THE ARITHMETIC RIEMANN–ROCH THEOREM AND THE JACQUET–LANGLANDS CORRESPONDENCE

GERARD FREIXAS I MONTPLET

Abstract. The arithmetic Riemann–Roch theorem refines both the algebraic geometric and differential geometric counterparts, and it is stated within the formalism of Arakelov geometry. For some simple Shimura varieties and automorphic vector bundles, the cohomological part of the formula can be understood via the theory of automorphic representations. Functoriality principles from this theory may then be applied to derive relations between arithmetic intersection numbers for different Shimura varieties. In this lectures we explain this philosophy in the case of modular curves and compact Shimura curves. This indicates that there is some relationship between the arithmetic Riemann–Roch theorem and trace type formulae, from which these functoriality principles are derived.

1. Introduction

The Riemann–Roch formula in Arakelov geometry is a local-to-global statement, that translates some arithmetic intersection numbers of hermitian vector bundles into cohomological invariants. In cases of arithmetic relevance, such as Shimura varieties, it is natural to apply the theorem to automorphic vector bundles. According to general conjectures (e.g. the Maillot-Rössler conjecture and the vast Kudla programme), one expects the arithmetic intersection numbers to be related with logarithmic derivatives of L-functions. One may hope that the arithmetic Riemann–Roch theorem provides a cohomological approach to settle cases of this principle. In this geometric setting, the cohomological side of the formula affords an automorphic translation, so that the theory of automorphic representations may be of some use. Unfortunately this approach has actually not been fruitful. In a parallel with the trace formula, arithmetic intersection numbers seem to be analogue to the geometric side, supposed to be easier to deal with than the spectral side, itself analogous to the cohomological part of Riemann–Roch. However, in the theory of automorphic forms, a fruitful idea has been to compare trace formulae, in order to relate automorphic representations for different groups. It is then tempting to combine these relations, when

2000 *Mathematics Subject Classification.* Primary: 14C40. Secondary: 11G18.

they exist, to relate as well arithmetic intersection numbers for different Shimura varieties. While this does not provide the evaluation of these numerical invariants, it indicates some structural phenomenon that goes beyond the conjectural predictions alluded to above, and that has not been much explored. In these notes, we exemplify this "philosophy" in the case of modular and Shimura curves. This is an excuse to review the arithmetic Riemann–Roch theorem of Gillet–Soulé, as well as a variant for modular curves (more generally, arithmetic surfaces with "cusps" and "elliptic fixed points"). Also, we explain in the classical language of modular forms the content of the Jacquet–Langlands correspondence. All these wonderfully combine to provide a relation between arithmetic self-intersection numbers of sheaves of modular forms on modular and Shimura curves.

2. Riemann–Roch theorem for arithmetic surfaces and hermitian line bundles

2.1. **Riemann–Roch formulae in low dimensions.** As a matter of motivation, we recall the statement of the Hirzebruch–Riemann–Roch theorem for compact complex manifolds of dimensions 1 and 2 and line bundles on them.

Let X be a compact Riemann surface and L a holomorphic line bundle on X. To the line bundle L we can associate two integer valued invariants. The first and easiest one is the degree $\deg L$. It is known that L affords non-trivial meromorphic sections, and for such a section s

$$\deg L = \deg(\operatorname{div} s) = \sum_{p \in X} \operatorname{ord}_p(s).$$

The notation $\operatorname{ord}_p(s)$ stands for the order of vanishing or pole of s at the point $p \in X$. This sum is of course finite, and does not depend on the particular choice of s by the residue theorem: the divisor of any meromorphic function has degree zero. The second topological invariant is the holomorphic Euler–Poincaré characteristic:

$$\chi(X, L) = \dim H^0(X, L) - \dim H^1(X, L).$$

The coherent cohomology groups $H^0(X, L)$ and $H^1(X, L)$ are actually finite dimensional $\mathbb{C}$-vector spaces. They can be defined as Čech cohomology. For later motivation, let us provide a geometric differential interpretation of these spaces. Recall the Dolbeault complex of the holomorphic line bundle L, given by the $\overline{\partial}$ operator defining the holomorphic structure of L:

$$A^{0,0}(X, L) \xrightarrow{\overline{\partial}} A^{0,1}(X, L).$$

If $s \in A^{0,0}(X, L)$ is a smooth section of L, and e is a holomorphic trivialization of L on some analytic open subset $U \subset X$, then $s = fe$

for some smooth function f on U and

$$\overline{\partial} s_{|U} = (\overline{\partial} f) \otimes e \in A^{0,1}(U, L).$$

The coherent cohomology of L may then be canonically identified with Dolbeault cohomology:

$$H^0(X, L) \simeq \ker \overline{\partial}, \quad H^1(X, L) \simeq \frac{A^{0,1}(X, L)}{\operatorname{Im} \overline{\partial}}.$$

The Riemann–Roch formula in this setting relates the numerical invariants we attached to L:

Theorem 2.1 (Riemann–Roch)**.** *The degree and Euler–Poincaré characteristic of* L *are related by*

$$\begin{aligned} \chi(L) &= \deg L + \frac{1}{2} \deg T_X, \\ &= \deg L + 1 - g \end{aligned}$$

where T_X *is the holomorphic tangent bundle and* g *is the topological genus of* X*.*

The extension of the Riemann–Roch formula to higher compact complex manifolds requires the theory of characteristic classes of holomorphic vector bundles. We won't do this here, and we just state the formula and its meaning in complex dimension 2. Hence, let X be a compact complex manifold of dimension 2. Let L be a holomorphic line bundle on X. Now the holomorphic Euler–Poincaré characteristic is

$$\chi(X, L) = \dim H^0(X, L) - \dim H^1(X, L) + \dim H^2(X, L).$$

In the usual formulation, this invariant is computed by the *Hirzebruch–Riemann–Roch* formula, in terms of the characteristic classes as follows:

$$\chi(X, L) = \int_X (\operatorname{ch}(L) \operatorname{td}(T_X))^{(2)}$$

The index 2 indicates that we only take the codimension 2 contribution of this product of characteristic classes. This can be expanded in terms of Chern classes as

$$\chi(X, L) = \int_X \left\{ c_1(L)^2 + \frac{1}{2} c_1(T_X)\, c_1(L) + \frac{1}{12}(c_1(T_X)^2 + c_2(T_X)) \right\}.$$

Let us assume for the sake of simplicity that X is projective. If L and M are line bundles on X, we can find respective meromorphic sections s and t whose divisors $\operatorname{div} s = \sum_i m_i D_i$ and $\operatorname{div} t = \sum_j n_j E_j$ have smooth components and pairwise transversal intersections (one implicitly invokes Bertini's theorem for that). Then

$$\int_X c_1(L)\, c_1(M) = \sum_{i,j} m_i n_j \#(D_i \cap E_j).$$

This explains the meaning of the first three terms in the Hirzebruch–Riemann–Roch formula above, once we recall that for a vector bundle E we have $c_1(E) = c_1(\det E)$. The last term involving $c_2(T_X)$ is the Euler number of X, related to the topological Euler-Poincaré characteristic through

$$\chi_{\mathrm{top}}(X, \mathbb{C}) = \sum_{p=0}^{2} (-1)^p \dim H^p(X, \mathbb{C}) = \int_X c_2(T_X).$$

It is a sort of obstruction to the existence of non-vanishing global holomorphic vector fields.

Let us give an example of use of the Hirzebruch–Riemann–Roch formula. Let X be a $K3$ surface. This is a simply connected compact Kähler surface, whose canonical bundle $\omega_X = \wedge^2 \Omega^1_X$ is trivial (i.e. X has a nowhere vanishing holomorphic 2-form). On the one hand, the Hirzebruch–Riemann–Roch theorem applied to the trivial line bundle $\mathcal{O}_X$ gives

$$\chi(X, \mathcal{O}_X) = \frac{1}{12} \int_X (c_1(T_X)^2 + c_2(T_X)) = \frac{1}{12} \int_X c_2(T_X),$$

since $c_1(T_X) = -c_1(\omega_X) = 0$. On the other hand, $\chi(X, \mathcal{O}_X) = 2$. Indeed, $H^0(X, \mathcal{O}_X) = \mathbb{C}$, $H^1(X, \mathcal{O}_X) = 0$ by the Hodge decomposition and simply connectedness (i.e. $H^1(X, \mathbb{C}) = 0$), and by Serre duality and $K3$ assumption

$$H^2(X, \mathcal{O}_X) \simeq H^0(X, \omega_X)^\vee \simeq H^0(X, \mathcal{O}_X)^\vee = \mathbb{C}.$$

Hence we derive $\int_X c_2(T_X) = 24$, and therefore

$$\chi_{\mathrm{top}}(X, \mathbb{C}) = 24.$$

Again using the simply connectedness and the Hodge decomposition, we infer from this equality

$$h^{1,1} = \dim H^1(X, \Omega^1_X) = 19.$$

Let us end this section with a question. Assume now that X is the set of complex points of a proper flat scheme $\mathcal{X}$ over $\mathbb{Z}$. Assume as well that $\mathcal{L}$ is an invertible sheaf on $\mathcal{X}$. The coherent cohomology groups $H^p(\mathcal{X}, \mathcal{L})$ are $\mathbb{Z}$-modules of finite type. As such, they have a well-defined rank. By flat base change, the rank is computed after base changing to $\mathbb{C}$, and hence

$$\sum_p (-1)^p \operatorname{rank} H^p(\mathcal{X}, \mathcal{L}) = \sum_p (-1)^p \dim H^p(\mathcal{X}_{\mathbb{C}}, \mathcal{L}_{\mathbb{C}})$$

can be obtained from the Hirzebruch–Riemann–Roch theorem. One may wonder what additional information we can catch by making use of the integral structure. The arithmetic Riemann–Roch theorem takes this structure into account.

2.2. **Arithmetic intersections on arithmetic surfaces.** Let $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic surface, i.e. $\mathcal{X}$ is a regular scheme, flat and projective over $\mathbb{Z}$, of Krull dimension 2. The set of complex points $\mathcal{X}(\mathbb{C})$ has the structure of a possibly non-connected Riemann surface, equipped with an anti-holomorphic involution $F_\infty\colon \mathcal{X}(\mathbb{C}) \to \mathcal{X}(\mathbb{C})$ induced by complex conjugation. Recall that a (smooth) hermitian line bundle $\overline{\mathcal{L}}$ over $\mathcal{X}$ consists in giving a line bundle $\mathcal{L}$ together with a smooth hermitian metric on the associated holomorphic line bundle $\mathcal{L}_{\mathbb{C}}$ on $\mathcal{X}(\mathbb{C})$, invariant under the natural action of F_∞. For a pair of hermitian line bundles $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$, we proceed to recall the construction of their *arithmetic intersection number*

$$(\overline{\mathcal{L}} \cdot \overline{\mathcal{M}}) \in \mathbb{R},$$

to be compared with the geometric intersection number

$$\int_X c_1(L)\, c_1(M) \in \mathbb{Z}$$

of two line bundles L, M on a projective complex surface X.

Let ℓ and m be non-trivial rational sections of $\mathcal{L}$ and $\mathcal{M}$ respectively, such that $\operatorname{div}\ell$ and $\operatorname{div} m$ are disjoint on the generic fiber $\mathcal{X}_{\mathbb{Q}}$. It is always possible to find such sections, by the projectivity assumption on $\mathcal{X}$. Let us write

$$\operatorname{div}\ell = \sum m_i D_i, \quad \operatorname{div} m = \sum n_j E_j.$$

The D_i and E_j are pairwise generically disjoint. We define finite arithmetic intersection numbers $(D_i \cdot E_j)_{\mathrm{fin}}$ as follows. If D_i is a vertical divisor, hence a variety over $\mathbb{F}_p$ for some prime number p, then we put

$$(D_i \cdot E_j)_{\mathrm{fin}} = (\deg c_1(\mathcal{O}(E_j)) \cap [D_i]) \cdot \log p.$$

hence we consider the degree of $\mathcal{O}(E_j)$ restricted to the projective curve D_i over $\mathbb{F}_p$, weighter by $\log p$. Assume now that D_i is horizontal, and let $x \in D_i \cap E_j$ be an intersection point. We denote by $k(x)$ its (finite) residue field. If $f, g \in \mathcal{O}_{\mathcal{X},x}$ are local equations for D_i and E_j, then the local ring $\mathcal{O}_{\mathcal{X},x}/(f,g)$ has finite length, and we put

$$(D_i \cdot E_j)_{\mathrm{fin},\,x} = \lg \frac{\mathcal{O}_{\mathcal{X},x}}{(f,g)} \cdot \log(\#k(x)).$$

We define

$$(D_i \cdot E_j)_{\mathrm{fin}} = \sum_{x \in D_i \cap E_j} (D_i \cdot E_j)_{\mathrm{fin},\,x}.$$

Finally, we put

$$(\ell, m)_{\mathrm{fin}} = \sum_{i,j} m_i n_j (D_i \cdot E_j)_{\mathrm{fin}} \in \mathbb{R}.$$

Next we introduce the archimedean contribution to the arithmetic intersection pairing:

$$(\ell, m)_\infty = \int_{\mathcal{X}(\mathbb{C})} (-\log \|m_{\mathbb{C}}\| \frac{i}{2\pi} \overline{\partial}\partial \log \|\ell_{\mathbb{C}}\|^2 - \log \|\ell_{\mathbb{C}}\| \delta_{\operatorname{div} m_{\mathbb{C}}})$$
$$= \int_{\mathcal{X}(\mathbb{C})} (-\log \|m_{\mathbb{C}}\| c_1(\overline{\mathcal{L}}_{\mathbb{C}}) - \log \|\ell_{\mathbb{C}}\| \delta_{\operatorname{div} m_{\mathbb{C}}}) \in \mathbb{R},$$

where $c_1(\overline{\mathcal{L}}_{\mathbb{C}})$ is the first Chern form of the hermitian line bundle $\overline{\mathcal{L}}_{\mathbb{C}}$. The arithmetic intersection number of $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ is obtained by adding the finite and archimedean intersection pairings above:

$$(\overline{\mathcal{L}} \cdot \overline{\mathcal{M}}) = (\ell, m)_{\text{fin}} + (\ell, m)_\infty \in \mathbb{R}.$$

It can be easily checked, applying the product formula, that the construction does not depend on the choice of sections ℓ, m. Furthermore, by Stokes' theorem, the arithmetic intersection number is symmetric. It also behaves bilinearly with respect to the tensor product of hermitian line bundles. In the particular case $\overline{\mathcal{L}} = \overline{\mathcal{M}}$, the quantity $(\overline{\mathcal{L}} \cdot \overline{\mathcal{L}})$ is also written $(\overline{\mathcal{L}}^2)$, and is called the *arithmetic self-intersection number* of $\overline{\mathcal{L}}$. It actually equals, by definition, to the *height of* $\mathcal{X}$ *with respect to* $\overline{\mathcal{L}}$, which is also denoted $h_{\overline{\mathcal{L}}}(\mathcal{X})$.

2.3. **The determinant of cohomology and the Quillen metric.** Let X be a compact Riemann surface and L a line bundle on X. We define the determinant of the cohomology of L as

$$\det H^\bullet(X, L) = \wedge^{\text{top}} H^0(X, L) \otimes \wedge^{\text{top}} H^1(X, L)^\vee.$$

Recall that the cohomology $H^\bullet(X, L)$ can be computed as the cohomology of the Dolbeault complex

$$A^{0,0}(X, L) \xrightarrow{\overline{\partial}} A^{0,1}(X, L).$$

In particular, $H^0(X, L)$ can be realized inside $A^{0,0}(X, L)$. If the line bundles T_X and L are equipped with smooth hermitian metrics, one can realize $H^1(X, L)$ inside $A^{0,1}(X, L)$ as well. For if $\overline{\partial}^*$ denotes the formal adjoint of $\overline{\partial}$ with respect to the functional L^2 hermitian products on $A^{0,p}(X, L)$, induced from the choices of metrics (i.e. $\langle \overline{\partial} s, t \rangle = \langle s, \overline{\partial}^* t \rangle$), then

$$H^1(X, L) \simeq \ker \overline{\partial}^*.$$

Through these realizations $H^p(X, L) \subset A^{0,p}(X, L)$, the cohomology spaces inherit L^2 hermitian products. The determinant of cohomology $\det H^\bullet(X, L)$ carries an induced hermitian norm, called the L^2 metric and written h_{L^2} or $\| \cdot \|_{L^2}$. For the sake of completeness, let us just say that the volume form μ needed for the L^2 pairings is normalized to be, locally in holomorphic coordinates z,

$$\mu = \frac{i}{2\pi} \frac{dz \wedge d\overline{z}}{\|dz\|^2}.$$

Let us define $\Delta^{0,0}_{\overline{\partial},L} = \overline{\partial}^{*}\overline{\partial}$ and $\Delta^{0,1}_{\overline{\partial},L} = \overline{\partial}\overline{\partial}^{*}$. These are elliptic differential operators of second order, positive and essentially self-adjoint, with discrete spectrum accumulating only to ∞. The construction of the L^2 metric involves only the 0 eigenspaces of these operators. The Quillen metric instead involves the rest of the spectrum. Let us write

$$0 < \lambda_1 \leq \lambda_2 \leq \dots$$

for the strictly positive eigenvalues of $\Delta^{0,1}_{\overline{\partial},L}$ (or equivalently $\Delta^{0,0}_{\overline{\partial},L}$), repeated according to multiplicities. We define the spectral zeta function

$$\zeta^{0,1}_{\overline{\partial},L}(s) = \sum_n \frac{1}{\lambda_n^s},$$

which can be shown to be absolutely convergent, hence holomorphic, for $\mathrm{Re}(s) > 1$. It can be meromorphically continued to the whole complex plane, and $s = 0$ is a regular point. Actually, one proves asymptotic expansions for the spectral theta function

$$\theta(t) := \mathrm{tr}(e^{-t\Delta^{0,1}_{\overline{\partial},L}}) - \dim\ker\Delta^{0,1}_{\overline{\partial},L} = \sum_n e^{-t\lambda_n}, \quad t > 0,$$

as $t \to +\infty$ and $t \to 0^+$, that justify the Mellin transform identity

$$\zeta^{0,1}_{\overline{\partial},L}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \theta(t) t^{s-1} dt$$

and lead to the meromorphic continuation properties above. The zeta regularized determinant of $\Delta^{0,1}_{\overline{\partial},L}$ is then defined to be

$$\det\Delta^{0,1}_{\overline{\partial},L} = \exp\left(-\frac{d}{ds}\mid_{s=0}\zeta^{0,1}_{\overline{\partial},L}(s)\right) \text{``} = \prod_n \lambda_n\text{''}.$$

The Quillen metric is obtained by rescaling the L^2 metric:

$$h_Q := (\det\Delta^{0,1})^{-1} h_{L_2}.$$

Let now $\mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic surface and $\overline{\mathcal{L}}$ a hermitian line bundle. Assume that $T_{\mathcal{X}(\mathbb{C})}$ is endowed with a hermitian metric, with the usual invariance property under the action of F_∞. The groups $H^p(\mathcal{X}, \mathcal{L})$ are $\mathbb{Z}$-modules of finite rank, and by means of 2 term free resolutions, one can define their determinants $\det H^p(\mathcal{X}, \mathbb{Z})$. These are free $\mathbb{Z}$-modules of rank 1. We put

$$\det H^\bullet(\mathcal{X}, \mathcal{L}) = \det H^0(\mathcal{X}, \mathcal{L}) \otimes \det H^1(\mathcal{X}, \mathcal{L})^\vee.$$

This construction commutes with base change, and in particular we can endow $\det H^\bullet(\mathcal{X}, \mathcal{L})$ with the Quillen metric after base changing to $\mathbb{C}$. We usually denote $\det H^\bullet(\mathcal{X}, \mathcal{L})_Q$ to indicate the resulting hermitian line bundle over $\mathbb{Z}$. Then we can attach to this object a numerical invariant called the *arithmetic degree*: if e is any basis of the $\mathbb{Z}$-module $\det H^{\bullet}(\mathcal{X},\mathcal{L})_Q$, then the arithmetic degree is

$$\widehat{\deg}\ \det H^{\bullet}(\mathcal{X},\mathcal{L})_Q = -\log\|e\|_Q \in \mathbb{R}.$$

Observe this is well-defined, since e is unique up to sign. The arithmetic degree of the determinant of cohomology is the arithmetic counterpart of the Euler–Poincaré characteristic in the complex geometric setting.

2.4. **The arithmetic Riemann–Roch theorem of Gillet–Soulé.** Let $\pi\colon \mathcal{X}\to \operatorname{Spec}\mathbb{Z}$ be an arithmetic surface, $\overline{\mathcal{L}}$ a hermitian line bundle and fix a F_∞ invariant hermitian metric on $T_{\mathcal{X}(\mathbb{C})}$. The arithmetic Riemann–Roch formula of Gillet–Soulé computes the arithmetic degree $\widehat{\deg}\ \det H^{\bullet}(\mathcal{X},\mathcal{L})_Q$ in terms of arithmetic intersections, in a formally analogous expression to the Hirzebruch–Riemann–Rcoh theorem in dimension 2. To state the theorem, we briefly need to discuss the relative dualizing sheaf and the analogue of the Euler class.

Because the morphism $\pi\colon \mathcal{X}\to\operatorname{Spec}\mathbb{Z}$ is projective and $\mathcal{X}$ and $\operatorname{Spec}\mathbb{Z}$ are regular schemes, there is a relative dualizing line bundle $\omega_{\mathcal{X}/\mathbb{Z}}$. Its complexification is dual to $T_{\mathcal{X}(\mathbb{C})}$ and hence we can endow it with the dual hermitian metric. The line bundle can be explicitly constructed from any factorization

$$\begin{array}{ccc} \mathcal{X} & \xhookrightarrow{\ j\ } & \mathbb{P}^N_{\mathbb{Z}} \\ & \searrow & \downarrow{\scriptstyle p} \\ & & \operatorname{Spec}\mathbb{Z}. \end{array}$$

The immersion j is regular, and its conormal sheaf is thus a vector bundle N_j on $\mathcal{X}$. The relative cotangent bundle Ω_p is a locally free rank N vector bundle on $\mathbb{P}^N_{\mathbb{Z}}$. One can then prove

$$\omega_{\mathcal{X}/\mathbb{Z}} = \det j^*\Omega_p \otimes \det N_j$$

is a dualizing sheaf. Using the exact sequence

$$0 \longrightarrow N_j^\vee \longrightarrow j^*\Omega_p \longrightarrow \Omega_{\mathcal{X}/\mathbb{Z}} \longrightarrow 0$$

and the theory of Bott–Chern secondary classes, one can define an arithmetic second Chern class $\widehat{c}_2(\Omega_{\mathcal{X}/\mathbb{Z}})$. It is actually defined independently of any metrized datum. Its arithmetic degree can be expressed in terms of localized Chern classes

$$\delta_\pi = \widehat{\deg}\,\widehat{c}_2(\Omega_{\mathcal{X}/\mathbb{Z}}) = \sum_p \deg c_2^{\mathcal{X}_{\mathbb{F}_p}}(\Omega_{\mathcal{X}/\mathbb{Z}_p})\log p.$$

The localized classes $c_2^{\mathcal{X}_{\mathbb{F}_p}}(\Omega_{\mathcal{X}/\mathbb{Z}_p}) \in \mathrm{CH}^2_{\mathcal{X}_{\mathbb{F}_p}}(\mathcal{X})$ measure the bad reduction of π at p. If π is semi-stable, then its degree is the number of singular points in the geometric fiber of π at p.

We can now state the arithmetic Riemann–Roch theorem of Gillet–Soulé for hermitian line bundles on arithmetic surfaces.

Theorem 2.2 (Gillet–Soulé)**.** *Let $\pi\colon \mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic surface, $\overline{\mathcal{L}}$ a hermitian line bundle, and fix a F_∞ invariant metric h on $T_{\mathcal{X}(\mathbb{C})}$. Endow the relative dualizing sheaf $\omega_{\mathcal{X}/\mathbb{Z}}$ with the metric dual to h. Then there is an equality of real numbers*

$$\begin{aligned}12\,\widehat{\deg}\,\det H^\bullet(\mathcal{X},\mathcal{L})_Q - \delta_\pi =&(\overline{\omega}^2_{\mathcal{X}/\mathbb{Z}}) + 6(\overline{\mathcal{L}}\cdot\overline{\mathcal{L}}\otimes\overline{\omega}^{-1}_{\mathcal{X}/\mathbb{Z}})\\ &- (2g-2)\#\pi_0(\mathcal{X}(\mathbb{C}))\left(\frac{\zeta'(-1)}{\zeta(-1)}+\frac{1}{2}\right),\end{aligned}$$

where g is the genus of any connected component of $\mathcal{X}(\mathbb{C})$.

A particular relevant case of the theorem is the *arithmetic Noether formula*, obtained by specializing to $\overline{\mathcal{L}} = \overline{\mathcal{O}}_{\mathcal{X}}$ the trivial hermitian line bundle:

$$\begin{aligned}12\,\widehat{\deg}\,\det H^\bullet(\mathcal{X},\mathcal{O})_Q - \delta_\pi =&(\overline{\omega}^2_{\mathcal{X}/\mathbb{Z}})\\ &- (2g-2)\#\pi_0(\mathcal{X}(\mathbb{C}))\left(\frac{\zeta'(-1)}{\zeta(-1)}+\frac{1}{2}\right).\end{aligned}$$

3. An arithmetic Riemann–Roch formula for modular curves

3.1. **The setting.** A natural geometric situation of arithmetic interest to which apply the arithmetic Riemann–Roch theorem, is the case of integral models of compactified modular curves. Let $\mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic surface such that

$$\mathcal{X}(\mathbb{C}) = \bigsqcup_j (\Gamma_i\backslash\mathbb{H}\cup\{\text{cusps}\}),$$

where the $\Gamma_i \subset \mathsf{PSL}_2(\mathbb{R})$ are congruence subgroups (e.g. $\Gamma_0(N)$, $\Gamma_1(N)$ or $\Gamma(N)$). In the arithmetic Riemann–Roch theorem, we need to fix a hermitian metric on the holomorphic tangent bundle. In the present setting, it could be tempting to choose a Poincaré type metric provided by the uniformization by $\mathbb{H}$: if $\tau = x + iy$ is the usual complex coordinate on the upper half plane, then the tensor

$$\frac{|d\tau|^2}{(\operatorname{Im}\tau)^2}$$

defines a $\mathsf{PSL}_2(\mathbb{R})$ invariant metric, and it is unique with this property, up to a constant. However, the quotient metric on each factor $\Gamma_i\backslash\mathbb{H}\cup\{\text{cusps}\}$ has singularities. The obvious ones are at the cusps, where the metric is not even defined. Also, the groups Γ_i may have fixed points on $\mathbb{H}$, producing conical type metric singularities on the quotient. Observe these features are not specific of congruence subgroups, but this is a general fact for fuchsian groups of the first kind, i.e. discrete subgroups Γ of $\mathsf{PSL}_2(\mathbb{R})$ such that $\Gamma\backslash\mathbb{H}$ is a complex algebraic curve.

Let Γ be a fuchsian group of the first kind. The serious difficulty we have to face happens at the level of spectral theory. Let us work with the Poincaré metric, and the trivial hermitian line bundle on $\Gamma\backslash\mathbb{H}$. The corresponding laplace operator on $A^{0,0}(\Gamma\backslash\mathbb{H}) := A^{0,0}(\mathbb{H})^{\Gamma}$ is, up to a constant, induced by the $\mathsf{PSL}_2(\mathbb{R})$ invairant operator

$$-y^2\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}\right).$$

This is an elliptic positive differential operator of order 2, essentially self-adjoint (with respect to the natural L^2 structure on $A^{0,0}(\Gamma\backslash\mathbb{H})$), but it has discrete as well as continuous spectrum. Hence the definition of the spectral zeta function and the regularized determinant do not make sense. And even if we can find a sensible definition, the results of Gillet–Soulé don't automatically apply.

In this section we discuss a version of the arithmetic Riemann–Roch theorem that applies to the previous setting and the trivial hermitian line bundle (so it is actually a version of the arithmetic Noether formula). We consider an arithmetic surface $\pi\colon \mathcal{X}\to \operatorname{Spec}\mathcal{O}_K$ over the ring of integers of a number field K, together with generically disjoint sections

$$\sigma_1,\ldots,\sigma_n\colon \operatorname{Spec}\mathcal{O}_K\to\mathcal{X},$$

such that

$$\mathcal{X}(\mathbb{C})=\bigsqcup_{\tau\colon K\hookrightarrow\mathbb{C}}(\Gamma_\tau\backslash\mathbb{H}\cup\{\sigma_1(\tau),\ldots,\sigma_r(\tau)\}),$$

and $\sigma_{r+1}(\tau),\ldots,\sigma_n(\tau)$ are elliptic fixed points of orders $e_{r+1},\ldots,e_n\geq 2$, respectively. The Poincaré hermitian metric induces a log-singular metric on the $\mathbb{Q}$-line bundle

$$\omega_{\mathcal{X}/\mathcal{O}_K}(\sum_i(1-e_i^{-1})\sigma_i),$$

where we put $e_i=\infty$ for $i=1,\ldots,r$ (i.e. we declare the cusps have infinite order). We will indicate the choice of this metric by an index "hyp". Hence there is still a well-defined arithmetic intersection number

$$(\omega_{\mathcal{X}/\mathcal{O}_K}(\sum_i(1-e_i^{-1})\sigma_i)^2_{\mathrm{hyp}})\in\mathbb{R},$$

according to the formalism developed by Bost and Kühn, and later generalized by Burgos–Kramer–Kühn to any dimension. This will be the main numerical invariant on the right hand side of the arithmetic Riemann–Roch formula. The need of twisting by the sections σ_i will be compensated by a suitable "boundary" contribution.

3.2. **Renormalized metrics (Wolpert metrics).** Let Γ be a fuchsian group of the first kind, and endow the quotient $\Gamma\backslash\mathbb{H}$ with the metric induced by $|d\tau|^2/(\operatorname{Im}\tau)^2$. We discuss the existence of canonical coordinates at cusps and elliptic fixed points, that serve to renormalize the singularities of the quotient metric.

Recall that a cusp of Γ corresponds to a point in $\mathbb{P}^1(\mathbb{R})$ with non-trivial stabilizer under Γ. This stabilizer is conjugated in $\mathsf{PSL}_2(\mathbb{R})$ to the group generated by the translation $\tau \mapsto \tau + 1$. It follows that for a cusp p of $\Gamma\backslash\mathbb{H} \cup \{\text{cusps}\}$, there exists a holomorphic coordinate z such that the hyperbolic metric tensor becomes

$$\frac{|dz|^2}{(|z| \log |z|)^2}.$$

The coordinate z is unique up to a constant of modulus one, and hence the assignment

$$\|dz\|_{W,p} = 1$$

is a well-defined hermitian metric on the holomorphic cotangent space of X at p, $\omega_{X,p}$. In the theory of modular forms, this variable z is usually denoted q, and appears in the so-called q-expansions (Fourier series expansions).

Elliptic fixed points correspond to points in $\mathbb{H}$ whose stabilizer is non-trivial. The stabilizer is then conjugated in $\mathsf{PSL}_2(\mathbb{R})$ to a finite group of rotations centered at $i \in \mathbb{H}$. A neighborhood of an elliptic fixed point q in $\Gamma\backslash\mathbb{H}$ is thus isometric to a quotient $D(0, \varepsilon)/\mu_k$, where the disk is endowed with the hyperbolic metric $|dw|^2/(4(1 - |w|^2)^2)$ and $\mu_k = \langle e^{2\pi i/k} \rangle$ acts by multiplication. This quotient can again be identified to $D(0, \varepsilon)$, via the map $w \mapsto z = w^k$. On the quotient, the hyperbolic metric tensor becomes

$$\frac{|dz|^2}{4|z|^{2-2/k}(1 - |z|^{2/k})^2}.$$

Again, such a coordinate is unique up to a factor of modulus one, and the assignment

$$\|dz\|_{W,q} = 1$$

defines a hermitian metric on $\omega_{X,q}$.

The renormalized hyperbolic metrics defined above for cusps and elliptic fixed points were first introduced by Wolpert (in the case of cusps). We call them *Wolpert metrics*.

Let now $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ be an arithmetic surface and $\sigma_1, \ldots, \sigma_n$ be sections corresponding to elliptic fixed points of orders e_i or cusps, as before. For every section σ_i, the line bundle $\psi_i := \sigma_i^*(\omega_{\mathcal{X}/\mathcal{O}_K})$ can be endowed (after base change through $K \hookrightarrow \mathbb{C}$) with the corresponding Wolpert metric. We indicate this choice of hermitian metric by an index W. We define a $\mathbb{Q}$-hermitian line bundle

$$\psi_W = \sum_i (1 - e_i^{-2}) \psi_{i,W}.$$

The arithmetic degree of ψ_W is a measure of how far the transcendental canonical coordinates just discussed are from being formal algebraic. Actually, this can be used to construct heights on moduli spaces of curves with marked points.

3.3. **A Quillen type metric.** Let $X = \Gamma\backslash\mathbb{H} \cup \{\text{cusps}\}$ be a compact Riemann surface, where Γ is a fuchsian group of the first kind. We endow $\mathbb{H}$ with the hyperbolic metric, and we would like to define a Quillen type metric on the determinant of the cohomology of the trivial line bundle $\mathcal{O}_X$.

The L^2 metric poses no problem. Indeed, there is a well-defined L^2 metric on $H^0(X, \mathcal{O}_X) = \mathbb{C}$, given by taking $\|1\|^2$ as the volume, which is finite. We normalize the volume form so that

$$\|1\|^2 = 2g - 2 + \sum_i (1 - e_i^{-1}).$$

For $H^1(X, \mathcal{O}_X)$, we invoke the Serre duality isomorphism

$$H^1(X, \mathcal{O}_X) \simeq H^0(X, \omega_X)^\vee$$

and introduce the L^2 scalar product on the latter given by

$$\langle \alpha, \beta \rangle = \frac{i}{2\pi} \int_X \alpha \wedge \overline{\beta}.$$

Let Δ_{hyp} be the hyperbolic laplacian, acting on $\mathcal{C}^\infty(\mathbb{H})^\Gamma$ as

$$\Delta_{\text{hyp}} = -y^2 \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right).$$

The spectrum of Δ_{hyp} can be classified into three types:

- *cuspidal spectrum.* It consists of eigenvalues $\lambda > 0$ whose eigenvectors are L^2 functions (with respect to the hyperbolic volume form) with vanishing Fourier coefficients at cusps. The cuspidal spectrum constitutes a possibly finite discrete set.
- *continuous spectrum.* It arises from scattering theory. Let $\Gamma_0 \subset \Gamma$ be the stabilizer of a cusp. We define a corresponding Eisenstein series

  $$E_0(\tau, s) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \text{Im}(\gamma\tau)^s,$$

  which absolutely convergens for $\text{Re}(s) > 1$. It is not L^2 with respect to the hyperbolic measure, and satisfies

  $$\Delta_{\text{hyp}} E_0(\tau, s) = s(1 - s) E_0(\tau, s).$$

  It can be shown that $E_0(\tau, s)$ has a meromorphic continuation to $s \in \mathbb{C}$. Its residues are contained in the real interval $(1/2, 1]$, and $s = 1$ is a simple pole with constant residue. The points on $\text{Re}(s) = 1/2$ are regular. Finally, if we put all the Eisenstein series for all cusps in a vector $\mathcal{E}(\tau, s)$, then there is a functional equation

  $$\mathcal{E}(\tau, s) = \Phi(s)\mathcal{E}(\tau, 1 - s),$$

where $\Phi(s)$ is a square matrix with meromorphic function entries, satisfying $\Phi(s)\Phi(1-s) = \mathrm{id}$ and such that $\Phi(s)$ is unitary for $\mathrm{Re}(s) = 1/2$. The matrix $\Phi(s)$ is called the scattering matrix, and can be computed from the Fourier expansions of the Eisenstein series at cusps. More precisely, if $E_1, \ldots, E_r$ are all the Eisenstein series, then the Fourier expansion of E_i at the j-th cusp has the form

$$y^s + \varphi_{ij}(s)y^{1-s} + \rho_{ij}(\tau, s),$$

where ρ_{ij} is L^2 for the hyperbolic measure. Then $\Phi(s) = (\varphi_{ij}(s))$. Finally, the continuous spectrum is formed by $1/4+t^2$, where $t \in \mathbb{R}$.

- *residual spectrum.* It arises from residues of Eisenstein series. If $E_0(\tau, s)$ is an Eisenstein series with a pole at $s_0 \in (1/2, 1]$, then
$$u(\tau) = \mathrm{res}_{s=s_0} E_0(\tau, s)$$
is an L^2, non-cuspidal, eigenfunction of Δ_{hyp}, with eigenvalue $s_0(1 - s_0)$. These constitute a finite set of eigenvalues.

The cuspidal spectrum and the residual spectrum together form the discrete spectrum of Δ_{hyp}. The spectral theorem asserts that for any L^2 function f on $\Gamma\backslash\mathbb{H}$, there is an expansion (valid in L^2)

$$f(\tau) = \sum_i \langle f, \varphi_j\rangle_{L^2}\varphi_j(\tau) + \sum_j \frac{1}{4\pi}\int_{-\infty}^{\infty} \langle f, E_j(\tau, \frac{1}{2} + it)\rangle_{L^2} E_j(\tau, \frac{1}{2} + it)dt,$$

where φ_j are orthonormal L^2 eigenfunctions for the discrete spectrum, and the $E_j(\tau, s)$ constitute the finite set of Eisenstein series.

We introduce a spectral zeta function that takes into account both the discrete spectrum $\{\lambda_n\}_n$ and the continuous spectrum. Let $\varphi(s) = \det \Phi(s)$ be the determinant of the scattering matrix.

Definition 3.1. We define the hyperbolic spectral zeta function by

$$\zeta_{\mathrm{hyp}}(\Gamma, s) = \sum_{\lambda_n > 0} \frac{1}{\lambda_n^s} - \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\varphi'(1/2 + it)}{\varphi(1/2 + it)}(1/4 + t^2)^{-s}dt + 4^{s-1}\,\mathrm{tr}(\Phi(\frac{1}{2})).$$

This expression is inspired by the spectral side of the Selberg trace formula applied to a suitable test function. Actually, by means of the Selberg trace formula, one can shown that $\zeta_{\mathrm{hyp}}(s)$ is holomorphic for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to $\mathbb{C}$. Moreover it is holomorphic at $s = 0$. We then define the *zeta regularized determinant of the hyperbolic laplacian*

$$\det \Delta_{\mathrm{hyp},\Gamma} = \exp\left(-\frac{d}{ds}\,|_{s=0}\, \zeta_{\mathrm{hyp}}(\Gamma, s)\right).$$

Finally, we define a Quillen type metric by mimicking the usual definition:

$$h_{Q,\mathrm{hyp}} = (\det \Delta_{\mathrm{hyp},\Gamma})^{-1} h_{L^2}.$$

Remark 3.2. In the compact case, the Quillen metric thus defined agrees with the known one up to an explicit constant. This is because the Dolbeault laplacian on functions differs by a constant from the scalar hyperbolic laplacian. For the sake of a cleaner presentation, we decided to silence this normalization issue.

3.4. **An arithmetic Riemann–Roch formula.** Let $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ be an arithmetic surface, with geometrically connected fibers. We suppose given sections $\sigma_1, \ldots, \sigma_n$, such that

$$\mathcal{X}(\mathbb{C}) = \bigsqcup_{\tau\colon K \hookrightarrow \mathbb{C}} (\Gamma_\tau \backslash \mathbb{H} \cup \{\sigma_1(\tau), \ldots, \sigma_r(\tau)\}),$$

for some fuchsian groups of the first kind Γ_τ having $\sigma_1(\tau), \ldots, \sigma_r(\tau)$ as cusps and $\sigma_{r+1}(\tau), \ldots, \sigma_n(\tau)$ as elliptic fixed points of orders e_i. For the cusps we put $e_i = \infty$. We endow $\mathcal{X}(\mathbb{C})$ with the singular hyperbolic metric.

Theorem 3.3 (Freixas–von Pippich)**.** *With the notations and assumptions as above, there is an equality of real numbers*

$$\begin{aligned} 12\,\widehat{\deg}\,\det H^\bullet(\mathcal{X}, \mathcal{O}_\mathcal{X})_Q - \delta + \widehat{\deg}\,\psi_W =& (\omega_{\mathcal{X}/\mathcal{O}_K}(\sum_i (1 - e_i^{-1})\sigma_i)^2_{\mathrm{hyp}}) \\ &- [K\colon \mathbb{Q}]C(g, \{e_i\}), \end{aligned}$$

for some explicit constant $C(g, \{e_i\})$ depending only on the genus g and the orders e_i (i.e. the type of the groups Γ_τ).

The proof of the theorem is long and technical, and combines basic facts of arithmetic intersection theory, glueing properties of determinants of laplacians, and explicit computations of determinants of laplacians on "model" hyperbolic cusps and cones. The latter are inspired by the physics literature. The explicit value of the constant is actually relevant in some applications, but in these lectures we prefer to focus on the rest of the terms of the arithmetic Riemann–Roch formula.

Remark 3.4. In later computations, we will appeal to a weak version of the theorem, where instead of working with an arithmetic surface over $\mathcal{O}_K$, we directly work with a smooth projective curve over K. The consequence for the numerical invariants will be that they are then only well defined modulo the $\mathbb{Q}$ vector space spanned by the real numbers

$\log p$, for p prime. Indeed, one may choose an auxiliary regular model over $\mathcal{O}_K$ and apply the theorem. The numerical invariants for two different models differ by an element in $\mathbb{Q} \otimes_{\mathbb{Z}} \log |\mathbb{Q}^\times| \hookrightarrow \mathbb{R}$.

4. Modular and Shimura curves

4.1. **Modular curves.** Modular curves are moduli spaces of elliptic curves, possibly with some extra structure. The point of departure is the mapping

$$\begin{aligned} \mathbb{H} &\longrightarrow \{\text{elliptic curves}\} \\ \tau &\longmapsto \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}). \end{aligned}$$

It is known that every elliptic curve over $\mathbb{C}$ is isomorphic to a torus as above. The isomorphism relation on such corresponds to the action of $\mathsf{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$. The quotient $\mathbb{H}/\mathsf{PSL}_2(\mathbb{Z})$ is the open modular curve, whose points are thus in bijection with isomorphism classes of elliptic curves over $\mathbb{C}$. The j-invariant of elliptic curves defines a biholomorphic map

$$\mathbb{H}/\mathsf{PSL}_2(\mathbb{Z}) \xrightarrow{\sim} \mathbb{C}.$$

The cusp compactification of $\mathbb{H}/\mathsf{PSL}_2(\mathbb{Z})$ corresponds to $\mathbb{C}\cup\{\infty\}$, hence to $\mathbb{P}^1_{\mathbb{C}}$. A holomorphic neighborhood of the cusp has holomorphic coordinate $q = e^{2\pi i\tau}$, and it is best understood as parametrizing elliptic curves in the form

$$\mathbb{C}^\times/q^{\mathbb{Z}}.$$

It is thus reasonable to declare that the cusp $q = 0$ corresponds to the torus $\mathbb{C}^\times$. Or equivalently, to a so-called *generalized elliptic curve*: the singular nodal genus one curve

$$\mathbb{P}^1_{\mathbb{C}}/\{0 \sim \infty\},$$

together with its multiplicative algebraic group structure when deprived from the singular point $0 \sim \infty$. This complex geometric picture can be formulated over $\operatorname{Spec}\mathbb{Z}$, and gives raise first to a Deligne-Mumford stack $\overline{\mathcal{M}}_1$, then to the *coarse moduli scheme* of generalized elliptic curves $\mathbb{P}^1_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$, with the cusp at ∞ as a section.

In these lectures we will be mostly interested in the moduli of elliptic curves with a torsion point of exact order N, and slight variants. Fix $N \geq 1$ an integer. Over a general base scheme S, we consider elliptic curves $E \to S$ (i.e. smooth proper schemes over S, with geometrically connected fibers of dimension 1 and a relative group scheme structure) together with a section $P\colon S \to E$, generating a finite flat subgroup of order N. The coarse moduli of elliptic curves with a point of order N "classifies" such couples $(E/S, P)$ up to isomorphism. One can prove that it defines a proper flat normal scheme $Y_1(N)$ over $\operatorname{Spec}\mathbb{Z}$, with geometrically connected fibers of pure dimension 1. It is smooth over

$\mathbb{Z}[1/N]$. Moreover, forgetting the point of order N defines a finite flat morphism

$$j\colon Y_1(N) \longrightarrow \mathbb{A}^1_{\mathbb{Z}} \subset \mathbb{P}^1_{\mathbb{Z}}.$$

One can define a compactification of $Y_1(N)$ by taking the integral closure of $\mathbb{P}^1_{\mathbb{Z}}$ with respect to the morphism j. The compactification actually affords a moduli interpretation as a coarse moduli scheme of generalized elliptic curves. We won't need this description. For our purposes, it will be enough to know that $X_1(N) \setminus Y_1(N)$ is a relative Cartier divisor over $\mathbb{Z}$, that becomes rational (i.e. given by sections) over $\mathrm{Spec}[\zeta_N]$. Finally, if $N \geq 5$, then $X_1(N)$ is actually a fine moduli scheme. In applications, we will stick to the restriction $N \geq 5$, and we will actually consider $X_1(N)$ and its variants as defined over $\mathbb{Q}$.

It is easy to see from the moduli interpretation that $Y_1(N)(\mathbb{C})$ can be uniformized as

$$Y_1(N)(\mathbb{C}) = \mathbb{H}/\Gamma_1(N),$$

where

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) \mid a-1 \equiv c \equiv 0 \mod N \right\},$$

or rather its image in $\mathsf{PSL}_2(\mathbb{Z})$. When $N \geq 5$, $\Gamma_1(N)$ has no torsion and is actually identified with a fuchsian group.

For the purpose of defining Hecke operators later on, and in some intermediary steps, we need a variant of the geometric objects above. Instead of moduli of elliptic curves with a point of exact order N, we add to the data a cyclic subgroup of order M prime to N. Namely, we classify triples $(E/S, P, C)$ where P is an S-point of order N of E and $C \subset E[M]$ is a cyclic finite flat subgroup of order M. The outcome is a proper, normal and flat scheme $X_1(N,M)$ over $\mathrm{Spec}\,\mathbb{Z}$, with geometrically connected fibers and smooth over $\mathrm{Spec}\,\mathbb{Z}[1/NM]$. Again, in applications we will actually consider it to be defined over $\mathbb{Q}$. Over the complex numbers, it can be presented as

$$(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))/\Gamma_1(N) \cap \Gamma_0(M),$$

where now

$$\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod M \right\}.$$

We will write $\Gamma_1(N,M) = \Gamma_1(N) \cap \Gamma_0(M)$.

4.2. **Modular forms.** Classically, a modular form of weight k for $\Gamma_1(N)$ (or more generally $\Gamma_1(N,M)$) is a holomorphic map

$$f\colon \mathbb{H} \longrightarrow \mathbb{C},$$

such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \quad \text{for} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N),$$

that extends holomorphically to the cusps. For the cusp at ∞, this condition is stated as follows. Since $f(\tau+1) = f(\tau)$ by the equivariance property above, f has a Fourier series expansion in $q = e^{2\pi i\tau}$

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n.$$

We require that $a_n = 0$ for $n < 0$. We moreover say that f is a cusp form if $a_0 = 0$. We will only need weight 2 cusp forms, so that in the following we restrict to modular forms of even weight. These can be best understood as differential forms on $X_1(N)(\mathbb{C})$. First observe that the equivariance property of a modular form of weight $2k$, say f, is equivalent to the invariance of the tensor $f(\tau)d\tau^{\otimes k}$. In the coordinate q, this tensor becomes

$$\sum_{n\geq 0} a_n q^n \left(\frac{1}{2\pi i}\frac{dq}{q}\right)^{\otimes k}.$$

We thus see that modular forms are global holomorphic sections of the sheaf

$$(\omega_{X_1(N)}(\mathbf{cusps}))^{\otimes k},$$

and cusp forms are global holomorphic sections of the sheaf

$$\omega_{X_1(N)}^{\otimes k}((k-1)\,\mathbf{cusps}).$$

We wrote **cusps** for the divisor of cusps. Typical notations are

$$M_{2k}(\Gamma_1(N)) = H^0(X_1(N)(\mathbb{C}), (\omega_{X_1(N)}(\mathbf{cusps}))^{\otimes k})$$

for the space of modular forms of weight $2k$ and

$$S_{2k}(\Gamma_1(N)) = H^0(X_1(N)(\mathbb{C}), \omega_{X_1(N)}^{\otimes k}((k-1)\,\mathbf{cusps}))$$

for the subspace of cusp forms.

The spaces of modular and cusp forms have rational and integral structures provided by the rational and integral models of $X_1(N)$. In particular, in weight 2, the space of cusp forms has a rational structure

$$S_{2k}(\Gamma_1(N), \mathbb{Q}) := H^0(X_1(N)_{\mathbb{Q}}, \omega_{X_1(N)/\mathbb{Q}}).$$

It can be seen that this is exactly the $\mathbb{Q}$-vector space of cusp forms with rational Fourier coefficients (at all cusps). The same would be true for any subfield of $\mathbb{C}$, and this is known as the *q-expansion principle.* Due to its relation to the arithmetic Riemann–Roch theorem for modular curves, we will focus on this space from now on.

The space $S_2(\Gamma_1(N), \mathbb{Q})$ has an action of the algebra of Hecke correspondences and diamond operators. This is an algebra of endomorphisms of the Jacobian $J_1(N)/\mathbb{Q}$ of $X_1(N, M)$, constructed as follows. First, for the Hecke operators, let M be an integer prime to N. We introduce the auxiliary curve $X_1(N,M)$. There are two natural morphisms from $X_1(N,M)$ to $X_1(N)$. The first one, that we call α, is just forgetting the cyclic subgroup of order M. The second one is given by

$$\beta\colon (E/S,P,C)\mapsto (E/C/S,P \mod C),$$

where E/C is the well-defined quotient of the (possibly generalized) elliptic curve E by the finite flat subgroup C, and $P \mod C$ is the induced S-point of order N. In the language of correspondences, the M-th Hecke operator T_M is $\alpha_*\circ\beta^*$. For the diamond operators, let $d\in(\mathbb{Z}/N\mathbb{Z})^\times$. There is an induced automorphism

$$\begin{aligned}\langle d\rangle\colon X_1(N)&\longrightarrow X_1(N)\\ (E/S,P)&\longmapsto (E/S,dP).\end{aligned}$$

The Hecke operators together with the diamond operators span a free $\mathbb{Z}$-subalgebra of finite type of $\mathrm{End}_{\mathbb{Q}}(J_1(N))$, formed by pairwise commuting endomorphisms. It is called the Hecke algebra, and denoted $\mathbb{T}_N$. By functoriality, the Hecke algebra acts on

$$H^0(J_1(N),\Omega^1_{J_1(N)/\mathbb{Q}})\simeq H^0(X_1(N),\omega_{X_1(N)/\mathbb{Q}}).$$

The action of the Hecke algebra can also be described on Fourier expansions by well-known classical expressions that we won't review here. An analogous construction carries over to modular curves $X_1(N,M)$. An important property that we can't skip is that after extension scalars to $\overline{\mathbb{Q}}$, spaces of cusp forms have bases of simultaneous eigenfunctions for the Hecke algebra.

To conclude this section, we discuss the notion of d-new forms. From a modular curve $X_1(N,d)$, there are several morphisms down to $X_1(N,d')$, for $d'\mid d$, through which cusp forms can be pulled-backed. The resulting cusp forms are called d-old. Let us fix d' such a divisor. For any divisor of d/d', say m, we have a morphism

$$\begin{aligned}X_1(N,d)&\longrightarrow X_1(N,d/m)\\ (E/S,C)&\longmapsto (E/C[m]/S,C/C[m]),\end{aligned}$$

where as above we denote $C[m]$ for the m-torsion part of C. Then, since d' divides d/m, we have a forgetful map

$$\begin{aligned}X_1(N,d/m)&\longrightarrow X_1(N,d')\\ (E/S,C)&\longmapsto (E/S,C[d']).\end{aligned}$$

By composing these two arrows, we obtain a so-called *degeneracy* morphism depending both on d' and m, and denoted $\gamma_{d',m}$. In terms of the degeneracy maps, the d-old subspace of $S_2(\Gamma_1(N,d),\mathbb{Q})$ is

$$S_2(\Gamma_1(N,d),\mathbb{Q})^{d-old}:=\sum_{d'\mid d}\ \sum_{m\mid(d/d')}\gamma^*_{d',m}S_2(\Gamma_1(N,d'),\mathbb{Q}).$$

Actually, the sum can be seen to be direct. The d-new quotient of $S_2(\Gamma_1(N,d),\mathbb{Q})$ is defined to be

$$S_2(\Gamma_1(N,d),\mathbb{Q})^{d-new} := S_2(\Gamma_1(N,d),\mathbb{Q})/S_2(\Gamma_1(N,d),\mathbb{Q})^{d-old}.$$

Actually, the $\mathbb{Q}$-vector space of d-new forms can be realized inside $S_2(\Gamma_1(N,d),\mathbb{Q})$ in such a way that, with respect to the natural hermitian structure on $H^0(X_1(N,d),\omega_{X_1(N,d)/\mathbb{Q}})$, we have

$$S_2(\Gamma_1(N,d),\mathbb{Q}) = S_2(\Gamma_1(N,d),\mathbb{Q})^{d-new} \overset{\perp}{\oplus} S_2(\Gamma_1(N,d),\mathbb{Q})^{d-old}.$$

4.3. **Shimura curves and quaternionic modular curves.** We consider "compact" counterparts of modular curves, arising from arithmetic quaternionic groups. Moduli theoretically, they classify abelian surfaces with a faithful action of a maximal order in an indefinite quaternion algebra over $\mathbb{Q}$. They share many features with modular curves (integral models, Hecke operators, etc). They have the advantage of being automatically "compact", and at the same time the disadvantage of not having q-expansions, precisely due to the lack of cusps.

Let B be an indefinite quaternion division algebra over $\mathbb{Q}$, so that $B\otimes_{\mathbb{Q}}\mathbb{R}$ is isomorphic to the matrix algebra $M_2(\mathbb{R})$. For every finite prime p, the quaternion algebra $B\otimes_{\mathbb{Q}}\mathbb{Q}_p$ is either a matrix algebra or a division algebra. In the later case, we say that B is ramified at p. The set of primes where B is ramified is finite and of even cardinality. The discriminant of B is the product of all such primes, and is denoted $\mathrm{disc}(B)$. Let $N\geq 1$ be an integer prime to $\mathrm{disc}(B)$. We can choose an order $\mathcal{O}$ in B (hence a subring providing a lattice in B), with the following properties:

- for every $p \nmid \mathrm{disc}(B)N$,
$$\mathcal{O}\otimes\mathbb{Z}_p \xrightarrow{\sim} M_2(\mathbb{Z}_p).$$
- for every $p \mid N$,
$$\mathcal{O}\otimes\mathbb{Z}_p \xrightarrow{\sim} \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{Z}_p) \mid a-1\equiv c\equiv 0 \mod N\right\}.$$
- for every prime $p\mid \mathrm{disc}(B)$, $\mathcal{O}\otimes\mathbb{Z}_p$ is a maximal order in $B\otimes\mathbb{Q}_p$.

The analogue of the group $\Gamma_1(N)$ in this setting will be $\Gamma_1^B(N):=\mathcal{O}^{\times,1}$, namely the subgroup of the units in $\mathcal{O}$ of reduced norm 1. Actually the notation $\Gamma_1^B(N)$ is ambiguous in that it does not render the choice of $\mathcal{O}$ explicit. The group $\Gamma_1^B(N)$ can be realized into $\mathsf{SL}_2(\mathbb{R})$: take its image under a fixed algebra isomorphism $B\otimes\mathbb{R}\simeq M_2(\mathbb{R})$. As a group of fractional linear transformations of $\mathbb{H}$, it is cocompact. Moreover, if $N\geq 5$ it is torsion free as well. The quotient

$$X_1^B(N) := \mathbb{H}/\Gamma_1^B(N)$$

is thus a compact Riemann surface, and is called a Shimura curve. It can be seen that its points are in bijective correspondence with complex abelian surfaces A with a faithful action of a maximal order of B, together with a level structure of type $\Gamma_1(N)$. We won't make these structures more explicit. It will suffice to say that the Shimura curve $X_1^B(N)$ has a smooth projective model over $\operatorname{Spec}\mathbb{Z}[1/N\operatorname{disc}(B)]$, and in particular over $\mathbb{Q}$. For $N \geq 5$, we have again a fine moduli space.

The spaces of quaternionic modular forms are defined in analogy to the classical modular forms. We are particularly interested in weight 2 quaternionic forms and their rational structures:

$$S_2(\Gamma_1^B(N), \mathbb{Q}) = H^0(X_1^B(N), \omega_{X_1^B(N)/\mathbb{Q}}).$$

In contrast with modular curves, these rational structures can't be read in Fourier expansions, since there are no cusps at our disposal. Finally, an algebra of Hecke operators can as well be defined in the quaternionic setting, in a similar way as for modular curves. We won't give further details.

5. The Jacquet–Langlands correspondence and the arithmetic Riemann–Roch theorem

The cohomological side of the arithmetic Riemann–Roch theorem for modular (or Shimura) curves can be interpreted in terms of automorphic forms. As we saw in the previous chapter, the global sections of the canonical sheaf on a compactified modular curve correspond to cusp forms of weight 2. Similarly, the holomorphic analytic torsion is the contribution of the non-holomorphic modular forms, commonly known as Maass forms. Both holomorphic and non-holomorphic modular forms can be seen as vectors in spaces of automorphic representations. One can thus expect that general principles in the theory of (global) automorphic representations, combined with the arithmetic Riemann–Roch theorem, can be useful for a better understanding of some arithmetic intersection numbers. In these lectures we explain the relationship between the arithmetic Riemann–Roch theorem and the Jacquet–Langlands correspondence relating automorphic representations of $\mathsf{GL}_{2/\mathbb{Q}}$ to those of $B^\times$, for a division quaternion algebra B over $\mathbb{Q}$. We will however not enter into the details of automorphic representations in order to keep the size of this course reasonable, and only state the consequences we need in the classical language of modular forms.

5.1. **On the Jacquet–Langlands correspondence for weight 2 forms.** We fix an integer $N \geq 1$, and B an indefinite division quaternion algebra over $\mathbb{Q}$, whose discriminant $d = \operatorname{disc}(B)$ is prime to N. We deal with rational weight 2 cusp forms for $\Gamma_1(N, d)$ and rational quaternionic modular forms for $\Gamma_1^B(N)$. Recall that the notation for the latter hides several choices of orders in B. Recall as well that the spaces of such classical or quaternionic modular forms

$$S_2(\Gamma_1(N,d),\mathbb{Q}) = H^0(X_1(N,d),\omega_{X_1(N,d)/\mathbb{Q}}),$$
$$S_2(\Gamma_1^B(N),\mathbb{Q}) = H^0(X_1^B(N),\omega_{X_1^B(N)/\mathbb{Q}}),$$

come equipped with the action of the respective Hecke algebras.

Theorem 5.1 (Jacquet–Langlands, Faltings)**.** *There is a $\mathbb{Q}$-linear and Hecke equivariant isomorphism*

$$H^0(X_1(N,d),\omega_{X_1(N,d)/\mathbb{Q}})^{d-new} \xrightarrow{\sim} H^0(X_1^B(N),\omega_{X_1^B(N)/\mathbb{Q}}),$$

compatible with the natural hermitian structures up to $\mathbb{Q}^\times$.

Let us say some words about the proof of this theorem. First of all, over $\overline{\mathbb{Q}}$, the spaces of weight 2 forms can be decomposed according to characters of the Hecke algebra, themselves corresponding to spaces of Hecke new eigenforms of some level dividing dN. These can be encoded in terms of automorphic representations with a given *conductor* and *central character* (nebentypus). The global Jacquet–Langlands correspondence actually establishes a "natural" bijection at the level of automorphic representations. In this bijection the automorphic representations of $\mathsf{GL}_{2/\mathbb{Q}}$ are restricted to be *cuspidal*. The correspondence is compatible with its local version, and at primes dividing $\mathrm{disc}(B)$ it relies on the theory of the Weil representation. At the other places, the local correspondence is just the "identity". One can derive from this the behaviour of conductors and central characters through the correspondence. The central character is preserved. In the direction from $B^\times$ to $\mathsf{GL}_{2/\mathbb{Q}}$, the conductor gets multiplied by $\mathrm{disc}(B)$. In the classical language, these facts are summarized in a Hecke equivariant isomorphism as above, except that it is a priori only defined over $\overline{\mathbb{Q}}$. The assertion of the field of definition being $\mathbb{Q}$ and the compatibility with the hermitian structures requires the input of Faltings isogeny theorem. The Jacquet–Langlands correspondence can actually be used to relate the ℓ-adic Tate modules of the abelian varieties $J_1(N,d)^{d-new}$ and $J_1^B(N)$. The d-new quotient $J_1(N,d)^{d-new}$ of $J_1(N,d)$ is defined in a similar way to $S_2(\Gamma_1(N,d),\mathbb{Q})$. These abelian varieties are acted upon by Hecke algebras. After decomposing the Jacobians into Hecke isotypical components, one sees there is an isomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules

$$T_\ell J_1(N,d)^{d-new} \xrightarrow{\sim} T_\ell J_1^B(N).$$

Then Faltings isogeny theorem implies that the jacobians above are $\mathbb{Q}$-isogenous, and this easily implies our claim. In this argument we silenced some key steps, mostly the Galois representation attached to a Hecke eigensystem (through the Eichler–Shimura construction) and the compatibility with the Galois action.

An immediate corollary of the theorem is the relation between arithmetic degrees of modular forms:

Corollary 5.2. *We have an identity in* $\mathbb{R}/\log|\mathbb{Q}^\times|$

$$\widehat{\deg}\ H^0(X_1(N,d),\omega_{X_1(N,d)/\mathbb{Q}})^{d-new}_{L^2} = \widehat{\deg}\ H^0(X_1^B(N),\omega_{X_1^B(N)/\mathbb{Q}})_{L^2}.$$

Some comments are in order. First of all, by the very definition of the spaces of newforms, the equality of the statement is equivalent to

$$\begin{aligned}&\widehat{\deg}\ H^0(X_1^B(N),\omega_{X_1^B(N)/\mathbb{Q}})_{L^2}\\&\quad=\sum_{d'|d}\mu(d')\sigma(d/d')\ \widehat{\deg}\ H^0(X_1(N,d'),\omega_{X_1(N,d')/\mathbb{Q}})_{L^2},\end{aligned}$$

where μ is the Möbius function and σ is the divisor counting function. Second, if one works over $\operatorname{Spec}\mathbb{Z}[1/Nd]$, thanks to the relation of the quantities above to Faltings heihgts of Jacobians together with an observation due to Prasanna, one can refine the identity to an equality of real numbers modulo $\log p's$, for p in a controlled finite set of primes with a precise arithmetic meaning (a part from $p \mid Nd$, one has to take into account Eisenstein primes).

Finally, it is immediate from the corollary and the volume computations for modular and Shimura curves that:

Corollary 5.3.

$$\begin{aligned}&12\ \widehat{\deg}\det H^\bullet(X_1^B(N),\mathcal{O}_{X_1^B(N)})_{L^2}\\&\quad=12\sum_{d'|d}\mu(d')\sigma(d/d')\ \widehat{\deg}\det H^\bullet(X_1(N,d'),\mathcal{O}_{X_1(N,d')})_{L^2}\end{aligned}$$

in $\mathbb{R}/\log|\mathbb{Q}^\times|$.

5.2. **The Jacquet–Langlands correspondence for Maass forms.** We are now concerned with eigenspaces of the hyperbolic laplacian acting on functions of modular and Shimura curves. Let $\lambda > 0$ be a positive real number, and define

$$V_\lambda(\Gamma_1(N,d)) = \text{cuspidal eigenspace of } \Delta_{\mathrm{hyp}} \text{of eigenvalue } \lambda.$$

Hence, this is a finite dimension complex vector space spanned by non-holomorphic cusp forms, proper under the action of the hyperbolic laplacian, with eigenvalue λ. In an analogous way to classical modular forms, one can define an action of the Hecke algebra on $V_\lambda(\Gamma_1(N,d))$. The notion of d-new and d-old forms makes sense as well. The space $V_\lambda(\Gamma_1(N,d))$ is contained in the L^2 functional space with respect to the hyperbolic measure. There is an orthogonal decomposition

$$V_\lambda(\Gamma_1(N,d)) = V_\lambda(\Gamma_1(N,d))^{d-new} \overset{\perp}{\oplus} V_\lambda(\Gamma_1(N,d))^{d-old}.$$

Similarly we define spaces $V_\lambda(\Gamma_1^B(N))$ for the quaternionic modular group $\Gamma_1^B(N)$, with no need of any cuspidality condition.

The theory of automorphic representations works as well to decompose the spaces V_λ in irreducible modules for the action of the Hecke algebra. The Jacquet–Langlands correspondence applies too: it actually makes no distinction between automorphic representations arising from holomorphic modular forms or non-holomorphic ones.

Theorem 5.4. *There is a Hecke equivariant isomorphism of finite dimensional complex vector spaces*

$$V_\lambda(\Gamma_1(N,d))^{d-new} \xrightarrow{\sim} V_\lambda(\Gamma_1^B(N)).$$

In particular, we have the relation

$$\dim V_\lambda(\Gamma_1^B(N)) = \sum_{d'|d} \mu(d')\sigma(d/d') \dim V_\lambda(\Gamma_1(N,d')).$$

To relate spectral zeta functions, the consequence stated for the multiplicities of eigenspaces is the only we will need: the compatibility with the action of Hecke algebras does not play a role in the arithmetic Riemann–Roch theorem. Tow more pieces are needed. The spectral zeta function in the modular curve a priori involves the residual spectrum and the scattering matrices of the vector of Eisenstein series (i.e. the continuous spectrum), while for the Shimura curve there are no such contributions. It turns our that the residual spectrum for congruence subgroups is actually trivial: the only poles of the Eisenstein series happen at $s = 1$. And the spectral zeta function requires only strictly positive eigenvalues! For the scattering matrices, we have the following fact:

Lemma 5.5. *Denote by* $\Phi(\Gamma, s)$ *the scattering matrix for a fuchsian group* Γ*, and* $\varphi(\Gamma, s)$ *its determinant. Then we have*

$$\sum_{d'|d} \mu(d')\sigma(d/d') \operatorname{tr}(\Phi(\Gamma_1(N,d'), s) = 0,$$

$$\prod_{d'|d} \varphi(\Gamma_1(N,d'), s)^{\mu(d')\sigma(d/d')} = 1.$$

The lemma follows from explicit computations of scattering matrices, or simply by relating the cusps (and their stabilizers) for a group $\Gamma_1(X, d')$ with those coming from smaller levels through the degeneracy mappings. By taking logarithmic derivatives on the second relation, we find

$$\sum_{d'|d} \mu(d')\sigma(d/d') \int_{-\infty}^{\infty} \frac{\varphi'(\Gamma_1(N,d'), 1/2+it)}{\varphi(\Gamma_1(N,d'), 1/2+it)} (1/4+t^2)^{-s} dt = 0.$$

Recall now the definition of the spectral zeta function (Definition 3.1). The theorem and the lemma together imply:

Theorem 5.6. *Let $\zeta_{\mathrm{hyp}}(\Gamma, s)$ be the hyperbolic spectral zeta function for a fuchsian subgroup Γ. Then*

$$\zeta_{\mathrm{hyp}}(\Gamma_1^B(N), s) = \sum_{d'|d} \mu(d')\sigma(d/d')\zeta_{\mathrm{hyp}}(\Gamma_1(N, d'), s).$$

In particular, for the regularized zeta determinants

$$\det \Delta_{\mathrm{hyp},\Gamma_1^B(N)} = \prod_{d'|d} (\det \Delta_{\mathrm{hyp},\Gamma_1(N,d')})^{\mu(d')\sigma(d/d')}.$$

Together with Corollary 5.3 we obtain:

Corollary 5.7. *There is an equality in $\mathbb{R}/\log|\mathbb{Q}^\times|$*

$$\begin{aligned}12\, &\widehat{\deg} \det H^\bullet(X_1^B(N), \mathcal{O}_{X_1^B(N)})_Q \\ &= 12 \sum_{d'|d} \mu(d')\sigma(d/d')\, \widehat{\deg} \det H^\bullet(X_1(N,d'), \mathcal{O}_{X_1(N,d')})_Q.\end{aligned}$$

5.3. **Relating arithmetic intersection numbers.** The discussion of the previous paragraphs easily leads to the following conclusion.

Theorem 5.8. *Let $N \geq 5$ and B an indefinite division quaternion algebra of discriminant d coprime to B. There is a relation between arithmetic intersection numbers*

$$\frac{(\omega_{X_1(N)/\mathbb{Q}}(\boldsymbol{cusps})^2_{\mathrm{hyp}})}{\deg \omega_{X_1(N)/\mathbb{Q}}(\boldsymbol{cusps})} = \frac{(\omega^2_{X_1^B(N)/\mathbb{Q},\mathrm{hyp}})}{\deg \omega_{X_1^B(N)/\mathbb{Q}}} \quad \text{in } \mathbb{R}/\mathbb{Q}\log|\mathbb{Q}^\times|,$$

*where **cusps** is the reduced boundary divisor with support $X_1(N) \setminus Y_1(N)$.*

In the statement we wrote $\mathbb{Q}\log|\mathbb{Q}^\times|$ for the $\mathbb{Q}$ vector space spanned by $\log|\mathbb{Q}^\times|$, namely $\mathbb{Q} \otimes_{\mathbb{Z}} \log|\mathbb{Q}^\times|$. As for arithmetic degrees of spaces of cusp forms, the theorem can be refined to an equality of real numbers up to some $\log p$'s, for p running over a controlled finite set of primes. The assumption $N \geq 5$ is made to avoid the presence of elliptic fixed points and simplify the discussion.

Let us say some words about the proof. First of all, by functoriality properties of arithmetic intersection numbers, quotients as in the statement of the theorem are independent of the level. In particular, we have

$$\frac{(\omega_{X_1(N)/\mathbb{Q}}(\mathbf{cusps})^2_{\mathrm{hyp}})}{\deg \omega_{X_1(N)/\mathbb{Q}}(\mathbf{cusps})} = \frac{(\omega_{X_1(N,d')/\mathbb{Q}}(\mathbf{cusps})^2_{\mathrm{hyp}})}{\deg \omega_{X_1(N,d')/\mathbb{Q}}(\mathbf{cusps})}$$

for any $d' \mid d$. These arithmetic intersection numbers appear in the arithmetic Riemann-Roch theorem for the modular curves $X_1(N, d')$, up to a small detail: the cusp divisor **cusps** becomes rational only after base changing to $\mathbb{Q}(\zeta_N)$. Nevertheless, again by functoriality properties of arithmetic intersection numbers, the quotients above are also invariant under extension of the base field. Therefore, we can work over $\mathbb{Q}(\zeta_N)$ instead of $\mathbb{Q}$ and assume that **cusps** is formed by rational points. Then, after Corollary 5.7 and the arithmetic Riemann–Roch theorem for modular curves, we are reduced to showing that for every cusp σ (written as a section for coherence of notations) of $X_1(N,d')$ we have

$$\widehat{\deg}\ \sigma^*(\omega_{X_1^B(N,d')/\mathbb{Q}(\zeta_N)})_W = 0 \quad \text{in } \mathbb{R}/\mathbb{Q}\log|\mathbb{Q}^\times|.$$

To simplify the exposition, we will proceed for the cusp at ∞. Observe we are allowed to increase the level N, by the previous remarks. It is then known that for N big enough, the canonical sheaf of $X_1(N,d')$ is ample. We can then find a global section s of $\omega_{X_1^B(N,d')/\mathbb{Q}(\zeta_N)}^{\otimes k}$ that does not vanish at the cusp ∞. Fix an embedding $\mathbb{Q}(\zeta_N)\subset\mathbb{C}$. The Fourier expansion of s at the cusp ∞ then looks like

$$s = (\sum_{n\geq 0} a_n q^n) dq^{\otimes k},$$

with $a_n\in\mathbb{Q}(\zeta_N)$ and $a_0\neq 0$. This is a consequence of the q-expansion theorem, if we think of s as a modular form. Moreover, for any automorphism τ of $\mathbb{Q}(\zeta_N)$, the conjugate section s^τ has q-expansion

$$s^\tau = (\sum_{n\geq 0} \tau(a_n) q^n) dq^{\otimes k}.$$

By construction, the Wolpert norm of $\sigma^*(dq)$ is 1. We thus see that

$$\widehat{\deg}\ \sigma^*(\omega_{X_1^B(N,d')/\mathbb{Q}(\zeta_N)})_W^{\otimes k} = -\log|N_{\mathbb{Q}(\zeta_N)/\mathbb{Q}}(a_0)| = 0 \quad \text{in } \mathbb{R}/\log|\mathbb{Q}^\times|,$$

and hence the claim. For the proof of the theorem to be complete, one actually needs the precise value of the topological constant $C(g,\{e_i\})$ in the arithmetic Riemann–Roch theorem, and check the needed relations between the constants for the groups $\Gamma_1(N,d')$ and $\Gamma_1^B(N)$. We don't provide the details since this is deprived of any conceptual interest.

CNRS – Institut de Mathématiques de Jussieu, 4 Place Jussieu, 75005 Paris, France

E-mail address: gerard.freixas@imj-prg.fr