (φ, A).

(Arakelov) Degree of E: $\deg E = -\operatorname{Log} H(E) = -\int_{V(K)} \operatorname{Log} |\det A_v|_v \, d\lambda(v)$

Slope of E: $\mu(E) = \dfrac{\deg E}{\dim E}$ $\quad (E \neq \{0\})$

Examples

- $H(K^n) = 1$ $\quad \deg K^n = \mu(K^n) = 0$
- If $(\Lambda, \|\cdot\|)$ is an Euclidean lattice then $H(E_\Lambda) = \operatorname{vol}(\Lambda)$.

(Indeed $H(E_\Lambda) = |\det A|$ where $\|x_1 e_1 + \dots + x_n e_n\| = \left| A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right|$, ${}^t A A = (\langle e_i, e_j \rangle)_{i,j}$).

- If K is a number field, we have $H(E) = \prod_{v \in V(K)} |\det A_v|^{\frac{[K_v : \mathbb{Q}_v]}{[K:\mathbb{Q}]}}$
- $H(E) = H(\det E)$, $\deg E = \deg \det E$

Proposition : Let E and E' be rigid adelic spaces over k and $F \subset E$ a linear subspace endowed with its induced rigid structure.

Then
$$H(E/F) = \frac{H(E)}{H(F)} \qquad (\deg E = \deg F + \deg E/F)$$

$$H(E^\vee) = H(E)^{-1} \qquad (\deg E^\vee = -\deg E)$$

$$H(E \oplus E') = H(E)H(E') \qquad (\deg E \oplus E' = \deg E + \deg E')$$

$$H(E \otimes E') = H(E)^{\dim E'} H(E')^{\dim E} \qquad (\mu(E \otimes E') = \mu(E) + \mu(E'))$$

$$H(F^\perp) = H(F)/H(E) \qquad (\deg F^\perp = \deg F - \deg E)$$

$$H(\Lambda^\ell E) = H(E)^{\binom{n-1}{\ell-1}} \qquad (\mu(\Lambda^\ell E) = \ell\mu(E))$$

$$\mu(S^\ell E) = \ell\mu(E) + \frac{1}{2}\frac{1}{\binom{\ell+n-1}{n-1}} \sum_{\substack{i_j \in \mathbb{N} \\ i_1 + \dots + i_n = \ell}} \operatorname{Log} \frac{\ell!}{i_1! \dots i_n!}$$

Defining $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$, one can prove $\mu(S^\ell E) = \ell\mu(E) + \frac{1}{2}(H_n - 1)\,\ell(1 + o_{\ell \to \infty}(1))$

Moreover, if k' is an algebraic extension of k, then $H(E \otimes_k k') = H(E)$,

$\deg E \otimes_k k' = \deg E$, $\mu(E \otimes_k k') = \mu(E)$.

Ref: [GR2017, Prop. 3.6] , [Ga 2008, lemme 7.3] , [GR 2013, § 2.7]

Proposition : Let F and G be linear subspaces of a rigid adelic space over k.

Then $H(F+G)\,H(F \cap G) \leq H(F)\,H(G)$

that is, $\deg F + \deg G \leq \deg(F+G) + \deg F \cap G$.

Proof: Let $\iota : F/F \cap G \longrightarrow F+G/G$ be the natural isomorphism.

For all $v \in V(K)$ and $x \in (F/F\cap G) \otimes K_v$, we have $\|\iota(x)\|_{F+G/G, v} \leq \|x\|_{F/F \cap G, v}$.

In particular, if $e_1, \dots, e_m$ is an orthonormal basis of $F/F\cap G \otimes_K K_v$ then

$$\|(\det \iota)_v (e_1 \wedge \dots \wedge e_m)\|_{\det F+G/G, v} = \| \iota_v(e_1) \wedge \dots \wedge \iota_v(e_m)\|_{\det F+G/G, v}$$

$$\leq \prod_{i=1}^{m} \|\iota_v(e_i)\|_{F+G/G, v} \leq \prod_{i=1}^{m} \|e_i\|_{F/F\cap G, v} = 1$$

(Note: if v is ultrametric the word "orthonormal" means $\|x_1e_1 + \dots + x_me_m\| = \max_{1\leq i \leq m} |x_i|_v$).

Thus $\dfrac{H(F+G)\, H(F\cap G)}{H(F)\, H(G)} = \dfrac{H(F+G/G)}{H(F/F\cap G)} = H\left(\left(\det F/F\cap G\right)^\vee \otimes \det(F+G/G)\right)$

$$= \exp \int_{V(K)} \operatorname{Log} |(\det \iota)_v|_v \, d\lambda(v) \leq \exp 0 = 1. \quad \square$$

Heights of points

Let E/K be an adelic space.

Definition: The adelic space E is said to integrable if, for all $x \in E \setminus \{0\}$, the function $V(K) \to \mathbb{R}$, $v \mapsto \operatorname{Log} \|x\|_{E,v}$ is λ-integrable.

A rigid adelic space is integrable as well as a finite ε-tensor product of rigid adelic spaces.

Definition: Let E/K be integrable and $x \in E$.

The height of x, $H_E(x)$, is the ~~posi~~ nonnegative real number:

- $x = 0$ $\quad H_E(x) = 0$
- $x \neq 0$ $\quad H_E(x) = \exp \int_{V(K)} \operatorname{Log} \|x\|_{E,v} \, d\lambda(v)$

By the product formula it is a projective height: $H_E(\lambda x) = H_E(x)$ $\forall \lambda \in K \setminus \{0\}$.

Examples

- $H_{\mathbb{Q}^n}(x_1,\dots,x_n) = (x_1^2+\dots+x_n^2)^{1/2}\ \mathrm{pgcd}(x_1,\dots,x_n)^{-1}$ $\quad (x_1,\dots,x_n) \in \mathbb{Z}^n\setminus\{0\}$

- $(\Lambda, \|.\|)$ Euclidean lattice $\quad x \in E_\Lambda$

  Then $\exists\, d_x \in \mathbb{Q}\setminus\{0\}$; $H_{E_\Lambda}(x) = \|d_x x\|$.

- When $\dim E = 1$, $\quad H_E(x) = H(E) \quad \forall x \in E\setminus\{0\}$

- When K is a number field, $\quad H_E(x) = \prod_{v\in V(K)} \|x\|_{E,v}^{\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}}$

- Let F be the hyperplane $a_0x_0+\dots+a_nx_n = 0$ of K^{n+1} (for $(a_0,\dots,a_n)\in K^{n+1}\setminus\{0\}$)

  Then $H(F) = H_{K^{n+1}}(a_0,\dots,a_n)$ $\quad$ (consequence of $H(F) = H(F^\perp)$).

Note that when E is a rigid adelic space, the height is invariant by scalar extension:

$$H_{E\otimes_K K'}(x) = H_E(x) \quad \forall x\in E \quad \forall K'/K \text{ algebraic extension.}$$

Proposition : Convexity inequality for heights

Let $E_1,\dots,E_N$ be integrable adelic spaces over K $(N\in\mathbb{N}\setminus\{0\})$ and $x_i\in E_i$ $\forall i\in\{1,\dots,N\}$.

Then $E_1\oplus\dots\oplus E_N$ is integrable and

$$\left(\sum_{i=1}^N H_{E_i}(x_i)^2\right)^{1/2} \leq H_{E_1\oplus\dots\oplus E_N}(x_1,\dots,x_N)^2$$

Proof: For the height inequality, adapt the proof of [GR 2013, lemma 2.2].

As for the integrability, it is sufficient to do it for $N=2$.

Observe that for positive real numbers a and b, we have

$$|\log(a+b)| \leq \log\max\left(a+b, \frac{1}{a+b}\right) \leq \log\max\left(a+b, \frac{1}{a}+\frac{1}{b}\right) \leq \log 2 + |\log a| + |\log b|$$

and $$|\log\max(a,b)| \leq \log\max\left(a,\frac{1}{a}\right) + \log\max\left(b,\frac{1}{b}\right) = |\log a| + |\log b|.$$

The conclusion comes from the definition of $E_1\oplus E_2$. $\square$

3) Minima and slopes

a) Successive minima

Ref : [GR 2017]

Let E/K be a rigid adelic with dimension $n \geq 1$. We denote $\Lambda_1(E) = \inf\left(H_E(x)\,;\, x \in E \setminus \{0\}\right)$.

We define three types of successive minima associated to E (there are many others in the literature!). Here $i \in \{1, \dots, n\}$.

Bost-Chen minima: $\Lambda^{(i)}(E) = \sup\left(\Lambda_1\left(E/F\right)\,;\, \underset{\text{linear subspace}}{F \subset E},\ \dim F \leq i-1\right)$

Roy-Thunder minima: $\Lambda_i(E) = \inf\left(\max\left(H_E(x_1), \dots, H_E(x_i)\right)\,;\, \begin{array}{l} x_1, \dots, x_i \in E \\ \text{linearly independent} \\ \text{over } K \end{array}\right)$

Zhang minima: $Z_i(E) = \inf\left(\sup_{x \in S} H_E(x)\,;\, S \subset E,\ \dim \mathrm{Zar}(S) \geq i\right)$

$\mathrm{Zar}(S)$ means the Zariski closure of $K.S = \{ax\,;\, a \in K, x \in S\}$ in E

We have
$$\begin{array}{ccccccccc}
0 < \Lambda^{(1)}(E) & \leq & \Lambda^{(2)}(E) & \leq & \cdots & \leq & \Lambda^{(n)}(E) & < +\infty \\
\| & & \rotatebox{90}{$\geq$} & & & & \rotatebox{90}{$\geq$} & \\
\Lambda_1(E) & \leq & \Lambda_2(E) & \leq & \cdots & \leq & \Lambda_n(E) & < +\infty \\
& & \rotatebox{90}{$\geq$} & & & & \rotatebox{90}{$\geq$} & \\
\| & & & & & & & \\
Z_1(E) & \leq & Z_2(E) & \leq & \cdots & \leq & Z_n(E) & \leq +\infty
\end{array}$$

A field K is a Northcott field if, for all $B > 0$, the set $\{x \in K\,;\, H_{K^2}(1,x) \leq B\}$ is finite (for instance, a number field is a Northcott field).

It can be proved that $\forall n \geq 2$, $\forall E$ rigid with $\dim E = n$, $Z_n(E) < +\infty$ if and only if K is not a Northcott field (see [GR 2017, Proposition 4.4]).

Examples

- $\Lambda^{(1)}(K^n) = \Lambda^{(2)}(K^n) = \dots = \Lambda^{(n)}(K^n) = \Lambda_1(K^n) = \Lambda_2(K^n) = \dots = \Lambda_n(K^n) = 1$

- When $(\Lambda, \|.\|)$ is an Euclidean lattice, $\Lambda_i(E_\Lambda) = \lambda_i(\Lambda, \|.\|)$ (defined in the introduction)

- If K contains infinitely many roots of unity (e.g. $K = \bar{\mathbb{Q}}$) then $Z_i(K^n) = \sqrt{i}$, for all $i \in \{1, \dots, n\}$.
  (consequence of the convexity inequality, see p. 11).

- let $A_n = \{(x_0, \dots, x_n) \in K^{n+1} ; x_0 + x_1 + \dots + x_n = 0\} \subset K^{n+1}$ and $\ell \in \mathbb{N} \setminus \{0\}$. $1 \leq \ell \leq n$

  Then $\Lambda_1(\Lambda^\ell A_n) = \sqrt{\ell + 1}$

  (adapt [GR 2013, Proposition 7.2]).

In the following, to unify notation, we shall use $\lambda_i^*(E)$ with $* \in \{BC, \Lambda, Z\}$ to indicate $\lambda_i^{BC}(E) = \Lambda^{(i)}(E)$, $\lambda_i^{\Lambda}(E) = \Lambda_i(E)$, $\lambda_i^{Z}(E) = Z_i(E)$.

Basic properties: $E_{/K}$ rigid adelic with $\dim E = n$, $i \in \{1, \dots, n\}$

1) $\forall K'_{/K}$ algebraic extension $\lambda_i^*(E \otimes_K K') \leq \lambda_i^*(E)$ $\forall * \in \{\Lambda, Z\}$

   (What is true for $* = BC$?)

2) If $F \subset E$ is a linear subspace then $\lambda_i^*(E) \leq \lambda_i^*(F)$ $\forall i \in \{1, \dots, \dim F\}$ $\forall * \in \{BC, \Lambda, Z\}$

Proposition. let $N \in \mathbb{N} \setminus \{0\}$ and $E_1, \dots, E_N$ be rigid adelic spaces.

Then $\Lambda_1(E_1 \oplus \dots \oplus E_N) = \min(\Lambda_1(E_i) ; 1 \leq i \leq N)$

Proof: Convexity inequality p. 11.

b) Slopes

Ref: [BC2013], [Bo1995], [Ch2010], [Bo2005]

Let E be a rigid adelic space over K and $n = \dim E$.

Fact: There exists a positive constant $c(E)$ such that $H(F) \geq c(E)$ for every subspace $F \subset E$,

Proof: Let (φ, A) a couple defining the adelic structure of E. There exists $a \in \mathbb{A}_{\mathbb{Q}}^{\times}$, $a = (a_p)_{p \in V(\mathbb{Q})}$, such that $|a_p|_v^{-1} \, |\varphi_v(x)|_v \leq \|x\|_{E,v} \leq |a_p|_v \, |\varphi_v(x)|_v$ for all v/p, $p \in V(\mathbb{Q})$ and $x \in E \otimes_K K_v$. Put $|a| = \exp \int_{V(K)} \operatorname{Log} |a_p|_v \, d\mu(v)$. For every $F \subset E$ with dimension ℓ, we have $H(F) \geq |a|^{-\ell} H(\varphi(F)) = |a|^{-\ell} H(\det \varphi(F))$. Since $\det \varphi(F)$ is a non zero vector of $\Lambda^{\ell} K^n$, which is isometric to $K^{\binom{n}{\ell}}$, we have $H(\varphi(F)) \geq 1$. $\square$

In other words $\{ \deg F \,;\, F \subset E \}$ is bounded from above.

This fact allows to define some positive real numbers associated to E:

$$\sigma_i(E) = \inf \{ H(F) \,;\, \underset{\text{linear subspace}}{F \subset E} \text{ and } \dim F = i \} \qquad i = 0, 1, \dots, n$$

For instance, $\sigma_0(E) = 1$, $\sigma_1(E) = \lambda_1(E)$ and $\sigma_n(E) = H(E)$

We have $\sigma_{n-1}(E) = \lambda_1(E^\vee) H(E)$ and, more generally, $\sigma_{n-i}(E) = \sigma_i(E^\vee) H(E)$ for all $i \in \{0, \dots, n\}$ (use the isometry $E/F \simeq (F^\perp)^\vee$).

At last $\sigma_i(E) \geq \lambda_1(\Lambda^i E)$ (is this inequality can be strict ?).

Canonical polygon: Let $P_E : [0, n] \to \mathbb{R}$ denote the piecewise linear function delimiting from above the convex hull of

$$\{ (\dim F, \deg F) \,;\, \underset{\text{linear subspace}}{F \subset E} \} \subset \mathbb{R}^2.$$

Of course, we can replace the latter set by the (finite) set

$$\{(i, -\mathrm{Log}\, \sigma_i(E)) \; ; \; i \in \{0, \dots, n\}\}.$$

The function P_E is by definition a concave function and its slopes

$$\mu_i(E) = P_E(i) - P_E(i-1) \qquad (i \in \{1, \dots, n\})$$

form a non decreasing sequence $\mu_1(E) \geq \mu_2(E) \geq \dots \geq \mu_n(E)$.

The greatest slope $\mu_1(E)$ is also denoted $\mu_{max}(E)$ and the smallest slope $\mu_n(E)$ is $\mu_{min}(E)$.

This terminology is a bit more justified by the following result.

<u>Key lemma:</u> $\mu_{max}(E) = \max\left(\mu(F) \; ; \; \{0\} \subsetneq \underset{\text{linear subspace}}{F \subset E}\right)$

More precisely there exists a (unique) of E, denoted E_{des}, such that $\mu(E_{des}) = \mu_{max}(E)$ and E_{des} contains every linear subspace $F \subset E$ satisfying $\mu(F) = \mu_{max}(E)$.

The subscript « des » means « destabilizing ».

<u>Proof:</u> (We follow [BC 2013, Proposition 2.2]).

Let us temporarily denote by c the supremum of slopes $\mu(F)$ when F browses non zero linear subspaces of E. This is a real number since $\deg F$ is bounded from above by a constant depending on E. Actually, if $m = \dim F$, we have $\mu(F) = \dfrac{\deg F}{m} \leq \dfrac{P_E(m)}{m} = \dfrac{\mu_1(E) + \dots + \mu_m(E)}{m} \leq \mu_1(E)$ and so $c \leq \mu_1(E)$. On the other hand, for every $F \subset E$, we have $\deg F \leq (\dim F)\, c$. Since $m \mapsto mc$ is a concave (linear) function we deduce $P_E(m) \leq mc$ for all $m \in [0, n]$. Thus $\mu_1(E) = P_E(1) \leq c$ and we get $\mu_1(E) = c = \sup(\mu(F) \; ; \; 0 \neq F \subset E)$.

Let us now prove the maximality property of $\mu(E)$. We proceed by induction on n.

The statement is clear for $n=1$ since $\mu_{max}(E) = \mu(E)$ in this case.

Assume the existence of E_{des} when $\dim E \leq n-1$.

Let E of dimension n. If $\mu_{max}(E) = \mu(E)$ then $E_{des} = E$ is the winner.

Otherwise the set $\{F \subset E \ ; \ F \neq 0 \text{ and } \mu(F) > \mu(E)\}$ is non-empty and we can choose F in it with maximal dimension. By induction hypothesis (since $F \neq E$), there exists F_{des} such that $\mu(F_{des}) = \mu_{max}(F)$ and, for all $G \subset F$ with $\mu(G) = \mu_{max}(F)$, we have $G \subset F_{des}$.

Let G be a non zero linear subspace of E. If $G \not\subset F$ then $\dim(F+G) > \dim F$ and by maximality property of F, we have $\mu(F+G) \leq \mu(E)$.

If we put this information in the inequality

$$\deg F + \deg G \leq \deg(F+G) + \deg F \cap G \qquad \text{(see p. 9)}$$

we get $\quad (\dim F)\,\mu(F) + (\dim G)\,\mu(G) \leq \dim(F+G)\,\mu(E) + (\dim F\cap G)\,\mu_{max}(F)$

and so $\quad (\dim G)\,\mu(G) \leq \dim(F+G)\underbrace{(\mu(E) - \mu(F))}_{<0} + (\dim(F+G) - \dim F)\,\mu(F) + (\dim F\cap G)\,\underbrace{\mu_{max}(F)}_{}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underbrace{\mu(F)}_{\leq \mu_{max}(F)}$

then $\quad \mu(G) < \mu_{max}(F)$.

If $G \subset F$ we have of course $\mu(G) \leq \mu_{max}(F)$.

So, for all $0 \neq G \subset E$, $\mu(G) \leq \mu_{max}(F)$ and we have $\mu_{max}(E) = \mu_{max}(F)$.

The space $E_{des} = F_{des}$ satisfies the required property. $\square$

<u>Definition</u>: A rigid adelic space E is said to be semistable if

$$\mu(E) = \mu_{max}(E) \qquad (\text{that is, } E_{des} = E).$$

In this case the canonical polygon is a straight line.

<u>Examples</u>: $E = K^n$ or $E = A_n$ are semistable.

This key lemma allows to define a unique filtration of linear subspaces of E:

$$\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{N-1} \subsetneq E_N = E$$

such that E_{i+1}/E_i is semistable for every $i \in \{0, 1, \ldots, N-1\}$:

$$E_1 = E_{des}, \quad E_{i+1}/E_i = \left(E/E_i\right)_{des} \quad i \in \{0, 1, \ldots, N-1\}.$$

This filtration is called the <u>Harder - Narasimhan filtration</u> of E ("HN-filtration" in shortened writing).

By definition $\mu\left(E_{i+1}/E_{i-1}\right) < \mu\left(E_i/E_{i-1}\right)$ and using $\deg E_{i+1}/E_i =$ $\deg E_{i+1}/E_{i-1} - \deg E_i/E_{i-1}$, we deduce that

$$\mu\left(E_N/E_{N-1}\right) < \mu\left(E_{N-1}/E_{N-2}\right) < \cdots < \mu(E_1).$$

<u>Theorem</u>: let $E_0 = \{0\} \subsetneq E_1 \subsetneq \cdots \subsetneq E_N = E$ be the HN-filtration of E.

let $m_i = \dim E_i$ $\forall i$. Then $m_1, \ldots, m_N$ are (exactly) the points at which P_E is not differentiable and $P_E(m_i) = \deg E_i$ $\forall i \in \{0, \ldots, N\}$.

Moreover, for every $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, m_i - m_{i-1}\}$, we have

$$\mu_{m_{i-1}+j}(E) = \mu\left(E_i/E_{i-1}\right)$$

<u>Lemma</u>: let $x \in [0, n]$ such that P_E is not differentiable at x.

Then $x \in \mathbb{N}$ and there exists a unique linear subspace $F_x \subset E$ with dimension x such that $P_E(x) = \deg F_x$.

<u>Proof of the lemma</u>:

By definition of P_E, which is a linear function on each interval $(h, h+1)$ for $h \in \{0, \ldots, n-1\}$, the real number x is necessarily an integer.

Since obviously $F_0 = \{0\}$ and $F_n = E$ we can assume $x \in \{1, \ldots, n-1\}$.

Since P_E is not differentiable at x and by construction of P_E, we have $P_E(x) = \sup(\deg F \; ; \; F \subset E, \dim F = x)$. Then, let us choose linear subspaces A and B of E, of dimension x, such that $P_E(x) \leq \deg A + \varepsilon$ and $P_E(x) \leq \deg B + \varepsilon$ for

$$\varepsilon = \frac{1}{4} \min\left(\frac{P_E(s) - P_E(i)}{s-i} - \frac{P_E(j) - P_E(h)}{j-h} \; ; \; \begin{matrix} 0 \leq i < s \leq h < j \leq n \\ i, s, h, j \in \mathbb{N} \end{matrix} \right) > 0$$

(the minimum is taken over non zero quantities only; then it is positive by concavity of P_E).

Defining $j = \dim(A+B)$ and $i = \dim A \cap B$ and using $\deg A + \deg B \leq \deg(A+B) + \deg A \cap B$ we get $2P_E(x) - 2\varepsilon \leq P_E(j) + P_E(i)$ that is, if $x \neq i$,

$$\frac{P_E(x) - P_E(i)}{x-i} - \frac{P_E(j) - P_E(x)}{j-x} \leq 2\varepsilon \qquad (\text{since } x - i = j - x)$$

contradicting the definition of ε. Thus $x - i = j - x = 0$, i.e., $A = B$.

That proves that $\{ A \subset E ; \dim A = x \text{ and } P_E(x) \leq \deg A + \varepsilon \}$ (for the above ε) is a singleton $\{F_x\}$. □

Proof of the theorem

Let $f_0 = 0 < f_1 < \dots < f_M = n$ be the non differentiability points of P_E and $F_0 = \{0\} \subsetneq F_1 \subsetneq \dots \subsetneq F_M = E$ be the corresponding subspaces given by the lemma ($P_E(f_i) = \deg F_i$). For all linear subspace $F \subset E$ and $i \in \{1, \dots, M\}$ such that $F \not\subset F_{i-1}$, the concavity of P_E gives

$$\frac{P_E(\dim(F + F_{i-1})) - P_E(f_{i-1})}{\dim(F+F_{i-1}) - f_{i-1}} \leq \frac{P_E(f_i) - P_E(f_{i-1})}{f_i - f_{i-1}}$$

and this inequality is strict if $\dim(F + F_{i-1}) > f_i$.

In other words, if we bound from below $P_E(\dim(F+F_{i-1}))$ by $\deg(F+F_{i-1})$ we get $\mu\left(\frac{F+F_{i-1}}{F_{i-1}}\right) \leq \mu\left(\frac{F_i}{F_{i-1}}\right)$ and $F=F_i$ is maximal for this property (the inequality being strict if ~~many~~ $\dim(F+F_{i-1}) > \dim F_i$).

Thus $F_i/F_{i-1} = \left(E/F_{i-1}\right)_{des}$. Since $F_0=\{0\}$, the sequence $(F_i)_i$ satisfies the same definition as the HN-filtration of E and so $N=M$ and $F_i=E_i$.

The equality $\mu_{m_{i-1}+j}(E) = \mu\left(E_i/E_{i-1}\right)$ comes from the fact that $\mu_{m_{i-1}+1}(E) = \cdots = \mu_{m_i}(E)$ (since P_E is a line on $[m_{i-1}, m_i]$)

and
$$\sum_{j=1}^{m_i-m_{i-1}} \mu_{m_{i-1}+j}(E) = \sum P_E(m_{i-1}+j) - P_E(m_{i-1}+j-1)$$
$$= P_E(m_i) - P_E(m_{i-1}) = \deg E_i - \deg E_{i-1}$$
$$= \deg E_i/E_{i-1} = (m_i - m_{i-1})\,\mu\left(E_i/E_{i-1}\right).$$

□

From this theorem we can deduce a minimax formula for $\mu_i(E)$.

<u>Proposition</u>: let E be a rigid adelic space over K and $i \in \{1, \ldots, \dim E\}$.

Then $\mu_i(E) = \max_A \min_B \mu\left(A/B\right) = \min_B \max_A \mu\left(A/B\right)$

where $B \subset A$ browse linear subspaces of E with $\dim B \leq i-1 < \dim A$.

<u>Proof</u> let $\alpha_i(E) = \sup_A \inf_B \mu\left(A/B\right)$ with $B \subset A$ and $\dim B \leq i-1 < \dim A$.

~~The~~ equality $\alpha_i(E) = \mu_i(E)$ will prove the sup inf is a max min. ~~Max~~

let $E_0 = \{0\} \subsetneq E_1 \subsetneq \cdots \subsetneq E_N = E$ be the HN-filtration of E.

let $h \in \{0, \ldots, N-1\}$ such that $\dim E_h \leq i-1 < \dim E_{h+1}$.

We have $\mu_i(E) = \mu\left(E_{h+1}/E_h\right) = \max_{A \supsetneq E_h} \mu\left(A/E_h\right) \geq \max_A \inf_B \mu\left(A/B\right) = \alpha_i(E)$.

On the other hand the concavity of P_E gives

$$\mu\left(E_{h+1}/B\right) \geq \frac{P_E(\dim E_{h+1}) - P_E(\dim B)}{\dim E_{h+1} - \dim B} \geq \mu\left(E_{h+1}/E_h\right) = \mu_i(E)$$

(for $B \subset E_{h+1}$ of dimension $\leq i-1$). We conclude with

$\mu_i(E) \geq \inf\limits_B \mu\left(E_{h+1}/B\right)$. The same method can be applied for $\inf\limits_B \sup\limits_A \mu(A/B)$. $\square$

In particular this proposition gives ($n = \dim E$)

$$\mu_n(E) = \mu_{min}(E) = \min\left(\mu\left(E/F\right) \; ; \; F \subsetneq E\right).$$

<u>Some properties of P_E</u>

Let E be a rigid adelic space over K, $n = \dim E$.

1) If L is a rigid adelic space over K of dimension 1 then $P_{E\otimes L} = P_E + \deg L$

2) $\forall x \in [0, n]$, $P_{E^\vee}(x) = P_E(n-x) - \deg E$.

In particular, $\forall i \in \{1, \ldots, n\}$, $\mu_i(E^\vee) = -\mu_{n+1-i}(E)$

3) Let K'/K be an algebraic extension. Then $P_{E\otimes_K K'} = P_E$.

In particular $\forall i \in \{1, \ldots, n\}$, $\mu_i(E\otimes_K K') = \mu_i(E)$.

This remarkable last property suggests than μ_i's are absolute minima (over $\overline{K}$).

<u>Proof</u>

1) For all subspace $F \subset E$ with dimension m, we have $\dim F\otimes_K L = m$ and $\deg F + \deg L = \deg F\otimes L \leq P_{E\otimes L}(m)$. So $\deg F \leq P_{E\otimes L}(m) - \deg L$ and since the function $m \mapsto P_{E\otimes L}(m) - \deg L$ is concave, we deduce $P_E(m) \leq P_{E\otimes L}(m) - \deg L$. The reverse inequality is obtained replacing E by $E\otimes L$ and L by $L^\vee$ ($L\otimes L^\vee \simeq K$).

2) Since P_E is a linear function on each interval $[i, i+1]$, $i \in \{0, \dots, n-1\}$, it is enough to prove the equality for $x = m \in \{0, \dots, n\}$. For a subspace $F \subset E$ with dimension m, the isometric isomorphism $E/F \simeq (F^\perp)^\vee$ yields $\deg F - \deg E = \deg F^\perp$ and $\deg F \leq \deg E + P_{E^\vee}(n-m)$ and then

$$P_E(m) \leq \deg E + P_{E^\vee}(n-m) \quad \text{since } m \mapsto P_{E^\vee}(n-m) + \deg E \text{ is concave.}$$

For the reverse inequality, replace E by $E^\vee$, m by $n-m$ and use $E^{\vee\vee} \simeq E$.

3) For every subspace $F \subset E$ with dimension m, we saw that $\deg F = \deg F \otimes_k k'$ and so $\deg F \leq P_{E \otimes k'}(m)$ and then $P_E(m) \leq P_{E\otimes k'}(m)$.

For the reverse inequality, we may assume that k'/k is Galois (actually we could choose $k' = \bar{k}$). Let $\{0\} = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_N = E \otimes_k k'$ be the HN-filtration of $E \otimes_k k'$ and $d_i = \dim F_i$ $\forall i$. Let $e_1, \dots, e_n$ be a k-basis of E. For every $\sigma \in Gal(k'/k)$, the correspondence $\iota_\sigma : E \otimes_k k' \longrightarrow E \otimes_k k'$ which sends $\sum_{i=1}^n x_i e_i$ $(x_i \in k')$ to $\sum_{i=1}^n \sigma(x_i) e_i$ is a bijection that preserves dimension and degree of subspaces of $E \otimes_k k'$. Using lemma and Theorem on page 17, we deduce $\iota_\sigma(F_i) = F_i$ $\forall i$.

Now, let us fix $i \in \{1, \dots, n-1\}$. Even if it means permuting the vectors $e_1, \dots, e_n$, we can find a k'-basis $f_1, \dots, f_{d_i}$ of F_i and scalars $\alpha_{j,h} \in k'$, $1 \leq j \leq d_i$, $d_{i+1} \leq h \leq n$, such that

$$(*) \quad \forall j \in \{1, \dots, d_i\}, \quad f_j = e_j + \sum_{h=d_i+1}^{n} \alpha_{j,h} e_h \qquad \text{(Gaussian elimination)}$$

The Galois closure k'_0 of the field generated by k and all $\alpha_{j,h}$'s is both a subfield of k' and a finite extension of k. So we can consider its normalized trace function $Tr : k'_0 \to k$ $(Tr(1) = 1)$.

Since $\iota_\sigma(F_i) = F_i$ for all $\sigma \in \mathrm{Gal}(K'_0/K)$, the vector

$$\mathrm{Tr}\, f_j := \frac{1}{[K'_0 : K]} \sum_{\sigma \in \mathrm{Gal}(K'_0/K)} \iota_\sigma(f_j) \quad \text{belongs to } F_i \text{ for all } j \in \{1, .., d_i\}.$$

With $(*)$, $\mathrm{Tr}\, f_j$ is written $e_j + \sum_{h=d_i+1}^{n} \mathrm{Tr}(\alpha_{j,h})\, e_h$ and so $\mathrm{Tr}\, f_j \in E$.

Moreover $\{\mathrm{Tr}\, f_1, .., \mathrm{Tr}\, f_{d_i}\}$ is a free family and $G_i = \mathrm{Vect}_K(\mathrm{Tr} f_1, .., \mathrm{Tr} f_{d_i})$ has dimension d_i. Since $G_i \subset F_i$ we get $F_i = G_i \otimes_K K'$. In particular $P_{E\otimes K'}(d_i) = \deg F_i = \deg G_i \leq P_E(d_i)$ and so $P_{E\otimes K'}(d_i) = P_E(d_i)$ $\forall i$.

Since $P_{E\otimes K'}$ is linear on $[d_i, d_{i+1}]$ and $P_E \leq P_{E\otimes K'}$ are both concave functions, we deduce $P_E = P_{E\otimes K'}$ on $[d_i, d_{i+1}]$ and, finally, on $[0, n]$.

$$\Big(x \in [d_i, d_{i+1}], \quad \frac{P_E(x) - P_E(d_i)}{x - d_i} \underset{P_E \text{ concavity}}{\geq} \frac{P_E(d_{i+1}) - P_E(d_i)}{d_{i+1} - d_i} = \frac{P_{E\otimes K'}(d_{i+1}) - P_{E\otimes K'}(d_i)}{d_{i+1} - d_i}$$

$$\underset{P_{E\otimes K'} \text{ linear}}{=} \frac{P_{E\otimes K'}(x) - P_{E\otimes K'}(d_i)}{x - d_i} = \frac{P_{E\otimes K'}(x) - P_E(d_i)}{x - d_i}$$

and so $P_E(x) \geq P_{E\otimes K'}(x)$. $\Big)$

$\square$

In general P_E seems difficult to compute, even its first value $P_E(1) = \mu_{max}(E)$.

For example, here is an

<u>Exercice</u>: Let $n, \ell \in \mathbb{N}\setminus\{0\}$. Prove that

$$-\log \lambda_1(S^\ell(K^n)) = \mu_{max}(S^\ell(K^n)) = \max_{\substack{i_1 + \dots + i_n = \ell \\ i_j \in \mathbb{N}}} \frac{1}{2} \log \frac{\ell!}{i_1! \dots i_n!}$$

$$= \frac{1}{2} \log \frac{\ell!}{\lambda!^n (\lambda+1)^{\ell - n\lambda}} \quad \text{with } \lambda = \left[\frac{\ell}{n}\right] \text{ (floor function).}$$

In particular $S^\ell(K^n)$ is semistable if and only if ℓ or n equals 1.

(see [GR 2013]).

To conclude this paragraph, let us mention the maximal slope counterpart of the equality $\Lambda_1(E_1 \oplus \cdots \oplus E_N) = \min\limits_{1 \leq i \leq N} \Lambda_1(E_i)$ (see p. 13)

Proposition: let $N \in \mathbb{N} \setminus \{0\}$ and $E_1, \ldots, E_N$ be rigid adelic spaces over K.

Then
$$\mu_{max}(E_1 \oplus \cdots \oplus E_N) = \max_{1 \leq i \leq N} \mu_{max}(E_i) \quad \text{and}$$
$$\mu_{min}(E_1 \oplus \cdots \oplus E_N) = \min_{1 \leq i \leq N} \mu_{min}(E_i).$$

Proof. It is enough to treat the case $N = 2$.

Since $E_i \subset E_1 \oplus E_2$, $i = 1, 2$, we have $\max(\mu_{max}(E_1), \mu_{max}(E_2)) \leq \mu_{max}(E_1 \oplus E_2)$.

Now let F be a linear subspace of $E_1 \oplus E_2$. Define $F_1 = F \cap E_1$ and $F_2 = \mathrm{Im}(F \to E_2)$ where $F \to E_2$ is the restriction to F of the projection $E_1 \oplus E_2 \to E_2$. We get an isomorphism $p : F/F_1 \to F_2$ such that $\|p(x)\|_{F_2, v} \leq \|x\|_{F/F_1, v}$ for all $x \in (F/F_1) \otimes K_v$ and $v \in V(K)$.

Thus $|\det p|_v \leq 1$ for all v and $\mu(F/F_1) \leq \mu(F_2)$.

Writing $n_1 = \dim F_1$ and $n_2 = \dim F_2$, we deduce $\mu(F) \leq \dfrac{n_1 \mu(F_1) + n_2 \mu(F_2)}{n_1 + n_2}$

and so $\mu(F) \leq \max(\mu_{max}(E_1), \mu_{max}(E_2))$. Choosing $F = (E_1 \oplus E_2)_{\mathrm{des}}$ gives the result. For the minimal slope, one can easily check that the map

$$\begin{aligned} E_1^\vee \oplus \cdots \oplus E_N^\vee &\longrightarrow (E_1 \oplus \cdots \oplus E_N)^\vee \\ (\varphi_1, \ldots, \varphi_N) &\longmapsto \big((x_1, \ldots, x_N) \mapsto \varphi_1(x_1) + \cdots + \varphi_N(x_N)\big) \end{aligned}$$
is isometric.

In particular maximal slopes of $\bigoplus\limits_{i=1}^N E_i^\vee$ and $\Big(\bigoplus\limits_{i=1}^N E_i\Big)^\vee$ are the same and we conclude using $\mu_{max}(E^\vee) = -\mu_{min}(E)$.

$\square$

4) Comparisons between minima and slopes

a) Lower bounds

Fact: let E be a rigid adelic space over K. Then $1 \leq \Lambda_1(E)\, e^{\mu_1(E)}$.

Proof: For all $x \in E \setminus \{0\}$, $-\log H_E(x) = \deg K.x \leq P_E(1) = \mu_1(E)$

and $\Lambda_1(E) = \inf\left(H_E(x)\ ;\ x \in E\setminus\{0\}\right)$. $\square$

Corollary 1: For all $i \in \{1, \dots, \dim E\}$, we have $1 \leq \Lambda^{(i)}(E)\, e^{\mu_i(E)}$

In particular $1 \leq \lambda_i^*(E)\, e^{\mu_i(E)}$ $\forall *$ (since $\Lambda^{(i)}(E) \leq \Lambda_i(E) \leq Z_i(E)$)

Proof: let $F \subset E$ be a linear subspace with dimension $\leq i-1$.

The fact gives $1 \leq \Lambda_1(E/F)\, e^{\mu_1(E/F)}$. We bound from above $\Lambda_1(E/F)$ by $\Lambda^{(i)}(E) = \sup\left(\Lambda_1(E/F)\ ;\ \dim F \leq i-1\right)$ and then we take the minimum over F using $\mu_i(E) = \min\limits_{\dim F \leq i-1} \mu_{max}(E/F)$. $\square$

Corollary 2 (Hadamard inequality): $H(E) \leq \Lambda^{(1)}(E) \cdots \Lambda^{(n)}(E)$

Proof: Multiply the previous inequalities for $i = 1, \dots, n$ and use

$\mu_1(E) + \dots + \mu_n(E) = \deg E = -\log H(E)$. $\square$

Often the weaker inequality $H(E) \leq \Lambda_1(E) \cdots \Lambda_n(E)$ is used.

b) Upper bounds

Recall $\lambda_i^*(E) = \Lambda^{(i)}(E)$, $\Lambda_i(E)$ or $Z_i(E)$ according to $* = BC, \Lambda$ or Z.

Let us define several constants ($n \geq 1$ integer)

- $$c_I(n,k) = \sup_{\dim E = n} \Lambda_1(E)\, H(E)^{-\frac{1}{n}} = \sup_{\dim E = n} \Lambda_1(E)\, e^{\mu(E)}$$

- $$c_{II}^*(n,k) = \sup_{\dim E = n} \left(\frac{\lambda_1^*(E) \cdots \lambda_n^*(E)}{H(E)} \right)^{1/n}$$

- $$\forall i \in \{1, \dots, n\}, \quad c_i^*(n,k) = \sup_{\dim E = n} \lambda_i^*(E)\, e^{\mu_i(E)}$$

Here the suprema are taken over rigid adelic spaces over k with dimension n.

One can prove that we could take the suprema over hyperplane of the standard space k^{n+1} (see [GR 2017, §4.8]) and obtain the same numbers.

These constants can be infinite (see below).

Some simple observations

1) $c_I(n,\mathbb{Q}) = c_{II}^{\wedge}(n,\mathbb{Q})$ is the square root of the Hermite constant mentioned at the beginning of the text.

2) $c_I(n,k) \leq c_{II}^{BC}(n,k) \leq c_{II}^{\wedge}(n,k) \leq c_{II}^{Z}(n,k)$

3) $c_{II}^*(n,k)^n \leq \prod_{i=1}^{n} c_i^*(n,k)$

4) $\forall i \in \{1, \dots, n\}, \quad c_i^*(n,k) \leq c_{II}^*(n,k)^n$

5) $n \mapsto c_I(n,k)^n$ is nondecreasing (Take $E \oplus L$ with $\dim L = 1$ and $H(L) = \Lambda_1(E)$)

__Question:__ Is $n \mapsto c_I(n,k)$ a nondecreasing function?

To my knowledge, it is not known for the Hermite constant. We shall see that it is true for $k = \overline{\mathbb{Q}}$.

We nonetheless have the

<u>Mordell inequality</u> : $c_I(n+1,K) \leq c_I(n,K)^{\frac{n}{n-1}}$ for $n \geq 2$ integer.

By induction we deduce the exponential bound $c_I(n,K) \leq c_I(2,K)^{n-1}$.

<u>Proof</u> : let E be a rigid adelic space of dimension $n+1$. Let $\varepsilon > 0$ and $x \in E \setminus \{0\}$ such that $H_E(x) \leq \Lambda_1(E) + \varepsilon$. The hyperplane $F = \{x\}^\perp \subset E^\vee$ satisfies $\Lambda_1(E^\vee) \leq \Lambda_1(F) \leq c_I(n,K)\, H(F)^{1/n}$. Since $F \simeq (E/K\cdot x)^\vee$, we have

$$H(F) = H_E(x)/H(E) \leq \frac{\Lambda_1(E)}{H(E)} + \frac{\varepsilon}{H(E)}.$$

Replacing in the previous inequality and letting $\varepsilon \to 0$ lead to $\Lambda_1(E^\vee) \leq c_I(n,K)\left(\frac{\Lambda_1(E)}{H(E)}\right)^{1/n}$.

We apply this to $E^\vee$ instead of E and we combine both inequalities to obtain $\Lambda_1(E) \leq c_I(n,K)^{\frac{n}{n-1}}\, H(E)^{\frac{1}{n+1}}$. $\square$

With a bit more pain, one can also prove $c_{II}^{Z}(n,K) \leq c_{II}^{Z}(2,K)^{2^n}$ (see [GR 2017, Proposition 4.14]).

Let us also mention the analogue of Minkowski theorem :

$$\forall n \geq 1, \quad c_I(n,K) = c_{II}^{\wedge}(n,K)$$

(in particular $c_I(n,K) = c_{II}^{BC}(n,K)$)

The proof is based on a deformation metric argument. To a rigid adelic space E over K, we associate another rigid adelic space E' such that

$$\Lambda_1(E') \geq 1 \quad \text{and} \quad H(E') = \frac{H(E)}{\Lambda_1(E)\cdots\Lambda_n(E)}.$$

(see [GR 2017, Theorem 4.12]).

Definition: An algebraic extension $K/\mathbb{Q}$ is called a Siegel field if

$$c_{II}^{\wedge}(n,K) < +\infty \quad \forall n \geq 1.$$

With the previous observations, K is a Siegel field if and only if $c_I(2,K) < +\infty$.

In a more elementary approach, K is a Siegel field if and only if there exists $\alpha > 0$ such that, for all $(a,b,c) \in K^3 \setminus \{0\}$ there exists $(x,y,z) \in K^3 \setminus \{0\}$ such that $ax+by+cz=0$ and $H_{K^3}(x,y,z) \leq \alpha H_{K^3}(a,b,c)^{1/2}$.

Examples of Siegel fields:

- $\mathbb{Q}$, number field (Minkowski)
- $\overline{\mathbb{Q}}$ (Zhang [Zh 1995] and Roy & Thunder [RT 1996])
- Hilbert class field towers of number fields

A finite extension of a Siegel field is still a Siegel field.

Theorem [GR, 2017]:

1) $\forall n \geq 1$, $c_{II}^{2}(n,K) < +\infty \iff K$ is a Siegel field of infinite degree

2) A Northcott field is a Siegel field if and only if it is a number field

2) is a direct consequence of 1): If K is both a Northcott and Siegel field then $Z_i(E) = +\infty$ $\forall i \in \{2, .., \dim E\}$ so $c_{II}^{2}(n,K) = +\infty$ as soon as $n \geq 2$ and so $K/\mathbb{Q}$ is finite.

Besides the implication $\Rightarrow$ in 1) is easy enough: $c_{II}^{\wedge}(n,K) \leq c_{II}^{2}(n,K)$ and $Z_i(E) = +\infty$ for $2 \leq i \leq \dim E$ when K is a number field.

So the striking part of this theorem is that it suffices to be a Siegel field of infinite degree to have $c_{II}^{2}(2,K) < +\infty$.

The proof rests on a deformation metric argument at ~~some~~ ultrametric place, much more subtle than for Minkowski theorem (see [GR 2017, §4.6]).

To be a bit more precise, let us define the impurity index $u(K)$ of an algebraic extension $K/\mathbb{Q}$. If $v \in V(K) \setminus V_\infty(K)$ we denote $\lambda(v)$ the measure of the singleton $\{v\}$, p_v is the prime number associated to v, e_v the ramification index at v and f_v its residual degree.

Then <u>the impurity index of K</u> is

$$u(K) = \sup_{N \geq 1} \inf \left\{ p_v^{\frac{\lambda(v)}{e_v}} \; ; \; v \in V(K) \setminus V_\infty(K) \, , \, p_v^{f_v} \geq N \right\}$$

with the conventions : • $p_v^{\frac{\lambda(v)}{e_v}} = 1$ if $e_v = \infty$ (or $\lambda(v) = 0$)

• $p_v^{f_v} \geq N$ is true if $f_v = \infty$

We can check that $u(K) < +\infty \iff [K:\mathbb{Q}] = +\infty$

Note that $u(\overline{\mathbb{Q}}) = 1$ and, for all real number B, there exists K such that $B < u(K) < +\infty$.

<u>Proposition :</u> let E be a rigid adelic space over K with dimension n. For each $i \in \{1, \dots, n\}$, let α_i be a real number such that $0 < \alpha_i < Z_i(E)$. Then there exists a rigid adelic space E' over K with $\dim E' = n$ such that

$$\frac{\left(u(K) \, \Lambda_1(E')\right)^n}{H(E')} \geq \frac{\alpha_1 \cdots \alpha_n}{H(E)} .$$

This proposition leads to the bound $c_{II}^2(n,K) \leq u(K) \, c_I(n,K)$ $\forall n \geq 1$, thus giving 1) of the theorem.

In short all the constants $c_I(n,K)$, $c_{II}^{*}(n,K)$, $c_i^{*}(n,K)$ are finite if K is a Siegel field of infinite degree and, if $* \neq Z$, this remains true if K is a number field.

In the real life it is useful to have some concrete bounds for $c_{II}^{*}(n,K)$. In general it seems to be a difficult problem. Let us mention two cases:

1) If K is a number field of root discriminant $\delta_{K|\mathbb{Q}} = |\Delta_{K|\mathbb{Q}}|^{\frac{1}{[K:\mathbb{Q}]}}$,

then $c_{II}^{\wedge}(n,K) = c_I(n,K) \leq \left(n\,\delta_{K|\mathbb{Q}}\right)^{1/2}$.

2) If $K = \overline{\mathbb{Q}}$ then $c_I(n,\overline{\mathbb{Q}}) = c_{II}^{\wedge}(n,\overline{\mathbb{Q}}) = c_{II}^{Z}(n,\overline{\mathbb{Q}}) = \exp\frac{H_n - 1}{2}$

where $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$.

(For 2) use Zhang's inequalities $Z_n(E)\, e^{\mu(E)} \geq \exp\frac{H_n-1}{2} \geq c_{II}^{Z}(n,\overline{\mathbb{Q}})$ and $c_{II}^{Z}(n,\overline{\mathbb{Q}}) = c_I(n,\overline{\mathbb{Q}})$ since $u(\overline{\mathbb{Q}}) = 1$).

(Related question: A bound of the type $Z_n(E)\, e^{\mu_{min}(E)} \geq \exp\frac{H_n-1}{2}$ for all rigid adelic space E with $\dim E = n$ is credible?)

The numbers $c_I(n,\overline{\mathbb{Q}})$ are the only (analogous) Hermite constants computed for all integers $n \geq 1$.

Let us discuss now in more detail about the constants $c_i^*(n,K)$

Proposition: For all integer $n \geq 1$, $c_1^*(n,K) = \sup_{\dim E = n} \Lambda_1(E)\, e^{\mu_1(E)}$ does not depend on $*$

and $c_1^*(n,K) = \max_{1 \leq i \leq n} c_I(i,K)$

Proof: The first statement is clear since $\lambda_1^*(E) = \Lambda_1(E)$ $\forall *$.

As for the equality with the maximum, let E be a rigid adelic space over K with dimension n, E_{des} its destabilizing space and $d = \dim E_{des}$

We have $\Lambda_1(E) \leq \Lambda_1(E_{des})$ and $\mu(E) \leq \mu_{max}(E) = \mu(E_{des}) = \mu_1(E)$.

So $\Lambda_1(E)\, e^{\mu_1(E)} \leq \Lambda_1(E_{des})\, e^{\mu(E_{des})} \leq c_I(d,K) \leq \max_{1 \leq i \leq n} c_I(i,K)$.

Conversely, let F be a rigid adelic space of dimension $i \in \{1, \dots, n-1\}$ and $G = L^{\oplus n-i}$ with L a rigid adelic line with $\Lambda_1(L) = \Lambda_1(F)$.

Then $\dim(F \oplus G) = n$, $\Lambda_1(F \oplus G) = \min(\Lambda_1(F), \Lambda_1(L)) = \Lambda_1(F)$ and $\mu(F) \leq \mu_1(F \oplus G)$ (since F is a subspace of $F \oplus G$). We get

$\Lambda_1(F)\, e^{\mu(F)} \leq \Lambda_1(F \oplus G)\, e^{\mu_1(F \oplus G)} \leq c_1^*(n,K)$, so $c_I(i,K) \leq c_1^*(n,K)$.

The desired equality follows. $\square$

Proposition: $\forall n \geq 1$, $\forall * \in \{BC, \Lambda, Z\}$, $c_1^*(n,K) \leq c_2^*(n,K) \leq \dots \leq c_n^*(n,K)$

In other words $c_n^*(n,K) = \sup_{\dim E = n} \lambda_n^*(E)\, e^{\mu_{min}(E)}$ controls everything

and $c_1^*(n,K) = \sup \lambda_1^*(E)\, e^{\mu_{max}(E)}$ in particular.

The proof rests on two lemmas.

Lemma 1: $c_i^*(n,K) \leq c_i^*(n+1,K)$

Proof: let E be a rigid adelic space of dimension n and $L = (K, \lambda_i^*(E).|.|)$

$(\Lambda_1(L) = \lambda_i^*(E))$. We have $\lambda_i^*(E) \leq \lambda_i^*(E\oplus L)$ and $\mu_i(E) \leq \mu_i(E\oplus L)$

since
$$\mu_i(E) = \min_{\dim H \leq i-1} \mu_{max}(E/H) = \min_{\dim H \leq i-1} \mu_{max}\left(\frac{E\oplus L}{H\oplus 0}\right) \leq \mu_i(E\oplus L).$$

So $\lambda_i^*(E)\, e^{\mu_i(E)} \leq c_i^*(n+1,K)$ and Lemma 1 follows. □

Rmk: With the convexity inequality for heights, we can prove $\lambda_i^*(E\oplus L) = \min(\lambda_i^*(E), \max(\lambda_{i-1}^*(E), \Lambda_1(L)$

Lemma 2: $n \longmapsto c_n^*(n,K)$ is nondecreasing.

Proof:

let L be such that $\Lambda_1(L) = e^{-\mu_{min}(E)}$ and $\dim L = 1$.

Then $\lambda_{n+1}^*(E\oplus L) \geq \lambda_n^*(E)$ and $\mu_{min}(E\oplus L) \underset{\substack{\uparrow\\ \text{see after}}}{=} \min(\mu_{min}(E), \mu_{min}(L)) = \mu_{min}(E)$

$\mu_{min}(L) = \deg L = \mu_{min}(E)$

Thus $\lambda_n^*(E)\, e^{\mu_{min}(E)} \leq \lambda_{n+1}^*(E\oplus L)\, e^{\mu_{min}(E\oplus L)} \leq c_{n+1}^*(n+1,K)$. □

Proof of the proposition

We proceed by induction on n assuming $c_i^*(j,K) \leq c_{i+1}^*(j,K)$ for all $j<n$ and $i+1\leq j$.

let E be a rigid adelic space of dimension n.

If $\mu_i(E) = \mu_{i+1}(E)$ then $\lambda_i^*(E)\, e^{\mu_i(E)} \leq \lambda_{i+1}^*(E)\, e^{\mu_{i+1}(E)} \leq c_{i+1}^*(n,K)$.

Otherwise the HN-filtration $0 \subsetneq E_1 \subsetneq \dots \subsetneq E_N = E$ of E is not trivial $(N\geq 2)$

- If $i+1 \leq \dim E_{N-1}$ then $\mu_i(E) = \mu_i(E_{N-1})$ $\quad (P_E|[0,\dim E_{N-1}] = P_{E_{N-1}})$

and
$$\begin{aligned}\lambda_i^*(E)\, e^{\mu_i(E)} &\leq \lambda_i^*(E_{N-1})\, e^{\mu_i(E_{N-1})} \leq c_i^*(\dim E_{N-1}, K)\\ &\leq c_{i+1}^*(\dim E_{N-1}, K) \quad \text{(induction hypothesis)}\\ &\leq c_{i+1}^*(n,K) \quad \text{(Lemma 1)}\end{aligned}$$

• If $i+1 > \dim E_{N-1}$ then $\dim E_{N-1} = i$ $\big($since $\mu_j(E) = \mu_{min}(E)$ for $j \geq \dim E_{N-1}+1$ and $\mu_i(E) \neq \mu_{i+1}(E)\big)$

We have $\mu_{i+1}(E) = \mu_{min}(E)$ and $\mu_i(E_{N-1}) = \mu_{min}(E_{N-1}) = \mu_i(E)$

We deduce
$$\lambda_i^*\, e^{\mu_i(E)} \leq \lambda^*_{\dim E_{N-1}}(E_{N-1})\, e^{\mu_{min}(E_{N-1})}$$
$$\leq c^*_{\dim E_{N-1}}(\dim E_{N-1}, k) = c_i^*(i,k)$$
$$\leq c^*_{i+1}(i+1,k) \quad (\text{Lemma } 2)$$
$$\leq c^*_{i+1}(n,k) \quad (\text{Lemma } 1).$$

In all cases, $\lambda_i^*(E)\, e^{\mu_i(E)} \leq c^*_{i+1}(n,k)$, which implies the proposition. $\square$

Actually, for $* = BC$, we have $c_1^{BC}(n,k) = c_2^{BC}(n,k) = \dots = c_n^{BC}(n,k)$.

Indeed, for all $F \subsetneq E$, we have $\Lambda_1(E/F)\, e^{\mu_{max}(E/F)} \leq c_1(n,k)$ $\big(= c_1^{BC}(n,k)\big)$

so $\Lambda_1(E/F)\, e^{\mu_{min}(E)} \leq c_1(n,k)$ and taking the supremum over F gives

$\Lambda^{(n)}(E)\, e^{\mu_{min}(E)} \leq c_1(n,k)$, then $c_n^{BC}(n,k) = c_1(n,k)$.

In summary
$$c_I(n,k) = c_{II}^{BC}(n,k) = c_{II}^{\wedge}(n,k)$$
$$\leq c_1^{BC}(n,k) = c_2^{BC}(n,k) = \dots = c_n^{BC}(n,k)$$
$$\|$$
$$c_1^{\wedge}(n,k) \leq c_2^{\wedge}(n,k) \leq \dots \leq c_n^{\wedge}(n,k)$$
$$\|$$
$$c_1^{Z}(n,k) \leq c_2^{Z}(n,k) \leq \dots \leq c_n^{Z}(n,k)$$
$$\|$$
$$\max\big(c_I(i,k)\,;\ 1 \leq i \leq n\big)$$

Other relations between these constants

Proposition: let E be a rigid adelic space of dimension n over K.

Then, for all $m \in \{1, \dots, n\}$ and $* \in \{BC, \wedge, Z\}$, we have

1) $\lambda_1^*(E) \dots \lambda_m^*(E)\, e^{P_E(m)} \leq c_{II}^*(n,K)^n$

2) $\lambda_1^*(E) \dots \lambda_m^*(E) \leq c_{II}^*(n,K)^m\, H(E)^{\frac{m}{n}}$

Question: Is $\lambda_1^*(E) \dots \lambda_m^*(E)\, e^{P_E(m)} \leq c_{II}^*(n,K)^m$ true ?

(Since $P_E(m) \geq m\mu(E)$ it would improve 1) and 2) of the proposition)

Question: $c_i^*(n,K) \leq c_{II}^*(n,K)^i$ for $1 \leq i \leq n$?

(Following [Bo 2005] it should be true if $n \mapsto c_{II}^*(n,K)$ is non decreasing)

Proof of the proposition: For 1) we use the definition of $c_{II}^*(n,K)$ and $\lambda_i^*(E)\, e^{\mu_i(E)} \geq 1$ for all $i \in \{m+1, \dots, n\}$. We get the result with $\deg E - \sum_{i=m+1}^{n} \mu_i(E) = P_E(m)$.
For 2) we still use the definition of $c_{II}^*(n,K)$ but, for $i \in \{m+1, \dots, n\}$, we bound from below $\lambda_i^*(E)$ by $\left(\lambda_1^*(E) \dots \lambda_m^*(E)\right)^{\frac{1}{m}}$. □

Concerning point 2) of the proposition, one can prove that if, for $i \in \{1, \dots, n\}$

$$a_i^*(n,K) = \sup_{\dim E = n} \frac{\left(\lambda_1^*(E) \dots \lambda_i^*(E)\right)^{1/i}}{H(E)^{1/n}}$$

then $c_I^*(n,K) = a_1^*(n,K) \leq a_2^*(n,K) \leq \dots$
$\dots \leq a_n^*(n,K) = c_{II}^*(n,K)$. In particular, when $c_I^*(n,K) = c_{II}^*(n,K)$ all these constants are equal and $c_{II}^*(n,K)$ is the best constant in 2).

There exist other constants in the litterature such as the <u>Rankin constant</u>

$$R(m,n) = \sup\left(\frac{\sigma_m(E)}{H(E)^{\frac{m}{n}}}\ ;\ \dim E = n\right) \quad \text{for } 1 \leq m \leq n \text{ integers.}$$

Exercice: • $R(m,n) = R(n-m,n)$, • $R(1,n) = c_I^*(n,K)$ • $R(m,n) \leq c_I(n,K)^m$

• $1 \leq i \leq m \leq n \Rightarrow R(i,n) \leq R(i,m)\, R(m,n)^{\frac{i}{m}}$ (Generalization of Mordell inequality)

• $a_i^*(n,K) \leq a_i^*(m,K)\, R(m,n)^{1/m}$ • $R(i,n) \leq a_i^{BC}(n,K)^i$.

Transfert theorems

Ref : [Ba 1993], [Pe 2002]

Let E be a rigid adelic space of dim n over k and let $i \in \{1, \dots, n\}$.

A transfer theorem consists of bounding from above $\lambda_i^*(E)\, \lambda_{n-i+1}^*(E^\vee)$

for $* \in \{BC, \wedge, Z\}$ $\quad \left(\lambda_i^*(E) = \Lambda^{(i)}(E),\ \Lambda_i(E) \text{ or } Z_i(E) \right)$.

<u>Definition</u> : $t_i^*(n, k) = \sup\limits_{\dim E = n} \lambda_i^*(E)\, \lambda_{n-i+1}^*(E^\vee) \qquad \left(= t_{n-i+1}^*(n,k) \right)$

Of course $t_i^{BC}(n,k) \leq t_i^{\wedge}(n,k) \leq t_i^{Z}(n,k)$

Note that $\lambda_i^*(E)\, \lambda_{n-i+1}^*(E^\vee) = \lambda_i^*(E)\, e^{\mu_i(E)} \times \lambda_{n-i+1}^*(E^\vee)\, e^{\mu_{n-i+1}(E^\vee)}$ is greater

than 1 and lower than $c_i^*(n,k)\, c_{n-i+1}^*(n,k)$.

Moreover $\quad \lambda_i^*(E)\, e^{\mu_i(E)} \leq \lambda_i^*(E)\, \lambda_{n-i+1}^*(E) \quad$ and $\quad c_i^*(n,k) \leq t_i^*(n,k)$.

That proves $t_i^*(n,k)$ is a real number (finite) as soon k is a Siegel field (for $* = BC$ or $\wedge$)

or a Siegel field of infinite degree (for all $*$).

<u>Question</u> : $t_i^*(n,k) \leq c_{\mathrm{II}}^*(n,k)^2$?

The square might be justified by several observations. For instance, for $k = \bar{\mathbb{Q}}$, we have

$c_{\mathrm{II}}^*(n, \bar{\mathbb{Q}}) = \exp \dfrac{H_n - 1}{2} \leq \sqrt{n}$ whereas $t_i^Z(n, \bar{\mathbb{Q}}) \geq \Lambda_i(k^n)\, \Lambda_{n-i+1}(k^n)^\vee \geq \sqrt{i}\, \sqrt{n-i+1}$

Moreover we have the following theorem by Banaszczyk (1993)

<u>Theorem</u> : $t_i^{\wedge}(n,k) \leq n\, |\Delta_{k|\mathbb{Q}}|$ if k is a number field of absolute discriminant $|\Delta_{k|\mathbb{Q}}|$.

Actually this theorem is proved for $k = \mathbb{Q}$ in the paper by Banaszczyk.

At last ~~we know~~ the answer to the question is true if $* = BC$.

Indeed $t_i^{BC}(n,k) \leq c_i^{BC}(n,k)\, c_{n-i+1}^{BC}(n,k) = c_1^*(n,k)^2$

(but we do not know if $c_1(n,k) = c_{\mathrm{I}}^{\wedge}(n,k)$?)

Outline of the proof

~~Hints for the case of a number field~~

- Scalar restriction. Let E be a rigid adelic space over K

Define $\mathrm{Res}_{K|\mathbb{Q}}(E) = E$ viewed as a $\mathbb{Q}$-vector space (with dimension $[K:\mathbb{Q}]\dim E$)

$$+ \quad \|x\|_\infty = \Big(\sum_{\sigma : K \hookrightarrow \mathbb{C}} \|x\|_{E,\sigma}^2\Big)^{1/2}$$

$$+ \quad \|x\|_p = \max_{v \mid p} \|x\|_{E,v}$$

Then $\mathrm{Res}_{K|\mathbb{Q}}(E)$ is a rigid adelic space over $\mathbb{Q}$ with height

$$H(\mathrm{Res}_{K|\mathbb{Q}} E) = H(E)^{[K:\mathbb{Q}]}\, |\Delta_{K|\mathbb{Q}}|^{\frac{n}{2}} \qquad (n = \dim E)$$

Lemma 4.29 of Gaelle [GR 2017]

- Define a ~~Hasse~~ rigid adelic space ω_K over K:

$\omega_K = \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{Q})$ viewed as a K-vector space $\lambda.\varphi : \underset{K}{x} \longmapsto \underset{\mathbb{Q}}{\varphi(\lambda x)}$.

$\mathrm{Tr}_{K|\mathbb{Q}}$ is a K-basis of ω_K ($\dim_K \omega_K = 1$) and we define rigid adelic metrics on ω_K

with $\|\mathrm{Tr}_{K|\mathbb{Q}}\|_\sigma = 1 \quad \forall \sigma : K \hookrightarrow \mathbb{C}$, $\quad \|\mathrm{Tr}_{K|\mathbb{Q}}\|_v = \inf\{ |\lambda|_v \,;\, \lambda^{-1}\mathrm{Tr}_{K|\mathbb{Q}} \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_v, \mathbb{Z}_p),\ \lambda \in K_v \setminus \{0\}\}$

Then $H(\omega_K) = |\Delta_{K|\mathbb{Q}}|^{-1}$

- $\mathrm{Res}_{K|\mathbb{Q}}(E^\vee \otimes \omega_K) \simeq (\mathrm{Res}_{K|\mathbb{Q}}(E))^\vee$ (as rigid adelic spaces over $\mathbb{Q}$)

- $\Lambda_i(E) \leq [K:\mathbb{Q}]^{-1/2}\, \Lambda_{(i-1)[K:\mathbb{Q}]+1}(\mathrm{Res}_{K|\mathbb{Q}} E)$ (Corollary 4.28 of Gaelle [GR 2017])

$\forall i \in \{1, \dots, n\}$

Conclusion: $\Lambda_i(E)\,\Lambda_{n-i+1}(E^\vee) = \Lambda_i(E)\,\dfrac{\Lambda_{n-i+1}(E^\vee \otimes \omega_K)}{H(\omega_K)}$

$$\leq \frac{1}{[K:\mathbb{Q}]} \underbrace{\Lambda_{i[K:\mathbb{Q}]}(\mathrm{Res}_{K|\mathbb{Q}} E)\ \Lambda_{(n-i)[K:\mathbb{Q}]}(\mathrm{Res}_{K|\mathbb{Q}} E)^\vee}_{\leq [K:\mathbb{Q}]\, n \text{ (Banaszczyk)}} |\Delta_{K|\mathbb{Q}}|$$

□

05/06/17

The following results are some generalizations of Pekker's theorem concerning the case $K = \overline{\mathbb{Q}}$.
(JNT 2008)

It applies only to $* = \wedge$.

<u>Theorem</u> : let $1 \leq i \leq n$ be integers

1) $t_1^\wedge(2,K) = t_2^\wedge(2,K) = c_I^\wedge(2,K)^2$

2) $t_i^\wedge(n,K) \leq t_1^\wedge(i,K)\; t_1^\wedge(n-i+1,K)$

3) $t_1^\wedge(n,K) \leq t_1^\wedge(n-1,K)\; t_1^\wedge(2,K)$

4) $t_i^\wedge(n,K) \leq t_1^\wedge(2,K)^{n-1}$

<u>Proof</u>

1) In dimension 2, $\Lambda_1(E) = \sigma_{2-1}(E) = \Lambda_1(E^\vee)\, H(E)$ so

$$\Lambda_1(E)\,\Lambda_2(E^\vee) = \frac{\Lambda_1(E^\vee)\,\Lambda_2(E^\vee)}{H(E^\vee)} \quad \text{and} \quad t_1^\wedge(2,K) = \sup_{\dim E = 2} \frac{\Lambda_1(E)\Lambda_2(E)}{H(E)} = c_{II}^\wedge(2,K)^2$$
$$= c_I^\wedge(2,K)^2 .$$

2) We shall use an ~~two~~ ~~lemmas~~ auxiliary result.

<u>Lemma</u> : let E be a rigid adelic space of dimension n over K. let $\varepsilon > 0$ and
let $x \in E$ such that $H_E(x) \leq (1+\varepsilon)\,\Lambda_n(E)$
There exists a hyperplane $F \subset E$ such that $H(F) \leq (1+\varepsilon)\,\sigma_{n-1}(E)$ and

$$H_E(x)\; H(F) \leq (1+\varepsilon)^2\, H(E)\; t_1^\wedge(n,K).$$

<u>Proof of Lemma 1</u> : let $\varphi \in E^\vee \setminus \{0\}$; $H_{E^\vee}(\varphi) \leq (1+\varepsilon)\,\Lambda_1(E^\vee)$ and consider $F = \operatorname{Ker}\varphi$.

Since $\left(E/\operatorname{Ker}\varphi\right)^\vee \simeq (\operatorname{Ker}\varphi)^\perp = K\cdot\varphi$, we have $H_{E^\vee}(\varphi) = H(F)/H(E)$. □

let us now deduce point 2) of the theorem.

let $G \subset E^\vee$ be a K-vector space of dimension $i-1$.

We apply Lemma 1 to $G^\perp$ (viewed as a subspace of E) and x a vector of $G^\perp$ such that
$\Lambda_{n-i+1}(G^\perp) \leq H_E(x) \leq (1+\varepsilon)\,\Lambda_{n-i+1}(G^\perp)$; there exists a hyperplane $A \subset G^\perp$ such that

$$\Lambda_{n-i+1}(G^\perp)\ H(A^\perp) \leq (1+\varepsilon)\ t_1^\wedge(n-i+1,k)\ H(G^\perp)$$

Apply again lemma 1 (in the same way) to $A^\perp \subset E^\vee$ ($\hookrightarrow \dim = i$) : there exists a hyperplane $B \subset A^\perp$ such that $\Lambda_i(A^\perp)\ H(B) \leq (1+\varepsilon)\ t_1^\wedge(i,k)\ H(A^\perp)$

We have $\dim B = i-1$ and $H(B) \geq \sigma_{i-1}(E^\vee)$. Moreover $\Lambda_{n-i+1}(G^\perp) \geq \Lambda_{n-i+1}(E)$ and $\Lambda_i(A^\perp) \geq \Lambda_i(E^\vee)$. ~~We get~~ By product, we get

$$\Lambda_{n-i+1}(E)\ \Lambda_i(E^\vee) \leq (1+\varepsilon)^2\ t_1^\wedge(n-i+1,k)\ t_1^\wedge(i,k)\ \frac{H(A^\perp)}{H(B)}\ \frac{H(G^\perp)}{H(A)}.$$

Since $H(G^\perp) = \dfrac{H(G)}{H(E)}$ and $H(A^\perp) = \dfrac{H(A)}{H(E)}$ we get

$$\frac{H(A^\perp)}{H(B)}\ \frac{H(G^\perp)}{H(A)} = \frac{H(G)}{H(B)} \leq \frac{H(G)}{\sigma_{i-1}(E^\vee)}.$$

We conclude with $H(G) \to \sigma_{i-1}(E^\vee)$ and $\varepsilon \to 0$.

3) Let E be a rigid adelic space of dimension n over k.

Let $\varphi, \psi \in E^\vee$ linearly independent vectors. Define $V = \operatorname{Ker}\varphi$ and $W = \operatorname{Ker}\psi$.

By lemma 1, there exist a hyperplane $V' \subset V$ and a hyperplane $W' \subset W$ such that

$$\Lambda_{n-1}(V)\ H(V') \leq (1+\varepsilon)\ t_1^\wedge(n-1,k)\ H(V) \qquad H(V) = H_{E^\vee}(\varphi)\ H(E)$$

$$\Lambda_{n-1}(W)\ H(W') \leq (1+\varepsilon)\ t_1^\wedge(n-1,k)\ H(W) \qquad H(W) = H_{E^\vee}(\psi)\ H(E)$$

By hypothesis $V \neq W$, so $V+W = E$ and $\Lambda_n(E) \leq \max(\Lambda_{n-1}(V), \Lambda_{n-1}(W))$

If G is one of V or W, we get

$$\Lambda_n(E)\ H(G) \leq (1+\varepsilon)\ t_1^\wedge(n-1,k)\ H(E)\ \max\left(H_{E^\vee}(\varphi), H_{E^\vee}(\psi)\right)$$

Choosing φ, ψ such that $\max(H_{E^\vee}(\varphi), H_{E^\vee}(\psi)) \leq (1+\varepsilon)\ \Lambda_2(E^\vee)$ and using $H(G^\perp) = \dfrac{H(G)}{H(E)}$,

we get $\Lambda_n(E)\ H(G^\perp) \leq (1+\varepsilon)^2\ t_1^\wedge(n-1,k)\ \Lambda_2(E^\vee)$.

So
$$\Lambda_n(E)\ \Lambda_1(E^\vee) \leq (1+\varepsilon)^2\ t_1^\wedge(n-1,k)\ \frac{\Lambda_1(E^\vee)\Lambda_2(E^\vee)}{H(G^\perp)} \leq (1+\varepsilon)^2\ t_1^\wedge(n-1,k)\ \frac{\Lambda_1(G^\perp)\Lambda_2(G^\perp)}{H(G^\perp)}$$
$$\leq (1+\varepsilon)^2\ t_1^\wedge(n-1,k)\ c_{II}^\wedge(2,k)^2. \quad \square$$

4) $t_1^\wedge(n,K) \le t_1^\wedge(2,K)^{n-1}$ is a direct consequence of 3).

The generalization to $t_i^\wedge(n,K)$ uses this inequality and assertion 2). □

Questions: i) Do we have similar results for $t_i^2(n,K)$?

ii) Is it reasonable to expect a bound for $t_i^\wedge(n,K)$ or $t_i^2(n,K)$ which is polynomial in n when K is a Siegel field (of infinite degree), as $\overline{\mathbb{Q}}$ for instance ?

5) Heights for morphisms and slopes / minima inequalities Ref [Bo 1995]

Until now, we have considered only rigid adelic spaces. Nevertheless it is useful to work with $\mathrm{Hom}(E,F)$ endowed with the operator norms (that is, with $E^\vee \otimes_\varepsilon F$) which is not rigid (Hermitian) in general.

Definition: let E and F be adelic spaces over K such that $E^\vee \otimes_\varepsilon F$ is integrable.

The height of $\varphi \in \mathrm{Hom}(E,F) \setminus \{0\}$ is

$$h(\varphi) = h(E,F;\varphi) = \int_{V(K)} \operatorname{Log} \|\varphi\|_{E^\vee\otimes_\varepsilon F,\, v} \, d\lambda(v)$$

We may also use $H(\varphi) = \exp h(\varphi)$.

Here $\|\varphi\|_{E^\vee\otimes_\varepsilon F, v} = \sup \left(\dfrac{\|\varphi(x)\|_{F,v}}{\|x\|_{E,v}} \; ; \; x \in E \otimes_K K_v \setminus \{0\} \right)$ is the operator norm (see p. 6)

Note that if $E' \subset E$ then $h(E',F;\varphi_{|E'}) \le h(E,F;\varphi)$.

There is also the Hilbert-Schmidt height for φ built with $\|\varphi\|_{E^\vee\otimes F, v}$, which is greater than $h(\varphi)$.

In this paragraph our aim is to compare minima and slopes of two (rigid) adelic spaces connected by a linear map.

In the following results, E and F are rigid adelic spaces over K and $\varphi : E \to F$ is a linear map

<u>Proposition</u>: If φ is an isomorphism then

1) $\deg E = \deg F + h(\det E, \det F ; \det \varphi)$

2) $\mu(E) \leq \mu(F) + h(\varphi)$

<u>Proof</u>: 1) $\det \varphi : \det E \to \det F$ is an isomorphism between rigid adelic lines and

$$\|\det \varphi\|_v = \frac{\|\det\varphi(x)\|_{\det F, v}}{\|x\|_{\det E, v}} \quad \text{for all } x \in (\det E)\otimes_K K_v \setminus \{0\} \text{ and } v \in V(K).$$

We take logarithms and we integrate over v to conclude.

2) Simple consequence of 1) and Hadamard's inequality $\|\det\varphi\|_v \leq \|\varphi\|^{\dim E}_{E^\vee \otimes_K F, v}$.

<u>Theorem</u>:

1) If $\varphi : E \to F$ is injective then

$$\mu_{max}(E) \leq \mu_{max}(F) + h(\varphi) \quad \text{and} \quad \lambda_1(F) \leq \lambda_1(E)\, H(\varphi).$$

2) More generally, if $\varphi \neq 0$ then $\forall i \in \{1, 2, \dots, rk\varphi\}$, $\forall * \in \{BC, \wedge, Z\}$

$$\mu_{i+\dim \ker\varphi}(E) \leq \mu_i(F) + h(\varphi) \quad \text{and} \quad \lambda_i^*(F) \leq \lambda^*_{i+\dim\ker\varphi}(E)\; H(\varphi)$$

<u>Proof</u>

1) Let $E_0 \subset E$ be a non zero linear subspace and $F_0 = \varphi(E_0)$.
Since φ is injective the induced map $\tilde{\varphi} : E_0 \to F_0$ is an isomorphism and $\mu(E_0) \leq \mu(F_0) + h(E_0, F_0 ; \tilde{\varphi}) \leq \mu_{max}(F) + h(\varphi)$. Taking the supremum of the left hand side over E_0 leads to the maximal slope inequality.
Now if $x \in E \setminus \{0\}$ then $\varphi(x) \in F \setminus \{0\}$ so $\lambda_1(F) \leq H_E(\varphi(x)) \leq H_E(x)\, H(\varphi)$ and we take the infimum over x to replace $H_E(x)$ by $\lambda_1(E)$.

2) Let $F_0 \subset F$ be a linear subspace with dimension $\leq i-1$ and $E_0 = \bar{\varphi}^{-1}(F_0)$.

We have $\dim E_0 \leq \dim \operatorname{Ker}\varphi + i - 1$ and the induced map

$\bar{\varphi} : E/E_0 \longrightarrow F/F_0$ is injective. So, via the minimax formula for slopes,

$$\mu_{i+\dim \operatorname{Ker}\varphi}(E) \leq \mu_{max}\left(E/E_0\right) \leq \mu_{max}\left(F/F_0\right) + h\left(E/E_0, F/F_0 ; \bar{\varphi}\right)$$

$$\leq \mu_{max}\left(F/F_0\right) + h(E, F ; \varphi).$$

We conclude taking the infimum over F_0 with dimension $\leq i-1$, which replaces $\mu_{max}(F/F_0)$ by $\mu_i(F)$. As for the analogous inequality for λ_i^*, we distinguish the three cases $* = BC, \Lambda, Z$.

For $* = BC$, we proceed as above: $\Lambda_1\left(F/F_0\right) \leq \Lambda_1\left(E/E_0\right) H(\bar{\varphi})$

Since $\dim E_0 \leq i + \dim \operatorname{Ker}\varphi - 1$ we have $\Lambda_1\left(E/E_0\right) \leq \Lambda^{(i+\dim \operatorname{Ker}\varphi)}(E)$.

So $$\Lambda^{(i)}(F) = \sup_{\dim F_0 \leq i-1} \Lambda_1\left(F/F_0\right) \leq \Lambda^{(i+\dim \operatorname{Ker}\varphi)}(E) \, H(\varphi).$$

For $* = \Lambda$ or Z we get an injective map from φ doing the quotient by $\operatorname{Ker}\varphi$, which yields $\lambda_i^*(F) \leq \lambda_i^*(E/\operatorname{Ker}\varphi) H(\varphi)$ and we use $\lambda_i^*(E/\operatorname{Ker}\varphi) \leq \lambda^*_{i+\dim \operatorname{Ker}\varphi}(E)$

(for $* = \Lambda$ for instance, it means that if $\{e_1, e_2, \ldots, e_{i+\dim \operatorname{Ker}\varphi}\} \subset E$ are a free family then at least i of the images of the vectors e_j in $E/\operatorname{Ker}\varphi$ are also linearly independent).

$\square$

<u>Corollary:</u>

1) If $\varphi \neq 0$ then $\mu_{min}(E) \leq \mu_{max}(F) + h(\varphi)$

and $\lambda_1^*(F) \leq \lambda^*_{\dim E}(E) \, H(\varphi)$.

2) If φ is surjective then

i) $\mu_{\min}(E) \leq \mu_{\min}(F) + h(\varphi)$

ii) $\lambda^*_{\dim F}(F) \leq \lambda^*_{\dim E}(E)\, H(\varphi)$

3) If φ is surjective then $\mu_{\max}(F) \leq \deg F - (\dim F - 1)\,\mu_{\min}(E) + (\dim F - 1)\, h(\varphi)$

Proof : 1) Take $i = \mathrm{rk}\,\varphi$ in Theorem 2) and bound from below i by 1 (with $\dim E = \mathrm{rk}(\varphi) + \dim \ker \varphi$)

2) Idem but keep $i = \mathrm{rk}\,\varphi = \dim F$ since φ is surjective.

3) Observe $\deg F = \mu_{\max}(F) + \mu_2(F) + \dots + \mu_{\dim F}(F) \geq \mu_{\max}(F) + (\dim F - 1)\,\mu_{\min}(F)$

and use 2-i).

□

Remark : One can prove that if φ is injective then, for all $i \in \{1, \dots, \overset{\dim E}{\overset{\shortparallel}{n}}\}$,

$(*)$ $\quad P_E(i) \leq P_F(i) + h(\Lambda^i E, \Lambda^i F; \Lambda^i \varphi)$.

For this, observe that, for all $v \in V(K)$, $i \mapsto \dfrac{\|\Lambda^i \varphi\|_v}{\|\Lambda^{i-1}\varphi\|_v}$ is a nondecreasing function, so $\left(h(\Lambda^i \varphi) - h(\Lambda^{i-1}\varphi)\right)_{i=1,\dots,\mathrm{rk}\,\varphi}$ is a decreasing sequence and

$i \mapsto P_F(i) + h(\Lambda^i E, \Lambda^i F; \Lambda^i \varphi)$ is a concave function.

The case $i=1$ in $(*)$ corresponds to the first statement of the theorem.

Tensor product Ref : [BC 2013], [GR 2013]

We would like to conclude this course by raising the problem of the behaviour of minima and slopes (the first ones) with respect to tensor product.

We ~~proved~~ saw that $\mu(E \otimes F) = \mu(E) + \mu(F)$ for rigid adelic spaces E and F over K.

What happens for $\Lambda_1(E \otimes F)$ and $\mu_1(E \otimes F)$?

(Here $E \otimes F$ is the tensor product of rigid adelic spaces, not the injective tensor product $E \otimes_\varepsilon F$).

First observations:

- $\Lambda_1(E \otimes F) \leq \Lambda_1(E)\,\Lambda_1(F)$
- $\mu_{max}(E) + \mu_{max}(F) \leq \mu_{max}(E \otimes F)$

Proof:

- $\| x \otimes y \|_{E \otimes F, v} \leq \| x \|_{E, v} \, \| y \|_{F, v} \qquad v \in V(K) \quad x \in E \underset{K}{\otimes} K_v \quad y \in F \otimes K_v$

  Then $H_{E \otimes F}(x \otimes y) \leq H_E(x)\, H_F(y) \qquad x \in E \;,\; y \in F$

  Since $\begin{matrix} x \neq 0 \\ y \neq 0 \end{matrix} \Rightarrow x \otimes y \neq 0$, we get $H_{E \otimes F}(x \otimes y) \geq \Lambda_1(E \otimes F)$.

- $\mu_{max}(E) = \mu(E_{des})$ and $\mu(E_{des}) + \mu(F_{des}) = \mu(E_{des} \otimes F_{des})$
  $\mu_{max}(F) = \mu(F_{des})$ $\qquad \leq \mu_{max}(E \otimes F)$

$\square$

The problem is wether these inequalities are equalities.

For Λ_1 the answer is clear: not, in general.

Actually it has been proved by Steinberg that, for any $n \geq 292$, there exists a rigid adelic space E over $\mathbb{Q}$ with dimension n such that $\Lambda_1(E \otimes E) \neq \Lambda_1(E)^2$. Coulangeon [Co 2000] obtained similar results for some imaginary quadratic fields k.

Gaudron and Rémond proved that for every integers $n, m \geq 2$, there exist rigid adelic spaces E and F over $\overline{\mathbb{Q}}$ with $\dim E = n$ and $\dim F = m$ such that $\Lambda_1(E \otimes F) \neq \Lambda_1(E)\,\Lambda_1(F)$ ([GR 2013, Theorem 1.5]).

Here we shall give a proof (due to G. Rémond) of the following statement:

Proposition: There exists a rigid adelic plane E over $\overline{\mathbb{Q}}$ $(\dim E = 2)$ such that $\Lambda_1(E \otimes E^\vee) \neq \Lambda_1(E)\,\Lambda_1(E^\vee)$.

From this proposition it is quite easy to obtain the general case $\dim E = n$, $\dim F = m$.

Proof (Gaël Rémond)

We saw that $c_I(2,\overline{\mathbb{Q}}) = \exp \frac{H_2 - 1}{2} = \exp \frac{1}{4}$. Choose $\varepsilon > 0$ such that

$e > \frac{2}{(1-\varepsilon)^4}$. Let E be a rigid adelic plane over $\overline{\mathbb{Q}}$ such that

$$\frac{\Lambda_1(E)}{H(E)^{1/2}} \geq c_I(2,\overline{\mathbb{Q}})(1-\varepsilon) \quad (\text{definition of } c_I(2,\overline{\mathbb{Q}})).$$

Since $\dim E = 2$, we have $\Lambda_1(E^\vee) = \frac{\sigma_{2-1}(E)}{H(E)} = \frac{\Lambda_1(E)}{H(E)}$ so $\frac{\Lambda_1(E^\vee)}{H(E^\vee)^{1/2}} = \frac{\Lambda_1(E)}{H(E)^{1/2}}$

From this equality, we deduce $\Lambda_1(E)\,\Lambda_1(E^\vee) = \left(\frac{\Lambda_1(E)}{H(E)^{1/2}}\right)^2 \geq \left(\exp\frac{1}{2}\right)(1-\varepsilon)^2$

Besides considering a basis $\{e_1, e_2\}$ of E and the identity map $x = e_1 \otimes e_1^\vee + e_2 \otimes e_2^\vee$

we have $\Lambda_1(E \otimes E^\vee) \leq H_{E\otimes E^\vee}(x) = \sqrt{2}$. The choice of ε implies

$$\Lambda_1(E \otimes E^\vee) \leq \sqrt{2} < \left(\exp\frac{1}{2}\right)(1-\varepsilon)^2 \leq \Lambda_1(E)\,\Lambda_1(E^\vee).$$

$\square$

It may be that this phenomenon does not occur for the maximal slope

Bost's conjecture: For all rigid adelic spaces E and F over K,

we have $\mu_{\max}(E \otimes F) = \mu_{\max}(E) + \mu_{\max}(F)$.

Since the maximal slope is invariant by scalar extension, we can always assume $K = \overline{K}$.
This conjecture is known to be true when a group G acts isometrically on E in a geometrically irreducible way (the only G-stable subspaces of $E \otimes_K \overline{K}$ are $\{0\}$ and $E \otimes_K \overline{K}$) (see for instance [BC 2013, Proposition 1.14] or [GR 2013, §5.1]). Moreover Bost and Chen proved this conjecture when $(\dim E)(\dim F) \leq 9$. ([BC 2013])

Here we shall prove a weaker result, also due to Bost and Chen (2013)

<u>Theorem</u> : Let E and F be rigid adelic spaces over K

Let $n = \dim E$ and $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$

Then $\mu_{max}(E \otimes F) \leq \mu_{max}(E) + \mu_{max}(F) + \frac{1}{2}(H_n - 1)$

<u>Lemma 1</u> : For every rigid adelic space E and integrable adelic space F over K,

we have $\Lambda_1(F) \leq \Lambda^{(\dim E)}(E^*) \; \Lambda_1(E^\vee \otimes_\mathcal{E} F)$.

<u>Proof</u> : It is the corollary 1) on page 40 with $* = BC$, extended to an integrable (not only rigid) space F (same proof).

We take $\varphi \in E^\vee \otimes_\mathcal{E} F$ such that $H(\varphi) \to \Lambda_1(E^\vee \otimes_\mathcal{E} F)$ and we use

$\Lambda_1(F) \leq \Lambda^{(\dim E)}(E) \; H(\varphi)$. □

<u>Lemma 2</u> : For all rigid adelic spaces A and B, we have

$$\exp\left(-\mu_{max}(A) - \mu_{max}(B)\right) \leq \Lambda_1(A \otimes_\mathcal{E} B)$$

<u>Proof</u>

If $\varphi \in A \otimes_\mathcal{E} B = \mathrm{Hom}(A^\vee, B)$, $\varphi \neq 0$, we saw $\mu_{min}(A^\vee) \leq \mu_{max}(B) + h(\varphi)$ (Corollary p. 40). We conclude with $-\mu_{max}(A) = \mu_{min}(A^\vee)$ and $h(\varphi) \to \log \Lambda_1(A \otimes_\mathcal{E} B)$. □

<u>Lemma 3</u> : For all rigid adelic spaces A, B, E, we have

$$1 \leq \Lambda_1(E^\vee \otimes_\mathcal{E} A \otimes_\mathcal{E} B) \; \Lambda^{(\dim E)}(E) \; \exp\left(\mu_{max}(A) + \mu_{max}(B)\right)$$

<u>Proof</u> : Replace F by $A \otimes_\mathcal{E} B$ in Lemma 1 and use Lemma 2.

<u>Lemma 4</u> : For all rigid adelic spaces E and F over K, we have

$$\mu_{max}(E \otimes F) \leq \mu_{max}(F) + \log \Lambda^{(\dim E)}(E^\vee).$$

Proof: Replace E by $E^\vee$ and take $A = F$ and $G = (E \otimes F)^\vee_{des}$ in Lemma 3. We get

$$1 \leq \Lambda_1 \left(E \otimes_\mathcal{E} F \otimes_\mathcal{E} (E\otimes F)^\vee_{des}\right) \; \Lambda^{(\dim E)}(E^\vee) \; \exp\left(\mu_{max}(F) - \mu_{max}(E\otimes F)\right)$$

Then observe that
$$\Lambda_1 \left(E \otimes_\mathcal{E} F \otimes_\mathcal{E} (E\otimes F)^\vee_{des}\right) \leq \Lambda_1 \left(E\otimes F \otimes_\mathcal{E} (E\otimes F)^\vee_{des}\right) \leq 1$$

since the inclusion map $(E\otimes F)_{des} \hookrightarrow E\otimes F$ yields a vector with operator norms less than 1 (actually $\Lambda_1(A \otimes A^\vee_{des}) = 1$ $\forall A$ rigid adelic space).

$\square$

Proof of the theorem

Use lemma 4 and $\Lambda^{(\dim E)}(E^\vee)\, e^{\mu_{min}(E^\vee)} \leq c_n^{BC}(n, \bar{\mathbb{Q}})$.

We have $\mu_{min}(E^\vee) = -\mu_{max}(E)$ and we saw (p. 29 and p. 32) that $c_n^{BC}(n, \bar{\mathbb{Q}}) = \exp\left(\frac{1}{2}(H_n - 1)\right)$. $\square$

End.

References

[Ba 1993] W. Banaszczyk. New bounds in some transference theorems in the geometry of numbers. *Math. Ann.*, 296:625–635 (1993).

[Bo 2005] T. Borek. Successive minima and slopes of Hermitian vector bundles over number fields. *J. Number Theory*, 113:380–388 (2005).

[Bo 1995] J.-B. Bost. Périodes et isogénies des variétés abéliennes sur les corps de nombres (d'après D. Masser et G. Wüstholz). *Séminaire Bourbaki*. Volume 237 d'*Astérisque*, 115–161. Société Mathématique de France, 1996.

[BC 2013] J.-B. Bost and H. Chen. Concerning the semistability of tensor products in Arakelov geometry. *J. Math. Pures Appl. (9)*, 99:436–488 (2013).

[Ch 2010] H. Chen. Harder-Narasimhan categories. *J. Pure Appl. Algebra*, 214:187–200 (2010).

[Co 2000] R. Coulangeon. Tensor products of Hermitian lattices. *Acta Arith.*, 92:115–130 (2000).

[Ga 2008] É. Gaudron. Pentes des fibrés vectoriels adéliques sur un corps global. *Rend. Semin. Mat. Univ. Padova*, 119:21–95 (2008).

[GR 2013] É. Gaudron and G. Rémond. Minima, pentes et algèbre tensorielle. *Israel J. Math.*, 195:565–591 (2013).

[GR 2017] É. Gaudron and G. Rémond. Corps de Siegel. *J. reine angew. Math*, 726:187-247 (2017).

[Ma 2003] J. Martinet. *Perfect lattices in Euclidean spaces.* Grundlehren der Mathematischen Wissenschaften 327, Springer 2003.

[Mi 1910] H. Minkowski. *Geometrie der Zahlen.* Teubner 1910.
`http://gallica.bnf.fr/ark:/12148/bpt6k99643x`

[Pe 2008] A. Pekker. On successive minima and the absolute Siegel's lemma. *J. Number Theory*, 128:564–575 (2008).

[RT 1996] D. Roy and J. Thunder. An absolute Siegel's lemma. *J. reine angew. Math.*, 476:1-26 (1996). Addendum et erratum. *ibid.* 508:47-51 (1999).

[Zh 1995] S. Zhang. Positive line bundles on arithmetic varieties. *J. Amer. Math. Soc.*, 8:187–221 (1995).

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