The height of CM points on orthogonal Shimura varieties and Colmez conjecture

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1 Introduction

These are the notes for a 6 hours course given by the author during the summer school "Géométrie d'Arakelov", organized by the Institut Fourier (Grenoble) in the summer of 2017. They are based on the paper "Faltings Heights of Abelian Varieties with Complex Multiplication" [AGHMP2] by myself, Eyal Goren, Ben Howard and Keerthi Madapusi Pera and on notes by myself and Eyal Goren. No new results are presented. The goal is to describe the strategy to reduce the proof of an averaged version of Colmez's conjecture to a conjecture of Bruinier, Kudla and Yang, an instance of what is known as the *Kudla's programme*. Note that this weaker version of Colmez's conjecture has been used by Tsimerman [Ts] to provide an unconditional proof of the André-Ort conjecture for abelian varieties of Hodge type. Around the same time as [AGHMP2] also X. Yuan and S.-W. Zhang [YZ] proved, using different techniques, the averaged form of Colmez's conjecture.

2 The average Colmez conjecture

Let $E \subset \mathbb{C}$ be a CM field of degree $2d$ with totally real subfield F of degree g. Let A be an abelian variety over $\mathbb{C}$ of dimension d with action of the ring of integers $\mathcal{O}_E$ of E. One knows that A can be defined over a number field K and, since A has potentially everywhere good reduction, we may further assume that A extends to an abelian scheme $\mathcal{A}$ over the ring of integers $\mathcal{O}_K$ of K. We denote by $\omega_{\mathcal{A}}$ the $\wedge^d$-power of the invariant differentials of A. It is a projective $\mathcal{O}_K$-module of rank 1. Given a generator s of $\omega_{\mathcal{A}} \otimes_{\mathcal{O}_K} K$ define

$$h_\infty^{\mathrm{Falt}}(A,s) = \frac{-1}{2[K:\mathbb{Q}]} \sum_{\sigma\colon K\to\mathbb{C}} \log \| \int_{A^\sigma(\mathbb{C})} s^\sigma \wedge \overline{s^\sigma} \|.$$

here for every embedding $\sigma\colon K \to \mathbb{C}$ we let A^σ and s^σ be the base change of A and the section s to $\mathbb{C}$. When we write $A^\sigma(\mathbb{C})$ we consider the underlying complex analytic structure so that the integral makes sense. Define

$$h_f^{\mathrm{Falt}}(A,s)=\frac{1}{[K:\mathbb{Q}]}\sum_{\mathfrak{p}\subset\mathcal{O}_K}\mathrm{ord}_{\mathfrak{p}}(s)\cdot\log \mathrm{N}(\mathfrak{p}).$$

where $\mathrm{ord}_{\mathfrak{p}}(s)$ is defined as the order with respect to a generator of $\omega_{\mathcal{A}}\otimes_{\mathcal{O}_K}\mathcal{O}_{K,\mathfrak{p}}$ as $\mathcal{O}_{K,\mathfrak{p}}$-module. Finally define

$$h^{\mathrm{Falt}}(A)=h_f^{\mathrm{Falt}}(A,s)+h_\infty^{\mathrm{Falt}}(A,s),$$

the *Faltings height* or *modular height* of A. As the notation suggests it is independent of the choice of the section s thanks to the product formula. It is the arithmetic degree of the metrized line bundle $\omega_{\mathcal{A}}$ over $\mathrm{Spec}(\mathcal{O}_K)$ where the metrics at infinity are defined using integration as above. Thanks to the normalization factor $\frac{1}{[K:\mathbb{Q}]}$ it is invariant under field extension.

The fact that A has an action of $\mathcal{O}_E$ singles out a subset $\Phi\subset\mathrm{Hom}(E,\mathbb{C})$: the action of E on $\mathrm{H}^0(A,\Omega^1_{A/\mathbb{C}})$ decomposes into a sum of d one dimensional eigenspaces for the action of E and on each of them E acts via an embedding $E\to\mathbb{C}$. We let Φ be the subset of embeddings $E\subset\mathbb{C}$ appearing in this way. It is called the *CM type* of A. It is a subset of cardinality d and $\mathrm{Hom}(E,\mathbb{C})=\Phi\amalg\overline{\Phi}$ (here $\overline{\Phi}$ stands for the image of Φ under complex conjugation on $\mathbb{C}$) . We then have the following Theorem of Colmez, [Col, Théorème 0.3]:

Theorem 2.1. *Under the assumption that A is an abelian variety of dimension d with action of the ring of integers $\mathcal{O}_E$ of E and with CM type Φ, the Faltings height $h^{\mathrm{Falt}}(A)$ depends only on the pair (E,Φ), and not on the choice of the abelian variety A.*

We write $h^{\mathrm{Falt}}_{(E,\Phi)}$ for the quantity $h^{\mathrm{Falt}}(A)$ as it is independent of the choice of A.

In the same paper Colmez provided a conjectural formula that computes $h^{\mathrm{Falt}}_{(E,\Phi)}$ in terms special values of L-functions of Artin characters. When $d=1$, so E is a quadratic imaginary field, Colmez's conjecture is a form of the famous Chowla-Selberg formula:

$$h^{\mathrm{Falt}}_{(E,\Phi)}=-\frac{1}{2}\frac{L'(\varepsilon,0)}{L(\varepsilon,0)}-\frac{1}{2}\log(2\pi)+\frac{1}{4}\log|D_E|$$

where D_E is the fundamental discriminant of E and $L(\varepsilon,s)$ is the L-function associated to the quadratic character ε defined by E. Colmez verified its correctness, up to rational multiples of $\log(2)$, for E an abelian extension of $\mathbb{Q}$. Obus [Ob] removed this error term. When $d=2$, Yang [Ya] was the first to prove the formula for non-abelian extensions.

In these lectures we'll be interested in an averaged form of Colmez's conjecture where we fix the CM field E but we sum over the set $\mathrm{CM}(E)$ of all CM types Φ (i.e., subsets Φ of $\mathrm{Hom}(E,\mathbb{C})$ of cardinality d such that $\mathrm{Hom}(E,\mathbb{C})=\Phi\amalg\overline{\Phi}$). His conjectural formula amounts to the following:

Colmez's averaged conjecture:

$$\frac{1}{2^d}\sum_{\Phi} h^{\mathrm{Falt}}_{(E,\Phi)} = -\frac{1}{2}\cdot\frac{L'(\chi,0)}{L(\chi,0)} - \frac{1}{4}\cdot\log\left|\frac{D_E}{D_F}\right| - \frac{d}{2}\cdot\log(2\pi)$$
$$= -\frac{1}{2}\cdot\frac{\Lambda'(\chi,0)}{\Lambda(\chi,0)} - \frac{d}{4}\log(16\pi^3 e^{\gamma}),$$

where $\chi\colon \mathbb{A}_F^\times \to \{\pm 1\}$ is the quadratic Hecke character determined by the extension E/F, $L(\chi,s)$ is the associated L-function, $\Lambda(\chi,s) = |\frac{D_E}{D_F}|^{s/2}\Gamma_{\mathbb{R}}(s+1)^d L(\chi,s)$ is the completed L-function (so that $\Lambda(\chi,1-s)=\Lambda(\chi,s)$), D_E and D_F are the discriminants of E and F respectively and $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant.

2.1 The strategy

Let us a reformulate Colmez's conjecture more precisely. Let $\mathbb{Q}^{\mathrm{CM}} \subset \mathbb{C}$ be the composite of all algebraic CM extensions of $\mathbb{Q}$. Denote by $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/\mathbb{Q})$ and write $c \in \mathcal{G}$ for the complex conjugation. Let $\mathcal{CM}^0$ be the $\mathbb{Q}$-vector space of locally constant, *central* functions (i.e., functions constant on conjugacy classes) $a\colon \mathcal{G} \to \mathbb{Q}$ such that the function $\mathcal{G} \ni g \mapsto a(g)+a(cg)$ is constant. Any such a is a $\mathbb{C}$-linear combination $a = \sum_\eta a(\eta)\eta$ of Artin characters. Since $c\in\mathcal{G}$ is a central element any such character satisfies $\eta(c) = \pm\eta(\mathrm{Id})$. The assumption that $a(g)+a(cg)$ is constant implies that for all non-trivial η for which $a(\eta)\neq 0$ we have $\eta(c) = -\eta(\mathrm{Id})$ so that $L(\eta,0)\neq 0$. In particular to $a\in\mathcal{CM}^0$ we can associate the complex number

$$Z(a) = -\sum_\eta a(\eta)\left(\frac{L'(\eta,0)}{L(\eta,0)} + \frac{\log(f_\eta)}{2}\right)$$

where f_η is the Artin conductor of η.

Now start with a CM field $E\subset\mathbb{Q}^{\mathrm{CM}}$ and a CM type Φ. Define the locally constant function on $\mathcal{G}$:

$$a_{(E,\Phi)}(\sigma) = |\Phi\cap\sigma\circ\Phi| \quad \forall \sigma\in\mathcal{G}.$$

The average

$$a^0_{(E,\Phi)} = \frac{1}{[\mathcal{G}:\mathrm{Stab}(\Phi)]}\sum_{\tau\in\mathcal{G}/\mathrm{Stab}(\Phi)} a_{(E,\Phi)}$$

lies in $\mathcal{CM}^0$. In fact

$$a_{(E,\Phi)}(\sigma) + a_{(E,\Phi)}(c\sigma) = |\Phi|$$

is independent of $\sigma\in\mathcal{G}$ and hence also $a^0_{(E,\Phi)}(\sigma)+a^0_{(E,\Phi)}(c\sigma)$. Here $\mathrm{Stab}(\Phi)\subset\mathcal{G}$ is the subgroup stabilizing the CM type Φ. We then have.

Colmez's conjecture: If A is an abelian variety with CM by the ring of integers $\mathcal{O}_E$ of E and with CM type Φ, we have $h^{\mathrm{Falt}}_{(E,\Phi)} = Z(a^0_{(E,\Phi)})$.

We will use this (conjectural) combinatorial expression in two ways. First of all one can compute the following equality of virtual representations

$$\frac{1}{[E:\mathbb{Q}]}\sum_{\Phi} a^0_{(E,\Phi)} = 2^{d-2}\left(\mathbf{1} + \frac{1}{d}\mathrm{Ind}^{\mathcal{G}}_{\mathcal{G}_F}(\chi)\right),$$

which provides upon taking $Z(\_)$ the expression appearing on the right hand side of Colmez's averaged conjecture. Here $\mathcal{G}_F = \mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/\mathbb{Q})$.

Secondly, we relate the averaged sum on the CM types to the height of the total reflex algebra $E^\sharp$ and the reflex CM type $\Phi^\sharp$. This will play a crucial role in the sequel. We let $E^\sharp$ be the étale $\mathbb{Q}$-algebra defined, via Grothendieck's formalism of Galois theory, by the $\mathrm{Gal}(\overline{Q}/\mathbb{Q})$-set $\mathrm{CM}(E)$ of all CM types on E, i.e., $E^\sharp$ is characterized by the fact that $\mathrm{Hom}(E^\sharp, \overline{\mathbb{Q}}) \cong \mathrm{CM}(E)$ as $\mathrm{Gal}(\overline{Q}/\mathbb{Q})$-sets. We let $\Phi^\sharp \subset \mathrm{Hom}(E^\sharp, \overline{\mathbb{Q}})$ consist of all $\Phi \in \mathrm{CM}(E)$ such that the given embedding $\iota_0 \colon E \to \mathbb{C}$ lies in Φ. In fact one can prove that $E^\sharp = \prod_i E'_i$ is a product of CM fields (as many as the orbits of $\mathrm{Gal}(\overline{Q}/\mathbb{Q})$ on $\mathrm{CM}(E)$ and $\Phi^\sharp \subset \mathrm{Hom}(E^\sharp, \overline{\mathbb{Q}}) = \amalg_i \mathrm{Hom}(E'_i, \overline{\mathbb{Q}})$ is the disjoint union $\Phi^\sharp = \amalg_i \Phi'_i$ of CM types for the E'_i. Furthermore if E'_i corresponds to the orbit of $\Phi_i \in \mathrm{CM}(E)$ then (E, Φ_i) and (E'_i, Φ'_i) is a reflex pair). Then

$$a^0_{(E^\sharp,\Phi^\sharp)} = \frac{1}{[E:\mathbb{Q}]}\sum_{\Phi\in\mathrm{CM}(E)} a^0_{(E,\Phi)}. \tag{1}$$

Colmez further proves that $h^{\mathrm{Falt}}_{(E,\Phi)} = \mathrm{ht}(a^0_{(E,\Phi)})$ for a unique $\mathbb{Q}$-linear map $\mathrm{ht}\colon \mathcal{CM}^0 \to \mathbb{Q}$. Thus $h^{\mathrm{Falt}}_{(E^\sharp,\Phi^\sharp)} = \frac{1}{[E:\mathbb{Q}]}\sum_{\Phi\in\mathrm{CM}(E)} h^{\mathrm{Falt}}_{(E,\Phi)}$. Colmez's averaged conjecture amounts then to prove that

Colmez's averaged conjecture revisited:

$$[E:\mathbb{Q}]h^{\mathrm{Falt}}_{(E^\sharp,\Phi^\sharp)} = -\frac{1}{2}\cdot\frac{\Lambda'(\chi,0)}{\Lambda(\chi,0)} - \frac{d}{4}\log(16\pi^3 e^\gamma)$$

Our strategy: Thanks to Chowla-Selberg formula we may and will assume through this text that $d \geq 2$. We will define a certain normal scheme $\mathcal{Y}_0$, finite over $\mathrm{Spec}(\mathcal{O}_E)$, carrying an abelian scheme $\mathcal{A}^\sharp$ with action of $\mathcal{O}_{E^\sharp}$ and CM type $\Phi^\sharp$. In particular $\mathcal{Y}_0$ will carry the metrized line bundle $\omega_{\mathcal{A}^\sharp}$ whose degree, divided by the degree of $\mathcal{Y}_0$, will compute $h^{\mathrm{Falt}}_{(E^\sharp,\Phi^\sharp)}$ by definition.

On the other hand we will also define auxiliary morphisms $\mathcal{Y}_L \to \mathcal{M}_L$ where $\mathcal{Y}_L \to \mathcal{Y}_0$ is a finite morphism and $\mathcal{M}_L$ are certain models of Shimura varieties of orthogonal type such that the metrized line bundle $\omega_{\mathcal{A}^\sharp}$ over $\mathcal{Y}_L$ is realized as the pull–back of *arithmetic Heegner divisors* on $\mathcal{M}_L$. Using this and work of Bruinier, Kudla and Yang [BKY] we will get a way to compute $h^{\mathrm{Falt}}_{(E^\sharp,\Phi^\sharp)}$ as the arithmetic intersection between these Heegner divisors and $\mathcal{Y}_L$, providing the RHS in the formula for Colmez's averaged conjecture (revisited).

3 Shimura varieties of orthogonal type and CM cycles

3.1 GSpin Shimura varieties

Let V be a $\mathbb{Q}$ vector space of dimension $n+2$ with $n \geq 0$, and a quadratic form

$$Q \colon V \to \mathbb{Q}$$

which is non degenerate, of signature $(n,2)$. Consider the associated bilinear form

$$[\_,\_] \colon V \times V \to \mathbb{Q}, \quad [x,y] = Q(x+y) - Q(x) - Q(y).$$

We have the associated Clifford algebra $C(V) = C(V,Q)$. It is a $\mathbb{Q}$-algebra, with the natural inclusion

$$V \hookrightarrow C(V)$$

satisfying the following universal property: for any $\mathbb{Q}$ algebra R with a $\mathbb{Q}$-linear map $j \colon V \to R$ such that

$$j(v)j(v) = Q(v)$$

there exists a unique homomorphism of $\mathbb{Q}$-algebras

$$C(V) \to R$$

such that the composite with the inclusion $V \subset C(V)$ is j. In particular for any v and $w \in V$, we have

$$v \cdot w + w \cdot v = [v,w] \in C(V),$$

where $v \cdot w$ (and $w \cdot v$) is the product in $C(V)$.

The construction of the Clifford algebra is quite straightforward. In fact,

$$C(V) := (\bigoplus_{n=0}^{\infty} V^{\otimes m})/(v \otimes v - Q(v) | v \in V),$$

the quotient of the tensor algebra of V by the two sided ideal generated by the elements $v \otimes v - Q(v)$ for $v \in V$. As such ideal is generated by elements lying in even degree (in the tensor algebra considered with its natural grading), the $\mathbb{Z}/2\mathbb{Z}$-grading on the tensor algebra (into even and odd tensors) passes to the Clifford algebra that correspondingly splits into a direct sum

$$C(V) = C^+(V) \oplus C^-(V).$$

Note that $C^+(V)$ is a subalgebra while $C^-(V)$ is just a two-sided module for $C^+(V)$. Furthermore we have the following formulas:

$$\dim_{\mathbb{Q}}(C(V)) = 2^{n+2}, \quad \dim_{\mathbb{Q}}^{+}(C(V)) = \dim_{\mathbb{Q}}(C^-(V)) = 2^{n+1}$$

(Recall that V has dimension n+2)

Next we construct $\mathrm{GSpin}(V, Q)$. Given a commutative $\mathbb{Q}$-algebra R, its R-valued points are

$$\mathrm{GSpin}(V)(R) := \{x \in (C^+(V) \otimes_{\mathbb{Q}} R)^* : x(V \otimes_{\mathbb{Q}} R)x^{-1} \subset V \otimes_{\mathbb{Q}} R\}.$$

In particular any $x \in \mathrm{GSpin}(V)(R)$ acts on $V \otimes_{\mathbb{Q}} R$ and, since for any $y \in V \otimes_{\mathbb{Q}} R$

$$Q(y) = Q(xyx^{-1}),$$

such action factors through $\mathrm{O}(V, Q)(R)$. Notice that the units of R form a subgroup of the center of $\mathrm{GSpin}(V)(R)$ and in particular they act trivially on $V \otimes_{\mathbb{Q}} R$. Furthermore one can prove that the action of $\mathrm{GSpin}(V, Q)$ on V factors through the special orthogonal group $\mathrm{SO}(V, Q)$ and one gets an exact sequence of algebraic groups:

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V, Q) \longrightarrow 0 \tag{2}$$

3.2 Examples of GSpin groups

3.2.1 Example 1: the case $n = 0$

In this case

$$V = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2,$$

So $C^-(V) = V$ and $C^+(V)$ is algebra of dimension 2:

$$C^+(V) = \mathbb{Q} \oplus \mathbb{Q}e_1 \cdot e_2.$$

Denote $x = e_1 \cdot e_2$, let $a_1 = Q(e_1)$ and $a_2 = Q(e_2)$. They are negative rational numbers and if we write $b = [e_1, e_2] \in \mathbb{Q}$, we have

$$x^2 = e_1e_2e_1e_2 = -e_1^2e_2^2 + [e_1, e_2]e_1e_2 = -a_1a_2 + bx,$$

so we have

$$x^2 - bx + a_1a_2 = 0.$$

As an algebra, this gives

$$C^+(V) = \mathbb{Q}[x]/(x^2 - bx + a_1a_2),$$

which is an imaginary quadratic field K, and we see that

$$\mathrm{GSpin}(V) = C^+(V)^\times = \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m.$$

In particular its base change to $\mathbb{R}$ is the so called Deligne torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{R}}$.

3.2.2 Example 2: the case $n = 1$

Consider the $\mathbb{Q}$-vector space

$$V \subset M_{2\times 2}(\mathbb{Q}) = \{x \in M_{2\times 2} : Tr(v) = 0\}$$

Fix some $N \in \mathbb{N}$ such that $N \geq 1$ Let Q_N be the quadratic form

$$A \mapsto N \cdot \det A$$

Then

$$\mathrm{GSpin}(V) \cong \mathrm{GL}_2$$

where GL_2 acts on V by conjugation.

3.3 Hermitian symmetric spaces

Fix the algebraic group $G := \mathrm{GSpin}(V, Q)$. As a first step in order to construct a Shimura variety, we need to construct a Hermitian symmetric space. It admits several realizations:

1. as a complex manifold $D_{\mathbb{C}} = \{z \in V_{\mathbb{C}} \backslash \{0\} : Q(z) = 0, [z, \overline{z}] < 0\}/\mathbb{C}^* \subset \mathbb{P}(V_{\mathbb{C}})$
2. as a Riemannian manifold $D_{\mathbb{R}} = \{$ Negative definite oriented planes $H \subset V_{\mathbb{R}}\}$
3. using Deligne torus $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{R}}$, and realize

$$D = G(\mathbb{R}) \text{ conjugacy class of } h : \mathbb{S} \to G_{\mathbb{R}}$$

Let us explain how we can go back and forth between these incarnations.

Given an $H = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ as 2), we let $z = e_1 + ie_2$, then take the line $[z]$ to get the realization 1. To get realization 3, we simply take $\mathbb{S} \cong \mathrm{GSpin}H \hookrightarrow G_{\mathbb{R}}$ using that $H \subset V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$.

Example 1. The Hermitian space has two connected components

$$D_{\mathbb{R}} = \{ \text{ two possible orientations on } V_{\mathbb{R}} = \mathbb{R}e_1 \oplus \mathbb{R}e_2\} = \{\pm \text{ points }\}.$$

Example 2. We have

$$D_{\mathbb{R}} \cong \mathbb{H}^+ \sqcup \mathbb{H}^- \subset \mathbb{C}$$

which are the Poincaré upper and lower half planes. The inverse of the map is given as

$$\mathbb{R} \cdot Re \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \oplus \mathbb{R} \cdot \mathrm{im} \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \leftarrow z = x + iy.$$

Pick $[z] \in D_{\mathbb{C}}$, then

$$V_{\mathbb{C}} = \mathbb{C}z \oplus (\mathbb{C}z \oplus \mathbb{C}\overline{z})^{\perp} \oplus \mathbb{C}\overline{z}$$

and the tangent space of Q in $\mathbb{P}(V_{\mathbb{C}})$ at $[z]$ can be computed as the Zariski tangent space at $[z]$, namely the set of lines $[z + \delta\epsilon + \gamma\epsilon\overline{z}]$, with $\delta \in (\mathbb{C}z \oplus \mathbb{C}\overline{z})^{\perp}$ and ϵ a formal variable with square $\epsilon^2 = 0$, such that $Q([z + \delta\epsilon + \gamma\epsilon\overline{z}]) = 0$, i.e, if and only if $\gamma = 0$. Thus the tangent space of $D_{\mathbb{C}}$ at $[z]$ is isomorphic to $(\mathbb{C}z \oplus \mathbb{C}\overline{z})^{\perp}$ and $\dim D_{\mathbb{C}} = n$.

3.4 GSpin-Shimura varieties

Given V and Q and the algebraic group $G := \mathrm{GSpin}(V, Q)$ as in the previous section, define the complex manifold:

$$M_K(\mathbb{C}) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K$$

for some compact open K subgroup of the adelic points $G(\mathbb{A}_f)$ of G.

Given a quadratic lattice $L \subset V$, i.e., a lattice on which Q is integral valued, one can construct a compact open subgroup K_L by taking

$$K_L = G(\mathbb{A}_f) \cap C^+(\widehat{L})^\times \subset C^+(V)^\times(\mathbb{A}_f)$$

where $\widehat{L} := L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. We will be especially interested in the case that L is maximal among the integral lattices. This will guarantee the existence of good integral models over $\mathbb{Z}$ for the Shimura variety $M_{K_L}(\mathbb{C})$. If the compact open subgroup is of the type K_L for some lattice L, we simply write $M_L(\mathbb{C})$.

Now let us look at the examples again.

Example 1. $M(\mathbb{C})$ consists of finitely many points.

Example 2. We have

$$V = M_{2\times 2}(\mathbb{Q})^{Tr=0}$$

and Q_N makes $\mathrm{GSpin}(V) \cong \mathrm{GL}_2$, and one can take

$$L := \{\begin{pmatrix} a & \frac{-b}{N} \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{Z}\}$$

And one can check that

$$K_L \cong \pi_p \widetilde{K}_p$$

where

$$\widetilde{K}_p = \{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{Z}_p) : \gamma \in N\mathbb{Z}_p\}$$

In this case one sees that

$$M_2(\mathbb{C}) \cong Y_0(N)(\mathbb{C})$$

which is the modular curve of level $\Gamma_0(N)$, classifying cyclic isogenies $\rho \colon E \to E'$ of degree N of elliptic curves.

Warning: The case of elliptic curves is misleading as it might appear that $M_L(\mathbb{C})$ has a moduli interpretation. This is not the case if the dimension $n+2$ of V is large. In this case $M_L(\mathbb{C})$ does *not* in general represent a PEL type moduli problem, i.e., does not classify abelian varieties with given polarization, endomorphisms and level structures. As we will see, this is the source of complications when one attempts to provide integral models for $M_L(\mathbb{C})$.

4 Extra structures on GSpin-Shimura Varieties

Recall the notation. We fixed a vector space V over $\mathbb{Q}$ of dimension $n+2$, and a quadratic form $QV \to \mathbb{Q}$ of signature $(n,2)$, with a maximal quadratic lattice $L \subset V$. We let $G = \mathrm{GSpin}(V)$, and then defined

$$M_L(\mathbb{C}) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K_L$$

for a particular choice of compact open subgroup K_L associated to L.

We have a natural functor

$$\{\text{ Algebraic Representations of } G\} \to \{\text{ Local Systems of } \mathbb{Q} \text{ vector spaces on } M_L(\mathbb{C})\}$$

given by

$$(G \to GL(W)) \mapsto W_{\mathrm{Betti},\mathbb{Q}} \longrightarrow M_L(\mathbb{C})$$

where

$$W_{\mathrm{Betti},\mathbb{Q}} := G(\mathbb{Q})\backslash(W \times D) \times G(\mathbb{A}_f)/K_L$$

Note that this gives us a unique pair $(W_{\mathrm{dR}}, \nabla)$ of a locally free $\mathcal{O}_{M_L(\mathbb{C})}$-module with integral connection. Namely $W_{\mathrm{dR}} := W_{\mathrm{Betti},\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_{M_L(\mathbb{C})}$ with the connection $\nabla = 1 \otimes d$. Then $(W_{\mathrm{dR}}, \nabla)$ is characterized as the vector bundle with integrable connection such that

$$W_{\mathrm{dR}}^{\nabla=0} = W_{Betti,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}.$$

We further have the following extra properties:

a. For any z in the symmetric space D, the map

$$h_z \colon \mathbb{S} \to G_{\mathbb{R}} \to GL(W_{\mathbb{R}})$$

induces a map

$$\mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^* \to \mathrm{GL}(W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{GL}(W_{\mathbb{C}})$$

and the fiber $W_{\mathrm{dR},z}$ at z has a bigraduation $\oplus_{p,q} W_{\mathrm{dR},z}^{p,q}$ obtained by the decomposition of $W_{\mathbb{C}}$ according to the action of $\mathbb{C}^* \times \mathbb{C}^*$.

b. W_{dR} is endowed with a decreasing filtration $\mathrm{Fil}^J(W_{\mathrm{dR}}) \subset W_{\mathrm{dR}}$ by holomorphic sub-bundles of W_{dR}, defined pointwise by

$$\mathrm{Fil}^J(W_{\mathrm{dR},z}) := \oplus_{p \geq J} W_{\mathrm{dR},z}^{p,q}.$$

A $\mathbb{Q}$-local system on $M_L(\mathbb{C})$ with these properties is called *a variation of $\mathbb{Q}$-Hodge structures*. In particular, $W_{\mathrm{Betti},\mathbb{Q}}$ is a variation of $\mathbb{Q}$-Hodge structures. Consider in particular the homomorphism

$$G \to \mathrm{SO}(V)$$

given by

$$x \mapsto \{y \mapsto xyx^{-1}\}$$

as a map $V \to V$ inside $\mathrm{SO}(V)$. Now we get as before a variation of $\mathbb{Q}$-Hodge structures V_{Betti} and even a variation of $\mathbb{Z}$-Hodge structures $\mathbb{V}_{\mathrm{Betti}}$. In particular we get a vector bundle with connection V_{dR}. They are all endowed with a quadratic form Q_{Betti} and Q_{dR} respectively.

For any $(z,g) \in M_L(\mathbb{C})$ with $z \in D_{\mathbb{C}}$ and $g \in G(\mathbb{A}_f)$, where $D_{\mathbb{C}}$ is the incarnation of the symmetric space as the isotropic lines in $V_{\mathbb{C}}$, the morphism h_z defines a decomposition

$$V_{\mathbb{C}} = \mathbb{C}_z \oplus (\mathbb{C}_z \oplus \mathbb{C}_{\overline{z}})^{\perp} \oplus \mathbb{C}_{\overline{z}} \subset \mathrm{End}(H_{1,\mathrm{dR}}(A_z))$$

The filtration is given by

$$\mathrm{Fil}^1(\mathbb{V}_{\mathrm{dR},z}) = \mathbb{C}z$$

$$\mathrm{Fil}^0(\mathbb{V}_{\mathrm{dR},z}) = \mathbb{C}z \oplus (\mathbb{C}z \oplus \mathbb{C}\bar{z})^{\perp}$$

$$\mathrm{Fil}^{-1}(\mathbb{V}_{\mathrm{dR},z}) = \mathbb{C}z \oplus (\mathbb{C}z \oplus \mathbb{C}\bar{z})^{\perp} \oplus \mathbb{C}\bar{z}.$$

5 The big CM points

We follow here [BKY]. Let E be a CM filed of degree $2d$ with totally real subfield F (of degree d). We label the embeddings $\{\sigma_0, \ldots, \sigma_{d-1}\} = \mathrm{Hom}(F, \mathbb{R})$. For every integer $0 \leq i \leq d-1$ label by σ_i and $\overline{\sigma}_i \colon E \to \mathbb{C}$ the two conjugate emebddings of E extending $\sigma_i \colon F \to \mathbb{R}$.

Let $\lambda \in F$ be an element such that $\sigma_0(\lambda) < 0$ and $\sigma_i(\lambda) > 0$ for $1 \leq i \leq d-1$. Consider the quadratic space $W = E$ of dimension 2 over F with bilinear pairing

$$B_W \colon W \times W \longrightarrow F, \quad (x,y) \mapsto B_W(x,y) = \mathrm{Tr}_{E/F}(\lambda x \bar{y}).$$

It is negative definite at one place and positive definite at the others. Namely, for all embedding $\sigma_i \colon F \to \mathbb{R}$ with $i \neq 0$ the induced bilinear form $B_W \colon W_{\mathbb{R}} \times W_{\mathbb{R}} \longrightarrow \mathbb{R}$ is positive definite, and it is negative definite at the remaining place σ_0. We let

$$Q_W(x) = \frac{1}{2} B_W(x,x)$$

the associated quadratic form. The even Clifford algebra $C^+(W)$, over F, is identified with E.

One then lets

$$V = \mathrm{Res}_{F/\mathbb{Q}}(W).$$

That is, V is simply E, viewed as a quadratic $\mathbb{Q}$-vector space of dimension $2d$, equipped with the form

$$B_V(x,y) = \mathrm{Tr}_{E/\mathbb{Q}}(\lambda x \bar{y}).$$

Example Let $L = \mathfrak{a}$ be a fractional ideal of E. Let $\bar{L} := \bar{\mathfrak{a}}$ be the image of $\mathfrak{a}$ under complex conjugation. Then B_V is integrally valued on L if and only if $\lambda\mathfrak{a}\bar{\mathfrak{a}} \subset \mathfrak{D}_{E/\mathbb{Q}}^{-1}$. In this case $L^\vee = (\lambda\mathfrak{D}_{E/\mathbb{Q}}\bar{\mathfrak{a}})^{-1}$ and

$$L^\vee/L \cong \mathcal{O}_E/\lambda\mathfrak{D}_{E/\mathbb{Q}}\mathrm{Norm}_{E/F}(\mathfrak{a}) \tag{3}$$

Definition 5.1. Given a quadratic lattice $L \subset V$ we say that the prime p is *good for* L if the following conditions hold. For every prime $\mathfrak{p}$ of F over p set

$$L_\mathfrak{p} = (L \otimes \mathbb{Z}_p) \cap V_\mathfrak{p}.$$

We demand that

- For every $\mathfrak{p} \mid p$ unramified in E, the $\mathbb{Z}_p$-lattice $L_\mathfrak{p}$ is $\mathcal{O}_{E,\mathfrak{p}}$-stable and self-dual for the induced $\mathbb{Z}_p$-valued quadratic form.
- For every $\mathfrak{p} \mid p$ ramified in E, the $\mathbb{Z}_p$-lattice $L_\mathfrak{p}$ is maximal for the induced $\mathbb{Z}_p$-valued quadratic form, and there exists an $\mathcal{O}_{E,\mathfrak{q}}$-stable lattice $\Lambda_\mathfrak{p} \subset V_\mathfrak{p}$ such that

  $$\Lambda_\mathfrak{p} \subset L_\mathfrak{p} \subsetneq \mathfrak{d}_{E_\mathfrak{q}/F_\mathfrak{p}}^{-1}\Lambda_\mathfrak{p}.$$

  where $\mathfrak{q} \subset \mathcal{O}_E$ is the unique prime above $\mathfrak{p}$.

All but finitely many primes are good: Choose any $\mathcal{O}_E$-stable lattice $\Lambda \subset L$. Then, for all but finitely many primes p, $\Lambda_{\mathbb{Z}_p} = L_{\mathbb{Z}_p}$ will be self-dual and hence good. We let $D_{L,\mathrm{bad}}$ be the product of all primes that are *not* good for L.

Note that there are maps

$$\prod_j C^+(W_{\sigma_j}) \longrightarrow \otimes_j C^+(W_{\sigma_j}) \longrightarrow C^+(\bigoplus_j W_{\sigma_j}).$$

(The tensor product is over $\mathbb{R}$ or more generally any field over which F splits completely.) The first map is multiplicative and multi-linear and the second map is a ring homomorphism. The ring $C^+(W)$ has a basis over F given by $\{1, e_1e_2\}$. A general element of it has the form $a + be_1e_2$; via the isomorphism $C^+(W) \otimes_\mathbb{Q} \mathbb{R} \cong \prod_j C^+(W_{\sigma_j})$ and the maps above, the image of this element in $C^+(\bigoplus_j W_{\sigma_j})$ is $\prod_j(\sigma_j(a) + \sigma_j(b)e_1^je_2^j)$ (product in the Clifford algebra). In particular, $a \in F$ is mapped to $\mathrm{Norm}_{F/\mathbb{Q}}(a)$ and e_1e_2 to $\prod_j e_1^je_2^j$.

Passing to GSpin, making use of $\mathrm{GSpin}_F(W) = C^+(W)^\times$, we conclude a homomorphism of groups over $\mathbb{R}$ (or any field splitting F):

$$\prod_j \mathrm{GSpin}_\mathbb{R}(W_{\sigma_j}) \longrightarrow \mathrm{GSpin}_\mathbb{R}(\bigoplus_j W_{\sigma_j}). \tag{4}$$

The homomorphism (4) descends to a homomorphism of algebraic groups over $\mathbb{Q}$

$$g\colon \operatorname{Res}_{F/\mathbb{Q}}(\operatorname{GSpin}_F(W)) \longrightarrow \operatorname{GSpin}(V) \tag{5}$$

(Cf. [BKY, §2]). Let T be the image in $\operatorname{GSpin}(V)$ of this homomorphism, it is a torus and there is an exact sequence (loc. cit.)

$$1 \longrightarrow T_F^{\mathrm{Nm}=1} \longrightarrow T_E \xrightarrow{g} T \longrightarrow 1.$$

Here $T_E = \operatorname{Res}_{E/\mathbb{Q}}(\mathbb{G}_{m,E}) = \operatorname{Res}_{F/\mathbb{Q}}(\operatorname{GSpin}_F(W))$ and $T_F^{\mathrm{Nm}=1}$ is the subgroup of norm 1 elements of $T_F = \operatorname{Res}_{F/\mathbb{Q}}(\mathbb{G}_{m,F})$.

Note that there is a unique, up to isomorphism, rank 2 non-split torus over $\mathbb{R}$ and, thus, there is an isomorphism

$$h\colon \mathbb{S} \cong \operatorname{GSpin}(W_{\sigma_0}) \subset \operatorname{Res}_{F/\mathbb{Q}}(\operatorname{GSpin}_F(W))(\mathbb{R}). \tag{6}$$

This gives a point $z_0 \in D_{\mathbb{R}}$. Fixing a quadratic lattice $L \subset V$ we get the Shimura variety $M_L(\mathbb{C})$ as explained in §3.4. Taking a compact open subgroup $K_T \subset K_L \cap T(\mathbb{A}_f)$ we can define the Shimura variety

$$Y_L(\mathbb{C}) = T(\mathbb{Q})\backslash\{z_0\} \times T(\mathbb{A}_f)/K_T,$$

which consists of finitely many points, and a homomorphism

$$Y_L(\mathbb{C}) \to M_L(\mathbb{C}) = M_L(\mathbb{C}) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K_L,$$

whose image is called the *big CM cycle* associated to (E, σ_0, λ) (the specific subgroup K_T we will consider depends only on L). It is defined over its reflex field E and we let $\mathcal{Y}_L$ be the normalization of $\operatorname{Spec}(\mathcal{O}_E)$ in Y_L.

Remark 5.2. The world big suggests the existence of a *small CM cycle*. This is the case and it is constructed starting from a quadratic imaginary extension of $\mathbb{Q}$ instead of a CM filed extension of degree $n-2$. There are interesting conjectures in this setting as well in the spirit of Kudla's programme, elaborated by Bruinier and Yang in [BY]. These conjectures have been proven under some mild assumptions in [AGHMP1].

5.1 The total reflex algebra

Write

$$V_{\mathbb{C}} = E \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\rho \in \operatorname{Hom}(E,\mathbb{C})} \mathbb{C} \cdot e_\rho. \tag{7}$$

where the e_ρ's are orthogonal idempotents of the algebra $E \otimes_{\mathbb{Q}} \mathbb{C}$.

Lemma 5.3. *Define for $\rho \in \operatorname{Hom}(E,\mathbb{C})$ an element of $C^+(V_{\mathbb{C}})$,*

$$\delta_\rho = \frac{1}{\rho(\lambda)} e_\rho e_{\bar{\rho}}.$$

1. *There are* $2d$ *such elements* δ_ρ*. They all commute.*

2. $\delta_\rho^2 = \delta_\rho$.

3. $\delta_\rho \delta_{\bar\rho} = 0$.

4. $\delta_\rho e_\alpha = \begin{cases} e_\alpha \delta_\rho & \alpha \notin \{\rho, \bar\rho\} \\ 0 & \alpha = \bar\rho \\ e_\alpha & \alpha = \rho, \end{cases}$ *and* $e_\alpha \delta_\rho = \begin{cases} \delta_\rho e_\alpha & \alpha \notin \{\rho, \bar\rho\} \\ 0 & \alpha = \rho \\ e_\alpha & \alpha = \bar\rho. \end{cases}$

Using the decomposition

$$V_{\mathbb{C}} = E \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_j (\mathbb{C} e_{\rho_j} \oplus \mathbb{C} e_{\bar\rho_j})$$

we may identify V with a subspace of $C^+(V)$ via the map (defined over $\mathbb{C}$)

$$j \colon V \to C^+(V), \quad \sum_\rho x_\rho e_\rho \mapsto (x_{\rho_j} \delta_{\rho_j} + x_{\bar\rho_j} \delta_{\bar\rho_j})_j \in C^+(V_{\mathbb{C}}). \tag{8}$$

Denote by Φ the set of CM types on E.

Lemma 5.4. *For every* $\phi \in \Phi$ *define the following element of* $C^+(V_{\mathbb{C}})$*:*

$$\Delta_\phi := \prod_{\rho \in \phi} \delta_\rho$$

1. *There are* 2^d *such elements* Δ_ϕ*. They all commute.*

2. $\Delta_\phi^2 = \Delta_\phi$.

3. $\Delta_{\phi_1} \Delta_{\phi_2} = 0$ *if* $\phi_1 \neq \phi_2$.

4. *for* $\alpha \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, $\alpha(\Delta_\phi) = \Delta_{\alpha\phi}$.

5. $\sum_{\phi \in \Phi} \Delta_\phi = 1$.

In particular the $\overline{\mathbb{Q}}$-span of the Δ_Φ in $C^+(V) \otimes \overline{\mathbb{Q}}$ is an étale subalgebra. Taking $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariants we realize $E^\sharp$ as a $\mathbb{Q}$-subalgebra of $C^+(V)$. We write $V^\sharp := E^\sharp$ if we consider $E^\sharp$ simply as a $\mathbb{Q}$-vector space. Using the identification (8) and Lemma 5.3 we get an action, induce by left multiplication on $C^+(V)$:

$$\ell \colon V \to \mathrm{End}(V^\sharp).$$

Using (4) and viewing $j(V) \subset \prod_j C^+(W_{\sigma_j})$, we also obtain a multiplicative homomorphism, called the **complete reflex norm**, that factors through T

$$T_E \longrightarrow T_{E^\sharp} \hookrightarrow \mathrm{GSpin}(V), \qquad \sum_\rho x_\rho e_\rho \mapsto \prod_j (x_{\rho_j} \delta_{\rho_j} + x_{\bar\rho_j} \delta_{\bar\rho_j}). \tag{9}$$

Corollary 5.5. *Let* $h\colon \mathbb{S} \xrightarrow{\cong} \mathrm{GSpin}(W_{\sigma_0}) \subset \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSpin}_F(W))(\mathbb{R}) = (E \otimes_{\mathbb{Q}} \mathbb{R})^\times$ *be as in (6). There is a unique way to choose* h *so that* $g \circ h(i)$*, where* g *is the homomorphism (5), is the element*

$$\varphi(ie_{\rho_0} - ie_{\bar\rho_0} + \sum_{\rho \notin \{\rho_0, \bar\rho_0\}} e_\rho).$$

That is,

$$g \circ h(i) = \sum_{\phi \in \Phi} \epsilon(\phi) \cdot i \cdot \Delta_\phi, \qquad \epsilon(\phi) = \begin{cases} 1 & \rho_0 \in \phi \\ -1 & \rho_0 \notin \phi. \end{cases}$$

In the following, we shall denote $g \circ h\colon \mathbb{S} \longrightarrow \mathrm{GSpin}(V_{\mathbb{R}})$ *simply by* h*.*

5.2 Extra structure on the Big CM cycle

We start with

Lemma 5.6. *The action of* $E = C^+(W)$ *on* $W = C^-(W)$ *gives an action of* E *on* V*, as* V *is equal to* W *only viewed as a rational vector space. This action will be denoted* $\beta \cdot v, \beta \in E, v \in V$*. On the other hand, through the homomorphisms*

$$E^\times \xrightarrow{g} \mathrm{GSpin}(V) \xrightarrow{\pi} \mathrm{SO}(V),$$

we get another action of $E^\times$ *on* V *that we call* $\rho(\beta)v, \beta \in E, v \in V$ $(\rho = \pi \circ g)$*. The actions are related as follows:*

$$\rho(\beta)v = (\beta\bar\beta^{-1}) \cdot v.$$

Proof. We denote the action of $E^\times$ on W, by $\beta \cdot w, \beta \in E, w \in W$, and the action of E obtained via $E^\times \longrightarrow C^+(W)^\times \longrightarrow \mathrm{SO}(W)$ by $\rho(\beta)w$ are related by the formula $\rho(\beta)w = (\beta\bar\beta^{-1}) \cdot w$. But, $\rho(\beta)w = \beta w \beta^{-1} = (\beta\bar\beta^{-1})w = (\beta\bar\beta^{-1}) \cdot w$. The lemma now follows by applying restriction of scalars and the commutativity of the following diagram:

$$\begin{array}{ccccc} C^+(W) \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow{\cong} & \prod_j C^+(W_j) & \longrightarrow & \mathrm{GSpin}(V) \\ & & \downarrow & & \downarrow \\ & & \prod_j \mathrm{SO}(W_j) & \longrightarrow & \mathrm{SO}(V). \end{array}$$

□

Proposition 5.7. *The endomorphism ring of* V *as a rational Hodge structure is precisely* E *with the "dot action". Moreover,* E*, viewed in* $\mathrm{End}(V)$ *via the "dot action" is the* $\mathbb{Q}$*-linear span of the image of* $T(\mathbb{Q})$*.*

Proof. Let T_E be the $\mathbb{Q}$-torus associated to E and let $\gamma\colon T_E \longrightarrow T_E$ be the morphism given by $\alpha \mapsto \alpha/\bar\alpha$ on $\mathbb{C}$-points. This morphism is defined over $\mathbb{Q}$ and its image, by Lemma 5.6, is nothing but $T_{\mathrm{so}}(\mathbb{Q}) \subseteq \mathrm{GL}(V)$. One can check that E is the $\mathbb{Q}$-span of the elements of $T(\mathbb{Q})$, namely $\{\alpha/\bar\alpha : \alpha \in E^\times\}$. □

Notice that V has the extra structure of E-vector space that defines endomorphisms of $V_{\mathrm{Betti}}|_{Y(\mathbb{C})}$. In fact, for any $z \in Y(\mathbb{C})$ the E action on $V_{\mathrm{Betti},\mathbb{Q},z}$ induces a decomposition

$$V_{\mathrm{Betti},\mathbb{Q},z} \otimes_{\mathbb{Q}} \mathbb{C} = V_{\mathrm{dR},z} = \oplus_{i=0}^{d} V_{\mathrm{dR},z}(\sigma_i) \oplus V_{\mathrm{dR},z}(\overline{\sigma}_i)$$

where $V_{\mathrm{dR},z}(\sigma)$ is the 1-dimensional $\mathbb{C}$-vector space on which E acts via $\sigma\colon E \to \mathbb{C}$. Then

$$\mathrm{Fil}^1 V_{\mathrm{dR},z} = V_{\mathrm{dR},z}(\sigma_0), \mathrm{Gr}^{-1} V_{\mathrm{dR},z} = V_{\mathrm{dR},z}(\overline{\sigma}_0)$$

and

$$\mathrm{Gr}^0 V_{\mathrm{dR},z} = \oplus_{i=1}^{d} V_{\mathrm{dR},z}(\sigma_i) \oplus V_{\mathrm{dR},z}(\overline{\sigma}_i).$$

We summarize our findings. Write $V := E$, considered as $\mathbb{Q}$-vector space

- there is a morphism $T_E \to \mathrm{GSpin}(V)$, factoring through T, where $\alpha \in T_E(\mathbb{Q}) = E^{\times}$ acts on $V = E$ through multiplication by $\alpha\bar{\alpha}^{-1}$;
- we can realize $E^{\sharp}$ as a $\mathbb{Q}$-subalgebra of $C^{+}(V)$ such that the homomorphism $T \to \mathrm{GSpin}(V)$ factors via the subtorus $T_{E^{\sharp}} \subset \mathrm{GSpin}(V)$. We write $V^{\sharp} := E^{\sharp}$, considered as a $\mathbb{Q}$-vector space.
- there is a morphism $h\colon \mathbb{S} \to T$ inducing a Hodge structure on V where

  $$\mathrm{Fil}^1 V_{\mathbb{C}} = E \otimes_E^{\sigma_0} \mathbb{C}, \mathrm{Gr}^{-1} V_{\mathbb{C}} = E \otimes_E^{\overline{\sigma}_0} \mathbb{C}, \mathrm{Gr}^0 V_{\mathbb{C}} = \oplus_{i=1}^{d-1} (E \otimes_E^{\sigma_i} \mathbb{C} \oplus E \otimes_E^{\overline{\sigma}_i} \mathbb{C})$$

  and a Hodge structure of type $(-1,0)$ and $(0,-1)$ on $V^{\sharp}$ where, writing $V^{\sharp}_{\mathbb{C}} = E^{\sharp} \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{\Phi \in \mathrm{CM}(E)} \mathbb{C}\Delta_{\Phi}$ as a sum of idempotents,

  $$(V^{\sharp}_{\mathbb{C}})^{(-1,0)} = \oplus_{\sigma_0 \in \Phi \in \mathrm{CM}(E)} \mathbb{C}\Delta_{\Phi}, \quad (V^{\sharp}_{\mathbb{C}})^{(0,-1)} = \oplus_{\overline{\sigma}_0 \in \Phi \in \mathrm{CM}(E)} \mathbb{C}\Delta_{\Phi}$$
- there is an embedding $j\colon V \to C^{+}(V)$ such that multiplication on $C^{+}(V)$ induces a T-equivariant morphism $\ell\colon V \to \mathrm{End}(V^{\sharp})$.

We now consider a second level structure $K_0 \subset T(\mathbb{A}_f)$ defining a CM cycle

$$Y_0(\mathbb{C}) = T(\mathbb{Q}) \backslash \{z_0\} \times T(\mathbb{A}_f)/K_0.$$

We define $K_0 = \prod_p K_{0,p}$ where $K_{0,p} \subset T(\mathbb{Q}_p)$ sits in an exact sequence

$$1 \to \mathbb{Z}_p^{\times} \to K_{0,p} \to \frac{(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_E)^{\times}}{(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F)^{\times}} \to 1$$

via the exact sequence

$$1 \to \mathbb{G}_m \to T \to T_E/T_F \to 1$$

defined by the natural projection $T = T_E/T_F^{\mathrm{Nm}=1} \to T_E/T_F$. Then Y_0 is defined over E and we let $\mathcal{Y}_0$ be the normalization of $\mathrm{Spec}(\mathcal{O}_E)$ in Y_0.

Consider the integral structures $L_0 := \mathcal{O}_E \subset E = V$ and $L_0^\sharp := \mathcal{O}_{E^\sharp} \subset E^\sharp = V^\sharp$. Then K_0 preserves their profinite completions and the Hodge structure $V^\sharp$ of type $(-1,0)$ and $(0,-1)$ defines an abelian variety $A^\sharp$ over Y_0 as follows: given $z := (z_0, g) \in Y_0(\mathbb{C}) := T(\mathbb{Q})\backslash\{z_0\} \times T(\mathbb{A}_f)/K_0$ then $A_z^\sharp := V_\mathbb{R}^\sharp/(V^\sharp \cap g\hat{L}_0^\sharp g^{-1})$. It extends to an abelian scheme $\mathcal{A}^\sharp$ over $\mathcal{Y}_0$. We let $\mathbb{H}^\sharp$ be its first de Rham homology group. It is a filtered $\mathcal{O}_{\mathcal{Y}_0}$-module. Thanks to the choice of the compact open subgroup K_0 we have

Proposition 5.8. *$\mathcal{Y}_0$ is étale over $\mathrm{Spec}(\mathcal{O}_E)$. Furthermore the inclusion $\ell\colon V \to \mathrm{End}(V^\sharp)$ defines a strict morphism of filtered $\mathcal{O}_{\mathcal{Y}_0}$-module*

$$\ell\colon \mathbb{V}_{0,\mathrm{dR}} \to \mathrm{End}(\mathbb{H}^\sharp)$$

and the $\mathcal{O}_E$-action on L_0 extends to a $\mathcal{O}_E$-action on $\mathbb{V}_{0,\mathrm{dR}}$

Proof. The first claim follows from Shimura's reciprocity laws. The second claim follows from Kisin's theory. □

In particular we define the invertible $\mathcal{O}_{\mathcal{Y}_0}$-module $\omega_0 := \mathrm{Fil}^1\mathbb{V}_{0,\mathrm{dR}}$. We endow it with a metric at infinity given by induced by the standard Hermitian metric Q_0 on $V_\mathbb{C} = E \otimes_\mathbb{Q} \mathbb{C}$, $x \mapsto Q_0(x) := \mathrm{Tr}_{F/\mathbb{Q}}(x\bar{x})$.

6 Integral models

In order to proceed with the computation of the intersections numbers we want, we need models for $M_L(\mathbb{C})$ and the CM cycle $Y_L(\mathbb{C})$ over $\mathbb{Z}$.

6.1 Integral models of GSpin-Shimura varieties

By Deligne, we know that $M_L(\mathbb{C})$ is the complex analytic space associated to a quasi-projective variety M_L over a number field K called the *reflex field*. In the case of GSpin, for $n \geq 1$, the reflex field is $\mathbb{Q}$. For $n = 0$ it is the quadratic imaginary field $K = C^+(V)$.

We assume next that $n \geq 1$. Let Δ_L be the discriminant $\Delta_L := [L^\vee : L]$ where $L^\vee$ is the $\mathbb{Z}$-dual of L and the inclusion $L \subset L^\vee$ is defined using the bilinear form $[\_,\_]$. By work of Vasiu and Kisin (see in particular [Kis]) the scheme M_L has a canonical integral model $\mathcal{M}_{L,\mathbb{Z}[2^{-1}|\Delta_L|^{-1}]}$, smooth over $\mathbb{Z}[2^{-1}|\Delta_L|^{-1}]$. Also V_{dR} has a model $\mathbb{V}_{\mathrm{dR}}$ which is a locally free $\mathcal{O}_{\mathcal{M}_{L,\mathbb{Z}[2^{-1}|\Delta_L|^{-1}]}}$-module, endowed with a descending filtration by locally free submodules, $\mathrm{Fil}^\bullet\mathbb{V}_{\mathrm{dR}}$ and an integrable connection satisfying Griffiths' transversality. For the purpose of computing some arithmetic intersection we wish to have a model

$$\mathcal{M} \longrightarrow \mathrm{Spec}(\mathbb{Z})$$

over $\mathbb{Z}$, to which some of the extra structures described above extend as well.

If L is maximal among the quadratic lattices of V and is self dual at 2 Madapusi Pera [MP] constructed such a canonical integral model which has singular fibers at the primes dividing Δ_L. Unfortunately at primes whose square divides Δ_L this model is not well behaved: For our purposes the CM cycle Y_L will have a model $\mathcal{Y}_L$, finite over $\mathrm{Spec}(\mathcal{O}_E)$ but the morphism $Y_L \to M_L$ on the generic fiber does *not* in general extend to a a morphism from $\mathcal{Y}$ to the Madapusi Pera model. In [AGHMP2] we proceeded differently. Let p a prime dividing Δ_L and let $L \subset L^\diamond$ be an isometric embedding of quadratic lattices with signature $(n, 2)$ and $(n^\diamond, 2)$ respectively. We take $L^\diamond$ that is self-dual at p. By functoriality we will have a morphism of Shimura varieties $M_L \to M_{L^\diamond}$ and $M_{L^\diamond}$ will admit an integral model $\mathcal{M}_{L^\diamond,\mathbb{Z}[\Delta_{L^\diamond}^{-1}]}$ smooth over $\mathrm{Spec}(\mathbb{Z}[\Delta_{L^\diamond}^{-1}])$. We define $\mathcal{M}_{L,\mathbb{Z}[\Delta_{L^\diamond}^{-1}]}$ to be the normalization of $\mathcal{M}_{L^\diamond,\mathbb{Z}[\Delta_{L^\diamond}^{-1}]}$ in M_L. The restriction of the tautological bundle on $\mathcal{M}_{L^\diamond,\mathbb{Z}[\Delta_{L^\diamond}^{-1}]}$ defines a line bundle on $\omega_{L^\diamond}$ on $\mathcal{M}_{L,\mathbb{Z}[\Delta_{L^\diamond}^{-1}]}$. We have the following Proposition proven in [AGHMP2]

Proposition 6.1. *The models $\mathcal{M}_{L,\mathbb{Z}[\Delta_{L^\diamond}^{-1}]}$ and the line bundles $\omega_{L^\diamond}$ glue to a normal model $\mathcal{M}_L$ over* $\mathrm{Spec}(\mathbb{Z})$ *and a line bundle ω that agree with the construction on $\mathcal{M}_{L,\mathbb{Z}\left[2^{-1}|\Delta_L|^{-1}\right]}$ provided by Kisin and Vasiu.*

Via the uniformization map

$$\mathcal{D} \to M_L(\mathbb{C}), \quad z \mapsto (z, g)$$

the fiber of ω at $z \in \mathcal{D}$ is the isotropic line $\mathbb{C}z = V_{\mathbb{C}}^{(1,-1)} \subset V_{\mathbb{C}}$. We then endow ω with the metric $||z||^2 := -[z, \overline{z}]$, called the Petersson metric, we obtain a metrized line bundle

$$\widehat{\omega} \in \widehat{\mathrm{Pic}}(\mathcal{M}_L).$$

6.2 Integral models of big CM cycles

Consider now the big CM cycles $\mathcal{Y}_L$ (associated to the compact open subgroup $K_T := K_L \cap K_0$) and $\mathcal{Y}_0$ constructed in the previous sections. By construction of $\mathcal{M}_L$ the morphism $Y_L(\mathbb{C}) \to M_L(\mathbb{C})$ extends to a morphism $\mathcal{Y}_L \to \mathcal{M}_L$. As $K_T \subset K_0$ we get the following diagram

$$\begin{array}{ccc} \mathcal{Y}_L & \longrightarrow & \mathcal{M}_L \\ \downarrow & & \\ \mathcal{Y}_0 & & \end{array}.$$

In particular we have two metrized line bundles on $\mathcal{Y}_L$: the pull back $\widehat{\omega}_{\mathcal{Y}_L}$ of $\widehat{\omega}$ and the pull back $\widehat{\omega}_{0,\mathcal{Y}_L}$ of $\widehat{\omega}_0$.

For the first we will be able to compute the arithmetic degree using work of Bruinier, Kudla and Yang. The second is related to the Faltings' height of the abelian variety $A^\sharp$ which has an action of $\mathcal{O}_{E^\sharp}$. Let us start with this connection. Recall from Proposition 5.8 that we have a strict morphism of filtered $\mathcal{O}_{\mathcal{Y}_0}$-module

$$\ell \colon \mathbb{V}_{0,\mathrm{dR}} \to \mathrm{End}(\mathbb{H}^\sharp)$$

where $H^\sharp$ is the de Rham homology of $\mathcal{A}^\sharp$ (the extension of $A^\sharp$ to an abelian scheme over $\mathcal{A}^\sharp$). Then $\omega_0 = \mathrm{Fil}^1\mathbb{V}_{0,\mathrm{dR}}$ maps to the endomorphisms of $\mathbb{H}^\sharp$ sending $\mathrm{Gr}^{-1}\mathbb{H}^\sharp \to \mathrm{Fil}^0\mathbb{H}^\sharp$, i.e., upon taking determinants we have a map of $\mathcal{O}_{\mathcal{Y}_0}$-modules

$$\omega_0^{2^{d-1}} \otimes_{\mathcal{O}_{\mathcal{Y}_0}} \mathrm{det}\mathrm{Gr}^{-1}\mathbb{H}^\sharp \longrightarrow \mathrm{det}\mathrm{Fil}^0\mathbb{H}^\sharp.$$

By definition of the Hodge filtration of $\mathbb{H}^\sharp$ we have $(\mathrm{det}\mathrm{Gr}^{-1}\mathbb{H}^\sharp)^{-1} = \omega^\sharp$, the Hodge bundle of $\mathcal{A}^\sharp$. We have

Theorem 6.2. *The following holds*

$$\frac{1}{2^d}\sum_{\Phi} h^{\mathrm{Falt}}_{(E,\Phi)} = \frac{1}{4}\frac{\widehat{\deg}(\widehat{\omega}_0)}{\deg_{\mathbb{C}}(Y_0)} - \frac{1}{4}\log|D_F| + \frac{1}{2}d\cdot\log(2\pi).$$

Proof. We first use (1) to relate $\sum_\Phi h^{\mathrm{Falt}}_{(E,\Phi)}$ with $h^{\mathrm{Falt}}_{(E^\sharp,\Phi^\sharp)}$ via the formula $2dh^{\mathrm{Falt}}_{(E^\sharp,\Phi^\sharp)} = \sum_\Phi h^{\mathrm{Falt}}_{(E,\Phi)}$. Second we know that the degree of the metrized Hodge bundle $\widehat{\Omega}^\sharp$ of $\mathcal{A}^\sharp$ satisfies

$$\frac{\mathrm{deg}\widehat{\Omega}^\sharp}{\mathrm{deg}Y_0} = 2dh^{\mathrm{Falt}}_{(E^\sharp,\Phi^\sharp)}.$$

Finally we study the inclusion of invertible sheaves:

$$\omega_0^{2^{d-1}} \subset (\mathrm{det}\mathrm{Gr}^{-1}\mathbb{H}^\sharp)^{-2} \otimes_{\mathcal{O}_{\mathcal{Y}_0}} \mathrm{det}\mathbb{H}^\sharp \cong (\omega^\sharp)^2 \otimes_{\mathcal{O}_{\mathcal{Y}_0}} \mathrm{det}\mathbb{H}^\sharp.$$

One proves that the metrics on the two sides coincide; here $\mathrm{det}\mathbb{H}^\sharp$ is endowed with the standard metric given by integration over $A^\sharp(\mathbb{C})$ of top degree C^∞ de Rham classes. Its arithmetic degree is

$$\dim(A^\sharp)\cdot\mathrm{deg}Y_0\cdot\log(2\pi) = 2^d d\cdot\mathrm{deg}Y_0\cdot\log(2\pi).$$

We are left to study the cokernel of the displayed inclusion. A delicate algebra computation, see [AGHMP2, Prop. 9.4.1], shows that the difference of these line bundles has degree $2^{d-1}\mathrm{deg}Y_0\log|D_F|$ and the claim follows. □

Next we compare the metrized line bundles $\widehat{\omega}_{\mathcal{Y}_L}$ of $\widehat{\omega}$ and $\widehat{\omega}_{0,\mathcal{Y}_L}$.

Proposition 6.3. *We have* $\deg(\widehat{\omega}_{\mathcal{Y}_L}) \sim_L \deg(\widehat{\omega}_{0,\mathcal{Y}_L}) + \log|D_F|$ *where* $\sim_L$ *means equal up to rational linear combinations of log of primes dividing* $D_{L,\mathrm{bad}}$*; see Definition 5.1.*

Proof. Over $\mathbb{Q}$ the two sheaves coincide as they are associated to the same Hodge structure, namely V. Via this identification the metric on ω is the metric on ω_0 times $\sigma_0(\lambda)$ (recall that the quadratic form on V is defined by $x \mapsto \mathrm{Tr}_{F/\mathbb{Q}}(\lambda x\bar{x})$). We have only to check that over $\mathcal{Y}_L[D_{L,\mathrm{bad}}^{-1}]$ we have $\omega = \lambda\mathfrak{D}_{F/\mathbb{Q}}^{-1}\otimes_{\mathcal{O}_F}\omega_0$. Using Kisin's correspondence one is reduced to prove this statement for the lattices L and L_0 of V and the claim follows by local calculations (see the proof of [AGHMP2, Prop. 9.5.1] using Definition 5.1). □

7 The Bruinier, Kudla, Yang conjecture

Thanks to Theorem 6.2 and Proposition 6.3 and the $\mathbb{Q}$-linear independence of log of primes, in order to conclude the proof of the averaged version of Colmez's conjecture we need to prove the following

Theorem 7.1. *The degree of* $\widehat{\omega}_{\mathcal{Y}_L}$ *satisfies*

$$\frac{\deg \widehat{\omega}_{\mathcal{Y}_L}}{\deg Y_L} \sim_L -\frac{2\Lambda'(\chi,0)}{\Lambda(\chi,0)} - d\log(4\pi e^{\gamma}).$$

Moreover $\cap_L D_{L,\mathrm{bad}} = \emptyset$.

Here $\Lambda(\chi,0)$ is the complete L-function associated to the character χ. The fact that $\cap_L D_{L,\mathrm{bad}} = \emptyset$ can be turned via (3) into a class field theory question that we do no discuss here; we refer to [AGHMP2, Prop. 9.5.2] for details.

The main idea is then to realize $\widehat{\omega}$ as a combination of arithmetic divisors using Borcherds theory and results of Bruinier, Kudla, Yang that compute the contribution at infinity of the intersection of these arithmetic divisors with $\mathcal{Y}_L$. Bruinier, Kudla, Yang provided also conjectures for the contribution at finite places that we verify in [AGHMP2] in sufficiently many cases to get the result.

7.1 The Kuga-Stake abelian scheme

Consider the representation $C(V)$ of $G = \mathrm{GSpin}(V)$ via the inclusion $G \subset C^+(V)^*$ and the map $C^+(V)^* \,\mathrm{GL}(C(V))$ provided by left multiplication. As explained in §4 it provides a variation of Hodge structures: for any $z \in D$, we have

$$C(V)_z = C(V)_z^{-1,0} \oplus C(V)_z^{0,-1}$$

which is equivalent to giving a complex structure on $C(V_{\mathbb{R}})$ and, in particular, we obtain a complex abelian variety

$$A_z := C(V_{\mathbb{R}})/C(L),$$

called the *Kuga-Satake abelian variety*. It is proven in [AGHMP2] that it defines an abelian scheme

$$\mathcal{A} \longrightarrow \mathcal{M}_L.$$

The associated vector bundle with connection $C(V)_{\mathrm{dR}}$ extends to $\mathcal{M}_L$ and coincides the relative de Rham homology

$$C(V)_{\mathrm{dR}} = \mathcal{H}_{1,\mathrm{dR}}(\mathcal{A}),$$

the connection is the so called *Gauss-Manin connection* and the filtration is given by the Hodge filtration

$$0 \longrightarrow \mathrm{R}_1\pi_*(\mathcal{O}_{\mathcal{A}})^\vee \to H_{1,\mathrm{dR}}(\mathcal{A}) \longrightarrow \pi_*(\Omega^1_{\mathcal{A}})^\vee \longrightarrow 0.$$

Example 1. Recall that $n = 0$ and $C^+(L) \subset C^+(V) = K$ is an order in the quadratic imaginary field K. In this case

$$A_z = A_z^+ \times A_z^-$$

where A_z^+ is an elliptic curve with complex multiplication by $C^+(L)$ and

$$A_z^- = A_z^+ \otimes_{C^+(L)} L.$$

Example 2. In this case $V = M_{2\times 2}(\mathbb{Q})^{Tr=0}$, we have

$$M(\mathbb{C}) \cong Y_0(N)(\mathbb{C})$$

by $z \mapsto [E_z \to E_z']$and

$$A_z = A_z^+ \times A_z^-$$

and

$$A_z^+ = A_z^- = E_z \times E_z'.$$

7.2 Heegner divisors

In this section we will show how, given an element $\lambda \in V$ with $Q(\lambda) > 0$, we can construct a divisor in $M_L(\mathbb{C})$ as a Shimura subvariety. These will give the Heegner divisors mentioned in the introduction. The fact that we have such a large supply of easily constructed divisors, and in general of cycles of higher codimension obtained by intersection such divisors, make the theory of GSpin-Shimura varieties extremely rich.

Given λ as above, set $V_\lambda := \lambda^\perp \subset V$. This is a dimension $(n-1)+2$ subspace of V and $Q_\lambda := Q|_{V_\lambda}$ is a quadratic form of signature $(n-1, 2)$. Then we get a subgroup

$$G_\lambda = \mathrm{GSpin}(V_\lambda, Q_\lambda) \subset \mathrm{GSpin}(V) = G$$

The symmetric space D_λ for G_λ is identified with

$$D_\lambda = \left\{[z] \in D_{\mathbb{C}} \subset V_{\mathbb{C}} \backslash \{0\} : z \in V_{\lambda,G} = \lambda^\perp\right\}/\mathbb{C}^*$$

Let $L_\lambda := L \cap V_\lambda$ so we have $K_\lambda \subset G_\lambda(\mathbb{A}_f)$ and we get a GSpin-Shimura variety

$$M_\lambda(\mathbb{C}) = G_\lambda(\mathbb{Q}) \backslash D_\lambda \times G_\lambda(\mathbb{A}_f)/K_\lambda$$

together with a homomorphism

$$M_\lambda(\mathbb{C}) \to M_L(\mathbb{C}) = G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f)/K_L$$

Notice that such map is in general not an injection but the image of this map consists of divisors of $M_L(\mathbb{C})$. The discrepancy between $M_\lambda(\mathbb{C})$ and its image in $M_L(\mathbb{C})$ makes the intersection theory of Heegner divisors more involved. We will ignore this issue here for sake of simplicity and pretend that we can identify $M_\lambda(\mathbb{C})$ and its image. We refer to [AGHMP2] for the correct treatment using stacks.

Definition 7.2. For any $m \in \mathbb{N}_{>0}$ and every $\mu \in L^\vee/L$, let

$$Z(m,\mu)(\mathbb{C}) := \amalg_{g \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_L} \Gamma_g \backslash \big(\amalg_{\substack{\lambda \in \mu_g + L_g \\ Q(\lambda)=m}} \mathcal{D}_\lambda \big).$$

where $\Gamma_g = G(\mathbb{Q}) \cap gK_Lg^{-1}$, $L_g \subset V$ is the lattice $V \cap (g\widehat{L}g^{-1})$, and $\mu_g \in L_g^\vee/L_g$ is the class of $g\mu g^{-1}$.

Recall that $M_L(\mathbb{C}) = \amalg_{g \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_L} \Gamma_g \backslash \mathcal{D}$ so that we have a natural morphism

$$Z(m,\mu)(\mathbb{C}) \to M_L(\mathbb{C})$$

whose image is the union of the images of various $M_\lambda(\mathbb{C})$.

The image of $Z(m,\mu)(\mathbb{C})$ singles out points z of $M_L(\mathbb{C})$ where the $\mathbb{Z}$-Hodge structure $\mathbb{V}^\vee_{\mathrm{Betti},z}$ acquires a Hodge $(0,0)$-class λ_z of norm $Q_{\mathrm{Betti}}(\lambda_z) = m$ and whose class in $\mathbb{V}^\vee_{\mathrm{Betti},z}/\mathbb{V}_{\mathrm{Betti},z}$, which can be canonically identified with $L^\vee/L$, is $\mu \in L^\vee/L$. We elaborate.

Consider $\ell \colon V \hookrightarrow \mathrm{End}(C(V))$ given by

$$v \mapsto \ell_v := \{ \text{ left multiplication by } v \text{ on } C(V) \}.$$

It is a morphism in the category of representation of G which induces a morphism of Hodge structures

$$\ell_{\mathrm{Betti}} \colon \mathbb{V}_{\mathrm{Betti}} \hookrightarrow \mathrm{End}(\mathcal{H}_1(A,\mathbb{Q})) \tag{10}$$

and a morphism of vector bundle with connections over M_L

$$\ell_{\mathrm{dR}} \colon V_{\mathrm{dR}} \hookrightarrow \mathrm{End}(\mathcal{H}_{1,\mathrm{dR}}(A)). \tag{11}$$

We can give an intrinsic characterization of the image of $Z(m,\mu)(\mathbb{C})$ in $M_L(\mathbb{C})$; take $(z,g) \in M_L(\mathbb{C})$ and consider the corresponding element $\ell_{\mathrm{Betti}}(\lambda) \in \mathrm{End}(\mathrm{H}_{1,\mathrm{Betti}}(A_{(z,g)}))$. Then,

Proposition 7.3. *We have (z,g) is in the image of $\mathcal{D}_\lambda$ if and only if the pair $(\ell_{\mathrm{Betti}}(\lambda), \ell_{\mathrm{dR}}(\lambda))$ arise as the Betti and de Rham realizations respectively of an endomorphism $\ell_\lambda \in \mathrm{End}(A_{(z,g)})$.*

Proof. We have $(z,g) \in D_\lambda$ if and only if

$$\lambda \in z^\perp = \mathrm{Fil}^0\mathbb{V}_{\mathrm{dR},z} \subset \mathbb{V}_{\mathrm{dR},z}$$

if and only if the element $\ell_{\mathrm{dR}}(\lambda) \in \mathrm{End}(\mathrm{H}_{1,\mathrm{dR}}(A_{(z,g)}))$ of (11) lies in $\mathrm{Fil}^0\mathrm{End}(\mathrm{H}_{1,\mathrm{dR}}(A_{(z,g)}))$, i.e., $\ell_{\mathrm{dR}}(\lambda)$ preserves the Hodge filtration of $\mathrm{H}_{1,\mathrm{dR}}(A_{(z,g)})$. This is equivalent to require that the element $\ell_{\mathrm{Betti}}(\lambda)$ defines an endomorphism of $A_{(z,g)}$. □

As before take $\lambda \in V$ be an element with $Q(\lambda) > 0$. The next lemma shows that the images of $Y_L(\mathbb{C})$ and $M_\lambda(\mathbb{C})$ in $M_L(\mathbb{C})$ do not intersect. This will imply that for the associated arithmetic objects, i.e., the associated objects over $\mathbb{Z}$, we have *proper intersection*. This is not at all the case for the small CM points of Remark 5.2 where we have *proper intersection*; see [AGHMP1] for a discussion.

Lemma 7.4. *The intersection of the images $Y_L(\mathbb{C})$ and $M_\lambda(\mathbb{C})$ in $M_L(\mathbb{C})$ is empty.*

Proof. Assume that we have an element $z \in D_\lambda$ whose image on $(z, g) \in M_L(\mathbb{C})$ lies in the image of $Y_L(\mathbb{C})$. This is equivalent to saying that $\lambda \in V_{\mathrm{Betti},\mathbb{Q},z}$ is such that $\lambda \in (\mathbb{C}z \oplus \mathbb{C}\overline{z})^\perp$. But $V_{\mathrm{Betti},\mathbb{Q},z} = V = E$ as a $\mathbb{Q}$-vector space and we get that

$$V = E \cdot \lambda \subset (\mathbb{C}z \oplus \mathbb{C}\overline{z})^\perp$$

as the pairing on V is E-hermitian. This is clearly a contradiction as $(\mathbb{C}z \oplus \mathbb{C}\overline{z})^\perp \oplus (\mathbb{C}z \oplus \mathbb{C}\overline{z}) = V_\mathbb{C}$. □

7.3 Integral models of Heegner divisors

We use the previous section to define models of the Heegner divisors $Z(m, \mu)$ over $\mathbb{Q}$ and even their integral models $\mathcal{Z}(m, \mu)$ over $\mathbb{Z}$ as the functor representing pairs (ρ, f) where $\rho\colon S \to \mathcal{M}_L$ and $f \in \mathrm{End}^0(\mathcal{A} \times_{\mathcal{M}_L} S)$ is a rational endomorphism of the Kuga-Satake abelian scheme $\mathcal{A}_S := \mathcal{A} \times_{\mathcal{M}_L} S$ and

i. $f \circ f = [m]$ (multiplication by m map on $\mathcal{A}_S$);

ii. the endomorphism defined by f on the de Rham homology of $\mathcal{A}_S$, on the Tate module of $\mathcal{A}_S$ and on the crystal defined by $\mathcal{A}_S$ is in the image of a class of the de Rham realization, étale or crystalline realization of $L^\vee$ of class μ.

We explain what me mean in (ii). Over $\mathbb{Z}[\Delta_L^{-1}]$ the lattice $L \subset V$ defines a *motive* $\mathbb{V}$ over $\mathcal{M}_L$, namely a $\mathbb{Z}$-variation of Hodge structures over $M_L(\mathbb{C})$, a filtered vector bundle with connection over $\mathcal{M}_L$, a lisse ℓ-adic étale sheaf over $\mathcal{M}_L[\ell^{-1}]$, a crystal over $\mathcal{M}_L \otimes \mathbb{F}_p$. The map ℓ defines an embedding of motives $\mathbb{V} \subset \mathrm{End}(\mathbb{H})$ where $\mathbb{H}$ is the motive associated to the abelian scheme $\mathcal{A}$: its de Rham homology, its ℓ-adic Tate module, its Dieudonné module. In (ii) we ask that the realization of f lies in the image of $\mathbb{V}$.

To extend this notion to the whole of $\mathbb{Z}$ one works with auxiliary lattices $L \subset L^\diamond$, that are self dual at a given prime p and demands this condition working on $\mathcal{M}_{L^\diamond}$. A result analogous to Proposition 6.1 implies that we get a well-posed definition, independent of the auxiliary choice of $L^\diamond$.

It is proven in [AGHMP2, Prop. 4.5.8] that the models $\mathcal{Z}(m, \mu)$ have good properties, namely they do not have vertical components that would create troubles in computing arithmetic intersections:

Proposition 7.5. *If V has dimension ≥ 5 then the $\mathcal{Z}(m, \mu)$'s are flat over $\mathbb{Z}[1/2]$ and even over $\mathbb{Z}$ if L is self dual at 2.*

7.4 Heegner divisors and $\widehat{\omega}$

We now come to the main result express $\widehat{\omega}$ as a combination of arithmetic Heegner divisors. Thanks to Borcherds' theory in fact the Heegner divisors $\mathcal{Z}(m,\mu)$ are endowed with natural Green functions $\Phi_{m,\mu}$ (see [BY, (4.7)]). And we can consider the pair $\widehat{\mathcal{Z}}(m,\mu) := (\mathcal{Z}(m,\mu), \Phi_{m,\mu}) \in \widehat{CH}^1(\mathcal{M}_L)$. We then have the following fundamental result:

Theorem 7.6. *Suppose that $n \geq 3$. There are finitely many integers $c(-m,\mu)$ for $m \geq 0$ and $\mu \in L^\vee/L$ with $c(0,0) \neq 0$ and there exists a rational section Ψ of $\omega^{\otimes c(0,0)}$, defined over $\mathbb{Q}$, such that*

$$\begin{aligned}\widehat{\omega}^{\otimes c(0,0)} &= \widehat{\operatorname{div}}(\Psi) \\ &= \sum_{m,\mu} c(-m,\mu)\widehat{\mathcal{Z}}(m,\mu) - c_f(0,0)\cdot(0,\log(4\pi e^\gamma)) + \widehat{\mathcal{E}},\end{aligned}$$

where $(0,\log(4\pi e^\gamma))$ denotes the trivial divisor endowed with the constant Green function $\log(4\pi e^\gamma)$ (here $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant), and $\widehat{\mathcal{E}} = (\mathcal{E}, 0)$ is a divisor with the trivial Green function that decomposes

$$\mathcal{E} = \sum_{p \mid D_L} \mathcal{E}_p$$

such that $\mathcal{E}_p$ is supported on the special fiber $\mathcal{M}_{L,\mathbb{F}_p}$, and:

- *If p is odd and $p^2 \nmid D_L$ then $\mathcal{E}_p = 0$;*
- *If $n \geq 5$ then $\mathcal{E} = \mathcal{E}_2$ is supported on $\mathcal{M}_{L,\mathbb{F}_2}$.*
- *If $n \geq 5$ and $L_{(2)}$ is self-dual, then $\mathcal{E} = 0$.*

Given our lattice L coming from the CM field E will use the theorem first for the lattice L. We will need it it also for some auxiliary lattice $L \subset L^\diamond$, self dual at a given bad prime p. In this case some care is needed to assure that the Heegner divisors on $\mathcal{M}_{L^\diamond}$ do *not* contain the big CM cycle, i.e., that we have proper intersection. In particular $\mathcal{E}_p = 0$ and we will compute the contribution at p of $\deg(\widehat{\omega}_{\mathcal{Y}_L})$ using the expression of $\widehat{\omega}$ in terms of the divisors $\widehat{\mathcal{Z}}(m,\mu)$ provided by the theorem above.

7.5 Arithmetic intersection and special values

Write $[\mathcal{Y}_L : \widehat{\mathcal{Z}}]$ for the arithmetic degree of the base change of $\widehat{\mathcal{Z}}$ to $\mathcal{Y}_L$. The main result that finishes the proof of the averaged version of Colmez's conjecture is the following:

Theorem 7.7. *Consider the divisor $\widehat{\mathcal{Z}} := \sum_{m,\mu} c(-m,\mu)\widehat{\mathcal{Z}}(m,\mu)$ of theorem 7.6. Then*

$$\frac{[\mathcal{Y}_L : \widehat{\mathcal{Z}}]}{\deg(Y)} \sim_L -2\frac{\Lambda'(0,\chi)\cdot c(0,0)}{\Lambda(0,\chi)}$$

(recall that $\sim_L$ means equality up to a $\mathbb{Q}$-linear combinations of $\{\log(p) : p \mid D_{L,\mathrm{bad}}\}$).

There are two inputs in the proof of this result. The first is the computation of the contribution at infinity of the degree, provided by the following theorem of Bruinier, Kudla and Yang:

Theorem 7.8. *Let Φ be the Green function associated to the arithmetic divisor $\widehat{\mathcal{Z}}$ of the previous theorem.*

$$\frac{\Phi(\mathcal{Y}_L^\infty)}{2\deg(Y_L)} = \sum_{\substack{\mu\in L^\vee/L\\ m\geq 0}} \frac{a(m,\mu)\cdot c(-m,\mu)}{\Lambda(0,\chi)},$$

where $\mathcal{Y}^\infty = Y_L \times_{\mathbb{Q}} \mathbb{C}$ and $\Phi(\mathcal{Y}^\infty)$ is the weighted sum of the values of Φ

$$\Phi(\mathcal{Y}^\infty) = \sum_{y\in\mathcal{Y}^\infty(\mathbb{C})} \frac{\Phi(y)}{|\mathrm{Aut}(y)|}.$$

Here the $a(m,\mu)/\Lambda(0,\chi)$'s are the coefficients of the formal q-expansion of the restriction of the derivative of a suitable weight 1 Hilbert modular Eisenstein series introduced by Kudla [Ku]. In order to deduce Theorem 7.7 from the result of [BKY] one needs to:

1. compute the coefficients $a(m,\mu)$. Typically they are expressed as orbital integrals and one wants to get explicit quantities;

2. prove that the *finite* part of the intersection is $\mathcal{Z}(m,\mu)$ along $\mathcal{Y}_L$ is $a(m,\mu)/\Lambda(0,\chi)$.

Both calculations need to be done up to $\mathbb{Q}$-linear combinations of $\log(p)$'s for $p \mid D_{L,\mathrm{bad}}$. Computation (1) is due to Kudla and Yang [KY] (and [AGHMP2, §6] for the contribution at the prime $p=2$). Claim (2) is proven in §7 of [AGHMP2]. It is a remarkable instance of Kudla's programme relating generating series of (arithmetic) intersection numbers and automorphic forms.

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