Arakelov geometry, heights, equidistribution, and the Bogomolov conjecture

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Fermat's method of infinite descent studies the solutions to diophantine equations by constructing, from a given solution of a diophantine equation, a smaller solution, and ultimately deriving a contradiction. In order to formalize the intuitive notion of ''size'' of an algebraic solution of a diophantine equation, Northcott (1950) and Weil (1951) have introduced the notion of *height* of an algebraic point of an algebraic variety defined over a number field and established their basic functorial properties, using the decomposition theorem of Weil (1929). The *height machine* is now an important tool in modern diophantine geometry.

The advent of arithmetic intersection theory with Arakelov (1974) and, above all, its extension in any dimension by Gillet & Soulé (1990) (''Arakelov geometry'') has led Faltings (1991) to extend the concept further by introducing the height of a subvariety, defined in pure analogy with its degree, replacing classical intersection theory with arithmetic intersection theory. This point of view has been developed in great depth by Bost *et al* (1994) and Zhang (1995*a*).

Although I shall not use it in these notes, I also mention the alternative viewpoint of Philippon (1991) who defines the height of a subvariety as the height of the coefficients-vector of its ''Chow form''.

The viewpoint of adelic metrics introduced in Zhang (1995*b*) is strengthened by the introduction of Berkovich spaces in this context, based on Gubler (1998), and leading to the definition by Chambert-Loir (2006) of measures at all places analogous to product of Chern forms at the archimedean place.

We then present the equidistribution theorem of Szpiro *et al* (1997) and its extension by Yuan (2008).

Finally, we use these ideas to explain the proof of Bogomolov's conjecture, following Ullmo (1998); Zhang (1998).

Contents

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1. Arithmetic intersection numbers

1.1. — Let $\mathscr{X}$ be a proper flat scheme over $\mathbf{Z}$. For every integer $d \geqslant 0$, let $\mathrm{Z}_d(\mathscr{X})$ be the group of d-cycles on $\mathscr{X}$: it is the free abelian group generated by integral closed subschemes of dimension d.

Remark **(1.2)**. — Let $f : \mathscr{X} \to \mathrm{Spec}(\mathbf{Z})$ be the structural morphism. By assumption, f is proper so that the image of an integral closed subscheme Z of $\mathscr{X}$ is again an integral closed subscheme of $\mathrm{Spec}(\mathbf{Z})$. There are thus two cases:

(1) Either $f(Z) = \mathrm{Spec}(\mathbf{Z})$, in which case we say that Z is horizontal;
(2) Or $f(Z) = \{(p)\}$ for some prime number p, in which case we say that Z is vertical.

Theorem **(1.3)**. — *Let $n = \dim(\mathscr{X})$ and let $\overline{\mathscr{L}_1}, \dots, \overline{\mathscr{L}_n}$ be hermitian line bundles on $\mathscr{X}$. There exists a unique family of linear maps:*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathscr{L}_1}) \dots \widehat{c}_1(\overline{\mathscr{L}_d}) \mid \cdot\right) : \mathrm{Z}_d(\mathscr{X}) \to \mathbf{R},$$

for $d \in \{0, \dots, n\}$ satisfying the following properties:

(1) *For every integer $d \in \{0, \dots, n\}$, every integral closed subscheme Z of $\mathscr{X}$ such that $\dim(Z) = d$, every integer $m \neq 0$ and every regular meromorphic*[(1)] *section s of $\mathscr{L}_d^m|_Z$, one has*

(1.3.1)
$$m\,\widehat{\deg}\left(\widehat{c}_1(\overline{\mathscr{L}_1}) \dots \widehat{c}_1(\overline{\mathscr{L}_d}) \mid Z\right) = \widehat{\deg}\left(\widehat{c}_1(\overline{\mathscr{L}_1}) \dots \widehat{c}_1(\overline{\mathscr{L}_{d-1}}) \mid \mathrm{div}(s)\right) + \int_{Z(\mathbf{C})} \log \|s\|^{-1} c_1(\overline{\mathscr{L}_1}) \dots c_1(\overline{\mathscr{L}_d}).$$

(2) *For every closed point z of $\mathscr{X}$, viewed as a integral closed subscheme of dimension $d = 0$, one has*

(1.3.2)
$$\widehat{\deg}(Z) = \log(\mathrm{Card}(\kappa(z))).$$

Moreover, these maps are multilinear and symmetric in the hermitian line bundles $\overline{\mathscr{L}_1}, \dots, \overline{\mathscr{L}_n}$ and only depend on their isomorphism classes in $\widehat{\mathrm{Pic}}(\mathscr{X})$.

(1) that is, defined over a dense open subscheme of Z

Remark **(1.4)**. — This theorem should be put in correspondence with the analogous geometric result for classical intersection numbers. Let F be a field and let X be a proper scheme over F, let $n = \dim(X)$ and let $L_1, \dots, L_n$ be line bundles over X. The degree $\deg(c_1(L_1) \dots c_1(L_d) \mid Z)$ of an d-cycle Z in X is characterized by the relations:

(1) It is linear in Z;

(2) If $d = 0$ and Z is a closed point z whose residue field $\kappa(Z)$ is a finite extension of F, and $\deg(Z) = [\kappa(Z) : F]$;

(3) If Z is an integral closed subscheme of X of dimension d, m a non-zero integer, s a regular meromorphic section of L_d^m, then

$$\textbf{(1.4.1)} \qquad m \deg(c_1(L_1) \dots c_1(L_d) \mid Z) = \deg(c_1(L_1) \dots c_1(L_{d-1}) \mid \operatorname{div}(s)).$$

The additional integral that appears in the arithmetic degree takes into account the fact that $\operatorname{Spec}(\mathbf{Z})$ does not behave as a proper variety.

Example **(1.5)**. — Assume that Z is vertical and lies over a maximal ideal (p) of $\operatorname{Spec}(\mathbf{Z})$. Then Z is a proper scheme over $\mathbf{F}_p$ the following formula

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathscr{L}_1}) \dots \widehat{c}_1(\overline{\mathscr{L}_d}) \mid Z\right) = \deg\left(c_1(\overline{\mathscr{L}_1}) \dots c_1(\overline{\mathscr{L}_d}) \mid Z\right) \log(p)$$

follows from the inductive definition and shows that the height pairing corresponds to classical intersection theory.

Example **(1.6)**. — Assume that $d = 1$ and that Z is horizontal, so that Z is the Zariski-closure in $\mathscr{X}$ of a closed point $z \in \mathscr{X}_{\mathbf{Q}}$. Let $F = \kappa(z)$ and let $\mathfrak{o}_F$ be its ring of integers; by properness of $\mathscr{X}$, the canonical morphism $\operatorname{Spec}(F) \to \mathscr{X}$ with image z extends to a morphism $\varepsilon_z : \operatorname{Spec}(\mathfrak{o}_F) \to \mathscr{X}$, whose image is Z. Then

$$\left(\widehat{c}_1(\overline{\mathscr{L}} \mid Z\right) = \widehat{\deg}(\varepsilon_z^* \overline{\mathscr{L}}).$$

Proposition **(1.7)**. — *Let $f : \mathscr{X}' \to \mathscr{X}$ be a generically finite morphism of proper flat schemes over $\mathbf{Z}$, let Z be an integral closed subscheme of $\mathscr{X}$ and let $d = \dim(Z)$.*

(1) *If $\dim(f(Z)) < d$, then*

$$\left(\widehat{c}_1(f^*\overline{\mathscr{L}_1}) \dots \widehat{c}_1(f^*\overline{\mathscr{L}_d}) \mid Z\right) = 0;$$

(2) *Otherwise, $\dim(Z) = d$ and*

$$\left(\widehat{c}_1(f^*\overline{\mathscr{L}_1}) \dots \widehat{c}_1(f^*\overline{\mathscr{L}_d}) \mid Z\right) = \left(\widehat{c}_1(\overline{\mathscr{L}_1}) \dots \widehat{c}_1(\overline{\mathscr{L}_d}) \mid f_*(Z)\right),$$

where $f_*(Z) = [\kappa(Z) : \kappa(f(Z))] f(Z)$ is a d-cycle on $\mathscr{X}$.

Remark **(1.8)**. — Let $n = \dim(\mathscr{X})$ and assume that $\mathscr{X}$ is regular. As the notation suggest rightly, the arithmetic intersection theory of Gillet & Soulé (1990) allows another definition of the real number $\widehat{\deg}\left(\widehat{c}_1(\overline{\mathscr{L}_1}) \dots \widehat{c}_1(\overline{\mathscr{L}_n}) \mid \mathscr{X}\right)$ as the arithmetic degree of the 0-dimensional arithmetic cycle $\widehat{c}_1(\overline{\mathscr{L}_1}) \dots \widehat{c}_1(\overline{\mathscr{L}_n}) \in \widehat{\mathrm{CH}}_0(\mathscr{X})$.

In fact, while the theory of Gillet & Soulé (1990) imposes regularity conditions on $\mathscr{X}$, the definition of arithmetic product of classes of the form $\widehat{c}_1(\overline{\mathscr{L}})$ requires less stringent

conditions; in particular, the regularity of the generic fiber $\mathscr{X}_{\mathbf{Q}}$ is enough. More generally, for every birational morphism $f : Z' \to Z$ such that $Z'_{\mathbf{Q}}$ is regular, one has

$$\widehat{\deg}\left(\widehat{c}_1(\overline{\mathscr{L}_1}) \dots \widehat{c}_1(\overline{\mathscr{L}_n}) \mid Z\right) = \widehat{\deg}\left(\widehat{c}_1(f^*\overline{\mathscr{L}_1}) \dots \widehat{c}_1(f^*\overline{\mathscr{L}_n}) \mid Z'\right).$$

2. The height of a variety

2.1. — Let X be a proper $\mathbf{Q}$-scheme and let L be a line bundle on X. The important case is when the line bundle L is ample, an assumption which will often be implicit below; in that case, the pair (X, L) is called a polarized variety.

2.2. — Let $\mathscr{X}$ be a proper flat scheme over $\mathbf{Z}$ and let $\overline{\mathscr{L}}$ be a hermitian line bundle on $\mathscr{X}$ such that $\mathscr{X}_{\mathbf{Q}} = X$ and $\mathscr{L}_{\mathbf{Q}} = L$. Let Z be a closed integral subscheme of X and let $d = \dim(Z)$. Let $\mathscr{Z}$ be the Zariski-closure of Z in $\mathscr{X}$; it is an integral closed subscheme of $\mathscr{X}$ and $\dim(\mathscr{Z}) = d + 1$.

Definition **(2.3)**. — *The degree and the height of Z relative to $\overline{\mathscr{L}}$ are defined by the formulas (provided* $\deg_{\mathscr{L}}(Z) \neq 0$*).*

(2.3.1) $$\deg_{\mathscr{L}}(Z) = \deg(c_1(\mathscr{L})^d \mid Z)$$

(2.3.2) $$h_{\overline{\mathscr{L}}}(Z) = \widehat{\deg}\left(\widehat{c}_1(\overline{\mathscr{L}}^{d+1}) \mid \mathscr{Z}\right) / (d+1) \deg_{\mathscr{L}}(Z).$$

Note that the degree $\deg_{\mathscr{L}}(Z)$ is computed on X, hence only depends on L. Moreover, the condition that $\deg_{\mathscr{L}}(Z) \neq 0$ is satisfied (for every Z) when L is ample on X.

Proposition **(2.4)**. — *Let $f : \mathscr{X}' \to \mathscr{X}$ be a generically finite morphism of proper flat schemes over $\mathbf{Z}$, let Z be a closed integral subscheme of $\mathscr{X}'_{\mathbf{Q}}$ and let $d = \dim(Z)$. Assume that L is ample on X and that $\dim(f(Z)) = d$. Then f^*L is ample on Z and*

$$h_{f^*\overline{\mathscr{L}}}(Z) = h_{\overline{\mathscr{L}}}(f(Z)).$$

Proof. — This follows readily from proposition 1.7 and its analogue for geometric degrees. Indeed, when one compares formula (2.3.2) for Z and for $f(Z)$, both the numerator and the denominator get multiplied by $[\kappa(Z) : \kappa(f(Z))]$. □

Example **(2.5)**. — For every $x \in X(\overline{\mathbf{Q}})$, let $[x]$ denote its Zariski closure in X. The function $X(\overline{\mathbf{Q}}) \to \mathbf{R}$ given by $x \mapsto h_{\overline{\mathscr{L}}}([x])$ is a height function relative to the line bundle $\mathscr{L}_{\mathbf{Q}}$ on X.

Example **(2.6)**. — Let us assume that X is an abelian variety over a number field F, with everywhere good reduction, and let $\mathscr{X}$ be an $\mathfrak{o}_F$-abelian scheme such that $\mathscr{X}_{\mathbf{Q}} = X$. Let o be the origin of X and let $\varepsilon_o : \operatorname{Spec}(\mathfrak{o}_F) \to \mathscr{X}$ be the corresponding section. Let L be a line bundle on X with a trivialisation ℓ of $L|_o$. There exists a unique line bundle $\mathscr{L}$ on $\mathscr{X}$ such that $\mathscr{L}_{\mathbf{Q}} = L$ and such that the given trivialisation of $L|_o$ extends to a trivialisation of $\varepsilon_o^*\mathscr{L}$. Moreover, for every embedding $\sigma : F \hookrightarrow \mathbf{C}$ the theory of Riemann forms on complex tori endows L_σ with a canonical metric $\|\cdot\|_\sigma$ whose curvature form $c_1(L_\sigma, \|\cdot\|_\sigma)$ is invariant by translation and such that $\|\ell\|_\sigma = 1$; this is in fact the unique metric possessing these two properties. We let $\overline{\mathscr{L}}$ be the hermitian line bundle on $\mathscr{X}$ so defined.

The associated height function will be denoted by $\widehat{h}_L$: it extends the *Néron–Tate height* from $X(\overline{\mathbf{Q}})$ to all integral closed subschemes.

Assume that L is even, that is $[-1]^*L \simeq L$. Then $[n]^* \simeq L^{n^2}$ for every integer $n \geqslant 1$, and this isomorphism extends to an isomorphism of hermitian line bundles $[n]^*\overline{\mathscr{L}} \simeq \overline{\mathscr{L}}^{n^2}$. Consequently, for every integral closed subscheme Z of X, one has the following relation

$$\widehat{h}_L([n](Z)) = n^2\widehat{h}_L(Z). \tag{2.6.1}$$

Assume otherwise that L is odd, that is $[-1]^*L \simeq L^{-1}$. Then $[n]^* \simeq L^n$ for every integer $n \geqslant 1$; similarly, this isomorphism extends to an isomorphism of hermitian line bundles $[n]^*\overline{\mathscr{L}} \simeq \overline{\mathscr{L}}^n$. Consequently, for every integral closed subscheme Z of X, one has the following relation

$$\widehat{h}_L([n](Z)) = n\widehat{h}_L(Z). \tag{2.6.2}$$

Proposition **(2.7)**. — *Let $\mathscr{X}'$ be a proper flat scheme over $\mathbf{Z}$ such that $\mathscr{X}'_{\mathbf{Q}} = X$; let $\overline{\mathscr{L}'}$ be a hermitian line bundle on $\mathscr{X}'$ such that $\mathscr{L}'_{\mathbf{Q}} = L$. Assume that L is ample. Then there exists a real number c such that*

$$|h_{\overline{\mathscr{L}}}(Z) - h_{\overline{\mathscr{L}'}}(Z)| \leqslant c$$

for every integral closed subscheme Z of X.

Proposition **(2.8)**. — *Let us assume that L is ample. For every real number B, the set of integral closed subschemes Z of X such that $\deg_L(Z) \leqslant B$ and $h_{\overline{\mathscr{L}}}(Z) \leqslant B$ is finite.*

The case of closed points is Northcott's theorem, and the general case is due to Bost *et al* (1994).

3. Adelic metrics

3.1. — Let $S = \{2, 3, \ldots, \infty\}$ be the set of places of $\mathbf{Q}$.

Each prime number p is identified with the p-adic absolute value on $\mathbf{Q}$, normalized by $|p|_p = 1/p$; these places are said to be finite. We denote by $\mathbf{Q}_p$ the completion of $\mathbf{Q}$ for this p-adic absolute value and fix an algebraic closure $\overline{\mathbf{Q}_p}$ of $\mathbf{Q}_p$. The p-adic absolute value extends uniquely to $\overline{\mathbf{Q}_p}$; the corresponding completion is denoted by $\mathbf{C}_p$: this is an algebraically closed complete valued field.

The archimedean place is represented by the symbol ∞, and is identified with the usual absolute value on $\mathbf{Q}$; it is also called the infinite place. For symmetry of notation, we may write $\mathbf{Q}_\infty = \mathbf{R}$ and $\mathbf{C}_\infty = \mathbf{C}$, the usual fields of real and complex numbers.

3.2. — Let X be a proper scheme over $\mathbf{Q}$. Let $v \in S$ be a place of $\mathbf{Q}$.

Assume $v = \infty$. Then we set $X_\infty^{\mathrm{an}} = X(\mathbf{C}_\infty)/F_\infty$, the set of complex points of X modulo the action of complex conjugation F_∞.

Assume now that $v = p$ is a finite place. Then we set X_p^{an} to be the analytic space associated by Berkovich (1990) to the $\mathbf{Q}_p$-scheme $X_p = X_{\mathbf{Q}_p}$. It is a compact metrizable topological space, locally contractible (in particular locally arcwise connected). There is a canonical continuous map $X(\mathbf{C}_p) \to X_p^{\mathrm{an}}$; it identifies the (totally discontinuous) topological

space $X(\mathbf{C}_p)/\operatorname{Gal}(\mathbf{C}_p/\mathbf{Q}_p)$ with a dense subset of X_p^{an}. It is endowed with a sheaf in local rings $\mathscr{O}_{X_p^{\mathrm{an}}}$; for every open subset U of X_p^{an}, every holomorphic function $f \in \mathscr{O}_{X_p^{\mathrm{an}}}(U)$ admits an absolute value $|f| : U \to \mathbf{R}_+$.

We gather all places to gather and consider the topological space $X_{\mathrm{ad}} = \coprod_{v\in S} X_v^{\mathrm{an}}$, coproduct of the family $(X_v^{\mathrm{an}})_{v\in S}$. By construction, a function φ on X_{ad} consists in a family $(\varphi_v)_{v\in S}$, where φ_v is a function on X_v^{an}, for every $v \in S$.

3.3. — Let L be a line bundle on X; it induces a line bundle L_v^{an} on X_v^{an} for every place v.

A continuous v-adic metric on L_v^{an} is the datum, for every open subset U of X_v^{an} and every section s on L_v^{an} on U, one is given a continuous function $\|s\| : U \to \mathbf{R}_+$, subject to the requirements:

(1) For every subset V of U, one has $\|s|_V\| = \|s\|\,|_V$;
(2) For every holomorphic function $f \in \mathscr{O}_{X_v^{\mathrm{an}}}(U)$, one has $\|fs\| = |f|\,\|s\|$.

If L and M are line bundles on X equipped with v-adic metrics, then L^{-1} and $L \otimes M$ admit natural v-adic metrics, and the canonical isomorphism $L^{-1} \otimes L \simeq \mathscr{O}_X$ is an isometry.

The trivial line bundle $\mathscr{O}_X$ admits a canonical v-adic metric for which $\|f\| = |f|$ for every local section of $\mathscr{O}_X$. More generally, for every v-adic metric $\|\cdot\|$ on $\mathscr{O}_X$, $\varphi = \log \|1\|^{-1}$ is a continuous function on X_v^{an}, and any v-adic metric on $\mathscr{O}_X$ is of this form. The v-adically metrized line bundle associated with φ is denoted by $\mathscr{O}_X(\varphi)$.

If $\overline{L}$ is a line bundle endowed with an v-adic metric and $\varphi \in \mathscr{C}(X_v^{\mathrm{an}}, \mathbf{R})$, we denote by $\overline{L}(\varphi)$ the v-adically metrized line bundle $\overline{L} \otimes \mathscr{O}_X(\varphi)$. Explicitly, its v-adic metric is that of $\overline{L}$ multiplied by $e^{-\varphi}$.

Example **(3.4)**. — Let $\mathscr{X}$ be a proper flat scheme over $\mathbf{Z}$ such that $\mathscr{X}_{\mathbf{Q}} = X$, let d be a positive integer and let $\mathscr{L}$ be a line bundle on $\mathscr{X}$ such that $\mathscr{L}_{\mathbf{Q}} = L^d$. Let us show that this datum endows L with an p-adic metric, for every finite place $p \in S$.

Let thus fix a prime number p. There exists a canonical *specialization map*, $X_p^{\mathrm{an}} \to \mathscr{X} \otimes_{\mathbf{Z}} \mathbf{F}_p$; it is anticontinuous. For every open subset $\mathscr{U} \subset \mathscr{X} \otimes_{\mathbf{Z}} \mathbf{F}_p$, let $]\mathscr{U}[$ be the preimage of $\mathscr{U}$.

There exists a unique continuous metric on L_p^{an} such that for every open subscheme $\mathscr{U}$ of $\mathscr{X} \otimes_{\mathbf{Z}} \mathbf{Z}_p$ and every basis ℓ of $\mathscr{L}$ on $\mathscr{U}$, one has $\|\ell\| \equiv 1$ on $]\mathscr{U} \otimes \mathbf{F}_p[$. Explicitly, if s is a section of L_p^{an} on an open subset U of $]\mathscr{U} \otimes \mathbf{F}_p[$, there exists a holomorphic function $f \in \mathscr{O}_{X_p^{\mathrm{an}}}(U)$ such that $s^d = f\ell$ and $\|s\| = |f|^{1/d}$ on U.

Such p-adic metrics are called *algebraic*.

3.5. — An adelic metric on L is the datum, for every place $v \in S$, of a v-adic metric on the line bundle L_v^{an} on X_v^{an}, subject to the additional requirement that there exists a model $(\mathscr{X}, \mathscr{L})$ of (X, L) inducing the given p-adic metric for all but finitely many prime p.

If L and M are line bundles on X equipped with adelic metrics, then L^{-1} and $L \otimes M$ admit natural adelic metrics, and the canonical isomorphism $L^{-1} \otimes L \simeq \mathscr{O}_X$ is an isometry.

The trivial line bundle $\mathscr{O}_X$ admits a canonical adelic metric for which $\|f\| = |f|$ for every local section of $\mathscr{O}_X$. More generally, for every adelic metric $\|\cdot\|$ on $\mathscr{O}_X$, and every place $v \in S$, then $\varphi_v = \log \|1\|_v^{-1}$ is a continuous function on X_v^{an}, and is identically zero for all but finitely many places v; in other words, the function $\varphi = (\varphi_v) \in \mathscr{C}(X_{\mathrm{ad}}, \mathbf{R})$ has compact support.

Conversely, any adelic metric on $\mathscr{O}_X$ is of this form; The adelically metrized line bundle associated with φ is denoted by $\mathscr{O}_X(\varphi)$.

If $\overline{L}$ is a line bundle endowed with an adelic metric and $\varphi \in \mathscr{C}_c(X_v^{an}, \mathbf{R})$, we denote by $\overline{L}(\varphi)$ the adelically metrized line bundle $\overline{L} \otimes \mathscr{O}_X(\varphi)$. Explicitly, for every place v, its v-adic metric is that of $\overline{L}$ multiplied by $e^{-\varphi_v}$.

Remark **(3.6)**. — Let $(\mathscr{X}, \mathscr{L})$ and $(\mathscr{X}', \mathscr{L}')$ be two models of the polarized variety (X, L). Since X is finitely presented, there exists a dense open subscheme U of $\mathrm{Spec}(\mathbf{Z})$ such that the isomorphism $\mathscr{X}_{\mathbf{Q}} = X = \mathscr{X}'_{\mathbf{Q}}$ extends to an isomorphism $\mathscr{X}_U \simeq \mathscr{X}'_U$. Then, up to shrinking U, we may assume that the isomorphism $\mathscr{L}_{\mathbf{Q}} = L = \mathscr{L}'_{\mathbf{Q}}$ extends to an isomorphism $\mathscr{L}_U \simeq \mathscr{L}'_U$. In particular, for every prime number p such that $(p) \in U$, the p-adic norms on L induced by $\mathscr{L}$ and $\mathscr{L}'$ coincide.

3.7. — Let $\widehat{\mathrm{Pic}}(X_{\mathrm{ad}})$ be the abelian group of isometry classes of line bundles on X endowed with adelic metrics. It fits into an exact sequence

$$\Gamma(X, \mathscr{O}_X^\times) \to \mathscr{C}_c(X_{\mathrm{ad}}, \mathbf{R}) \to \widehat{\mathrm{Pic}}(X_{\mathrm{ad}}) \to \mathrm{Pic}(X) \to 0. \tag{3.7.1}$$

The morphism on the left is given by $u \mapsto (\log|u|_v^{-1})_{v \in S}$. It is injective up to torsion, as a consequence of Kronecker's theorem: if $|u|_v = 1$ for every place v, then there exists $m \geqslant 1$ such that $u^m = 1$. Its image is the kernel of the morphism $\mathscr{C}(X_{\mathrm{ad}}, \mathbf{R}) \to \widehat{\mathrm{Pic}}(X)$; indeed, an isometry $\mathscr{O}_X(\varphi) \to \mathscr{O}_X(\psi)$ is given by an element $u \in \Gamma(X, \mathscr{O}_X^\times)$ such that $\psi_v + \log|u|_v^{-1} = \varphi_v$, for every place $v \in S$.

We denote by $\widehat{c}_1(\overline{L})$ the isometry class in $\widehat{\mathrm{Pic}}(X)$ of an adelically metrized line bundle on X.

Remark **(3.8)**. — Let D be an effective Cartier divisor on X and let $\mathscr{O}_X(D)$ be the corresponding line bundle; let s_D be its canonical section. Assume that $\mathscr{O}_X(D)$ is endowed with an adelic metric.

Let $v \in S$ be a place of $\mathbf{Q}$. The function $\log \|s_D\|_v^{-1}$ is a continuous function on $X_v^{an} - |D|$, and is called a v-adic Green function for D. For every open subscheme U of X and any equation f of D on U, $g_D + \log|f|_v$ extends to a continuous function on U_v^{an}. Conversely, this property characterizes v-adic Green functions for D.

The family $g_D = (g_{D,v})$ is called an adelic Green function for D.

Lemma **(3.9)** (Chambert-Loir & Thuillier (2009), prop. 2.2)

Let $\mathscr{X}$ be a proper flat integral scheme over $\mathbf{Z}$, let $\overline{\mathscr{L}}$ be a hermitian line bundle on $\mathscr{X}$. Let $X = \mathscr{X}_{\mathbf{Q}}$ and let $L = \mathscr{L}_{\mathbf{Q}}$, endowed with the algebraic adelic metric associated with $(\mathscr{X}, \overline{\mathscr{L}})$. Assume that $\mathscr{X}$ is integrally closed in its generic fiber (for example, that it is normal).

Then the canonical map $\Gamma(\mathscr{X}, \mathscr{L}) \to \Gamma(X, L)$ is injective and its image is the set of sections s such that $\|s\|_v \leqslant 1$ for every finite place $v \in S$.

Equivalently, effective Cartier divisors on $\mathscr{X}$ correspond to v-adic Green functions which are nonnegative at all finite places v.

Proof. — Injectivity follows from the fact that $\mathscr{X}$ is flat, so that X is schematically dense in $\mathscr{X}$. Surjectivity is a generalization of the fact that an integrally closed domain is the intersection of its prime ideals of height 1. □

3.10. — Let $\|\cdot\|$ and $\|\cdot\|'$ be two adelic metrics on L. For every place v, the ratio of these metrics is a continuous function on X_v^{an}, and we let

$$\delta_v(\|\cdot\|, \|\cdot\|') = \sup_{x\in X_v^{\mathrm{an}}} \left|\log \frac{\|\cdot\|'}{\|\cdot\|}(x)\right|.$$

Since X_v^{an} is compact, this is a nonnegative real number. Moreover, for all but finitely many places v, it is equal to 0.

We then define the distance between the two given adelic metrics by

$$\delta(\|\cdot\|, \|\cdot\|') = \sum_{v\in S} \delta_v(\|\cdot\|', \|\cdot\|).$$

(It may be infinite.)

The set of adelic metrics on a given line bundle L is a real affine space, its underlying vector space is the subspace $\mathscr{C}_{\mathrm{c}}(X_{\mathrm{ad}}, \mathbf{R})$ of $\mathscr{C}(X_{\mathrm{ad}}, \mathbf{R}) = \prod_v \mathscr{C}(X_v^{\mathrm{an}}, \mathbf{R})$ consisting of families (φ_v) such that $\varphi_v \equiv 0$ for all but finitely many places $v \in S$.

The space $\mathscr{C}_{\mathrm{c}}(X_{\mathrm{ad}}, \mathbf{R})$ is the union of the subspaces $\mathscr{C}_U(X_{\mathrm{ad}}, \mathbf{R})$ of functions with (compact) support above a given finite set U of places of S. We thus endow it with its natural inductive limit topology.

Example **(3.11)** (Algebraic dynamics, Zhang (1995*b*)). — Let X be a proper $\mathbf{Q}$-scheme, let $f : X \to X$ be a morphism, let L be a line bundle on X such that $f^*L \simeq L^q$, for some integer $q \geqslant 2$. We fix such an isomorphism ε. The claim is that *there exists a unique adelic metric on L for which the isomorphism ε is an isometry.*

Let us first fix a place v and prove that there is a unique v-adic metric on L for which ε is an isometry. To that aim, let us consider, for any v-adic metric $\|\cdot\|$ on L, the induced v-adic metric on f^L and transfer it to L^q via ε. This furnishes a v-adic metric $\|\cdot\|^f$ on L such that ε is an isometry from $(L, f^*\|\cdot\|)$ to $(L, \|\cdot\|^f)^q$, and it is the unique v-adic metric on L satisfying this property. Within the real affine space of v-adic metrics on L, normed by the distance δ_v, and complete, the self-map $\|\cdot\| \mapsto \|\cdot\|^f$ is contracting with Lipschitz constant $1/q$. Consequently, the claim follows from Picard's theorem.

We also note that there exists a dense open subscheme U of $\mathrm{Spec}(\mathbf{Z})$, a model $(\mathscr{X}, \mathscr{L})$ of (X, L) over U such that $f : X \to X$ extends to a morphism $\varphi : \mathscr{X} \to \mathscr{X}$ and the isomorphism $\varepsilon : f^*L \simeq L^q$ extends to an isomorphism $\varphi^*\mathscr{L} \simeq \mathscr{L}^q$, still denoted by ε. This implies that for every finite place p above U, the canonical v-adic metric is induced by the model $(\mathscr{X}, \mathscr{L})$.

Consequently, the family $(\|\cdot\|_v)$ of v-adic metrics on L for which ε is an isometry is an adelic metric.

4. Arithmetic ampleness

Definition **(4.1)**. — *Let $\mathscr{X}$ be a proper scheme over $\mathbf{Z}$ and let $\overline{\mathscr{L}}$ be a hermitian line bundle on $\mathscr{X}$. One says that $\overline{\mathscr{L}}$ is relatively* ample *if there exists an embedding $\varphi : \mathscr{X} \hookrightarrow \mathbf{P}^N_{\mathbf{Z}}$, a metric with positive curvature on $\mathscr{O}_{\mathbf{P}^N}(1)$ and an integer $d \geqslant 1$ such that $\overline{\mathscr{L}}^d \simeq \overline{\mathscr{O}_{\mathbf{P}^N}(1)}$.*

5. Positivity

Definition **(5.1)**. — *Let $\mathscr{X}$ be a proper scheme over $\mathbf{Z}$ and let $\overline{\mathscr{L}}$ be a hermitian line bundle on $\mathscr{X}$. One says that $\overline{\mathscr{L}}$ is relatively* semipositive *if:*

(1) *For every vertical integral curve C on $\mathscr{X}$, one has $\deg_{\mathscr{L}}(C) \geqslant 0$;*
(2) *For every holomorphic map $f : \mathbf{D} \to \mathscr{X}(\mathbf{C})$, the curvature of $f^*\overline{\mathscr{L}}$ is semipositive.*

If $\overline{\mathscr{L}}$ is relatively semipositive, then $\mathscr{L}_{\mathbf{Q}}$ is nef.

Proposition **(5.2)**. — *Let X be a proper scheme over $\mathbf{Q}$; let $L_0, \dots, L_d$ be line bundles on X. Let $\mathscr{X}$, $\mathscr{X}'$ be proper flat schemes over $\mathbf{Z}$ such that $X = \mathscr{X}_{\mathbf{Q}} = \mathscr{X}'_{\mathbf{Q}}$, let $\overline{\mathscr{L}_0}, \dots, \overline{\mathscr{L}_d}$ (*resp. *$\overline{\mathscr{L}'_0}, \dots, \overline{\mathscr{L}'_d}$) be semipositive hermitian line bundles on $\mathscr{X}$* (resp. *$\mathscr{X}'$) such that $\mathscr{L}_{j,\mathbf{Q}} = \mathscr{L}'_{j,\mathbf{Q}} = L_j$; We write $\overline{L_0}, \dots, \overline{L_d}$* (resp. *$\overline{L'_0}, \dots, \overline{L'_d}$) for the corresponding adelically metrized line bundles on X. Then*

$$\Big|\widehat{\deg}\big(\widehat{c}_1(\overline{\mathscr{L}'_0})\dots\widehat{c}_1(\overline{\mathscr{L}'_d}) \mid Z\big) - \widehat{\deg}\big(\widehat{c}_1(\overline{\mathscr{L}_0})\dots\widehat{c}_1(\overline{\mathscr{L}_d}) \mid Z\big)\Big| \leqslant \sum_{j=0}^{d} \delta_j(\overline{L_j}, \overline{L'_j}) \deg\Big(c_1(L_0)\dots\widehat{c_1(L_j)}\dots c_1(L_d) \mid Z\Big),$$

where the factor $c_1(L_j)$ is omitted in the jth term.

Proof. — We first reduce to the case where $\mathscr{X} = \mathscr{X}'$ is normal. Moreover, by homogeneity, we also assume that $(\mathscr{L}_j)_{\mathbf{Q}} = (\mathscr{L}'_j)_{\mathbf{Q}}$ for every j. We then write

$$\begin{aligned}\widehat{\deg}\big(\widehat{c}_1(\overline{\mathscr{L}'_0})\dots\widehat{c}_1(\overline{\mathscr{L}'_d}) \mid Z\big) &- \widehat{\deg}\big(\widehat{c}_1(\overline{\mathscr{L}_0})\dots\widehat{c}_1(\overline{\mathscr{L}_d}) \mid Z\big)\\ &= \sum_{j=0}^{d} \widehat{\deg}\Big(\widehat{c}_1(\overline{\mathscr{L}'_0})\dots\widehat{c}_1(\overline{\mathscr{L}'_{j-1}})(\widehat{c}_1(\overline{\mathscr{L}'_j}) - \widehat{c}_1(\overline{\mathscr{L}_j}))\widehat{c}_1(\overline{\mathscr{L}_{j+1}})\dots\widehat{c}_1(\overline{\mathscr{L}_d}) \mid Z\Big)\end{aligned}$$

and bound the jth term as follows. Let s_j be the regular meromorphic section of $\mathscr{O}_X = \mathscr{L}'_j \otimes (\mathscr{L}_j)^{-1}$ corresponding to 1. By definition, one has

$$\begin{aligned}&\widehat{\deg}\Big(\widehat{c}_1(\overline{\mathscr{L}'_0})\dots\widehat{c}_1(\overline{\mathscr{L}'_{j-1}})(\widehat{c}_1(\overline{\mathscr{L}'_j}) - \widehat{c}_1(\overline{\mathscr{L}_j}))\widehat{c}_1(\overline{\mathscr{L}_{j+1}})\dots\widehat{c}_1(\overline{\mathscr{L}_d}) \mid Z\Big)\\ &= \widehat{\deg}\Big(\widehat{c}_1(\overline{\mathscr{L}'_0})\dots\widehat{c}_1(\overline{\mathscr{L}'_{j-1}})\widehat{c}_1(\overline{\mathscr{L}_{j+1}})\dots\widehat{c}_1(\overline{\mathscr{L}_d})(\widehat{c}_1(\overline{\mathscr{L}'_j}) - \widehat{c}_1(\overline{\mathscr{L}_j})) \mid Z\Big)\\ &= \widehat{\deg}\Big(\widehat{c}_1(\overline{\mathscr{L}'_0})\dots\widehat{c}_1(\overline{\mathscr{L}'_{j-1}})\widehat{c}_1(\overline{\mathscr{L}_{j+1}})\dots\widehat{c}_1(\overline{\mathscr{L}_d}) \mid \operatorname{div}(s_j|_Z)\Big)\\ &\quad + \int_{Z(\mathbf{C})} \log\|s_j\|^{-1} c_1(\overline{\mathscr{L}'_0})\dots c_1(\overline{\mathscr{L}'_{j-1}})c_1(\overline{\mathscr{L}_{j+1}})\dots c_1(\overline{\mathscr{L}_d}).\end{aligned}$$

Moreover, all components of $\mathrm{div}(s_j|_Z)$ are vertical. For every $j \in \{0, \dots, d\}$ and every $v \in S$, let $\delta_{j,v} = \delta_v(\overline{L_j}, \overline{L'_j})$ (this is zero for all but finitely many places v). Using the assumption that $|\log \|s_j\|_v| \leqslant \delta_v(\overline{L_j}, \overline{L'_j})$ for every place $v \in S$, the normality assumption on $\mathscr{X}$ implies that

$$\mathrm{div}(s_j) \leqslant \sum_{p \in S - \{\infty\}} \delta_{j,p} (\log p)^{-1} [\mathscr{X} \otimes \mathbf{F}_p].$$

Since the line bundles $\mathscr{L}_k$ and $\mathscr{L}'_k$ are semipositive, this implies the bound

$$\begin{aligned}
&\widehat{\deg}\Big(\widehat{c}_1(\overline{\mathscr{L}'_0}) \dots \widehat{c}_1(\overline{\mathscr{L}'_{j-1}}) \widehat{c}_1(\overline{\mathscr{L}_{j+1}}) \dots \widehat{c}_1(\overline{\mathscr{L}_d}) \mid \mathrm{div}(s_j|_Z)\Big) \\
&\quad = \sum_p \deg\big(c_1(\mathscr{L}'_0) \dots c_1(\mathscr{L}'_{j-1}) c_1(\mathscr{L}_{j+1}) \dots c_1(\mathscr{L}_d) \mid \mathrm{div}(s_j|_Z)\big) \log p \\
&\quad \leqslant \sum_p \delta_{j,p} \deg\big(c_1(\mathscr{L}'_0) \dots c_1(\mathscr{L}'_{j-1}) c_1(\mathscr{L}_{j+1}) \dots c_1(\mathscr{L}_d) \mid [\mathscr{X} \otimes \mathbf{F}_p]\big) \\
&\quad \leqslant \Big(\sum_p \delta_{j,p}\Big) \deg\big(c_1(L_0) \dots c_1(L_{j-1}) c_1(L_{j+1}) \dots c_1(L_d) \mid Z\big).
\end{aligned}$$

Similarly, the curvature forms $c_1(\overline{L_k})$ and $c_1(\overline{L'_k})$ are semipositive, so that the upper bound $\log \|s_j\|^{-1} \leqslant \delta_{j,\infty}$ implies

$$\begin{aligned}
&\int_{Z(\mathbf{C})} \log \|s_j\|^{-1} c_1(\overline{\mathscr{L}'_0}) \dots c_1(\overline{\mathscr{L}'_{j-1}}) c_1(\overline{\mathscr{L}_{j+1}}) \dots c_1(\overline{\mathscr{L}_d}) \\
&\qquad \leqslant \delta_{j,\infty} \int_{Z(\mathbf{C})} c_1(\overline{\mathscr{L}'_0}) \dots c_1(\overline{\mathscr{L}'_{j-1}}) c_1(\overline{\mathscr{L}_{j+1}}) \dots c_1(\overline{\mathscr{L}_d}) \\
&\qquad \leqslant \delta_{j,\infty} \deg\big(c_1(L_0) \dots c_1(L_{j-1}) c_1(L_{j+1}) \dots c_1(L_d) \mid Z\big).
\end{aligned}$$

Adding these contributions, we get one of the desired upper bound, and the other follows by symmetry. □

Definition **(5.3)**. — *An adelic metric on a line bundle L on X is said to be semipositive if it is a limit of a sequence of semipositive algebraic adelic metrics on L.*

Let $\overline{\mathrm{Pic}}^+(X)$ be the set of all isometry classes of line bundles endowed with a semipositive metric. It is submonoid of $\overline{\mathrm{Pic}}(X)$; moreover, its image in $\mathrm{Pic}(X)$ is the set of isomorphism classes of all nef line bundles on X.

Corollary **(5.4)**. — *Let Z be an integral closed subscheme of X, let $d = \dim(Z)$. The arithmetic degree maps extends uniquely to a continuous function $\overline{\mathrm{Pic}}^+(X)^{d+1} \to \mathbf{R}$. This extension is multilinear and symmetric.*

Proof. — This follows from proposition 5.2 and from the classical extension theorem of uniformly continuous maps. □

Definition **(5.5)**. — *Let X be a projective $\mathbf{Q}$-scheme and let L be a line bundle on X. An adelic metric on L is said to be* admissible *if there exists two line bundles endowed with semipositive adelic metrics, $\overline{M_1}$ and $\overline{M_2}$, such that $\overline{L} \simeq \overline{M_1} \otimes \overline{M_2}^{-1}$.*

More generally, we say that a v-adic metric on L is admissible if it is the v-adic component of an adelic metric on L The set of all adelically metrized line bundles on X is denoted by $\overline{\mathrm{Pic}}^{\mathrm{int}}(X)$; it is the subgroup generated by $\overline{\mathrm{Pic}}^{+}(X)$.

By construction, the arithmetic intersection product extends by linearity to $\overline{\mathrm{Pic}}^{\mathrm{adm}}(X)$. We use the notation $\widehat{\deg}(\widehat{c}_1(\overline{L}_0)\dots\widehat{c}_1(\overline{L}_d)\mid Z)$ for the arithmetic degree of a d-dimensional integral closed subscheme Z of X with respect to admissible adelically metrized line bundles $\overline{L}_0,\dots,\overline{L}_d$.

This gives rise to a natural notion of height parallel to that given in definition 2.3.

Example **(5.6)**. — Let us retain the context and notation of example 3.11. Let us moreover assume that L is ample and let us prove that the canonical adelic metric on L is semipositive.

We make the observation that if $\|\cdot\|$ is an algebraic adelic metric on L induced by a relatively semipositive hermitian line bundle $\overline{\mathscr{L}}$ on a proper flat model $\mathscr{X}$ of X, then the metric $\|\cdot\|^f$ is again relatively semipositive. Indeed, the normalization of $\mathscr{X}$ in the morphism $f : X \to X$ furnishes a proper flat scheme $\mathscr{X}'$ over $\mathbf{Z}$ such that $\mathscr{X}'_{\mathbf{Q}} = X$ and a morphism $\varphi : \mathscr{X}' \to \mathscr{X}$ that extends f. Then $\varphi^*\overline{\mathscr{L}}$ is a relatively semipositive hermitian line bundle on $\mathscr{X}'$, model of L^d, which induces the algebraic adelic metric $\|\cdot\|^f$ on L.

Starting from a given algebraic adelic metric induced by a relatively semipositive model (for example, a relatively ample one), the proof of Picard's theorem invoked in example 3.11 proves that the sequence of adelic metrics obtained by the iteration of the operator $\|\cdot\| \mapsto \|\cdot\|^f$ converges to the unique fixed point. Since this iteration preserves algebraic adelic metrics induced by a relatively semipositive model, the canonical adelic metric on L is semipositive, as claimed.

For a generalization of this construction, see theorem 4.9 of Yuan & Zhang (2017).

6. Measures

Definition **(6.1)**. — *Let X be a projective $\mathbf{Q}$-scheme. A function $\varphi \in \mathscr{C}(X_{\mathrm{ad}},\mathbf{R})$ is said to be admissible if the adelically metrized line bundle $\mathscr{O}_X(\varphi)$ is admissible.*

More generally, we say that a function $\varphi_v \in \mathscr{C}(X_v^{\mathrm{an}},\mathbf{R})$ is admissible if it is the v-adic component of an admissible family $\varphi = (\varphi_v)$. The set $\mathscr{C}_{\mathrm{adm}}(X_{\mathrm{ad}},\mathbf{R})$ of admissible families (φ_v) is a real subspace of $\mathscr{C}_{\mathrm{c}}(X_{\mathrm{ad}},\mathbf{R})$. One has an exact sequence

$$\Gamma(X,\mathscr{O}_X^\times) \to \mathscr{C}_{\mathrm{adm}}(X_{\mathrm{ad}},\mathbf{R}) \to \overline{\mathrm{Pic}}^{\mathrm{adm}}(X) \to \mathrm{Pic}(X) \to 0 \tag{6.1.1}$$

analogous to (3.7.1)

Proposition **(6.2)** (Gubler, 1998, theorem 7.12). — *For every place $v \in S$, the subspace $\mathscr{C}_{\mathrm{adm}}(X_v^{\mathrm{an}},\mathbf{R})$ is dense in $\mathscr{C}(X_v^{\mathrm{an}},\mathbf{R})$.*

The space $\mathscr{C}_{\mathrm{adm}}(X_{\mathrm{ad}},\mathbf{R})$ of admissible functions is dense in $\mathscr{C}_{\mathrm{c}}(X_{\mathrm{ad}},\mathbf{R})$.

Proof. — Observe that X_v^{an} is a compact topological space. By corollary 7.7 and lemma 7.8 of Gubler (1998), the subspace of $\mathscr{C}_{\mathrm{adm}}(X_v^{\mathrm{an}},\mathbf{R})$ corresponding to algebraic v-adic metrics on L separates points and is stable under sup and inf. The first part of the proposition thus follows from Stone's density theorem.

The second part follows from the first one and a straightforward argument. □

Theorem **(6.3)**. — *Let v be a place of S. Let Z be an integral closed subscheme of X, let $d = \dim(Z)$, let $\overline{L_1}, \dots, \overline{L_d}$ be admissible adelically metrized line bundles on X.*

(1) *There exists a unique measure $c_1(\overline{L_1}) \dots c_1(\overline{L_d})\delta_Z$ on X_{ad} such that*

$$\int_{X_v^{\mathrm{an}}} \varphi_0 c_1(\overline{L_1}) \dots c_1(\overline{L_d})\delta_Z = (\widehat{c}_1(\mathscr{O}_X(\varphi_0))\widehat{c}_1(\overline{L_1}) \dots \widehat{c}_1(\overline{L_d}) \mid Z)$$

for every compactly supported admissible function φ_0 on X_{ad}.

(2) *This measure is supported on Z_{ad}; its total mass is equal to*

$$\int_{X_v^{\mathrm{an}}} c_1(\overline{L_1}) \dots c_1(\overline{L_d})\delta_Z = \deg(c_1(L_1) \dots c_1(L_d) \mid Z).$$

If $\overline{L_1}, \dots, \overline{L_d}$ are semipositive, then this measure is is nonnegative.

(3) *The induced map $\overline{\mathrm{Pic}}_{\mathrm{adm}}(X)^d \to \mathscr{M}(X_{\mathrm{ad}})$ is d-linear and symmetric.*

(4) *Every admissible function is integrable for this measure.*

Proof. — Let us first assume that $\overline{L_1}, \dots, \overline{L_d}$ are semipositive. It then follows from the definition of the arithmetic intersection degrees that the map

$$\varphi_0 \mapsto (\widehat{c}_1(\mathscr{O}_X(\varphi_0))\widehat{c}_1(\overline{L_1}) \dots \widehat{c}_1(\overline{L_d}) \mid Z)$$

is a positive linear form on $\mathscr{C}_{\mathrm{adm}}(X_{\mathrm{ad}}, \mathbf{R})$. By the density theorem, it extends uniquely to a positive linear form on $\mathscr{C}_c(X_{\mathrm{ad}}, \mathbf{R})$, which then corresponds to an inner regular, locally finite, positive Borel measure on X_{ad}.

The rest of the theorem follows from this. □

Remark **(6.4)**. — (1) At archimedean places, the construction of the measure $c_1(\overline{L_1}) \dots c_1(\overline{L_d})\delta_Z$ shows that it coincides with the measure defined by Bedford & Taylor (1982) and Demailly (1985).

(2) At finite places, it has been first given in Chambert-Loir (2006). By approximation, the definition of the measure in the case of a general semipositive p-adic metric is then deduced from the case of algebraic metrics, given by a model $(\mathscr{X}, \mathscr{L})$, the measure $c_1(\overline{L_1}) \dots c_1(\overline{L_d})\delta_Z$ on X_p^{an} has finite support. Let us describe it when $Z = X$ and the model $\mathscr{X}$ (the general case follows). For each component $\mathscr{Y}$ of $\mathscr{X} \otimes \mathbf{F}_p$, there exists a unique point $y \in X_p^{\mathrm{an}}$ whose specialization is the generic point of $\mathscr{Y}$. The contribution of the point y to the measure is then equal to

$$m_{\mathscr{Y}} \deg(c_1(\mathscr{L}) \dots c_1(\mathscr{L}) \mid \mathscr{Y}),$$

where $m_{\mathscr{Y}}$ is the multiplicity of $\mathscr{Y}$ in the special fiber, that is, the length of the ideal (p) at the generic point of $\mathscr{Y}$.

Example **(6.5)**. — Let X be an abelian variety of dimension d over a number field F. Let $\overline{L}$ be an ample line bundle equipped with a canonical adelic metric; let us then describe the measure $c_1(\overline{L})^d$ on X_v^{an}, for every place $v \in S$. For simplicity, we assume that $F = \mathbf{Q}$.

(1) First assume $v = \infty$. Then X_∞^{an} is the quotient, under complex conjugation, of the complex torus $X(\mathbf{C})$, and the canonical measure on X_∞^{an} is the direct image of the unique Haar measure on $X(\mathbf{C})$ with total mass $\deg(c_1(L)^d \mid X)$.

(2) The situation is more interesting in the case of a finite place p.

If X has good reduction at p, that is, if it extends to an abelian scheme $\mathscr{X}$ over $\mathbf{Z}_p$, then the canonical measure is supported at the unique point of X_p^{an} whose specialization is the generic point of $\mathscr{X} \otimes \mathbf{F}_p$.

Let us assume, on the contrary, that X has (split) totally degenerate reduction. In this case, the uniformization theory of abelian varieties shows that X_p^{an} is the quotient of a torus $(\mathbf{G}_{\mathrm{m}}{}^d)^{\mathrm{an}}$ by a lattice Λ. The definition of $(\mathbf{G}_{\mathrm{m}}{}^d)^{\mathrm{an}}$ shows that this analytic space contains a canonical d-dimensional real vector space V, and V/Λ is a real d-dimensional torus $S(X_p^{\mathrm{an}})$ contained in X_p^{an}, sometimes called its skeleton. Gubler (2007) has shown that the measure $c_1(\overline{L})^d$ on X_p^{an} coincides with the Haar measure on $S(X_p^{\mathrm{an}})$ with total mass $\deg(c_1(L)^d \mid X)$.

The general case is a combination of these two cases.

Remark **(6.6)**. — At finite places, the theory described in this section defines measures $c_1(\overline{L}_1) \dots c_1(\overline{L}_d)\delta_Z$ without defininining the individual components $c_1(\overline{L}_1), \dots, c_1(\overline{L}_d), \delta_Z$.

In Chambert-Loir & Ducros (2012), we propose a theory of real differential forms and currents on Berkovich analytic spaces that allows a more satisfactory analogy with the theory at complex spaces. In particular, we provide an analogue of the Poincaré–Lelong equation, and a semipositive metrized line bundle possesses a curvature current (curvature form in the ''smooth'' case) whose product can be defined and coincides with the measure.

7. Volumes

7.1. — Let X be a proper $\mathbf{Q}$-scheme and let $\overline{L}$ be a line bundle endowed with an adelic metric.

The Riemann-Roch space $\mathrm{H}^0(X, L)$ is a finite dimensional $\mathbf{Q}$-vector space. For every place $v \in S$, we endow it with a v-adic semi-norm:

$$\|s\|_v = \sum_{x \in X_v^{\mathrm{an}}} \|s(x)\|$$

for $s \in \mathrm{H}^0(X, L)$. If X is reduced, then this is a norm; let then B_v be its unit ball.

Let $\mathbf{A}$ be the ring of adeles of $\mathbf{Q}$ and let μ be a Haar measure on $\mathrm{H}^0(X, L) \otimes \mathbf{A}$. Then $\prod_{v \in S} B_v$ has finite positive volume in $\mathrm{H}^0(X, L)_{\mathbf{A}}$, and one defines

$$\chi(X, \overline{L}) = -\log\left(\frac{\mu(\mathrm{H}^0(X, L) \otimes \mathbf{A}/\mathrm{H}^0(X, L))}{\prod_v \mu(B_v)}\right). \tag{7.1.1}$$

This does not depend on the choice of the Haar measure μ.

One also defines

$$\widehat{\mathrm{H}}^0(X, \overline{L}) = \{s \in \mathrm{H}^0(X, L)\,;\, \|s\|_v \leqslant 1 \text{ for all } v \in S\}. \tag{7.1.2}$$

This is a finite set. We then let

$$\widehat{\mathrm{h}}^0(X,\overline{L}) = \log\bigl(\mathrm{Card}(\widehat{\mathrm{H}}^0(X,\overline{L}))\bigr). \tag{7.1.3}$$

Lemma **(7.2)**. — *One has*

$$\chi(X,\overline{L}) \leqslant \widehat{\mathrm{h}}^0(X,\overline{L}).$$

Proof. — This follows from the adelic version of Minkowski's first theorem of Bombieri & Vaaler (1983). □

7.3. — The volume and the χ-volume of $\overline{L}$ are defined by the formulas

$$\widehat{\mathrm{vol}}(X,\overline{L}) = \limsup_{n\to\infty} \frac{\widehat{\mathrm{h}}^0(X,\overline{L}^n)}{n^{d+1}/(d+1)!} \tag{7.3.1}$$

$$\widehat{\mathrm{vol}}_\chi(X,\overline{L}) = \limsup_{n\to\infty} \frac{\chi(X,\overline{L}^n)}{n^{d+1}/(d+1)!}. \tag{7.3.2}$$

One thus has the inequality

$$\widehat{\mathrm{vol}}_\chi(X,\overline{L}) \leqslant \widehat{\mathrm{vol}}(X,\overline{L}). \tag{7.3.3}$$

In fact, it has been independently shown by Yuan (2009) and Chen (2010) that the volume is in fact a limit.

The relation between volumes and heights follows from the following result.

Lemma **(7.4)**. — [2] *Assume that L is big. Then, for every real number $t < \widehat{\mathrm{vol}}_\chi(X,\overline{L})/(d+1)\,\mathrm{vol}(X,L)$, the set of closed points $x \in X$ such that $h_{\overline{L}}(x) \leqslant t$ is not dense for the Zariski topology.*

Proof. — Consider the adelically metrized line bundle $\overline{L}(-t)$, whose metric at the archimedean place has been multiplied by e^t. It follows from ahe definition of the χ-volume that

$$\widehat{\mathrm{vol}}_\chi(X,\overline{L}(-t)) = \widehat{\mathrm{vol}}_\chi(X,\overline{L}) - (d+1)t\,\mathrm{vol}(X,L).$$

Indeed, for every finite place p, changing $\overline{L}$ to $\overline{L}(-t)$ does not modify the balls B_p in $\mathrm{H}^0(X,\overline{L}^n)\otimes_{\mathbf{Q}}\mathbf{Q}_p$, while it dilates it by the ratio e^{nt} at the archimedean place, so that its volume is multiplied by $e^{nt\dim(\mathrm{H}^0(X,L^n))}$.

Consequently,

$$\widehat{\mathrm{vol}}(X,\overline{L}(-t)) \geqslant \widehat{\underset{\chi}{\mathrm{vol}}}(X,\overline{L}(-t)) \geqslant \widehat{\underset{\chi}{\mathrm{vol}}}(X,\overline{L}) - (d+1)t\,\mathrm{vol}(X,L) > 0.$$

In particular, there exists an integer $n \geqslant 1$ and a nonzero section $s \in \mathrm{H}^0(X,\overline{L}^n)$ such that $\|s\|_p \leqslant 1$ for all finite places p, and $\|s\|_\infty \leqslant e^{-nt}$. Let now $x \in X$ be a closed point that is not contained in $|\mathrm{div}(s)|$; one then has

$$h_{\overline{L}}(x) = \sum_{v\in S}\int_{X_v^{\mathrm{an}}} \log\|s\|_v^{-1/n}\,\delta_v(x) \geqslant t,$$

whence the lemma. □

[2] Is there a version for $\widehat{\mathrm{vol}}$?

Theorem **(7.5)**. — *Assume that $\overline{L}$ is semipositive. Then one has*

$$\widehat{\mathrm{vol}}(X,\overline{L}) = \widehat{\mathrm{vol}}_\chi(X,\overline{L}) = \widehat{\deg}\left(\widehat{c}_1(\overline{L})^{d+1} \mid X\right). \tag{7.5.1}$$

This is the arithmetic Hilbert–Samuel formula, due to Gillet & Soulé (1988); Bismut & Vasserot (1989) when $X_{\mathbf{Q}}$ is smooth and the adelic metric of $\overline{L}$ is algebraic. Abbes & Bouche (1995) later gave an alternative proof. In the given generality, the formula is a theorem of Zhang (1995*a*,*b*).

Theorem **(7.6)**. — (1) *The function $\overline{L} \mapsto \widehat{\mathrm{vol}}(X,\overline{L})$ extends uniquely to a continuous function on the real vector space $\widehat{\mathrm{Pic}}(X) \otimes_{\mathbf{Q}} \mathbf{R}$.*

(2) *If $\widehat{\mathrm{vol}}(X,\overline{L}) > 0$, then $\widehat{\mathrm{vol}}$ is differentiable at $\overline{L}$.*

(3) *If $\overline{L}$ is semipositive, then $\widehat{\mathrm{vol}}$ and $\widehat{\mathrm{vol}}_\chi$ are differentiable at $\overline{L}$, with differential*

$$\overline{M} \mapsto (d+1)\,\widehat{\deg}(\widehat{c}_1(\overline{L})^d \widehat{c}_1(\overline{M}) \mid X).$$

[Add indications and references.]

8. Equidistribution

The main result of this section is the equidistribution theorem 8.4. It has been first proved in the case $v = \mathbf{C}$ by Szpiro *et al* (1997), under the assumption that the given archimedean metric is smooth and has a strictly positive curvature form, and the general case is due to Yuan (2008). However, our presentation derives it from a seemingly more general result, lemma8.2, whose proof, anyway, closely follows their methods. Note that for the application to Bogomolov's conjecture in §9, the initial theorem of Szpiro *et al* (1997) is sufficient.

Definition **(8.1)**. — *Let X be a proper $\mathbf{Q}$-scheme, let $\overline{L}$ be a big line bundle on X endowed with an admissible adelically metric. Let (x_n) be a sequence of closed points of X.*

(1) *One says that (x_n) is* generic *if for every strict closed subscheme Z of X, the set of all $n \in \mathbf{N}$ such that $x_n \in Z$ is finite; in other words, this sequence converges to the generic point of X.*

(2) *One says that (x_n) is* small *(relative to $\overline{L}$) if*

$$h_{\overline{L}}(x_n) \to h_{\overline{L}}(X).$$

Lemma **(8.2)**. — *Let X be a proper $\mathbf{Q}$-scheme, let $d = \dim(X)$, let $\overline{L}$ be a semipositive adelically metrized line bundle on X such that L is big. Let (x_n) be a generic sequence of closed points of X which is small relative to $\overline{L}$. For every line bundle $\overline{M}$ on X endowed with an admissible adelic metric, one has*

$$\lim_{n\to\infty} h_{\overline{M}}(x_n) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{L})^d \widehat{c}_1(\overline{M}) \mid X)}{\deg_L(X)} - \frac{d}{d+1} h_{\overline{L}}(X) \frac{\deg(c_1(L)^{d-1}c_1(M) \mid X)}{\deg_L(X)^2}.$$

Proof. — Since L is ample, $L^t \otimes M$ is ample for every large enough integer t, and the classical Hilbert-Samuel formula implies that

$$\frac{1}{t^d}\mathrm{vol}(X, L^t \otimes M) = \deg(c_1(L)^d \mid X) + dt^{-1}\deg(c_1(L)^{d-1}c_1(M) \mid X) + \mathrm{O}(t^{-2})$$

when $t \to \infty$. Since $\overline{L}$ is semipositive and L is ample, the main inequality of Yuan (2008) implies that

$$\frac{1}{t^{d+1}} \widehat{\mathrm{vol}}_\chi(X, \overline{L}^t \otimes \overline{M}) \geqslant \widehat{\mathrm{vol}}_\chi(X, \overline{L}) + (d+1) t^{-1} \widehat{\deg}(\widehat{c}_1(\overline{L})^d \widehat{c}_1(\overline{M}) \mid X) + \mathrm{O}(t^{-2}).$$

Consequently, when $t \to \infty$, one has

$$\begin{aligned}\frac{\widehat{\mathrm{vol}}_\chi(X, \overline{L}^t \otimes M)}{(d+1)\,\mathrm{vol}(X, L^t \otimes M)} \geqslant t \frac{\widehat{\mathrm{vol}}_\chi(X, \overline{L})}{(d+1)\,\mathrm{vol}(X, L)} + \\ + (d+1) \frac{\widehat{\deg}(\widehat{c}_1(\overline{L})^d \widehat{c}_1(\overline{M}) \mid X)}{\deg(c_1(L)^d \mid X)} \\ - d \frac{\widehat{\deg}(\widehat{c}_1(\overline{L})^{d+1} \mid X)}{\deg(c_1(L)^d \mid X)} \frac{\deg(c_1(L)^{d-1} c_1(M) \mid X)}{\deg(c_1(L)^d \mid X)} + \mathrm{O}(1/t).\end{aligned}$$

In order to apply the inequality

$$\liminf_n h_{\overline{L}^t \otimes \overline{M}} \frac{\widehat{\mathrm{vol}}_\chi(X, \overline{L}^t \otimes M)}{(d+1)\,\mathrm{vol}(X, L^t \otimes M)},$$

we observe that

$$\liminf_n h_{\overline{L}^t \otimes \overline{M}} = t \lim h_{\overline{L}}(x_n) + t \liminf_n h_{\overline{M}}(x_n).$$

When $t \to \infty$, we then obtain

$$\begin{aligned}\liminf_n h_{\overline{M}}(x_n) \geqslant (d+1) \frac{\widehat{\deg}(\widehat{c}_1(\overline{L})^d \widehat{c}_1(\overline{M}) \mid X)}{\deg(c_1(L)^d \mid X)} \\ - d \frac{\widehat{\deg}(\widehat{c}_1(\overline{L})^{d+1} \mid X)}{\deg(c_1(L)^d \mid X)} \frac{\deg(c_1(L)^{d-1} c_1(M) \mid X)}{\deg(c_1(L)^d \mid X)}.\end{aligned}$$

Applying this inequality for $\overline{M}^{-1}$ shows that $\limsup_n h_{\overline{M}}(x_n)$ is bounded above by its right hand side. The lemma follows. □

8.3. — Let X be a proper $\mathbf{Q}$-scheme. Let $v \in S$ be a place of $\mathbf{Q}$.

Let $x \in X$ be a closed point. Let $F = \kappa(x)$; this is a finite extension of $\mathbf{Q}$, and there are exactly $[F : \mathbf{Q}]$ geometric points on $X(\mathbf{C}_v)$ whose image is x, permuted by the Galois group $\mathrm{Gal}(\mathbf{C}_v/\mathbf{Q}_v)$. We consider the corresponding ''probability measure'' in $X(\mathbf{C}_v)$, giving mass $1/[F : \mathbf{Q}]$ to each of these geometric points, and let $\delta_v(x)$ be its image under the natural map $X(\mathbf{C}_p) \to X_v^{\mathrm{an}}$.

By construction, $\delta_v(x)$ is a probability measure on X_v^{an} with finite support, a (rigid) point of X_v^{an} being counted proportionaly to the number of its liftings to a geometric point supported by x.

Theorem **(8.4)**. — *Let X be a proper $\mathbf{Q}$-scheme, let $d = \dim(X)$, let $\overline{L}$ be a semipositive adelically metrized line bundle on X such that L is big. Let (x_n) be a generic sequence of closed points of X which is small relative to $\overline{L}$. Then for each place $v \in S$, the sequence of measures $(\delta_v(x_n))$ on X_v^{an} converges to the measure $c_1(\overline{L})^d / \deg(c_1(L)^d \mid X)$.*

Proof. — Let $\mu_{\overline{L}}$ denote the probability measure $c_1(\overline{L})^d/\deg_L(X)$ on X_v^{an} and let $f \in \mathscr{C}(X_v^{\mathrm{an}}, \mathbf{R})$ be an admissible function, extended by zero to an element of $\mathscr{C}_{\mathrm{adm}}(X_{\mathrm{ad}}, \mathbf{R})$. We apply lemma 8.2 to the metrized line bundle $\overline{M} = \mathscr{O}_X(f)$. For every closed point $x \in X$, one has

$$h_{\overline{M}}(x) = \int_{X_v^{\mathrm{an}}} f\delta_v(x).$$

Moreover,

$$\widehat{\deg}(\widehat{c}_1(\overline{L})^d\widehat{c}_1(\overline{M}) \mid X) = \frac{1}{\deg_L(X)}\int_{X_v^{\mathrm{an}}} fc_1(\overline{L})^d.$$

It thus follows from lemma 8.2 that

$$\lim_{n\to\infty}\int_{X_v^{\mathrm{an}}} f\delta_v(x_n) = \frac{1}{\deg_L(X)}\int_{X_v^{\mathrm{an}}} fc_1(\overline{L})^d.$$

The case of an arbitrary continuous function on X_v^{an} follows by density. □

9. The Bogomolov conjecture

9.1. — Let X be an abelian variety over a number field F and let L be a line bundle on X trivialized at the origin. Let us first explain how the theory of canonical adelic metrics allows to extend the Néron–Tate height to arbitrary integral closed subschemes. For alternative and independent presentations, see Philippon (1991), Gubler (1994), Bost *et al* (1994).

If L is even ($[-1]^*L \simeq L$), then it admits a unique adelic metric for which the canonical isomorphism $[n]^*L \simeq L^{n^2}$ is an isometry, for every integer n. Similarly, if L is odd ($[-1]^*L \simeq L^{-1}$), then it admits a unique adelic metric for which the canonical isomorphism $[n]^*L \simeq L^n$ is an isometry, for every integer n. In general, one can write $L^2 \simeq (L \otimes [-1]^*L) \otimes (L \otimes [-1]^*L^{-1})$, as the sum of an even and an odd line bundle, and this endows L with an adelic metric. This adelic metric is called the *canonical adelic metric* on L (compatible with the given trivialization at the origin).

If L is ample and even, then the canonical adelic metric on L is semipositive. This implies that the canonical adelic metric of an arbitrary even line bundle is admissible.

Assume that L is odd. Fix an even ample line bundle M. Up to extending the scalars, there exists a point $a \in X(F)$ such that $L \simeq \tau_a^*M \otimes M^{-1}$, where τ_a is the translation by a on X. Then there exists a unique isomorphism $L \simeq \tau_a^*M \otimes M^{-1} \otimes M_a^{-1}$ which is compatible with the trivialization at the origin, and this gives rise to an isometry $\overline{L} \simeq \tau_a^*\overline{M} \otimes \overline{M}^{-1} \otimes \overline{M}_a^{-1}$. In particular, the adelic metric of $\overline{M}$ is admissible. In fact, it follows from a construction of Künnemann that it is even semipositive, see Chambert-Loir (1999).

9.2. — In particular, let us consider an ample even line bundle L on X endowed with a canonical adelic metric. This furnishes a *height*

$$h_{\overline{L}}(Z) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{L})^{d+1} \mid Z)}{(d+1)\deg(c_1(L)^d \mid Z)},$$

for every integral closed subscheme Z of X, where $d = \dim(Z)$.

In fact, if $(\mathscr{X}, \overline{\mathscr{L}})$ is any model of (X, L), one has

$$h_{\overline{L}}(Z) = \lim_{n\to+\infty} n^{-2} h_{\overline{\mathscr{L}}}([n](Z)),$$

which shows the relation of the point of view of adelic metrics with Tate's definition of the Néron–Tate height, initially defined on closed points. This formula also implies that $h_{\overline{L}}$ is nonnegative.

More generally, if Z is an integral closed subscheme of $X_{\overline{F}}$, we let $h_{\overline{L}}(Z) = h_{\overline{L}}([Z])$, where $[Z]$ is its Zariski-closure in X (more precisely, the smallest closed subscheme of X such that $[Z]_{\overline{F}}$ contains Z).

Lemma **(9.3)**. — *The induced height function $h_{\overline{L}} : X(\overline{F}) \to \mathbf{R}$ is a positive quadratic form. It induces a positive definite quadratic form on $X(\overline{F}) \otimes \mathbf{R}$. In particular, a point $\in X(\overline{F})$ satisfies $h_{\overline{L}}(x) = 0$ if and only if x is a torsion point.*

Proof. — For $I \subset \{1, 2, 3\}$, let $p_I : X^3 \to X$ be the morphism given by $p_I(x_1, x_2, x_3) = \sum_{i\in I} x_i$. The cube theorem asserts that the line bundle

$$\mathscr{D}_3(L) = \bigotimes_{\varnothing\neq I\subset\{1,2,3\}} (p_I^* L)^{(-1)^{\mathrm{Card}(I)-1}}$$

on X^3 is trivial, and admits a canonical trivialisation. The adelic metric of $\overline{L}$ endows it with an adelic metric which satisfies $[2]^*\mathscr{D}_3(\overline{L}) \simeq \mathscr{D}_3(\overline{L})^4$, hence is the trivial metric. This implies the following relation on heights:

$$h_{\overline{L}}(x+y+z) - h_{\overline{L}}(y+z) - h_{\overline{L}}(x+z) - h_{\overline{L}}(x+y) + h_{\overline{L}}(x) + h_{\overline{L}}(y) + h_{\overline{L}}(z) \equiv 0$$

on $X(\overline{F})^3$. Consequently,

$$(x, y) \mapsto h_{\overline{L}}(x+y) - h_{\overline{L}}(x) - h_{\overline{L}}(y)$$

is a symmetric bilinear form on $X(\overline{F})$. Since it is even, $h_{\overline{L}}$ is a quadratic form on $X(\overline{F})$.

Since L is ample, $h_{\overline{L}}$ is bounded from below. The formula $h_{\overline{L}}(x) = h_{\overline{L}}(2x)/4$ then implies that $h_{\overline{L}}$ is nonnegative. By what precedes, it induces a positive quadratic form on $X(\overline{F})_{\mathbf{R}}$.

Let us prove that it is in fact positive definite. By definition, it suffices that its restriction to the subspace generated by finitely many points $x_1, \dots, x_m \in X(\overline{F})$ is positive definite. Let E be a finite extension of F such that $x_1, \dots, x_m \in X(E)$. On the other hand, Northcott's theorem implies that for every real number t, the set of $(a_1, \dots, a_m) \in \mathbf{Z}^m$ such that $h_{\overline{L}}(a_1x_1 + \cdots + a_mx_m) \leqslant t$ is finite. One deduces from that the asserted positive definiteness. □

Definition **(9.4)**. — *A torsion subvariety of $X_{\overline{F}}$ is a subvariety of the form $a + Y$, where $a \in X(\overline{F})$ is a torsion point and Y is an abelian subvariety of $X_{\overline{F}}$.*

Theorem **(9.5)**. — a) *Let Z be an integral closed subscheme of $X_{\overline{F}}$. One has $h_{\overline{L}}(Z) = 0$ if and only if Z is a torsion subvariety of $X_{\overline{F}}$.*

b) *Let Z be an integral closed subscheme of $X_{\overline{F}}$ which is not a torsion subvariety. There exists a positive real number δ such that the set*

$$\{x \in Z(\overline{F}) \,;\, h_{\overline{L}}(x) \leqslant \delta\}$$

is not Zariski-dense in $Z_{\overline{F}}$.

Assertion *a*) has been independently conjectured by Philippon (1991, 1995) and Zhang (1995*b*). Assertion *b*) has been conjectured by Bogomolov (1980) in the particular case where Z is a curve of genus $g \geqslant 2$ embedded in its jacobian variety; for this reason, it is called the ''generalized Bogomolov conjecture''. The equivalence of *a*) and *b*) is a theorem of Zhang (1995*b*). In fact, the implication *b*)⇒*a*) follows from theorem 7.5 and lemma 7.4.

Theorem 9.5 has been proved by Zhang (1998), following a breakthrough of Ullmo (1998) who treated the case of a curve embedded in its jacobian; their proof makes use of the equidistribution theorem. Soon after, David & Philippon (1998) gave an alternative proof; when Z is not a translate of an abelian subvariety, their proof provides a positive lower bound for $h_{\overline{L}}(Z)$ (in *a*)) as well as an explicit real number δ (in *b*)) which only depends on the dimension and the degree of Z with respect to L.

As a corollary of theorem 9.5, one obtains a new proof of the Manin–Mumford conjecture in characteristic zero, initially proved by Raynaud (1983).

Corollary **(9.6)**. — *Let X be an abelian variety over an algebraically closed field of characteristic zero, let Z be an integral closed subscheme of X which is not a torsion subvariety. Then the set of torsion points of X which are contained in Z is not Zariski-dense in X.*

Proof. — A specialization argument reduces to the case where X is defined over a number field F. In this case, the torsion points of X are defined over $\overline{F}$ and are characterized by the vanishing of their Néron–Tate height relative to an(y) ample line bundle L on X. It is thus clear that the corollary follows from theorem 9.5, *b*). □

9.7. — The proof of theorem 9.5, *b*), begins with the observation that the statement does not depend on the choice of the ample line L on X. More precisely, if $\overline{M}$ is another symmetric ample line bundle on X endowed with a canonical metric, then there exists an integer $a \geqslant 1$ such that $\overline{L}^a \otimes \overline{M}^{-1}$ is ample, as well as $\overline{M}^a \otimes \overline{L}^{-1}$. Consequently, $h_{\overline{L}} \geqslant a^{-1} h_{\overline{M}}$ and $h_{\overline{M}} \geqslant a^{-1} h_{\overline{L}}$. From these two inequalities, one deduces readily that the statement holds for $\overline{L}$ if and only if it holds for $\overline{M}$.

For a similar reason, if $f : X' \to X$ is an isogeny of abelian varieties, then the statements for X and X' are equivalent. Let indeed Z be an integral closed subvariety of $X_{\overline{F}}$ and let Z' be an irreducible component of $f^{-1}(Z)$. Then Z is a torsion subvariety of $X_{\overline{F}}$ if and only if Z' is a torsion subvariety of $X'_{\overline{F}}$. On the other hand, the relation $h_{f^*\overline{L}}(x) = h_{\overline{L}}(f(x))$ shows that $h_{f^*\overline{L}}$ has a strictly positive lower bound on Z' outside of a strict closed subset E' if and only if $h_{\overline{L}}$ has a strictly positive lower bound on Z outside of the strict closed subset $f(E')$.

9.8. — Building on that observation, one reduces the proof of the theorem to the case where the stabilizer of Z is trivial.

Let indeed X'' be the neutral component of this stabilizer and let $X' = X/X''$; this is an abelian variety. By Poincaré's complete reducibility theorem, there exists an isogeny $f : X' \times X'' \to X$. This reduces us to the case where $X = X' \times X''$ and $Z = Z' \times X''$, for some integral closed subscheme Z' of $X'_{\overline{F}}$. We may also assume that $\overline{L} = \overline{L}' \boxtimes \overline{L}''$. It it then clear that the statement for (X', Z') implies the desired statement for (X, Z).

Lemma **(9.9)**. — *Assume that* $\dim(Z) > 0$ *and that its stabilizer is trivial. Then, for every large enough integer* $m \geqslant 1$*, the morphism*

$$f : Z^m \to X_{\overline{F}}^{m-1}, \qquad (x_1, \dots, x_m) \mapsto (x_2 - x_1, \dots, x_m - x_{m-1})$$

is birational onto its image but not finite.

Proof. — For $x \in Z(\overline{F})$, write $Z_x = Z \cap (Z - x)$. Let $x = (x_1, \dots, x_m)$ be an $\overline{F}$-point of Z^m. Then a point $y = (y_1, \dots, y_m) \in Z(\overline{F})^m$ belongs to the fiber of x if and only if $y_2 - y_1 = x_2 - x_1, \dots$, that is, if and only if, $y_1 - x_1 = y_2 - x_2 = \cdots = y_m - x_m$. This identifies $f^{-1}(f(x))$ with the intersection $Z_{x_1} \cap \dots \cdots \cap Z_{x_m}$. If m is large enough and $x_1, \dots, x_m$ are well chosen, this intersection is equal to stabilizer of Z in $X_{\overline{F}}$, hence is reduced to a point. This shows that the morphism f is generically injective.

On the other hand, the preimage of the origin $(o, \dots, o)$ contains the diagonal of Z^m, which has strictly positive dimension by hypothesis. □

9.10. — For the proof of theorem 9.5, *b*), we now argue by contradiction and assume the existence of a generic sequence (x_n) in $Z(\overline{F})$ such that $h_{\overline{L}}(x_n) \to 0$.

Having reduced, as explained above, to the case where the stabilizer of Z is trivial, we consider an integer $m \geqslant 1$ such that the morphism $f : Z^m \to X_{\overline{F}}^{m-1}$ is birational onto its image, but not finite.

Since the set of strict closed subschemes of Z is countable, one can construct a generic sequence (y_n) in Z^m where y_n is of the form $(x_{i_1}, \dots, x_{i_m})$. One has $h_{\overline{L}}(y_n) \to 0$, where, by abuse of language, we write $h_{\overline{L}}$ for the height on X^m induced by the adelically metrized line bundle $\overline{L} \boxtimes \dots \overline{L}$ on X^m. This implies that $h_{\overline{L}}(Z) = 0$, hence the sequence (y_n) is *small*.

For every integer n, let $z_n = f(y_n)$. By continuity of a morphism of schemes, the sequence (z_n) is generic in $f(Z^m)$. Moreover, we deduce from the quadratic character of the Néron–Tate height $h_{\overline{L}}$ that $h_{\overline{L}}(z_n) \to 0$. In particular, $h_{\overline{L}}(f(Z^m)) = 0$, and the sequence (z_n) is small.

Fix an archimedean place σ of F. Applied to the sequences (y_n) and (z_n), the equidistribution theorem 8.4 implies the following convergences:

$$\lim_{n\to\infty} \delta_\sigma(y_n) \propto c_1(\overline{L} \boxtimes \cdots \boxtimes \overline{L})^{m \dim(Z)} \delta_{Z^m}$$
$$\lim_{n\to\infty} \delta_\sigma(z_n) \propto c_1(\overline{L} \boxtimes \cdots \boxtimes \overline{L})^{\dim(f(Z^m))} \delta_{f(Z^m)},$$

where, by $\propto$, I mean that both sides are proportional. (The proportionality ratio is the degree of Z^m, *resp.* of $f(Z^m)$, with respect to the indicated measure.) Since $f(y_n) = z_n$, we conclude that the measures

$$f_* c_1(\overline{L} \boxtimes \cdots \boxtimes \overline{L})^{m \dim(Z)} \delta_{Z^m} \quad \text{and} \quad c_1(\overline{L} \boxtimes \cdots \boxtimes \overline{L})^{\dim(f(Z^m))} \delta_{f(Z^m)}$$

on $f(Z^m)$ are proportional.

Recall that the archimedean metric of $\overline{L}$ has the property that it is smooth and that its curvature form $c_1(\overline{L})$ is a smooth positive $(1,1)$-form on $X_\sigma(\mathbf{C})$. Consequently, on a dense smooth open subscheme of $f(Z^m)$ above which f is an isomorphism, both measures are

given by differential forms, which thus coincide there. We can pull back them to Z^m by f and obtain a proportionality of differential forms

$$c_1(\overline{L} \boxtimes \cdots \boxtimes \overline{L})^{m \dim(Z)} \propto f^* c_1(\overline{L} \boxtimes \cdots \boxtimes \overline{L})^{m \dim(Z)}$$

on $Z_\sigma(\mathbf{C})^m$. At this point, the contradiction appears: the differential form on the left is strictly positive at every point, while the one on the right vanishes at every point of $Z_\sigma^m(\mathbf{C})$ at which f is not smooth.

This concludes the proof of theorem 9.5.

Remark **(9.11)**. — The statement of 9.5 can be asked in more general contexts that allow for canonical heights. The case of toric varieties has been proved by Zhang (1995*a*), while in that case the equidistribution result is first due to Bilu (1997). The case of semiabelian varieties is due to David & Philippon (2002), by generalization of their proof for abelian varieties; I had proved in Chambert-Loir (1999) the equidistribution result for almost-split semi-abelian varieties, but the general case is still open.

The general setting of algebraic dynamics (X, f) is unclear. For a polarized dynamical system as in 3.11, the obvious and natural generalization proposed in Zhang (1995*b*) asserts that subvarieties of height zero are exactly those whose forward orbit is finite. However, Ghioca and Tucker have shown that it does not hold; see Ghioca *et al* (2011) for a possible rectification. The case of dominant endomorphisms of $\mathbf{P}_1^n$ is a recent theorem of Ghioca *et al* (2017).

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