BEYOND HEIGHTS: SLOPES AND DISTRIBUTION OF RATIONAL POINTS

by

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Abstract. — The distribution of rational points of bounded height on algebraic varieties is far from uniform. Indeed the points tend to accumulate on thin subsets which are images of non-trivial finite morphisms. The problem is to find a way to characterise the points in these thin subsets. The slopes introduced by Jean-Benoît Bost are a useful tool for this problem. These notes will present several cases in which this approach is fruitful. We shall also describe the notion of locally accumulating subvarieties which arises when one considers rational points of bounded height near a fixed rational point.

Résumé. — La distribution des points rationnels de hauteur bornée sur les variétés algébriques est loin d'être uniforme les points peuvent s'accumuler sur l'image de variétés formant un ensemble mince. La difficulté est de pouvoir caractériser les points de ces ensembles accumulateurs. Les pentes de la géométrie d'Arakelov forment un outil utile pour attaquer cette problématique. Ces notes présenteront différents exemples où cette approche est efficace. On évoquera également la question des sous-variétés localement accumulatrices qui apparaissent lorsqu'on considère les points de hauteur bornée au voisinage d'un point rationnel.

Contents

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1. Introduction

For varieties with infinitely many rational points, one may equip the variety with a height and study asymptotically the finite set of rational points with a bounded height. The study of many examples shows that the distribution of rational points of bounded height on algebraic varieties is far from uniform. Indeed the points tend to accumulate on thin subsets which are images of non-trivial finite morphisms. It is natural to look for new invariants to characterise the points in these thin subsets. First of all, it is natural to consider all possible heights, instead of one relative to a fixed line bundle. But the geometric analogue described in section 5 suggests to go beyond heights to find a property similar to being very free for rational curves. The slopes introduced by Jean-Benoît Bost give the tool for such a construction. In section 6, we describe the notion of freeness which measures how free a rational point is. This section will present several cases in which this approach is fruitful. In section 7, we also describe its use in connection with the notion of locally accumulating subvarieties which arises when one considers rational points of bounded height near a fixed rational point.

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2. Norms and heights

2.1. Adelic metric. — In this chapter, I am going to use heights defined by an adelic metric, which I use in a more restrictive sense than in the rest of the volume. In fact, an adelic metric will be an analogue of the notion of Riemannian metric in the adelic setting. Let me fix some notation for the remaining of this chapter.

Notation 2.1. — The letter $\mathbf{K}$ denotes a number field. The set of places of $\mathbf{K}$ is denoted by $\mathrm{Val}(\mathbf{K})$, and its set of finite places by $\mathrm{Val}(\mathbf{K})_f$. Let w be a place of $\mathbf{K}$. We denote by $\mathbf{K}_w$ the completion of $\mathbf{K}$ at w. For an ultrametric place, $\mathscr{O}_w$ is the ring of integers of $\mathbf{K}_w$ and $\mathfrak{m}_w$ its maximal ideal. Let $v \in \mathrm{Val}(\mathbf{Q})$ denote the restriction of w to $\mathbf{Q}$. We consider the map $|\cdot|_w : \mathbf{K}_w \to \mathbf{R}_{\geqslant 0}$ defined by

$$|x|_w = |N_{\mathbf{K}_w/\mathbf{Q}_v}(x)|_v$$

for $x \in \mathbf{K}_w$, where $N_{\mathbf{K}_w/\mathbf{Q}_v}$ denotes the norm map. The Haar measure on the locally compact field $\mathbf{K}_w$ is normalized as follows:

a) $\int_{\mathscr{O}_w} \mathrm{d}x_w = 1$ for a non-archimedean place w;

b) $\mathrm{d}x_w$ is the usual Lebesgue measure if w is real;

c) $\mathrm{d}x_w$ is twice the usual euclidean measure for a complex place.

Remark 2.2. — The map $|\cdot|_w$ is an absolute value if w is ultrametric or real, it is the square of the modulus for a complex place. This choice of notation is motivated by the fact that $|\lambda|_w$ is the multiplier of the Haar measure for the change of variables $y = \lambda x$:

$$\mathrm{d}y_w = |\lambda|_w \mathrm{d}x_w$$

and we have the product formula:

$$\prod_{w \in \mathrm{Val}(\mathbf{K})} |x|_w = 1$$

for any $x \in \mathbf{K}^*$.

Terminology 2.3. — The varieties we consider are integral separated schemes of finite type over a field. We shall say that a variety V is *nice* if it is smooth, projective, and geometrically integral.

Notation 2.4. — Let X be a variety over $\mathbf{K}$.. For any commutative $\mathbf{K}$-algebra A, we denote by X_A the product $X \times_{\mathrm{Spec}(\mathbf{K})} \mathrm{Spec}(A)$ and by $X(A)$ the set of A-points which is defined as $\mathrm{Mor}_{\mathrm{Spec}(\mathbf{K})}(\mathrm{Spec}(A), X)$. For any place w of $\mathbf{K}$, we equip $X(\mathbf{K}_w)$ with the w-adic topology.

For the rest of this chapter, we denote by V a nice variety on the number field $\mathbf{K}$. The Picard group of V, denoted by $\mathrm{Pic}(V)$, is thought as the set of isomorphism classes of line bundles on V. A line bundle L is said to be *big* if a multiple of its class may be written as a sum of an ample class and an effective one.

Definition 2.5. — Let $\pi : E \to V$ be a vector bundle on V. For any extension $\mathbf{L}$ of $\mathbf{K}$ and any $\mathbf{L}$-point P of V, we denote by $E_P \subset E(\mathbf{L})$ the $\mathbf{L}$-vector space corresponding to the fiber $\pi^{-1}(P)$ of π at P. In this text, a *classical adelic norm on E* is a family $(||\cdot||_w)_{w \in \mathrm{Val}(\mathbf{K})}$ of continuous maps

$$||\cdot||_w : E(\mathbf{K}_w) \to \mathbf{R}_{\geqslant 0}$$

such that:

(i) If w is non-archimedean, for any $P \in V(\mathbf{K}_w)$, the restriction $||\cdot||_{w|E_P}$ is an ultrametric norm with values in $\mathrm{im}(|\cdot|_w)$;

(ii) If $\mathbf{K}_w$ is isomorphic to $\mathbf{R}$, then, for any P in $V(\mathbf{K}_w)$, the restriction $||\cdot||_{w|E_P}$ is a euclidean norm;

(iii) If $\mathbf{K}_w$ is isomorphic to $\mathbf{C}$, then, for any P in $V(\mathbf{K}_w)$, there exists a positive definite hermitian form ϕ_P on E_P such that

$$\forall y \in E_P, \quad ||y||_w = \phi_P(y, y);$$

(iv) There exists a finite set of places $S \subset \mathrm{Val}(\mathbf{K})$ containing the set of archimedean places and a model $\mathscr{E} \to \mathscr{V}$ of $E \to V$ over $\mathscr{O}_S$ such that for any place w in $\mathrm{Val}(\mathbf{K}) - S$ and any $P \in \mathscr{V}(\mathscr{O}_w)$

$$\mathscr{E}_P = \{\, y \in E_P \mid ||y||_w \leqslant 1 \,\},$$

where $\mathscr{E}_P$ denotes the $\mathscr{O}_w$-submodule of E_P defined by $\mathscr{E}$.

In the rest of this chapter, we shall say adelic norm for classical adelic norm. An *adelically normed vector bundle* is a vector bundle equipped with an adelic norm. We call *adelic metric* an adelic norm on the tangent bundle TV.

The point of using this type of norms is that you can do all the usual constructions:

Examples 2.6. — a) If E and F are vector bundles equipped with classical adelic norms, then we can define adelic norms on the dual $E^\vee$, the direct sum $E \oplus F$ and the tensor product $E \otimes F$.

b) If E is a vector bundle equipped with a classical norm, then we define a classical norm on the exterior product $\Lambda^m E$ in the following manner (see also chapter II, §2.2). Let $P \in V(\mathbf{K}_w)$. If w is an ultrametric space, then let

$$\mathscr{E}_P = \{\, y \in E_P \mid \|y\|_w \leqslant 1 \,\}.$$

The set $\mathscr{E}_P$ is a $\mathscr{O}_w$-submodule of E_P of maximal rank. Then we take on $\Lambda^m E_P$ the norm defined by the module $\Lambda^m \mathscr{E}_P$. In the archimedean case, we choose the norm on $\Lambda^m E_P$ so that if $(e_1, \dots, e_r)$ is an orthonormal basis of E_P then the family $(e_{k_1} \wedge e_{k_2} \wedge \dots \wedge e_{k_m})_{1 \leqslant k_1 < k_2 < \dots < k_m \leqslant r}$ is an orthonormal basis of $\Lambda^m E_P$.

c) It is possible to define pull-backs for morphisms of nice varieties over $\mathbf{K}$.

d) If $V = \operatorname{Spec}(\mathbf{K})$, then we may consider a vector bundle on V as a $\mathbf{K}$-vector space. Let E be a $\mathbf{K}$ vector space of dimension r equipped with an adelic norm $(\|\cdot\|)_{w \in \operatorname{Val}(\mathbf{K})}$. Then

$$\mathscr{E} = \{\, y \in E \mid \forall w \in \operatorname{Val}(\mathbf{K})_f, \|y\|_w \leqslant 1 \,\}$$

is a projective $\mathscr{O}_\mathbf{K}$ module of constant rank r.

If $r = 1$, by the product formula, the product

$$\prod_{w \in \operatorname{Val}(\mathbf{K})} \|y\|_w$$

is constant for $y \in E - \{0\}$. So we can define

$$\widehat{\deg}(E) = -\sum_{w \in \operatorname{Val}(\mathbf{K})} \log(\|y\|_w).$$

Let $\widehat{\operatorname{Pic}}(\operatorname{Spec}(\mathbf{K}))$ be the set of isomorphism classes of line bundles with an adelic norm on $\operatorname{Spec}(\mathbf{K})$. Let r_1 be the number of real places and r_2 the number of complex places. Let $\operatorname{Val}(\mathbf{K})_\infty \subset \operatorname{Val}(\mathbf{K})$ be the set of archimedean places. Let

$H \subset \mathbf{R}^{\mathrm{Val}(\mathbf{K})_\infty}$ be the hyperplane given by the equation $\sum_{w \in \mathrm{Val}(\mathbf{K})_\infty} X_w = 0$. Then the map

$$x \mapsto (\log(|x|_w))_{w \in \mathrm{Val}(\mathbf{K})_\infty}$$

induces a map from $\mathscr{O}_{\mathbf{K}}^*$ to H. Let $\boldsymbol{T}$ be the quotient of H by the image of this map. The group $\boldsymbol{T}$ is a compact torus of dimension $r_1 + r_2 - 1$ and we get an exact sequence

$$0 \longrightarrow \boldsymbol{T} \longrightarrow \widehat{\mathrm{Pic}}(\mathrm{Spec}(\mathbf{K})) \longrightarrow \mathrm{Pic}(\mathrm{Spec}(\mathscr{O}_{\mathbf{K}})) \times \mathbf{R} \longrightarrow 0$$

where a line bundle E equipped with an adelic norm $(\|\cdot\|_w)_{w \in \mathrm{Val}(\mathbf{K})}$ is sent on the pair $([\mathscr{E}], \widehat{\deg}(E))$ where $[\mathscr{E}]$ is the class of $\mathscr{E}$ in the ideal class group of $\mathscr{O}_{\mathbf{K}}$.

For arbitrary rank r, we may define:

$$\widehat{\deg}(E) = \widehat{\deg}(\Lambda^r(E)).$$

2.2. Arakelov heights

Definition 2.7. — For any vector bundle E over V equipped with an adelic norm, the corresponding *logarithmic height* is defined as the map $h_E : V(\mathbf{K}) \to \mathbf{R}$ given by $P \mapsto \widehat{\deg}(E_P)$, where E_P is the pull-back of E by the map $P : \mathrm{Spec}(\mathbf{K}) \to V$. The corresponding *exponential height* is defined by $H_E = \exp \circ h_E$.

Remark 2.8. — If $r = \mathrm{rk}(E)$, we have that $h_E = h_{\Lambda^r E} = h_{\det(E)}$. Therefore we do not get more than the heights defined by line bundles.

Example 2.9. — For any $w \in \mathrm{Val}(\mathbf{K})$, we may consider the map $\|\cdot\|_w : \mathbf{K}_w^{N+1} \to \mathbf{R}$ defined by

$$\|(y_0, \dots, y_N)\|_w = \max_{0 \leqslant i \leqslant N} |y_i|_w.$$

This does not define a classical norm on $\mathbf{K}_w^{N+1}$ in the sense above, however it defines a norm on the tautological line bundle as follows. Let $w \in \mathrm{Val}(\mathbf{K})$. The fibre of the tautological $\mathscr{O}_{\mathbf{P}^N_{\mathbf{K}}}(-1)$ over a point $P \in \mathbf{P}^N(\mathbf{K}_w)$ may be identified with the line corresponding to the point. By restricting $\|\cdot\|_w$ to these lines, we obtain an adelic norm $(\|\cdot\|_w)_{w \in \mathrm{Val}(\mathbf{K})}$ on $\mathscr{O}_{\mathbf{P}^N_{\mathbf{K}}}(-1)$ and by duality on $\mathscr{O}_{\mathbf{P}^N_{\mathbf{K}}}(1)$. If $(y_0, \dots, y_N) \in \mathbf{K}^{N+1} - \{0\}$, let P, also denoted by $[y_0 : \ldots : y_N]$, be the corresponding point in $\mathbf{P}^N(\mathbf{K})$. Then $y = (y_0, \dots, y_n) \in \mathscr{O}(-1)_P$ and we get the formula

$$H_{\mathscr{O}(-1)}(P) = \prod_{w \in \mathrm{Val}(K)} \|y\|_w^{-1}.$$

Thus $H_{\mathscr{O}(1)}(P) = \prod_{w\in\mathrm{Val}(\mathbf{K})} \|y\|_w$. In the case where $\mathbf{K} = \mathbf{Q}$ and $y_0, \dots, y_N$ are coprime integers, we have $\|(y_0, \dots, y_N)\|_v = 1$ for any finite place v and the height may be written as

$$H_{\mathscr{O}(1)}(P) = \max_{0\leqslant i\leqslant N} |y_i|$$

which is one of the naïve heights for the projective space.

Notation 2.10. — For any function $H : V(\mathbf{K}) \to \mathbf{R}$, any subset $W \subset V(\mathbf{K})$ and any positive real number B, we consider the set

$$W_{H\leqslant B} = \{ P \in V(\mathbf{K}) \mid H(P) \leqslant B \}.$$

Our aim is to study such sets for heights H as B goes to infinity. Let us motivate this study with a few pictures of such sets.

Examples 2.11. — Figure 1 represents rational points of bounded height in the projective plane. More precisely this drawing represents

$$\{ (x, y) \in \mathbf{Q}^2 \mid H_{\mathscr{O}(1)}(x : y : 1) < 40, |x| \leqslant 1 \text{ and } |y| \leqslant 1 \}.$$

Figure 2 represents rational points of bounded height in the one-sheeted hyperboloid defined by the equation $xy = zt$ in $\mathbf{P}^3_{\mathbf{Q}}$:

$$\{ P = (x, y) \in \mathbf{Q}^2 \mid H_{\mathscr{O}(1)}(xy : 1 : x : y) \leqslant 50, |x| \leqslant 1 \text{ and } |y| \leqslant 1 \}.$$

This quadric is the image of the Segre embedding

$$([u_1 : v_1], [u_2 : v_2]) \longmapsto [u_1u_2 : v_1v_2 : u_1v_2 : v_1u_2]$$

and therefore isomorphic to the product $\mathbf{P}^1_{\mathbf{Q}} \times \mathbf{P}^1_{\mathbf{Q}}$. The last picture represents

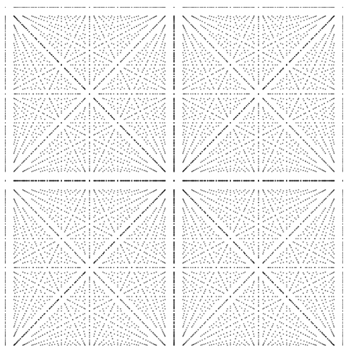

FIGURE 1. Projective plane

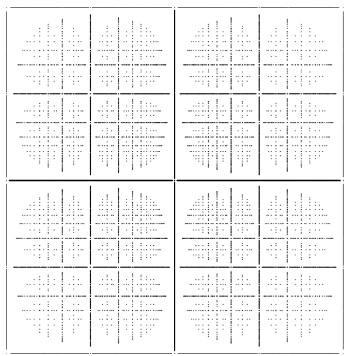

FIGURE 2. Hyperboloid

rational points of bounded height on the sphere:

$$\{ P = [x : y : z : t] \in \mathbf{P}^3(\mathbf{Q}) \mid H_{\mathscr{O}(1)}(P) \leqslant B \text{ and } x^2 + y^2 + z^2 = t^2 \}.$$

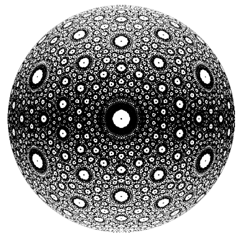

FIGURE 3. The sphere

Proposition 2.12. — *If L is a big line bundle and H a height relative to L, then there exists a dense open subset $U \subset V$ for Zariski topology such that for any $B \in \mathbf{R}_{\geqslant 0}$, the set $U(\mathbf{K})_{H \leqslant B}$ is finite.*

Proof. — It is enough to prove the result for a multiple of L. Thus we may assume that we can write L as $E + A$ where E is effective and A very ample. Taking U as the complement of the base locus of E, and choosing a basis $(s_0, \dots, s_N)$ of $\Gamma(V, L)$, we get an embedding

$$\varphi : U \longrightarrow \mathbf{P}^N_{\mathbf{K}}.$$

Using the height of example 2.9 on $\mathbf{P}^N_{\mathbf{K}}$, we get that

$$\frac{H(\varphi(x))}{H(x)} = \prod_{w \in \mathrm{Val}(\mathbf{K})} \max_{0 \leqslant i \leqslant N} \|s_i(x)\|_w.$$

Thus there exists a constant $C \in \mathbf{R}_{>0}$ such that

$$\forall x \in V(\mathbf{K}), \quad H(\varphi(x)) \leqslant CH(x).$$

Using Northcott theorem (see [**No1**], [**No2**]), the set of points of bounded height in the projective space is finite. A fortiori, the set $U(\mathbf{K})_{H \leqslant B}$ is finite. □

The height depends on the metric, but in a bounded way:

Proposition 2.13. — *Let H and H' be heights defined by adelic norms on a line bundle L then the quotient H/H' is bounded: there exist real constants* $0 < C < C'$ *such that*

$$\forall P \in V(\mathbf{K}), \quad C \leqslant \frac{H'(P)}{H(P)} < C'.$$

Proof. — The quotient of the norms $\frac{\|\cdot\|'_w}{\|\cdot\|_w}$ induces a continuous map from the compact set $V(\mathbf{K}_w)$ to $\mathbf{R}_{>0}$. Thus it is bounded from below and above. Moreover the adelic condition imposes that the norms coincide for all places outside a finite set. □

3. Accumulation and equidistribution

In this chapter, I shall first consider the distribution of rational points of bounded height on the variety.

3.1. Sandbox example: the projective space. — First, I have to explain what I mean by distribution. Let us for example consider the picture in figure 4. We have selected a "simple" open subset W in $\mathbf{P}^n(\mathbf{R})$, which is drawn in grey.

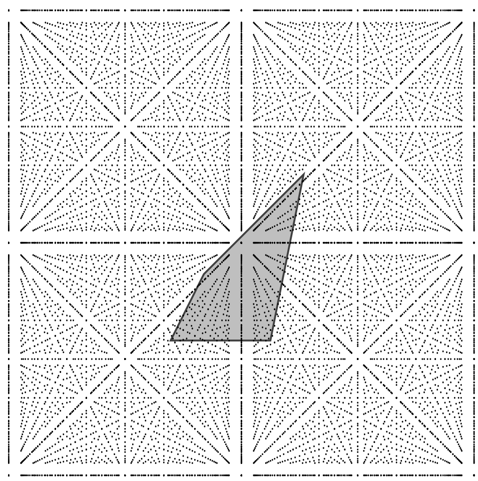

FIGURE 4. Open subset

We may then study asymptotically the proportion of rational points of bounded height in this open set. More precisely, one may formulate the following question:

Question 3.1. — *Does the quotient*

$$\frac{\sharp(W \cap \mathbf{P}^n(\mathbf{Q}))_{H \leqslant B}}{\sharp \mathbf{P}^n(\mathbf{Q})_{H \leqslant B}}.$$

have a limit as B goes to $+\infty$ and how can we interpret its value?

Similarly, let us fix some integer $M > 0$ and consider the reduction modulo M of the points. More precisely, let A be a commutative ring. The set of A-points of the projective space, denoted by $\mathbf{P}^n(A)$, is the set of morphisms from $\operatorname{Spec}(A)$ to $\mathbf{P}^n_{\mathbf{Z}}$. This defines a covariant functor from the category of rings to the category of sets. A $(n+1)$-tuple $(a_0, \dots, a_n)$ in A^{n+1} is said to be primitive if the generated ideal $(a_0, \dots, a_n)$ is A itself; this is equivalent to the existence of $(u_0, \dots, u_n) \in A^{n+1}$ such that $\sum_{i=0}^n u_i a_i = 1$. The group of invertible elements acts by multiplication on the set of primitive elements in A^{n+1}. Then the $\mathbf{Z}/M\mathbf{Z}$ points of the projective space $\mathbf{P}^n_{\mathbf{Z}}$ may be described as the orbits for the action of $(\mathbf{Z}/M\mathbf{Z})^*$ on the set of primitive elements in $(\mathbf{Z}/M\mathbf{Z})^{n+1}$. For any point P in $\mathbf{P}^n(\mathbf{Q})$, we may choose homogeneous coordinates $[y_0 : \ldots : y_n]$ so that $y_0, \dots, y_n$ are coprime integers. The reduction modulo M of P, is the point of $\mathbf{P}^n(\mathbf{Z}/M\mathbf{Z})$ defined by the primitive element $(\overline{y_0}, \dots, \overline{y_n})$, where $\bar{y}$ denotes the reduction modulo M of the integer y. This define a map

$$r_M : \mathbf{P}^n(\mathbf{Q}) \longrightarrow \mathbf{P}^n(\mathbf{Z}/M\mathbf{Z}).$$

This description of the reduction map generalises easily to any quotient of a principal ring. Then for any subset W of $\mathbf{P}^n(\mathbf{Z}/M\mathbf{Z})$, we may consider the question

Question 3.2. — *Does the quotient*

$$\frac{\sharp(r_M^{-1}(W))_{H \leqslant B}}{\sharp \mathbf{P}^N(\mathbf{Q})_{H \leqslant B}}$$

converges as B goes to infinity?

With the adelic point of view, we can see questions 3.1 and 3.2 as particular cases of the following more general question:

Question 3.3. — *Let $\mathbf{K}$ be a number field. Let*

$$\mathbf{P}^N(\mathbf{A_K}) = \prod_{w \in \operatorname{Val}(\mathbf{K})} \mathbf{P}^n(\mathbf{K}_w)$$

be the adelic projective space and let $f : \mathbf{P}^N(A_\mathbf{K}) \to \mathbf{R}$ be a continuous function. Does the quotient

$$S_B(f) = \frac{1}{\sharp \mathbf{P}^n(\mathbf{K})_{H\leqslant B}} \sum_{P\in\mathbf{P}^n(\mathbf{K})_{H\leqslant B}} f(P)$$

have a limit as B goes to infinity?

The answer is positive and we shall state it as a proposition:

Proposition 3.4. — *With the notations introduced in question 3.3,*

$$S_B(f) \xrightarrow[B\to+\infty]{} \int_{\mathbf{P}^n_\mathbf{K}(A_\mathbf{K})} f \boldsymbol{\mu}_{\mathbf{P}^n}$$

where $\boldsymbol{\mu}_{\mathbf{P}^n}$ is the probability measure given as the product $\prod_{w\in\mathrm{Val}(\mathbf{K})} \boldsymbol{\mu}_w$ where $\boldsymbol{\mu}_w$ is the borelian probability measure on $\mathbf{P}^n(\mathbf{K}_w)$ defined by:

— *If w is a non-archimedean place, let $\pi_k : \mathbf{P}^n(\mathbf{K}_w) \to \mathbf{P}^n(\mathscr{O}_w/\mathfrak{m}_w^k)$ be the reduction modulo $\mathfrak{m}_w^k$ then we equip $\mathbf{P}^n(\mathbf{K}_w)$ with the natural probability measure:*

$$\boldsymbol{\mu}_w(\pi_k^{-1}(W)) = \frac{\sharp W}{\sharp \mathbf{P}^n(\mathscr{O}_w/\mathfrak{m}_w^k)}$$

for any subset W of $\mathbf{P}^n(\mathscr{O}_w/\mathfrak{m}_w^k)$;

— *If w is archimedean, let $\pi : \mathbf{K}_w^{n+1} - \{0\} \to \mathbf{P}^n(\mathbf{K}_w)$ be the natural projection. Than μ_w is defined by*

$$\mu_w(U) = \frac{\mathrm{Vol}(\pi^{-1}(U)\cap B_{\|\cdot\|_w}(1))}{\mathrm{Vol}(B_{\|\cdot\|_w}(1))},$$

for any borelian subset U in $\mathbf{P}^n(\mathbf{K}_w)$, where $B_{\|\cdot\|_w}(1)$ denotes the ball of radius 1 for $\|\cdot\|_w$.

As a consequence, we may give a precise answer to questions 3.1 and 3.2:

Corollary 3.5. — *If W is an open subset of $\mathbf{P}^n(A_\mathbf{K})$ such that $\boldsymbol{\mu}_{\mathbf{P}^n}(\partial W) = 0$ then*

$$\frac{\sharp(W\cap \mathbf{P}^n(\mathbf{K}))_{H\leqslant B}}{\sharp \mathbf{P}^n(\mathbf{K})_{H\leqslant B}} \xrightarrow[B\to+\infty]{} \boldsymbol{\mu}_{\mathbf{P}^n}(W).$$

Sketch of the proof of proposition 3.4 for $\mathbf{K} = \mathbf{Q}$. — Take an open cube $\mathscr{C} = \prod_{i=0}^n]a_i, b_i[$ where a_i and b_i are real numbers with $a_i < b_i$ for $i \in \{0,\dots,n\}$, an integer $M \geqslant 1$ and an element $P_0 \in \mathbf{P}^n(\mathbf{Z}/M\mathbf{Z})$. We imbed $\mathbf{R}^n$ in $\mathbf{P}^n(\mathbf{R})$ and consider $\mathscr{C}$ as an open subset of the projective space. We choose a primitive element y_0 in $(\mathbf{Z}/M\mathbf{Z})^{n+1}$ representing P_0. We then want to estimate

$$\begin{aligned}
&\sharp\{\, P \in \mathbf{P}^n(\mathbf{Q}) \mid H(P) \leqslant B, P \in \mathscr{C} \text{ and } \pi_M(P) = P_0 \,\}\\
&= \frac{1}{2} \sum_{\lambda \in (\mathbf{Z}/M\mathbf{Z})^*} \sharp\{y \in \mathbf{Z}^{n+1} \mid y \text{ primitive}, \|y\|_\infty \leqslant B, y \in \pi^{-1}(\mathscr{C})\\
&\qquad\qquad \text{and } y \equiv \lambda y_0\, [M]\,\}\\
&= \frac{1}{2} \sum_{\substack{d>0\\ \lambda \in \mathbf{Z}/M\mathbf{Z}^*}} \mu(d) \sharp\{y \in (d\mathbf{Z})^{n+1} - \{0\} \mid \|y\|_\infty \leqslant B, y \in \pi^{-1}(\mathscr{C})\\
&\qquad\qquad \text{and } y \equiv \lambda y_0\, [M]\,\}
\end{aligned}$$

where $\mu : \mathbf{N} - \{0\} \to \{-1, 0, 1\}$ denotes the Möbius function. As y_0 is primitive, the set we obtained in the sum is empty if M and d are not coprime. Otherwise it is the intersection of the translation of a lattice of covolume $(dM)^{n+1}$, the cone $\pi^{-1}(\mathscr{C})$ and the ball $B_{\|\cdot\|_\infty}(B)$. Thus its cardinal may be approximated by

$$\frac{\mathrm{Vol}(\pi^{-1}(\mathscr{C}) \cap B_{\|\cdot\|_\infty}(1))B^{n+1}}{(dM)^{n+1}}$$

with an error term which is bounded up to a constant by $\left(\frac{B}{d}+1\right)^n$. Up to an error term left to the reader, we get that the sum is equivalent to

$$\frac{1}{2}\mathrm{Vol}(\pi^{-1}(\mathscr{C}) \cap B_{\|\cdot\|_\infty}(1)) \times \frac{\varphi(M)}{M^{n+1}\prod_{p|M}\left(1-\frac{1}{p^{n+1}}\right)} \times \frac{1}{\zeta_{\mathbf{Q}}(n+1)}B^{n+1}.$$

In this product, the term

$$\frac{\varphi(M)}{M^{n+1}\prod_{p|M}\left(1-\frac{1}{p^{n+1}}\right)}$$

is $\sharp(\mathbf{P}^n(\mathbf{Z}/M\mathbf{Z}))^{-1}$. In particular, this implies that

$$\frac{1}{2}\mathrm{Vol}(B_{\|\cdot\|_\infty}(1))/\zeta_{\mathbf{Q}}(n+1)$$

is the limit of $\sharp\mathbf{P}^n(\mathbf{Q})_{H \leqslant B}/B^{n+1}$ as B goes to infinity. $\square$

3.2. Adelic measure. — By choosing different norms on the anticanonical line bundle, and thus different heights on a variety, one realizes that the measure which gives the asymptotic distribution as B goes to infinity may be directly defined from the adelic norm on ω_V^{-1}, exactly as a Riemannian metric defines a volume form. This construction in fact applies to any nice variety equipped with an adelic metric.

***Construction 3.6.* —** Let V be a nice variety with a rational point. We fix an adelic norm $(\|\cdot\|_w)_{w\in\mathrm{Val}(\mathbf{K})}$ on $\omega_V^{-1} = \det(TV)$. The formula for the change of variables (see [**We**, §2.2.1]) proves that the local measures

$$(1) \qquad \left\| \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \cdots \wedge \frac{\partial}{\partial x_n} \right\|_w \mathrm{d}x_{1,w}\, \mathrm{d}x_{2,w} \ldots \mathrm{d}x_{n,w},$$

where $(x_1,\ldots,x_n) : \Omega \to \mathbf{K}_w^n$ is a local system of coordinates defined on an open subset Ω of $V(\mathbf{K}_w)$, does not depend on the choice of coordinates; therefore by patching together these measures, we get a measure $\boldsymbol{\omega}_{V,w}$ on $V(\mathbf{K}_w)$, which induces a probability measure

$$\boldsymbol{\mu}_{V,w} = \frac{1}{\boldsymbol{\omega}_{V,w}(V(\mathbf{K}_w))} \boldsymbol{\omega}_{V,w}.$$

Then the product

$$\boldsymbol{\mu}_V = \prod_{w\in\mathrm{Val}(\mathbf{K})} \boldsymbol{\mu}_{V,w}$$

is a probability measure on the adelic space $V(\boldsymbol{A}_{\mathbf{K}})$.

***Remark 3.7.* —** For the projective space, this construction gives the right asymptotic distribution for the points of bounded height. So it is natural to try to generalise to other varieties. To state precisely our question, we introduce the counting measure defined by the set of points of bounded height.

***Definition 3.8.* —** For any non-empty subset $W \subset V(\mathbf{K})$ we define, for B a real number bigger than the smallest height of a point of W,

$$\boldsymbol{\delta}_{W_{H\leqslant B}} = \frac{1}{\sharp W_{H\leqslant B}} \sum_{P\in W_{H\leqslant B}} \boldsymbol{\delta}_P,$$

where $\boldsymbol{\delta}_P$ denotes the Dirac measure at P on the adelic space.

Naïve equidistribution 3.9. *We shall say that the* naïve equidistribution (NE) *holds if the measure $\boldsymbol{\delta}_{V(\mathbf{K})_{H\leqslant B}}$ converges to $\boldsymbol{\mu}_V$ as B goes to infinity for the weak topology.*

Remark 3.10. — In other words, the naïve equidistribution holds if for any continuous function $f : V(\boldsymbol{A}_{\mathbf{K}}) \to \mathbf{R}$, one has the convergence

$$\int_{V(\boldsymbol{A}_{\mathbf{K}})} f \delta_{V(\mathbf{K})_{H \leqslant B}} \xrightarrow[B \to +\infty]{} \int_{V(\boldsymbol{A}_{\mathbf{K}})} f \boldsymbol{\mu}_V.$$

This equidistribution may seem to be overoptimistic and one may wonder whether there exists any case besides the projective space for which it is valid.

Theorem 3.11. — *If V is a generalized flag variety, that is a quotient G/P where G is a linear algebraic group over $\mathbf{K}$ and P a parabolic subgroup of G, then* (NE) *is true.*

Example 3.12. — Grassmannian are examples of such flag varieties. Any smooth quadric with a rational point is a generalized flag variety for the orthogonal group. Therefore any smooth quadric with a rational point satisfies the naive equidistribution.

Tools of the proof of theorem 3.11. — To prove this result one may use harmonic analysis on the adelic space $G/P(\boldsymbol{A}_{\mathbf{K}})$ and apply Langland's work on Eisenstein series (see [**Pe1**, corollaire 6.2.17], [**Lan**]). □

So we have solved the case of hypersurfaces of degree 2. In higher degrees, the equidistribution, when the number of variables is large enough, is an easy consequence of the very general result of Birch [**Bir**] based on the circle method. His result implies the following theorem:

Theorem 3.13. — *Let $V \subset \mathbf{P}^n_{\mathbf{Q}}$ be a smooth hypersurface of degree d such that $V(\boldsymbol{A}_{\mathbf{Q}}) \neq \varnothing$ with $n > (d-1)2^d$, then V satisfies* (NE).

Remark 3.14. — In fact, it applies to all the cases considered by Birch, that is for smooth complete intersection of m hypersurfaces of the same degree d if $n > m(m+1)(d-1)2^{d-1}$.

3.3. Weak approximation.

— The first indications of the naïveté of (NE) appear when one considers obvious consequences of it. Let us recall the definition of weak approximation:

Definition 3.15. — A nice variety V satisfies *weak approximation* if the rational points of V are dense in the adelic space $V(\boldsymbol{A}_{\mathbf{K}})$.

Remarks 3.16. — a) Let V be a nice variety with a rational point. If it satisfies the naïve equidistribution, then it satisfies weak approximation and therefore $V(\mathbf{K})$ is dense for Zariski topology.

This follows from the fact that for any real number B, the support of the measure $\delta_{V(\mathbf{K})_{H\leqslant B}}$ in $V(\boldsymbol{A}_{\mathbf{K}})$ is contained in the closure $\overline{V(\mathbf{K})}$ of the set of rational points. But the support of the measure $\boldsymbol{\mu}_V$ is the whole adelic space. Thus (NE) implies that $\overline{V(\mathbf{K})} = V(\boldsymbol{A}_{\mathbf{K}})$. Let then U be a non-empty open subset for Zariski topology. If V has an adelic point, the implicit function theorem ensures that for any place w, the set $U(\mathbf{K}_w)$ is a non-empty open subset of $V(\mathbf{K}_w)$. If $V(\mathbf{K})$ is dense in $V(\mathbf{K}_w)$, it follows that U contains a rational point and the rational points are Zariski dense.

b) So (NE) has to fail for any variety in which the rational points are not Zariski dense. In that case, one may consider the desingularisation of the closure of the rational points for Zariski topology and ask wether the principle holds for that variety. But even such a modified question fails because examples are known where rational points are dense for Zariski topology but the variety does not satisfy weak approximation.

Convention 3.17. — From now on, we assume that V is a nice variety in which the set of rational points $V(\mathbf{K})$ is Zariski dense.

About weak approximation, we are going to give a quick overview of the Brauer–Manin obstruction, which was introduced by Y. Manin in [**Ma**] to explain the previously known counterexamples to weak approximation (see also [**Pe3**] for a survey).

Construction 3.18. — For a nice variety V, we define its *Brauer group* as the cohomology group

$$\mathrm{Br}(V) = H^2_{\text{ét}}(V, \mathbf{G}_m)$$

which defines a contravariant functor from nice varieties to the category of abelian groups. In the case of the spectrum of a field of characteristic 0, we get the Brauer group of $\mathbf{L}$, which is defined in terms of Galois cohomology by

$$\mathrm{Br}(\mathbf{L}) = H^2(\mathrm{Gal}(\overline{\mathbf{L}}/\mathbf{L}), \mathbf{G}_m),$$

where $\overline{\mathbf{L}}$ is an algebraic closure of $\mathbf{L}$. Class field theory gives for any place w an injective morphism

$$\mathrm{inv}_w : \mathrm{Br}(\mathbf{K}_w) \longrightarrow \mathbf{Q}/\mathbf{Z}$$

which is an isomorphism if w is not archimedean, so that the sequence

$$0 \to \mathrm{Br}(\mathbf{K}) \to \bigoplus_{w \in \mathrm{Val}(\mathbf{K})} \mathrm{Br}(\mathbf{K}_w) \xrightarrow{\sum_w \mathrm{inv}_w} \mathbf{Q}/\mathbf{Z} \to 0 \tag{2}$$

is an exact sequence. Therefore we may define a pairing

$$\begin{aligned} \mathrm{Br}(V) \times V(\boldsymbol{A}_{\mathbf{K}}) &\longrightarrow \mathbf{Q}/\mathbf{Z} \\ (\alpha, (P_w)_{w \in \mathrm{Val}(\mathbf{K})}) &\longmapsto \sum_{w \in \mathrm{Val}(\mathbf{K})} \mathrm{inv}_w(\alpha(P_w)) \end{aligned}$$

where $\alpha(P_w)$ denotes the pull-back of α by the morphism $\mathrm{Spec}(\mathbf{K}_w) \to V$ defined by P_w. Let us denote by $\mathrm{Br}(V)^\vee$ the group $\mathrm{Hom}(\mathrm{Br}(V), \mathbf{Q}/\mathbf{Z})$ then the above pairing may be seen as a map

$$\eta : V(\boldsymbol{A}_{\mathbf{K}}) \longrightarrow \mathrm{Br}(V)^\vee.$$

If $P \in V(\mathbf{K})$ then the fact that (2) is a complex implies that

$$\sum_{w \in \mathrm{Val}(\mathbf{K})} \mathrm{inv}_w(\alpha(P)) = 0;$$

in other words, $\eta(P) = 0$. By arguments of continuity, one gets that

$$\overline{V(\mathbf{K})} \subset V(\boldsymbol{A}_{\mathbf{K}})^{\mathrm{Br}} = \{\, P \in V(\boldsymbol{A}_{\mathbf{K}}) \mid \eta(P) = 0 \,\}.$$

The element $\eta(P)$ is called the *Brauer–Manin obstruction* to weak approximation at P.

Remark 3.19. — Let $\overline{\mathbf{K}}$ be an algebraic closure of $\mathbf{K}$ and $\overline{V} = V_{\overline{\mathbf{K}}}$. Since we assume V to have a rational point, there is an exact sequence

$$0 \to \mathrm{Br}(\mathbf{K}) \to \ker(\mathrm{Br}(V) \to \mathrm{Br}(\overline{V})) \to H^1(\mathrm{Gal}(\overline{\mathbf{K}}/\mathbf{K}), \mathrm{Pic}(\overline{V})) \to 0.$$

Also the exponential map gives an exact sequence

$$\begin{aligned} &H^1(V_{\mathbf{C}}, \mathcal{O}_V) \to \mathrm{Pic}(\overline{V}) \to H^2(V(\mathbf{C}), \mathbf{Z}) \to \\ &\qquad\qquad H^2(V_{\mathbf{C}}, \mathcal{O}_V) \to \mathrm{Br}(\overline{V}) \to H^3(V(\mathbf{C}), \mathbf{Z})_{\mathrm{tors}} \end{aligned}$$

Thus assuming that $H^i(V, \mathcal{O}_V) = \{0\}$ for $i = 1$ and $i = 2$, which is automatic for Fano varieties by Kodaira's vanishing theorem, we get first that the geometric Picard of the variety is finitely generated. Thus the action of the Galois group on the Picard group is trivial over a finite extension of the ground field. Therefore, in this case, the groups $H^1(\mathrm{Gal}(\overline{\mathbf{K}}/\mathbf{K}), \mathrm{Pic}(\overline{V}))$ and $\mathrm{Br}(\overline{V})$ are finite. Hence the cokernel of the morphism $\mathrm{Br}(\mathbf{K}) \to \mathrm{Br}(V)$ is finite, which implies that $V(\boldsymbol{A}_{\mathbf{K}})^{\mathrm{Br}}$ is open and closed in the adelic space.

If one hopes that the Brauer–Manin obstruction to the weak approximation is the only one, then it is natural to define the measure induced by the probability measure μ_V on the space on which the obstruction is 0. Since we assume that the variety V has a rational point, the space $V(\boldsymbol{A}_{\mathbf{K}})^{\mathrm{Br}}$ is not empty. In that setting, we may give the following definition:

Definition 3.20. — The measure μ_V^{Br} is defined as follows: for any Borelian subset W of $V(\boldsymbol{A}_{\mathbf{K}})$

$$\mu_V^{\mathrm{Br}}(W) = \frac{\mu_V(W \cap V(\boldsymbol{A}_{\mathbf{K}})^{\mathrm{Br}})}{\mu_V(V(\boldsymbol{A}_{\mathbf{K}})^{\mathrm{Br}})}.$$

The following question then takes into account the Brauer–Manin obstruction to weak approximation:

Global equidistribution 3.21. *We shall say that* global equidistribution *holds if the measure $\delta_{V(\mathbf{K})_{H \leqslant B}}$ converges weakly to μ_V^{Br} as B goes to infinity.*

Potential counterexamples to global equidistribution have been known for quite a long time (see for example [**Se2**]), but Y. Manin was the first to consider accumulating subsets, which we will study at length in the next section.

3.4. Accumulating subsets. — In fact, the support of the limit of the measure $\delta_{V(\mathbf{K})_{H \leqslant B}}$ is, in general, much smaller than the closure $\overline{V}(\mathbf{K})$ of the set of rational points. Let me give a few examples.

3.4.1. *The plane blown up in one point.* — The blowing up of the projective plane at the point $P_0 = [0:0:1]$ may be described as the hypersurface V in the product $\mathbf{P}^2_{\mathbf{Q}} \times \mathbf{P}^1_{\mathbf{Q}}$ defined by the equation $XV = YU$, where X, Y, Z denote the coordinates on the first factor and U, V the coordinates on the second one. Let π be the projection on the first factor. Then $E = \pi^{-1}(P_0)$ is an exceptional divisor on V and the second projection pr_2 defines an isomorphism from E to $\mathbf{P}^1_{\mathbf{Q}}$. Let U be the complement of E in V. The projection π induces an isomorphism from U to $\mathbf{P}^2_{\mathbf{Q}} - \{P_0\}$. As an exponential height, we may use the map

$$\begin{aligned} H : V(\mathbf{Q}) &\longrightarrow \mathbf{R}_{>0} \\ (P, Q) &\longmapsto H_{\mathscr{O}_{\mathbf{P}^2_{\mathbf{Q}}}(1)}(P)^2 H_{\mathscr{O}_{\mathbf{P}^1_{\mathbf{Q}}}(1)}(Q). \end{aligned}$$

This example as been used as a sandbox case for the study of rational points of bounded height by many people, including J.-P Serre [**Se2**, §2.12], as well as V. V. Batyrev and Y. I. Manin [**BM**, proposition 1.6] and the results may be summarized as follows:

Proposition 3.22. — *On the exceptional line, the number of points of bounded height is given by*

$$\sharp E(\mathbf{Q})_{H\leqslant B} \sim \frac{2}{\zeta_{\mathbf{Q}}(2)} B^2$$

as B goes to infinity, whereas on its complement it is given by

$$\sharp U(\mathbf{Q})_{H\leqslant B} \sim \frac{8}{3\zeta_{\mathbf{Q}}(2)^2} B\log(B)$$

as B goes to infinity.

Remark 3.23. — Thus there are much more rational points on the exceptional line E than on the dense open subset U. In fact, since the points on the exceptional line are distributed as on $\mathbf{P}^1_{\mathbf{Q}}$, we get that the measure $\delta_{V(\mathbf{Q})_{H\leqslant B}}$ converges to μ_E for the weak topology.

On the other hand, if we only consider the rational points on the open set U, we get the right limit:

Proposition 3.24. — *The measure $\delta_{U(\mathbf{Q})_{H\leqslant B}}$ converges to μ_V for the weak topology as B goes to infinity.*

Remarks 3.25. — a) Let W be an infinite subset of $V(\mathbf{K})$. If the measure $\delta_{W_{H\leqslant B}}$ converges to μ_V for the weak topology, then, for any strict closed subvariety F in V, we have that

$$\sharp(W\cap F(\mathbf{Q}))_{H\leqslant B} = o(\sharp W_{H\leqslant B})$$

since we have $\mu_V(F(\boldsymbol{A}_{\mathbf{K}})) = 0$. Thus any strict closed subset with a strictly positive contribution to the number of points has to be removed to get equidistribution.

b) It may seem counterintuitive that by removing points, we get a measure with a larger support. But this comes from the fact that we divide the counting measure on U by a smaller term. From this example, it follows that it is natural to consider only the points outside a set of "bad" points. The problem is that this set of bad points might be quite big.

3.4.2. *The principle of Manin.* — The principle suggested by Manin and his collaborators in the founding papers [**BM**] and [**FMT**] is that, on Fano varieties, there should be an open subset on which the points of bounded height behave as expected. Let us give a precise expression for this principle, in a slightly more general setting. Since this principle deals with the number of points of bounded height rather than their distribution, we have to introduce another normalisation of the measures to get a conjectural value for the constant, which is defined as a volume.

Notation 3.26. — Let $\mathrm{NS}(V)$ be the Néron-Severi group of V, that is the quotient of the Picard group by the connected component of the neutral element. We put $\mathrm{NS}(V)_{\mathbf{R}} = \mathrm{NS}(V) \otimes_{\mathbf{Z}} \mathbf{R}$ and denote by $C_{\mathrm{eff}}(V)$ the closed cone in $\mathrm{NS}(V)_{\mathbf{R}}$ generated by the classes of effective divisors. We write $C_{\mathrm{eff}}(V)^{\vee}$ for the dual of the effective cone in the dual space $\mathrm{NS}(V)^{\vee}_{\mathbf{R}}$:

$$C_{\mathrm{eff}}(V)^{\vee} = \{ y \in \mathrm{NS}(V)^{\vee}_{\mathbf{R}} \mid \forall x \in C_{\mathrm{eff}}(V), \langle y, x \rangle \geqslant 0 \}.$$

To construct the constant, we shall restrict ourselves to a setting in which the local measures can be normalized using the action of the Galois group of $\mathbf{K}$ on the Picard group. Therefore, we make the following hypothesis:

Hypotheses 3.27. — From now on, V is a nice variety, which satisfies the following conditions:

(i) A multiple of the class of ω_V^{-1} is the sum of an ample divisor and a divisor with normal crossings;

(ii) The set $V(\mathbf{Q})$ is Zariski dense;

(iii) The groups $H^i(V, \mathscr{O}_V)$ are $\{0\}$ if $i \in \{1, 2\}$;

(iv) The geometric Brauer group $\mathrm{Br}(\overline{V})$ is trivial and the geometric Picard group $\mathrm{Pic}(\overline{V})$ has no torsion;

(v) The closed cone $C_{\mathrm{eff}}(\overline{V})$ is generated by the classes of a finite set of effective divisors.

Remark 3.28. — The first four conditions are satisfied by Fano varieties, that is varieties for which ω_V^{-1} is ample. The fifth has been conjectured by V. V. Batyrev for these varieties [**Ba**].

Construction 3.29. — We choose a finite set S of places containing the archimedean places and the places of bad reduction for V. Let $\mathbf{L}$ be a finite extension of $\mathbf{K}$ such that the Picard group $\mathrm{Pic}(V_{\mathbf{L}})$ is isomorphic to the geometric

Picard group $\mathrm{Pic}(\overline{V})$. We assume that S contains all the places which ramify in the extension $\mathbf{L}/\mathbf{K}$. With this assumption, for any place $w \in \mathrm{Val}(\mathbf{K}) - S$, let $\mathbf{F}_w$ be the residual field at w. The Frobenius lifts to an element $(w, \mathbf{L}/\mathbf{K})$ in $\mathrm{Gal}(\mathbf{L}/\mathbf{K})$ which is well defined up to conjugation (see [**Se1**, §1.8]). Then we can consider the local factors of the L function defined by the Picard group:

$$L_w(s, \mathrm{Pic}(\overline{V})) = \frac{1}{\det(1 - \sharp\mathbf{F}_w^{-s}(w, \mathbf{L}/\mathbf{K}) \,|\, \mathrm{Pic}(\overline{V}))},$$

where s is a complex number with $\mathfrak{R}(s) > 0$. If the real part of s satisfies $\mathfrak{R}(s) > 1$, then a theorem of Artin [**Art**, Satz 3] implies that the eulerian product

$$L_S(s, \mathrm{Pic}(\overline{V})) = \prod_{w \in \mathrm{Val}(\mathbf{K}) - S} L_w(s, \mathrm{Pic}(\overline{V}))$$

converges. For $w \in \mathrm{Val}(\mathbf{K})$, we define $\lambda_w = L_w(1, \mathrm{Pic}(\overline{V}))^{-1}$ if w does not belong to S and $\lambda_w = 1$ otherwise. We put $t = \mathrm{rk}(\mathrm{Pic}(V))$. It follows from the Weil's conjecture proven by P. Deligne [**Del**] that the product of measures

$$\boldsymbol{\omega}_V = \frac{\lim_{s \to 1} (s-1)^t L_S(s, \mathrm{Pic}(\overline{V}))}{\sqrt{d_{\mathbf{K}}}^{\dim(V)}} \prod_{w \in \mathrm{Val}(\mathbf{K})} \lambda_w \boldsymbol{\omega}_{V,w} \tag{3}$$

converges (see [**Pe1**, §2.1]), where $d_{\mathbf{K}}$ denotes the absolute value of the discriminant of $\mathbf{K}$. We may then define the *Tamagawa–Brauer–Manin volume of V* as

$$\tau^{\mathrm{Br}}(V) = \boldsymbol{\omega}_V(V(\boldsymbol{A}_{\mathbf{K}})^{\mathrm{Br}}).$$

We also introduce the constant

$$\alpha(V) = \frac{1}{(t-1)!} \int_{C^1_{\mathrm{eff}}(V)^\vee} e^{-\langle \omega_V^{-1}, y \rangle} \mathrm{d}y$$

which is a rational number under the hypothesis 3.27 (v), and the integer

$$\beta(V) = \sharp(\mathrm{Br}(V)).$$

Then the *empirical constant* associated to the chosen metric on V is the constant

$$C(V) = \alpha(V)\beta(V)\tau^{\mathrm{Br}}(V).$$

Batyrev–Manin principle 3.30. *Let V be a variety which satisfies the conditions 3.27. We say that V satisfies the* refined Batyrev–Manin principle *if there exists a dense open subset U of V such that*

$$\sharp U(\mathbf{K})_{H \leqslant B} \sim C(V) B \log(B)^{t-1} \tag{4}$$

as B goes to infinity.

For equidistribution, we may introduce the following notion

Relative equidistribution 3.31. *Let W be an infinite subset of $V(\mathbf{K})$, we say that* the points of W are equidistributed in V *if the counting measure $\delta_{W_{H\leqslant B}}$ converges to μ_V.*

Remark 3.32. — The relation between the Batyrev-Manin principle as stated here and the equidistribution may be described as follows: if the principle holds for a given open subset U for any metric on V, then the points of $U(\mathbf{K})$ are equidistributed on V. Conversely if the principle holds for a particular choice of the metric and an open subset U and if the points of $U(\mathbf{K})$ are equidistributed, then the principle holds for any choice of the metric (see [**Pe1**, §3]).

3.4.3. *The counterexample of V. V. Batyrev and Y. Tschinkel.* — This example was described in [**BT1**]. We consider the hypersurface V in $\mathbf{P}^3_{\mathbf{Q}} \times \mathbf{P}^3_{\mathbf{Q}}$ defined by the equation

$$\sum_{i=0}^{3} X_i Y_i^3 = 0.$$

We denote by $\mathscr{O}_V(a,b)$ the restriction to V of the line bundle $\mathrm{pr}_1^*(\mathscr{O}_{\mathbf{P}^3_{\mathbf{Q}}}(a)) \otimes \mathrm{pr}_2^*(\mathscr{O}_{\mathbf{P}^3_{\mathbf{Q}}}(b))$ Then the anticanonical line bundle on V is given by $\mathscr{O}_V(3,1)$ and therefore the function $H : V(\mathbf{Q}) \to \mathbf{R}$ defined by

$$H(P,Q) = H_{\mathscr{O}_{\mathbf{P}^3_{\mathbf{Q}}}(1)}(P)^3 H_{\mathscr{O}_{\mathbf{P}^3_{\mathbf{Q}}}(1)}(Q)$$

defines a height relative to the anticanonical line bundle on V. Let π be the projection on the first factor and for any $P \in \mathbf{P}^3(\mathbf{Q})$, let $V_P = \pi^{-1}(P)$ the fibre over P. If $P = [x_0 : x_1 : x_2 : x_2]$ with $\prod_{i=0}^{3} x_i \neq 0$, then the fibre V_P is a smooth cubic surface which contains 27 projective lines. The complement U_P of these 27 lines is defined over $\mathbf{Q}$. For cubic surfaces, it is expected that the Batyrev-Manin principle holds for any dense open subset contained in U_P. For any P as above, let $t_P = \mathrm{rk}(\mathrm{Pic}(V_P))$ be the rank of the Picard group of the cubic surface corresponding to P. Thus, according to (4), one expects that for any $U \subset U_P$, one has

$$\sharp U(\mathbf{Q})_{H\leqslant B} \sim C(V_P) B \log(B)^{t_P - 1}$$

as B goes to infinity. One can show that $t_P \in \{1,2,3,4\}$ and that $t_P = 4$ if all the quotients x_i/x_j are cubes, that is if P is in the image of the morphism c from $\mathbf{P}^3_{\mathbf{Q}}$ to $\mathbf{P}^3_{\mathbf{Q}}$ defined by $[x_0 : x_1 : x_2 : x_3] \mapsto [x_0^3 : x_1^3 : x_2^3 : x_3^3]$. But, on the other hand, by Lefschetz theorem, the application $(a,b) \mapsto \mathscr{O}_V(a,b)$ induces an isomorphims of groups from $\mathbf{Z}^2$ to $\mathrm{Pic}(V)$. Therefore, the principle of Batyrev and Manin would be satisfied for V if and only if there existed an open subset U of V such that

$$\sharp U(\mathbf{Q})_{H\leqslant B} \sim C(V)B\log(B)$$

as B goes to infinity. Since the rational points in the image of c are dense for Zariski topology, the open set U has to intersect an open set U_P for some P. Thus the principle can not hold for both the cubic surfaces and V itself.

Remarks 3.33. — a) In fact, V. V. Batyrev and Y. Tschinkel proved in [**BT1**] that any dense open set of V contains too many rational points over $\mathbf{Q}(j)$, where j is a primitive third root of unity. More recently, C. Frei, D. Loughran, and E. Sofos proved in [**FLS**] that it is in fact the case over any number field.

b) One may look at the set

$$T = \{ P \in \mathbf{P}^3(\mathbf{Q}) \mid \mathrm{rk}(\mathrm{Pic}(V_P)) > 1 \}$$

that is the set of points for which the rank of the Picard group is bigger than the generic one. As we are about to explain,

$$\sharp T_{H\leqslant B} = o(\sharp \mathbf{P}^3(\mathbf{Q})_{H\leqslant B})$$

which means that most of the fibers have a Picard group of rank one.

This example led to the introduction of a new kind of accumulating subsets, namely thin subsets (see J.-P. Serre [**Se3**, §3.1]).

Definition 3.34. — Let V be a nice variety over the number field $\mathbf{K}$. A subset $T \subset V(\mathbf{K})$ is said to be *thin*, if there exists a morphism of varieties $\varphi : X \to V$ which satisfies the following conditions:

(i) The morphism φ is generically finite;

(ii) The morphism φ has no rational section;

(iii) The set T is contained in the image of φ.

Remarks 3.35. — a) If E is an elliptic curve, the group $E(\mathbf{K})/2E(\mathbf{K})$ is a finite group. Let $(P_i)_{i\in I}$ be a finite family of points of $E(\mathbf{K})$ containing a representant for each element of $E(\mathbf{K})/2E(\mathbf{K})$. Then the morphism $\varphi : \coprod_{i\in I} E \to E$ which maps a point P in the i-th component to $P_i + 2P$ gives a surjective map onto the sets of rational points. This shows that $E(\mathbf{K})$ itself is thin.

b) In the example of Batyrev and Tschinkel, as T is a thin subset in $\mathbf{P}^3(\mathbf{Q})$, it follows from [**Se2**, §13, theorem 3] that

$$\sharp T_{H\leqslant B} = o(\sharp \mathbf{P}^3(\mathbf{Q})_{H\leqslant B}).$$

The set
$$V_T = \bigcup_{P\in T} V_P(\mathbf{Q})$$
is itself a thin subset of $V(\mathbf{Q})$. Conjecturally we may hope that
$$\sharp(V(\mathbf{Q}) - V_T)_{H\leqslant B} \sim C_H(V)B\log(B)$$
as B goes to infinity. In other words, the points on the complement of the accumulating subset should behave as expected. We shall explain below how a result of this kind was proven by C. Le Rudulier for a Hilbert scheme of the projective plane [**Ru**]. More recently, T. Browning and D.R. Heath-Brown [**BHB**] proved that for the hypersurface of $\mathbf{P}^3_{\mathbf{Q}} \times \mathbf{P}^3_{\mathbf{Q}}$ defined by the equation
$$\sum_{i=0}^{3} X_i Y_i^2$$
the number of points on the complement of an accumulating thin subset behaves as expected.

c) The work of B. Lehmann, S. Tanimoto and Y. Tschinkel [**LTT**] shows how common varieties with accumulating thin subsets probably are.

d) We may assume that φ is a proper morphism. Then $\varphi(X(\boldsymbol{A}_{\mathbf{K}})) \subset V(\boldsymbol{A}_{\mathbf{K}})$ is a closed subset. Under mild hypotheses, T. Browning and D. Loughran proved in [**BL**] that
$$\boldsymbol{\mu}_V(\varphi(X(\boldsymbol{A}_{\mathbf{K}}))) = 0.$$
Thus the existence of such a thin subset with a positive contribution to the asymptotic number of points is an obstruction to the global equidistribution of points.

3.4.4. *The example of C. Le Rudulier.* — C. Le Rudulier considers the Hilbert scheme V which parametrizes the points of degree 2 in $\mathbf{P}^2_{\mathbf{Q}}$ [**Ru**]. To describe this scheme, let us consider the scheme Y defined as the second symmetric product of $\mathbf{P}^2_{\mathbf{Q}}$:
$$Y = \mathrm{Sym}^2(\mathbf{P}^2_{\mathbf{Q}}) = (\mathbf{P}^2_{\mathbf{Q}})^2/\mathfrak{S}_2.$$
More precisely, we may define it as the projective scheme associated to the ring of invariant polynomials $\mathbf{Q}[X_1, Y_1, Z_1, X_2, Y_2, Z_2]^{\mathfrak{S}_2}$. Let us denote by Δ_Y the image of the diagonal Δ in Y. The scheme Y is singular along this diagonal and V may be seen as the blowing up of Y along the diagonal Δ_Y. From this point

of view, the variety V is a desingularization of Y. Let us define P as the blowing up of $(\mathbf{P}^2_{\mathbf{Q}})^2$ along the diagonal. We get a cartesian square

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\tilde{\pi}"'] \arrow[dr, phantom, "\square"] & (\mathbf{P}^2_{\mathbf{Q}})^2 \arrow[d, "\pi"] \\ V \arrow[r, "b"'] & Y \end{tikzcd}$$

We put $\Delta_V = b^{-1}(\Delta_Y)$ and $U_0 = V - \Delta_V$. Then the set

$$T = \tilde{\pi}(P(\mathbf{Q})) \cap U_0(\mathbf{Q})$$

is a Zariski dense thin accumulating subset. More precisely, C. Le Rudulier proves the following theorem:

***Theorem 3.36* (C. Le Rudulier). —** a) *Asymptotically the points of T give a positive contribution to the total number of points:*

$$\frac{\sharp T_{H\leqslant B}}{\sharp U_0(\mathbf{Q})_{H\leqslant B}} \xrightarrow[B\to+\infty]{} c$$

for a real number $c > 0$. But for any strictly closed subset $F \subset V$, one has

$$\sharp(F(\mathbf{Q}) \cap T)_{H\leqslant B} = o(U_0(\mathbf{Q})_{H\leqslant B}).$$

b) *On the complement of T, one has*

$$\sharp(U_0(\mathbf{Q}) - T)_{H\leqslant B} \sim C(V) B \log(B)$$

as $B \to +\infty$.

***Remarks 3.37.* —** a) It follows from this theorem that the set T is a thin subset which is not the union of accumulating subvarieties but which gives a positive contribution to the total number of points of bounded height on the variety. In the adelic space the closure of the points of T are contained in a closed subset F with a volume $\boldsymbol{\mu}_V(F)$ equal to 0. Therefore this thin accumulating subset is an obstruction to the equidistribution of the points on V.

b) Hopefully, in general, if ω_V^{-1} is "big enough", there should be a natural "small" subset T such that the points of bounded height on $W = V(\mathbf{K}) - T$ should behave as expected. The problem is to describe this subset T.

c) W. Sawin recently proved that, in the example described in theorem 3.36, the empiric formula [**Pe4**, formule empirique 6.13] is *false* [**Sa**]. However the approach described below involving several heights might still be correct.

d) In these notes, so far, we did not go into the distribution of the rational points of bounded height for a height associated to an ample line with a class which is not a multiple of ω_V^{-1}. The description in that case requires to introduce more complicated measures and we refer the interested reader to the work of V. V. Batyrev and Y. Tschinkel (see [**BT2**]).

4. All the heights

4.1. Heights systems. — A natural approach to select the points we wish to keep is to introduce more invariants. The rest of this chapter is devoted to such invariants. Let us start by considering other heights. Traditionally, most authors in arithmetic geometry consider only one height given by a given ample line bundle. However there are no reason to do so, and we may consider the whole information given by heights. In order to do this, let us introduce the notion of family of heights.

Definition 4.1. — Let L and L' be adelically normed line bundles on a nice variety V. Let $(||\cdot||_w)_{w\in\mathrm{Val}(\mathbf{K})}$ be the adelic norm on L. We say that L and L' are *equivalent* if there is an integer $M > 0$, a family $(\lambda_w)_{w\in\mathrm{Val}(\mathbf{K})}$ in $\mathbf{R}_{>0}^{(\mathrm{Val}(\mathbf{K}))}$, such that its support $\{\, w \in \mathrm{Val}(\mathbf{K}) \mid \lambda_w \neq 1\,\}$ is finite and $\prod_{w\in\mathrm{Val}(\mathbf{K})}\lambda_w = 1$, and an isomorphism of adelically normed line bundles from the line bundle $L^{\otimes M}$ equipped with the adelic norm $(\lambda_w||\cdot||_w^{\otimes M})_{w\in\mathrm{Val}(\mathbf{K})}$ to the adelically normed line bundle $L'^{\otimes M}$. We denote by $\mathscr{H}(V)$ the set of equivalence classes of adelically normed line bundles. It has a structure of group induced by the tensor product of line bundles, we call this group the group of *Arakelov heights on V*.

Remark 4.2. — The height introduced in definition 2.7 depends only on the equivalence class of the adelically normed line bundle $\det(E)$. From that point of view, the group $\mathscr{H}(V)$ does parametrize the heights on V. If V satisfies weak approximation and has an adelic point, then two distinct elements of $\mathscr{H}(V)$ define heights which differ at least at one rational point.

Example 4.3. — If V is a point, that is the spectrum of a field, then the height defines an isomorphism from $\mathscr{H}(V)$ to $\mathbf{R}_{>0}$. Indeed, it is surjective and if we take a representative L of an element of $\mathscr{H}(\mathrm{Spec}(\mathbf{K}))$ of height 1, then let y be an nonzero element of L. The unique morphism of vector spaces from $\mathbf{K}$ to L which maps 1 to y then induces an isomorphism from $\mathbf{K}$ equipped with the adelic norm $(||y||_w\,|\cdot|_w)_{w\in\mathrm{Val}(\mathbf{K})}$ to L.

Definition 4.4. — A *system of Arakelov heights* on our nice variety V is a section $\boldsymbol{s}$ of the forgetful morphism of groups

$$\boldsymbol{o} : \mathscr{H}(V) \longrightarrow \mathrm{Pic}(V).$$

Such a system defines a map

$$\boldsymbol{h} : V(\mathbf{K}) \to \mathrm{Pic}(V)^{\vee}_{\mathbf{R}}$$

constructed as follows: for any $P \in V(\mathbf{K})$ and any $L \in \mathrm{Pic}(V)$, the real number $\boldsymbol{h}(P)(L)$ is the logarithmic height of the point P relative to the Arakelov height $\boldsymbol{s}(L)$ (see definition 2.7). We shall call $\boldsymbol{h}(P)$ the *multiheight* of the point P. By abuse of language, a function of the form $P \mapsto \exp(\langle u, \boldsymbol{h}(P)\rangle)$ for some $u \in \mathrm{Pic}(V)_{\mathbf{R}}$ will also be called an exponential height on V.

Since $\mathrm{Pic}(V)$ is finitely generated, we may fix a system of Arakelov heights on our nice variety V. We still assume that V satisfies the hypotheses 3.27. Then one can study the multiheights of rational points.

Lemma 4.5. — *Under the hypotheses 3.27, there is a dense open subset U of V and an element $c \in \mathrm{Pic}(V)^{\vee}_{\mathbf{R}}$ such that*

$$\forall P \in U(\mathbf{K}), \quad \boldsymbol{h}(P) \in c + C_{\mathrm{eff}}(V)^{\vee}.$$

Proof. — Let $L_1, \dots, L_m$ be line bundles the classes of which generate the effective cone in $\mathrm{Pic}(V)_{\mathbf{R}}$. We may assume that they have nonzero sections. Let U be the complement of the base loci of these line bundles. Let $i \in \{1, \dots, m\}$. Then choosing a basis $(s_0, \dots, s_{N_i})$ of the space of sections of the line bundle L_i, we get a morphism from U to a projective space $\mathbf{P}^{N_i}_{\mathbf{K}}$. For any place w, there exist a constant c_w such that $\|s_j(x)\|_w \leqslant c_w$ for any $x \in V(\mathbf{K}_w)$ and any $j \in \{0, \dots, N_i\}$. Moreover we may take $c_w = 1$ outside a finite set of places. Therefore there exists a constant C such that for any $x \in U(\mathbf{K})$ there is an $j \in \{0, \dots, N_i\}$ with

$$0 < \prod_{w \in \mathrm{Val}(\mathbf{K})} \|s_j(x)\| \leqslant C.$$

It follows that there exists a constant $c_i \in \mathbf{R}$ such that $h_i(P) \geqslant c_i$ for any $P \in U(\mathbf{K})$. The statement of the lemma follows. □

Remark 4.6. — Let $C^{\circ}_{\mathrm{eff}}(V)^{\vee}$ be the interior of the dual cone $C_{\mathrm{eff}}(V)^{\vee}$. This lemma shows that it is quite natural to count the number of rational points in

$V(K)$ such that $\boldsymbol{h}(P) \in \mathscr{D}_B$ for some compact domain $\mathscr{D}_B \subset C^\circ_{\mathrm{eff}}(V)^\vee$ depending on a parameter $B \in \mathbf{R}_{>0}$. In the following, we shall consider domains of the form

$$\mathscr{D}_B = \mathscr{D}_1 + \log(B)u$$

where $u \in C^\circ_{\mathrm{eff}}(V)^\vee$ and $\mathscr{D}_1$ is a compact polyhedron in $\mathrm{Pic}(V)^\vee_{\mathbf{R}}$. In other words, we get a finite number of conditions of the form

$$aB \leqslant H(P) \leqslant bB$$

where H is an exponential height on V, in the sense of definition 4.4, and $a, b \in \mathbf{R}_{>0}$.

Notation 4.7. — We define the measure ν on $\mathrm{Pic}(V)^\vee_{\mathbf{R}}$ as follows: for a compact subset $\mathscr{D}$ of $\mathrm{Pic}(V)^\vee_{\mathbf{R}}$,

$$\nu(\mathscr{D}) = \int_{\mathscr{D}} e^{\langle \omega_V^{-1}, \boldsymbol{y}\rangle} \mathrm{d}\boldsymbol{y},$$

where the Haar measure $\mathrm{d}\boldsymbol{y}$ on $\mathrm{Pic}(V)^\vee_{\mathbf{R}}$ is normalised so that the covolume of the dual of the Picard group is one.

For any domain $\mathscr{D} \subset \mathrm{Pic}(V)^\vee_{\mathbf{R}}$, we define

$$V(\mathbf{K})_{\boldsymbol{h}\in\mathscr{D}} = \{ P \in V(\mathbf{K}) \mid \boldsymbol{h}(P) \in \mathscr{D} \}$$

With these notations, we may ask the following question:

Question 4.8. — *We assume that our nice variety V satisfies the conditions of the hypothesis 3.27. Let $\mathscr{D}_1$ be a compact polyhedron of* $\mathrm{Pic}(V)^\vee_{\mathbf{R}}$ *and u be an element of the open cone $C^\circ_{\mathrm{eff}}(V)^\vee$. For a real number $B > 1$, let $\mathscr{D}_B = \mathscr{D}_1 + \log(B)u$. Can we find a "small" subset T so that we have an equivalence of the form*

$$\sharp(V(\mathbf{K}) - T)_{h\in\mathscr{D}_B} \sim \beta(V)\nu(\mathscr{D}_1)\boldsymbol{\omega}_V(V(\boldsymbol{A}_{\mathbf{K}})^{\mathrm{Br}})B^{\langle \omega_V^{-1}, u\rangle} \tag{5}$$

as B goes to infinity?

Remarks 4.9. — a) One may note that in the right hand side of (5), one may use $\nu(\mathscr{D}_B) = \nu(\mathscr{D}_1)B^{\langle \omega_V^{-1}, u\rangle}$.

b) One can easily imagine variants of this question. For example, some methods from analytic number theory give much better error terms if ones use smooth functions instead of characteristic functions of sets. So it would be natural to consider a smooth function $\varphi : \mathrm{Pic}(V)^\vee_{\mathbf{R}} \to \mathbf{R}$ with compact support and ask whether we have

$$\sum_{P \in V(\mathbf{K})} \varphi(\boldsymbol{h}(P) - Bu) \sim \beta(V) \int_{\mathrm{Pic}(V)^\vee_{\mathbf{R}}} \varphi \mathrm{d}\nu \, \boldsymbol{\omega}_V(V(\boldsymbol{A}_{\mathbf{K}})^{\mathrm{Br}})B^{\langle \omega_V^{-1}, u\rangle}$$

as B goes to infinity.

c) Let us compare formula (5) with formula (4). First we may note that

$$\nu(\{y \in C_{\mathrm{eff}}(V)^\vee \mid \langle y, \omega_V^{-1}\rangle \leqslant \log(B)\}) \sim \alpha(V) B \log(B)^{t-1}$$

Thus using remark 4.9 a), formula (4) may be seen as integrating formula (5) over

$$\mathscr{D}_B = \{y \in C_{\mathrm{eff}}(V)^\vee \mid \langle y, \omega_V^{-1}\rangle \leqslant \log(B)\}.$$

In this context in which we consider all the possible heights, we may consider again the question of the global equidistribution.

Global equidistribution 4.10. *We shall say that the* global equidistribution holds for $\boldsymbol{h}$ *if, for any compact polyhedron* $\mathscr{D}_1$ *in* $\mathrm{Pic}(V)^\vee_{\mathbf{R}}$ *and any* u *in the open cone* $C^\circ_{\mathrm{eff}}(V)^\vee$*, the measure* $\boldsymbol{\delta}_{V(\mathbf{K})_{\boldsymbol{h}\in\mathscr{D}_B}}$ *converges weakly to* $\boldsymbol{\mu}_V^{\mathrm{Br}}$ *as* B *goes to infinity.*

Note that the expected limit probability measure is the same as before and does not depend on u.

4.2. Compatibility with the product. —

A positive answer to question 4.8 is compatible with the product of varieties in the following sense:

Proposition 4.11. — *Let* V_1 *and* V_2 *be nice varieties equipped with system of heights which satisfy the conditions 3.27. If the sets* $V_1(\mathbf{K}) - T_1$ *and* $V_2(\mathbf{K}) - T_2$ *satisfy the equivalences* (5) *for any compact polyhedra, then this is also true for the product*

$$(V_1(\mathbf{K}) - T_1) \times (V_2(\mathbf{K}) - T_2),$$

equipped with the induced system of heights.

If these varieties satisfy the global equidistribution 4.10, then so does their product.

Proof. — We put $W_i = V_i(\mathbf{K}) - T_i$ for $i \in \{1,2\}$. Let W be the product $W_1 \times W_2$. For $i \in \{1,2\}$, we denote by $\boldsymbol{h}_i$ the multiheight on V_i, and fix a compact polyhedron $\mathscr{D}_{i,1}$ in $\mathrm{Pic}(V_i)^\vee_{\mathbf{R}}$, as well as an element $u_i \in C^\circ_{\mathrm{eff}}(V_i)^\vee$. Let us first note that by [**Ha**, exercise III.12.6], the natural morphism induced by pull-backs $\mathrm{Pic}(V_1)\times\mathrm{Pic}(V_2) \to \mathrm{Pic}(V)$ is an isomorphism which maps the product $C_{\mathrm{eff}}(V_1) \times C_{\mathrm{eff}}(V_2)$ onto $C_{\mathrm{eff}}(V)$ and $(\omega_{V_1}^{-1}, \omega_{V_2}^{-1})$ on ω_V^{-1} (see [**Ha**, exercise II.8.3]). Therefore we identify these groups and consider $\mathscr{D}_1 = \mathscr{D}_{1,1} \times \mathscr{D}_{2,1}$ as a subset of $\mathrm{Pic}(V)^\vee_{\mathbf{R}}$ and $u = (u_1, u_2)$ as an element of $C^\circ_{\mathrm{eff}}(V)^\vee$. If we put $\mathscr{D}_B = \log(B)u + \mathscr{D}_1$, we have

$$\sharp W_{\boldsymbol{h}\in\mathscr{D}_B} = \sharp(W_1)_{\boldsymbol{h}_1\in\mathscr{D}_{1,B}} \times \sharp(W_2)_{\boldsymbol{h}_2\in\mathscr{D}_{2,B}}$$

and the result follows from the compatibility of equivalence with products. To extend the result to an arbitrary polyhedra $\mathscr{D}$, we find domains $\mathscr{D}'$ and $\mathscr{D}''$ which are finite unions of products of polyhedra with disjoint interiors such that $\mathscr{D}' \subset \mathscr{D} \subset \mathscr{D}''$ and use the fact that the equivalence is valid for such a finite union.

Similarly for the equidistribution, it is enough to count the points in open subsets U of $V(\boldsymbol{A}_{\mathbf{K}})^{\mathrm{Br}}$ which are of the form $U = U_1 \times U_2$ for open subsets U_1 and U_2 such that $\boldsymbol{\omega}_{V_1}(\partial U_1) = 0$ and $\boldsymbol{\omega}_{V_2}(\partial U_2) = 0$. But in that case,

$$\sharp(W \cap U)_{\boldsymbol{h} \in \mathscr{D}_B} = \sharp(W_1 \cap U_1)_{\boldsymbol{h}_1 \in \mathscr{D}_{1,B}} \times \sharp(W_2 \cap U_2)_{\boldsymbol{h}_2 \in \mathscr{D}_{2,B}}$$

and we may conclude in the same way. □

It is worthwile to note that this proof is much simpler than the proof of the compatibility of the principle of Batyrev and Manin for products (see [**FMT**, §1.1]). It illustrates the fact that in question 4.8 we cut out the "spikes" where the heights of the components of the points are very different.

4.3. Lifting to versal torsors. — Following Salberger [**Sal**], we shall now explain how the question lifts naturally to versal torsors (see also [**Pe2**]). Let us start by a quick reminder on versal torsors. In our setting, the geometric Picard is supposed to be without torsion, thus we shall restrict ourselves to torsors under algebraic tori.

Definition 4.12. — Let $\mathbf{L}$ be a field and $\mathbf{L}^s$ be a separable closure of $\mathbf{L}$. For any scheme X over $\mathbf{L}$, we write X^s for the product $X \times_{\mathrm{Spec}(\mathbf{L})} \mathrm{Spec}(\mathbf{L}^s)$.

An algebraic group G over a field $\mathbf{L}$ is said to be *of multiplicative type* if there exists an integer n such that G^s is isomorphic to a closed subgroup of $\mathbf{G}^n_{m,\mathbf{L}^s}$. A *torus* $\mathbf{T}$ over $\mathbf{L}$ is an algebraic group $\mathbf{T}$ over $\mathbf{L}$ such that $\mathbf{T}^s$ is isomorphic to a power $\mathbf{G}^n_{m,\mathbf{L}^s}$ of the multiplicative group.

The *group of characters* of an algebraic group G, denoted by $X^*(G^s)$ is the group of group homomorphisms from G^s to $\mathbf{G}_{m,\mathbf{L}^s}$. If G is of multiplicative type, it is a finitely generated $\mathbf{Z}$-module. If G is a torus, it is a free $\mathbf{Z}$-module of rank n. In both cases, it is equipped with an action of the absolute Galois group of $\mathbf{L}$, that is $\mathscr{G}_{\mathbf{L}} = \mathrm{Gal}(\mathbf{L}^s/\mathbf{L})$, which splits over a finite separable extension of $\mathbf{L}$.

Conversely, let us define a *Galois module L over* $\mathbf{L}$ (resp.a *Galois lattice L over* $\mathbf{L}$) as a finitely generated $\mathbf{Z}$-module (resp.a free $\mathbf{Z}$-module of finite rank) equipped with an action of the Galois group $\mathscr{G}_{\mathbf{L}}$ which splits over a finite extension. To a Galois module L, we may associate the monoid algebra $\mathbf{L}^s[L]$ and thus the algebraic variety

$$\mathbf{T} = \mathrm{Spec}(\mathbf{L}^s[L]^{\mathscr{G}_\mathbf{L}})$$

equipped with the algebraic group structure induced by the coproduct ∇ on $\mathbf{L}^s[L]$ defined by $\nabla(\lambda) = \lambda \otimes \lambda$ for any $\lambda \in L$. This algebraic group is an algebraic group of multiplicative type, which we shall say to be *associated to* L.

Example 4.13. — As a basic example, the group of characters of $\mathbf{G}_{m,\mathbf{L}}^n$ is $\mathbf{Z}^n$ with a trivial action of the Galois group and the torus associated with $\mathbf{Z}^n$ is isomorphic to $\mathbf{G}_{m,\mathbf{L}}^n$.

Remark 4.14. — These constructions are functorial and we get a contravariant equivalence of categories between the category of tori (resp. groups of multiplicative type) over $\mathbf{L}$ and the category of Galois lattices (resp. Galois modules) over $\mathbf{L}$.

Notation 4.15. — We shall denote by $\mathbf{T}_{\mathrm{NS}}$ the torus associated to the Galois lattice $\mathrm{Pic}(\overline{V})$.

We are going to use pointed torsors, that is torsors in the category of pointed schemes.

Definition 4.16. — Let G be an algebraic group over a field $\mathbf{L}$ and let X be an algebraic variety over $\mathbf{L}$. A *G-torsor T* over X is an algebraic variety T over $\mathbf{L}$ equipped with a faithfully flat morphism $\pi : T \to X$ and an action $\mu : G \times T \to T$ of G such that $\pi \circ \mu = \pi \circ \mathrm{pr}_2$ and the morphism given by $(g, y) \mapsto (gy, y)$ is an isomorphism from $G \times T$ to $T \times_X T$.

A *pointed variety* over $\mathbf{L}$ is a variety X over $\mathbf{L}$ equipped with a chosen rational point $x \in X(\mathbf{L})$. A *pointed torsor* over the pointed variety X is a torsor T over X equipped with a rational point $t \in T(\mathbf{L})$ such that $\pi(t) = x$.

Example 4.17. — For any line bundle L over X, we can define a $\mathbf{G}_{m,\mathbf{L}}$ torsor by considering $L^\times$ which is the complement of the zero section in L. Conversely for a nice variety X, given a $\mathbf{G}_m$ torsor T, we get a line bundle by considering the contracted product $T \times^{\mathbf{G}_{m,\mathbf{L}}} \mathbf{A}^1_\mathbf{L}$ which is the quotient $(T \times \mathbf{A}^1_\mathbf{L})/\mathbf{G}_{m,\mathbf{L}}$ where $\mathbf{G}_{m,\mathbf{L}}$ acts by $t.(y, a) = (t.y, t^{-1}.a)$. We get in that way the equivalence of category between the line bundles and the $\mathbf{G}_{m,\mathbf{L}}$-torsors over X.

4.3.1. *Versal and universal torsors.* — The versal torsors were introduced by J.-L. Colliot-Thélène and J.-J. Sansuc in the study of the Brauer-Manin obstruction for Hasse principle and weak approximation (see [**CTS1**], [**CTS2**], and [**CTS3**]) . For a survey on versal torsors, the reader may also look at [**Pe3**].

In topology, universal coverings for an unlaceable pointed space X answers a universal problem for coverings: it is a pointed covering E over X such that for any pointed covering $C \to X$ there exists a unique morphism $E \to C$ of pointed spaces over X (see [**Bki**, TA IV, §1, n^{o}3]). We could in fact restrict ourselves to Galois coverings, that is connected coverings with an automorphism group which acts transitively on the fibre over the marked point of X. Fixing a point in the space X is necessary to guarantee the unicity, up to a unique isomorphism, of the universal covering. The universal torsor is the answer to a similar problem for torsor under groups of multiplicative type.

Definition 4.18. — Let $\mathbf{L}$ be a field and $\overline{\mathbf{L}}$ be an algebraic closure of $\mathbf{L}$. Let X be a smooth and geometrically integral variety over $\mathbf{L}$ with a rational point such that all invertible functions on $\overline{X}$ are constant: $\Gamma(\overline{X}, \mathbf{G}_m) = \overline{\mathbf{L}}^*$. We see X as a pointed space by fixing a rational point $x \in X(\mathbf{L})$. Then a *universal torsor* is a pointed torsor T_u over the pointed space X under a group of multiplicative type $\mathbf{T}_u$ such that for any pointed torsor T over X under a group of multiplicative type $\mathbf{T}$, there is a unique morphism of group $\varphi : \mathbf{T}_u \to \mathbf{T}$ and a unique morphism $\psi : T_u \to T$ over X, compatible with the actions of $\mathbf{T}_u$ and $\mathbf{T}$ and the marked points.

Remarks 4.19. — a) If such a torsor exists it is by definition unique up to a unique isomorphism.

b) Using the cohomological characterisation of universal torsors [**CTS3**, §2], one may show that the extension of scalars of a universal torsor is also a universal torsor.

c) Let us assume that there exists a universal torsor T_u. Let x be the chosen point of X. For any line bundle L over X, we can consider the $\mathbf{G}_m$-torsor $L^\times$ and fix a point in its fibre over x. Thus there exists a unique morphism of pointed torsors from T_u to $L^\times$ compatible with a morphism $\mathbf{T}_u \to \mathbf{G}_m$. By duality, it corresponds to a homomorphism of groups from $\mathbf{Z}$ to the group of characters of $\mathbf{T}_u$. Moreover if $L^{\otimes n}$ is isomorphic to the trivial line bundle, the image of $n \in \mathbf{Z}$ in $X^*(\mathbf{T}_u)$ is trivial. Therefore, over $\mathbf{L}^s$, we get a homomorphism of groups from $\mathrm{Pic}(X^s)$ to $X^*(\mathbf{T}_u^s)$, which is compatible with the Galois actions.

Conversely, for any torsor T under a multiplicative group $\mathbf{T}$ and any group character $\chi : \mathbf{T} \to \mathbf{G}_m$, the contracted product $T \times^{\mathbf{T}} \mathbf{G}_{m,\mathbf{L}}$ is a $\mathbf{G}_m$ torsor over $\mathbf{L}$. We get a homomorphism of groups from $X^*(\mathbf{T}^s)$ to $\mathrm{Pic}(X^s)$. It is possible to deduce from such arguments that the character group of $\mathbf{T}_u$ over $\mathbf{L}^s$ has to be isomorphic to $\mathrm{Pic}(X^s)$.

Construction 4.20. — Let us now explain how it is possible to construct such universal torsors. We shall assume again hypothesis 3.27, and fix a rational point $x \in V(\mathbf{K})$. In that case the group $\mathbf{T}_u$ is canonically isomorphic to the Néron-Severi torus $\mathbf{T}_{\mathrm{NS}}$. Over $\overline{\mathbf{K}}$, the construction of remark 4.19 c) gives an isomorphism of $\mathbf{T}_{\mathrm{NS}}$-torsor from a universal torsor $\overline{T}_u$ to the product $L_1^\times \times_V \cdots \times_V L_t^\times$ where $([L_1], \dots, [L_t])$ is a basis of $\mathrm{Pic}(\overline{V})$. But the unicity of the universal torsor shows that, by marking $\overline{T}_u$ with a point in the fibre of x, there exists no non-trivial automorphism of $\overline{T}_u$ as a pointed torsor over X. By descent theory, $\overline{T}_u$ comes from a unique pointed $\mathbf{T}_{\mathrm{NS}}$-torsor T_u over X.

Remark 4.21. — In particular, as a non-pointed $\overline{\mathbf{T}}_{\mathrm{NS}}$-torsor over $\overline{V}$, the torsor $\overline{T}_u$ does not depend on the choice of the point x in $V(\mathbf{K})$. This is not true over $\mathbf{K}$.

Definition 4.22. — A *versal torsor* over V is a $\mathbf{K}$-form of the $\overline{\mathbf{T}}_{\mathrm{NS}}$-torsor $\overline{T}_u$.

Remark 4.23. — The automorphisms of $\overline{T}_u$ as a $\overline{\mathbf{T}}_{\mathrm{NS}}$-torsor over $\overline{V}$ are given by the action of $\mathbf{T}_{\mathrm{NS}}(\overline{\mathbf{K}})$. It follows that if we fix a rational point, and therefore a universal torsor T_u, the versal torsors are classified by the group of Galois cohomology $H^1(\mathbf{K}, \mathbf{T}_{\mathrm{NS}})$ and we get a map from $V(\mathbf{K})$ to $H^1(\mathbf{K}, \mathbf{T}_{\mathrm{NS}})$ which maps a point to the class of the corresponding universal torsor. In general this cohomology group is infinite. But Colliot-Thélène and Sansuc proved in [**CTS2**, proposition 2] that the image of the map is finite. In other words, there exists a finite family $(T_i)_{i \in I}$ of non-isomorphic versal torsors over V with a rational point such that

$$V(\mathbf{K}) = \coprod_{i \in I} \pi_i(T_i(\mathbf{K})),$$

where $\pi_i : T_i \to V$ is the structural morphism.

4.3.2. *Structures on versal torsors.* — Let T_u be a universal torsor over V. By definition of the torsors, there is a natural isomorphism

$$T_u \times_V T_u \xrightarrow{\sim} \mathbf{T}_{\mathrm{NS}} \times T_u$$

which shows that the pull-back of T_u to T_u is trivial. But from the universality of T_u it is possible to show that the pull-back of any pointed torsor under a group of multiplicative type is trivial [**CTS3**, proposition 2.1.1]. By this proposition, we also have that invertible functions on T_u are constant: $\Gamma(T_u, \mathbf{G}_m) = \mathbf{K}^*$.

Moreover, by [**Pe2**, lemme 2.1.10], ω_{T_u} is isomorphic to the pull-back of ω_V. We get the following assertion concerning *volume forms*, that is non-vanishing sections of ω_{T_u}.

Proposition 4.24. — *Let T be a versal torsor over V. Then up to multiplication by a constant there exists a unique volume form on T.*

Construction 4.25. — Let T be a versal torsor on V with a rational point. By the proposition, we may take a non-vanishing section ω of ω_T. For any place w of $\mathbf{K}$, the expression

$$\left|\left\langle \omega, \frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}\wedge\cdots\wedge\frac{\partial}{\partial x_n}\right\rangle\right|_w \mathrm{d}x_{1,w}\,\mathrm{d}x_{2,w}\ldots\mathrm{d}x_{n,w},$$

defines a local measure, which, like in construction 3.6, we may patch together to get a measure $\boldsymbol{\omega}_{T_u,w}$ on $T_u(\mathbf{K}_w)$.

We then choose a finite set S of places containing all the places of bad reduction for V, the archimedean places, as well as the ramified places in a extension splitting the action of the Galois group on the Picard group of V. Moreover, we may assume that any isomorphism class of versal torsors with a rational point has a model over the ring of S-integers $\mathscr{O}_S$ and that the projection maps $T(\mathbf{K}_w)\to V(\mathbf{K}_w)$ are surjective for $w\notin S$ ([**CTS3**, lemme 3.2.3]). Let us fix such a model $\mathscr{T}$ of our versal torsor T. Then for any place w outside a finite set of places, one can prove (see the proof of theorem 4.33 below) that

$$\boldsymbol{\omega}_{T,w}(\mathscr{T}(\mathscr{O}_w)) = L_w(1,\mathrm{Pic}(\overline{V}))^{-1}\boldsymbol{\omega}_{V,w}(V(\mathbf{K}_w)).$$

Using the arguments of construction 3.29, it follows that we can define the product of the measures

$$\boldsymbol{\omega}_T = \frac{1}{\sqrt{d_{\mathbf{K}}}^{\dim T}}\prod_{w\in\mathrm{Val}(\mathbf{K})}\boldsymbol{\omega}_{T,w}.$$

on the adelic space $T(\boldsymbol{A}_{\mathbf{K}})$. By the product formula, this measure does not change if we multiply ω by a nonzero constant. Thus we may call $\boldsymbol{\omega}_T$ the *canonical measure* on the adelic space of the versal torsor T.

Example 4.26. — For a smooth hypersurface V of degree d in $\mathbf{P}^N_{\mathbf{K}}$, with $N\geqslant 4$, any versal torsor is isomorphic to the cone over the hypersurface in $\mathbf{A}^{N+1}_{\mathbf{K}} - \{0\}$, and the canonical measure is given by the Leray form [**Le**, chapter IV, §1]. If F

is a homogeneous equation for V, then locally the measure may be defined as

$$\boldsymbol{\omega}_{T,w} = \frac{1}{|F(1,x_1,\dots,x_N)|_w}\mathrm{d}x_{1,w}\dots\mathrm{d}x_{N,w}.$$

Let us now turn to the lifting of heights to versal torsors. We have to take into account that the rank of the Picard group at a place w depends on w.

Construction 4.27. — We choose a system of representants $(T_i)_{i\in I}$ of the isomorphism classes of versal torsors over V which have a rational points over $\mathbf{K}$. For each $i \in I$, we also fix a point $y_i \in T_i(\mathbf{K})$. Let $\mathbf{L}$ be a Galois extension of $\mathbf{K}$ which splits the Picard group of V. Let $\boldsymbol{s}_{\mathbf{L}} : \mathrm{Pic}(V_{\mathbf{L}}) \to \mathscr{H}(V_{\mathbf{L}})$ be a system of heights over $\mathbf{L}$. We also fix a place w_0 of $\mathbf{K}$. Let $i \in I$. For any line bundle L over $\mathrm{Pic}(V_{\mathbf{L}})$ there exists a morphism $\phi_L : T_i \to L^\times$ over V, which is compatible with the character $\chi_L : (\mathbf{T}_{\mathrm{NS}})_{\mathbf{L}} \to \mathbf{G}_{m,\mathbf{L}}$ defined by L. This morphism is unique up to multiplication by a constant. Let us choose a representant $(\|\cdot\|_v)_{v\in\mathrm{Val}(\mathbf{L})}$ of $s_{\mathbf{L}}([L])$ defining the exponential height $H_{\mathbf{L}}$ on $V(\mathbf{L})$. For any $v \in \mathrm{Val}(\mathbf{L})$, we may then consider the map from $T_i(\mathbf{L}_v)$ to $\mathbf{R}$ given by

$$y \longmapsto \|y\|_v^L = \begin{cases} \dfrac{\|\phi_L(y)\|_v}{\|\phi_L(y_i)\|_v} \text{ if } v \not| w_0 \\ \dfrac{\|\phi_L(y)\|_v}{\|\phi_L(y_i)\|_v} H_L(\pi_i(y_i))^{-\frac{[\mathbf{L}_v:\mathbf{K}_{w_0}]}{[\mathbf{L}:\mathbf{K}]}} \text{ otherwise.} \end{cases}$$

This map does not depend on the choice of ϕ_L nor on the choice of the representant of $s_{\mathbf{L}}([L])$ and satisfies

$$\forall y \in T_i(\mathbf{L}), \quad H_L(\pi_i(y)) = \prod_{v\in\mathrm{Val}(\mathbf{L})} (\|y\|_v^L)^{-1}.$$

Moreover it satisfies the formula $\|t.y\|_v^L = |\chi_L(t)|_v \|y\|_v^L$, for $t \in \mathbf{T}_{\mathrm{NS}}(\mathbf{L}_v)$ and y in $T_i(\mathbf{L}_v)$. We get a map

$$\widetilde{\boldsymbol{h}}_v : T_i(\mathbf{L}_v) \longrightarrow (\mathrm{Pic}(V_{\mathbf{L}_v}))^\vee_{\mathbf{R}}$$

defined by the relations

$$\|y\|_v^L = q_v^{-\langle \widetilde{\boldsymbol{h}}_v(y),[L]\rangle}$$

for $y \in T_i(\mathbf{L}_v)$ and $[L] \in \mathrm{Pic}(V_{\mathbf{L}_v})$, with q_v the cardinal of the residue field $\mathbf{F}_v$ if v is ultrametric, $q_v = e$ for a real place and $q_v = e^2$ for a complex one. Let us now write V_w for $V_{\mathbf{K}_w}$. Using the inclusion $T_i(\mathbf{K}_w) \to \prod_{v|w} T_i(\mathbf{L}_v)$ and the projection $\mathrm{pr} : \prod_{v|w} \mathrm{Pic}(V_{L_v})^\vee_{\mathbf{R}} \to \mathrm{Pic}(V_w)^\vee_{\mathbf{R}}$, we define a map

$$\widetilde{\boldsymbol{h}}_w : T_i(\mathbf{K}_w) \longrightarrow \mathrm{Pic}(V_w)^\vee_{\mathbf{R}}.$$

so that the diagram

$$\begin{tikzcd}
\prod_{v|w} T_i(\mathbf{L}_v) \arrow[r] & \prod_{v|w} \mathrm{Pic}(V_{\mathbf{L}_v})^\vee_{\mathbf{R}} \arrow[d, "\frac{1}{[\mathbf{L}:\mathbf{K}]}\mathrm{pr}"] \\
T_i(\mathbf{K}_w) \arrow[u] \arrow[r, "\widetilde{\boldsymbol{h}}_w"] & \mathrm{Pic}(V_{\mathbf{K}_w})^\vee_{\mathbf{R}}
\end{tikzcd} \tag{6}$$

commutes.

If L is line bundle over X and if $(\|\cdot\|_v)_{v\in\mathrm{Val}(\mathbf{L})}$ is an adelic norm for the extension of scalars $L_{\mathbf{L}}$, then it induces an adelic norm on L defined by

$$\forall w \in \mathrm{Val}(\mathbf{K}), \forall y \in L(\mathbf{K}_w), \|y\|_w = \left(\prod_{v|w} \|y\|_v\right)^{\frac{1}{[\mathbf{L}:\mathbf{K}]}}.$$

Therefore the system of heights $\boldsymbol{s}_{\mathbf{L}}$ induces a system of heights $\boldsymbol{s} : \mathrm{Pic}(V) \to \mathscr{H}(V)$. For any point $y \in T_i(\mathbf{K})$ we have the formula

$$\boldsymbol{h}(\pi_i(y)) = \sum_{w\in\mathrm{Val}(\mathbf{K})} \log(q_w)\widetilde{\boldsymbol{h}}_w(y).$$

These construction enables us to lift a system of heights to versal torsors with a rational point.

4.3.3. *Lifting of the asymptotic formula.* — We now wish to express the asymptotic formula (5) at the torsor level. The fibre of the projection map $\pi_i : T_i(\mathbf{K}) \to V(\mathbf{K})$ is either empty or a principal homogeneous space under $\mathbf{T}_{\mathrm{NS}}(\mathbf{K})$. Therefore we now need to use the description of the rational points of the torus $\mathbf{T}_{\mathrm{NS}}$, as described in the work of Ono ([**Ono1**] and [**Ono2**]).

Definition 4.28. — Let $\mathbf{T}$ be an algebraic torus over $\mathbf{K}$. We denote by $W(\mathbf{T})$ the torsion subgroup of $\mathbf{T}(\mathbf{K})$. By an abuse of notation, for any place w of $\mathbf{K}$, we denote by $\mathbf{T}(\mathscr{O}_w)$ the maximal compact subgroup of $\mathbf{T}(\mathbf{K}_w)$. Let us put $K_{\mathbf{T}} = \prod_{w\in\mathrm{Val}(\mathbf{K})} \mathbf{T}(\mathscr{O}_w)$ which is a compact subgroup of $\mathbf{T}(\boldsymbol{A}_{\mathbf{K}})$. We also have that $W(\mathbf{T}) = K_{\mathbf{T}} \cap \mathbf{T}(\mathbf{K})$ For any place w, there is an injective morphism of groups

$$\log_w : \mathbf{T}(\mathbf{K}_w)/\mathbf{T}(\mathscr{O}_w) \longrightarrow X^*(\mathbf{T}_w)^\vee_{\mathbf{R}}$$

so that for any $t \in \mathbf{T}(\mathbf{K}_w)$ and any $\chi \in X^*(\mathbf{T}_w)$, we have $q_w^{\langle \log_w(t), \chi\rangle} = |\chi(t)|_w$. For almost all places w the image of $\log_w$ coincide with $X^*(\mathbf{T}_w)^\vee$. In fact, by [**Ono1**, theorem 4] and [**Ono2**, §3] there exists a finite set of places S_T such that the induced map gives an exact sequence

$$1 \longrightarrow \mathbf{T}(\mathscr{O}_{S_T}) \longrightarrow \mathbf{T}(\mathbf{K}) \longrightarrow \bigoplus_{w \in \mathrm{Val}(\mathbf{K}) - S_T} X^*(\mathbf{T})^\vee_w \longrightarrow 0 \tag{7}$$

and there is an exact sequence

$$1 \longrightarrow W(\mathbf{T}) \longrightarrow \mathbf{T}(\mathscr{O}_{S_T}) \xrightarrow[\log_{S_T}]{} \bigoplus_{w \in S_T} X^*(\mathbf{T}_w)^\vee_{\mathbf{R}}, \tag{8}$$

where $\log_{S_T}$ is the map defined by taking $\log_w$ for $w \in S_T$. For any $w \in S_T$, the extension of scalars defines a linear map $\pi_w : X^*(T_w)^\vee_{\mathbf{R}} \to X^*(T)^\vee_{\mathbf{R}}$. We then consider the linear map $\pi = \sum_{w \in S_T} \log(q_w)\pi_w$:

$$\bigoplus_{w \in S_{\mathbf{T}}} X^*(\mathbf{T}_w)^\vee_{\mathbf{R}} \longrightarrow X^*(\mathbf{T})^\vee_{\mathbf{R}}.$$

By the product formula, the image of $\mathbf{T}(\mathscr{O}_{S_T})$ is contained in $\ker(\pi)$. The image $M = \pi(\mathbf{T}(\mathscr{O}_{S_\mathbf{T}}))$ is a lattice in the $\mathbf{R}$-vector space $\ker(\pi)$. Let $(e_1, \dots, e_m)$ be a basis for this lattice and let

$$\Delta = \left\{ \sum_{i=1}^m t_i e_i,\ (t_i)_{1 \leqslant i \leqslant m} \in [0, 1[^m \right\}.$$

By construction, Δ is a fundamental domain for the action of $\mathbf{T}(\mathscr{O}_{S_\mathbf{T}})$ on $\ker(\pi)$.

Construction 4.29. — By increasing the finite set of places S introduced in construction 3.6, we assume that we may take $S_{\mathbf{T}_{\mathrm{NS}}} = S$ for the finite set of places considered in the last definition. In particular, we get that outside of S, the map

$$\mathbf{T}_{\mathrm{NS}}(\mathbf{K}_w) \longrightarrow \mathrm{Pic}(V_w)^\vee$$

is surjective. For each of the chosen torsors we may also fix models $\mathscr{T}_i$ over $\mathscr{O}_S$. We may further assume that, for a family of line bundles which generates $\mathrm{Pic}(V_\mathbf{L})$ and is invariant under the action of the Galois group $\mathrm{Gal}(\mathbf{L}/\mathbf{K})$, the heights are given by models of the corresponding line bundles and that the maps ϕ_L from the chosen versal torsors to a line bundle L of the family are defined over $\mathscr{O}_S$. We may also assume that the adelic metrics outside S are compatible with the action of the Galois group. For any i in I, we define the set $\Delta(T_i)$ as

$$\{y \in T_i(\boldsymbol{A_K}) \mid \mathrm{pr}((\widetilde{\boldsymbol{h}}_w(y_w))_{w\in S}) \in \Delta \text{ and } \forall w \notin S,\ y_w \in \mathscr{T}_i(\mathscr{O}_w)\},$$

where pr is a linear projection on $\ker(\pi)$.

Lemma 4.30. — *For any place $w \notin S$, the projection map $\mathscr{T}_i(\mathscr{O}_w) \to V(\mathbf{K}_w)$ is surjective and the map $\widetilde{\boldsymbol{h}}_w$ is characterized by the following two conditions:*

(i) *We have the relation $\widetilde{\boldsymbol{h}}_w(t.y) = -\log_w(t) + \widetilde{\boldsymbol{h}}_w(y)$ for any $t \in \mathbf{T}_{\mathrm{NS}}(\mathbf{K}_w)$ and any $y \in T_i(\mathbf{K}_w)$;*

(ii) *The integral points of $\mathscr{T}_i$ are given by*

$$\mathscr{T}_i(\mathscr{O}_w) = \{y \in T_i(\mathbf{K}_w) \mid \widetilde{\boldsymbol{h}}_w(y) = 0\}.$$

Proof. — Relation (i) follows from the formula for $||t.y||_w^{\mathbf{L}}$ and the description in (ii) from the fact that all maps are compatible with the models. By the choice of S, for any place $w \notin S$, the projection $\pi_i : T_i(\mathbf{K}_w) \to V(\mathbf{K}_w)$ is surjective. Moreover the functions $||\cdot||_w^{\mathbf{L}}$ are compatible with the action of the Galois group. By the diagram (6), it follows that $\widetilde{\boldsymbol{h}}_w(y)$ belongs to $\mathrm{Pic}(V_w)^\vee$. Since the map $\log_w$ is surjective, we may find in any fibre an element y such that $\widetilde{\boldsymbol{h}}_w(y) = 0$. By (ii), this element is an integral point. Since the map $\pi_i : \mathscr{T}_i(\mathscr{O}_w) \to V(\mathbf{K}_w)$ is surjective, conditions (i) and (ii) characterize $\widetilde{\boldsymbol{h}}_w$. □

Theorem 4.31. — *The set $\Delta(T_i) \cap T_i(\mathbf{K})$ is a fundamental domain for the action of $\mathbf{T}_{\mathrm{NS}}(\mathbf{K})$ modulo $W(\mathbf{T}_{\mathrm{NS}})$. In other words, it satisfies the following conditions:*

(i) *We have $T_i(\mathbf{K}) = \bigcup_{t\in\mathbf{T}_{\mathrm{NS}}(\mathbf{K})} t.\left(\Delta(T_i) \cap T_i(\mathbf{K})\right)$;*

(ii) *For any $t \in \mathbf{T}_{\mathrm{NS}}(\mathbf{K})$, we have*

$$\left(\Delta(T_i) \cap T_i(\mathbf{K})\right) \cap t.\left(\Delta(T_i) \cap T_i(\mathbf{K})\right) \neq \varnothing$$

if and only if $t \in W(\mathbf{T}_{\mathrm{NS}})$.

(iii) *For $t \in W(\mathbf{T}_{\mathrm{NS}})$, we have*

$$t.\left(\Delta(T_i) \cap T_i(\mathbf{K})\right) = \Delta(T_i) \cap T_i(\mathbf{K}).$$

Proof. — Let $y \in T_i(\mathbf{K})$. By the lemma, for any $w \notin S$, $\widetilde{\boldsymbol{h}}_w(y) \in \mathrm{Pic}(V_w)^\vee$. Thus, using the exact sequence (7), we get an element $t \in \mathbf{T}_{\mathrm{NS}}(\mathbf{K})$ such that $t.y \in \mathscr{T}_i(\mathscr{O}_w)$ for $w \notin S$. Using the exact sequence (8) and the definition of Δ, there is an element t' in $\mathbf{T}_{\mathrm{NS}}(\mathscr{O}_S)$ such that $(t't).y \in \Delta(T_i)$. Assertions (ii) and (iii) follow from the definition of Δ. $\square$

Notation 4.32. — For any $i \in I$, we define the map

$$\widetilde{\boldsymbol{h}} : \Delta(T_i) \longrightarrow \operatorname{Pic}(V)^{\vee}_{\mathbf{R}}$$

by the relation $\widetilde{\boldsymbol{h}}(y) = \pi\Big((\widetilde{\boldsymbol{h}}_w(y_w))_{w\in S}\Big)$.

Theorem 4.33. — *We assume conditions 3.27. Let W be a borelian subset of $V(\boldsymbol{A}_{\mathbf{K}})$. Let $\mathscr{D}$ be a borelian subset of* $\operatorname{Pic}(V)^{\vee}_{\mathbf{R}}$. *Then*

$$\begin{aligned}\beta(V)\nu(\mathscr{D})\boldsymbol{\omega}_V(W \cap V(\boldsymbol{A}_{\mathbf{K}})^{\mathrm{Br}})\\ = \frac{1}{W(\mathbf{T}_{\mathrm{NS}})}\sum_{i\in I}\boldsymbol{\omega}_{T_i}(\{y \in \Delta(T_i)\cap \pi_i^{-1}(W) \mid \widetilde{\boldsymbol{h}}(y) \in \mathscr{D}\}).\end{aligned}$$

Proof. — This proof follows the ideas of Salberger [**Sal**] (see also [**Pe2**, §3.5] for more details). If $(\xi_1, \dots, \xi_r)$ is a basis of $X^*(\mathbf{T}_{\mathrm{NS}}) = \operatorname{Pic}(V_{\mathbf{L}})$, then $\bigwedge_{i=1}^r \xi_i^{-1}\mathrm{d}\xi_i$ is a section of $\omega_{\mathbf{T}_{\mathrm{NS}}}$, which, up to sign, does not depend on the choice of the basis. This defines a canonical Haar measure $\boldsymbol{\omega}_{\mathbf{T}_{\mathrm{NS}},w}$ on $\mathbf{T}_{\mathrm{NS}}(\mathbf{K}_w)$ for any place w of $\mathbf{K}$. Let $w \in \operatorname{Val}(\mathbf{K}) - S$. Locally for w-adic topology, we may choose a section of $\pi_i : \mathscr{T}_i(\mathscr{O}_w) \to V(\mathbf{K}_w)$ and the measure $\boldsymbol{\omega}_{T_i,w}$ on $\mathscr{T}_i(\mathbf{K}_w)$ is locally isomorphic to the measure

$$L_w(1, X^*(\mathbf{T}_{\mathrm{NS}}))|\omega_V(t)|_w \boldsymbol{\omega}_{\mathbf{T}_{\mathrm{NS}},w} \times \lambda_w \boldsymbol{\omega}_{V,w}.$$

where ω_V is seen as a character of $\mathbf{T}_{\mathrm{NS}}$. Let us also consider the groups $\mathbf{T}_{\mathrm{NS}}(\boldsymbol{A}_{\mathbf{K}})^1$, defined as

$$\Big\{ (t_w)_{w\in\operatorname{Val}(\mathbf{K})} \in \mathbf{T}_{\mathrm{NS}}(\boldsymbol{A}_{\mathbf{K}}) \,\Big|\, \forall \xi \in X^*(\mathbf{T}_{\mathrm{NS}}),\ \prod_{w\in\operatorname{Val}(\mathbf{K})} |\xi(t_w)|_w = 1 \Big\},$$

and $\mathbf{T}_{\mathrm{NS}}(\mathbf{K}_S)^1$, defined as

$$\Big\{ (t_w)_{w\in S} \in \prod_{w\in S}\mathbf{T}_{\mathrm{NS}}(\mathbf{K}_w) \,\Big|\, \forall \xi \in X^*(\mathbf{T}_{\mathrm{NS}}),\ \prod_{w\in S} |\xi(t_w)|_w = 1 \Big\}.$$

The lattice $X^*(\mathbf{T}_{\mathrm{NS}})^{\vee}$ normalises the Haar measure on $X^*(\mathbf{T}_{\mathrm{NS}})^{\vee}_{\mathbf{R}}$ and therefore on the quotient $\prod_{w\in S}\mathbf{T}_{\mathrm{NS}}(\mathbf{K}_w)/\mathbf{T}_{\mathrm{NS}}(\mathbf{K}_S)^1$. Using the measure $\prod_{w\in S}\boldsymbol{\omega}_{\mathbf{T}_{\mathrm{NS}},w}$

on the product, we get a normalised Haar measure ω_{T^1} on $\mathbf{T}_{\mathrm{NS}}(\mathbf{K}_S)^1$. We consider the fibration

$$\widetilde{\boldsymbol{h}} \times \pi_i : \prod_{w \in S} T_i(\mathbf{K}_w) \longrightarrow \mathrm{Pic}(V)^\vee_{\mathbf{R}} \times \prod_{w \in S} V(\mathbf{K}_w),$$

which, over its image, is a principal homogeneous space under $\mathbf{T}_{\mathrm{NS}}(\mathbf{K}_S)^1$. By choosing a local adequate section of this fibration, we get that the measure $\prod_{w\in S} \boldsymbol{\omega}_{T_i,w}$ on $\prod_{w\in S} T_i(\mathbf{K}_w)$ is the measure induced by the product measure $\nu \times \prod_{w\in S} \boldsymbol{\omega}_{V,w}$ on the image and the measure ω_{T^1} on $\mathbf{T}_{\mathrm{NS}}(\mathbf{K}_S)^1$. Taking the product over all places, and multiplying by the normalisation terms, we get that

$$\frac{1}{\sharp W(\mathbf{T}_{\mathrm{NS}})} \boldsymbol{\omega}_{T_i}(\{ y \in \Delta(T_i) \cap \pi_i^{-1}(W) \mid \widetilde{\boldsymbol{h}}(y) \in \mathscr{D} \}) \\ = \tau(\mathbf{T}_{\mathrm{NS}}) \nu(\mathscr{D}) \boldsymbol{\omega}_V(\pi_i(T_i(\boldsymbol{A}_{\mathbf{K}})) \cap W),$$

where $\tau(\mathbf{T}_{\mathrm{NS}})$ is the Tamagawa number of $\mathbf{T}_{\mathrm{NS}}$, that is the normalized volume of the compact quotient $\mathbf{T}_{\mathrm{NS}}(\boldsymbol{A}_{\mathbf{K}})^1/\mathbf{T}_{\mathrm{NS}}(\mathbf{K})$ which is isomorphic to the product

$$\mathbf{T}_{\mathrm{NS}}(\mathbf{K}_S)^1/\mathbf{T}_{\mathrm{NS}}(\mathscr{O}_S) \times \prod_{w \notin S} \mathbf{T}_{\mathrm{NS}}(\mathscr{O}_w).$$

By Ono's theorem ([**Ono3**, §3]), the Tamagawa number of $\mathbf{T}_{\mathrm{NS}}$ is given by

$$\tau(\mathbf{T}_{\mathrm{NS}}) = \frac{\sharp H^1(\mathbf{K}, X^*(\mathbf{T}_{\mathrm{NS}}))}{\sharp Ш^1(\mathbf{K}, \mathbf{T}_{\mathrm{NS}})}$$

where $Ш^1(\mathbf{K}, \mathbf{T}_{\mathrm{NS}}) = \ker(H^1(\mathbf{K}, \mathbf{T}_{\mathrm{NS}}) \to \prod_{w \in \mathrm{Val}(\mathbf{K})} H^1(\mathbf{K}_w, \mathbf{T}_{\mathrm{NS}}))$. By definition, $\beta(V) = \sharp H^1(\mathbf{K}, X^*(\mathbf{T}_{\mathrm{NS}}))$. To conclude the proof, we use the crucial fact, first proven by Salberger [**Sal**], that for any $x \in V(\boldsymbol{A}_{\mathbf{K}})^{\mathrm{Br}}$, the number of $i \in I$ such that $x \in \pi_i(T_i(\boldsymbol{A}_{\mathbf{K}}))$ is precisely equal to $\sharp Ш^1(\mathbf{K}, \mathbf{T}_{\mathrm{NS}})$. □

Remarks 4.34. — a) Using theorems 4.31 and 4.33, we see that the equivalence formula (5) of question 4.8, reduces to an equivalence of the form

$$\sharp\{ y \in T_i(\mathbf{K}) \cap \Delta(T_i) \mid \widetilde{\boldsymbol{h}}(y) \in \mathscr{D}_B \} \\ \sim \boldsymbol{\omega}_{T_i}(\{ y \in \Delta(T_i) \mid \widetilde{\boldsymbol{h}}(y) \in \mathscr{D}_B \})$$

as $B \to +\infty$.

b) The conditions $y \in T_i(\mathscr{O}_w)$ for $w \in \mathrm{Val}(\mathbf{K}) - S$ correspond to an integrality condition combined with a gcd condition. For example, if V is a smooth complete intersection of dimension $\geqslant 3$ in the projective space $\mathbf{P}^N_{\mathbf{Q}}$, then the unique versal torsor T is the corresponding cone in $\mathbf{A}_{\mathbf{Q}}^{N+1}-\{0\}$ and the condition $(y_0,\dots,y_N)\in T(\mathbf{Z}_p)$ corresponds to $(y_0,\dots,y_N)\in\mathbf{Z}_p^{N+1}$ and $\gcd(y_0,\dots,y_N)=1$. Therefore to reduce to counting integral points in a bounded domain, the next step is to use a Moebius inversion formula to remove the gcd condition. Such an inversion formula is described in [**Pe2**, §2.3].

c) In the preceding description, we were not very careful about the choice of the finite set S of bad places. For practical reasons, to use this method, it is in fact more efficient to use a small set of bad primes.

d) The lifting to the versal torsors has been used in many cases, see for example [**Bre**] or [**BBP**]. For practical reasons, it is often simpler to consider an intermediate torsor corresponding to the Picard group $\mathrm{Pic}(V)$ (see for example the work of K. Destagnol [**Des**]). The main difference in the new approach described in this section is that the domain obtained after lifting does not have "spikes". In other words, the area of the boundary has a smaller rate of growth, which should remove some of the problems encountered when using a single height relative to the anticanonical line bundle.

4.4. Varieties of Picard rank one. — If the rank of $\mathrm{Pic}(V)$ is one, then without loss of generality formula (5) is reduced to estimating a difference of the form

$$\sharp(V(\mathbf{K})-T)_{H\leqslant bB}-\sharp(V(\mathbf{K})-T)_{H\leqslant aB} \tag{9}$$

as B goes to infinity, where H is a height relative to the anticanonical line bundle and a,b are real numbers with $0<a<b$. Therefore, in that case, a positive answer to question 4.8 is true if the principle of Batyrev and Manin is valid for $V(\mathbf{K})-T$. Similarly the global equidistribution in the sense of 4.10, follows from global equidistribution 3.21. However the knowledge of estimates for the difference (9) does not gives an estimate for $(V(\mathbf{K})-T)_{H\leqslant B}$, unless we have a uniform upper bound for the error term.

But several examples of Fano varieties of Picard rank one with acccumulating subvarieties are known in dimension $\geqslant 3$ (see the list given in [**BL**]). For example, if we consider a cubic volume, the projective lines it contains are parametrized by the Fano surface, which is of general type. Each of these rational lines has degree 2 and as we shall explain in section 6.4.1, these lines give a non negligible contribution to the total number of points thus contradicting the global equidistribution. In the case of a smooth complete intersection of two quadrics in $\mathbf{P}^5$, the situation is even worse since the projective lines it contains may be Zariski dense.

This shows that in higher dimension, even in the case of varieties with a Picard group of rank one, there might be accumulating subvarieties of codimension $\geqslant 2$ which are not detected by heights on line bundles. Thus one needs to go beyond heights. To help us in that direction we shall first consider the geometric analogue of this problem.

5. Geometric analogue

The geometric analogue of the study of rational points of bounded height is the study of rational curves of bounded degree. This is a very active subject in algebraic geometry, and we are going to give a very superficial survey of some particular aspects of this subject in this section. In fact, there is a very classical dictionary between number fields, global fields of positive characteristic and function fields of curves. To simplify the description, we shall mostly restrict ourselves to morphisms from $\mathbf{P}^1_k$ to a variety V defined over k.

Notation 5.1. — Let k be a field and let $\mathscr{C}$ be a smooth geometrically integral projective curve over k. In this section, we denote by $\mathbf{K} = k(\mathscr{C})$ the function field of $\mathscr{C}$. Let V be a nice variety over k. The image of the generic point gives a bijection between the set of rational point $V(\mathbf{K})$ and the set of morphisms $f : \mathscr{C} \to V$. From now on, we shall identify these sets. Let $f : \mathscr{C} \to V$ be a point of this space. Then the pull-back map is a morphism of groups $f^* : \mathrm{Pic}(V) \to \mathrm{Pic}(\mathscr{C})$. The composition $\deg \circ f^*$ is an element of $\mathrm{Pic}(V)^\vee$, which we call the *multidegree of f* and denote by $\mathbf{deg}(f)$.

The constructions of Grothendieck [**Gr**, §4.c] prove that for any $d \in \mathrm{Pic}(V)^\vee$, there exists a variety $\mathscr{H}om^d(\mathscr{C}, V)$ defined over k, which parametrizes the morphisms from $\mathscr{C}$ to V of multidegree d.

In that geometric setting, we want to describe *asymptotically* the geometric properties of the variety $\mathscr{H}om^d(\mathscr{C}, V)$ as the distance from d to the boundary of the dual of the effective cone goes to infinity. The problem is to give a framework for the asymptotic study of a variety. We shall use the framework given by the ring of integration which was introduced by Kontsevich (see also [**DL**]).

5.1. The ring of motivic integration. — Of course, the dimension of the variety $\mathscr{H}om^d(\mathscr{C}, V)$ goes to infinity as the multidegree d grows. But, as suggested by the work of J. Ellenberg [**Ell**], we could consider the stabilisation of cohomology groups. The ring of motivic integration enables us to consider the limit of a class associated to the variety.

Construction 5.2. — We denote by $\mathscr{M}_k$ the Grothendieck ring of varieties over k: as a group it is generated by the isomorphism classes of varieties over k, where the class of a variety V is denoted by $[V]$, with the relations

$$[V] = [F] + [U]$$

for any closed subvariety F of V, with $U = V - F$. We can then extend the definition of a class to non reduced schemes. Then $\mathscr{M}_k$ is equipped with the unique ring structure such that

$$[V_1] \times [V_2] = [V_1 \times_k V_2],$$

for any varieties V_1 and V_2 over k. We define the Tate symbol as $\boldsymbol{L} = [\mathbf{A}^1_k]$ and consider the localized ring $\mathscr{M}_{k,\mathrm{loc}} = \mathscr{M}_k[\boldsymbol{L}^{-1}]$. We then introduce a decreasing filtration on this ring where, for $i \in \mathbf{Z}$,

$$F^i \mathscr{M}_{k,\mathrm{loc}}$$

is the subgroup of $\mathscr{M}_{k,\mathrm{loc}} = \mathscr{M}_k[\boldsymbol{L}^{-1}]$ generated by symbols of the form $[V]\boldsymbol{L}^{-n}$ if $\dim(V) - n \leqslant -i$. We have the inclusion

$$F^i \mathscr{M}_{k,\mathrm{loc}}.F^j \mathscr{M}_{k,\mathrm{loc}} \subset F^{i+j} \mathscr{M}_{k,\mathrm{loc}},$$

for $i, j \in \mathbf{Z}$. Thus the inverse limit $\widehat{\mathscr{M}_k} = \varprojlim_i \mathscr{M}_{k,\mathrm{loc}}/F^i \mathscr{M}_{k,\mathrm{loc}}$ comes equipped with a structure of topological ring so that the natural map $\mathscr{M}_{k,\mathrm{loc}} \to \widehat{\mathscr{M}_k}$ is a morphism of rings.

Remark 5.3. — The morphism $\mathscr{M}_k \to \mathscr{M}_{k,\mathrm{loc}}$ is not injective (see [**Bo**]), so we loose information by looking at classes in $\widehat{\mathscr{M}_k}$.

With this ring we may formulate the analogue of question 4.8:

Question 5.4. — *We assume that the nice variety V over k is rationally connected, satisfies conditions (i) and (iii) to (v) of hypotheses 3.27 and that the rational points over $k(T)$ are Zariski dense. Does the symbol*

$$\left[\mathscr{H}\mathrm{om}^d(\mathbf{P}^1_k, V)\right] \boldsymbol{L}^{-\langle \omega_V^{-1}, d\rangle}$$

converges in $\widehat{\mathscr{M}_k}$ for $d \in \mathrm{Pic}(V)^\vee \cap C^\circ_{\mathrm{eff}}(V)^\vee$ as $\mathrm{dist}(d, \partial C_{\mathrm{eff}}(V)^\vee)$ goes to infinity and can we interpret the limit as some adelic volume?

5.2. A sandbox example: the projective space. — In the case of the projective space, it turns out that the symbol in fact stabilizes, and thus converges:

Proposition 5.5. — *If $d \geqslant 1$, then*

$$\left[\mathcal{H}om^d(\mathbf{P}^1_k, \mathbf{P}^n_k)\right]\boldsymbol{L}^{-(n+1)d} = \frac{\boldsymbol{L}^{n+1}-1}{\boldsymbol{L}-1}(1-\boldsymbol{L}^{-n}).$$

Proof. — In this proof, we shall describe the sets of k-points of our varieties and gloss over the description of the varieties themselves. So if we consider the set $W_d(k)$ of $(P_0, \dots, P_n) \in k[T]^{n+1}$ such that $\gcd_{0 \leqslant i \leqslant n}(P_i) = 1$ and $\max_{0 \leqslant i \leqslant n}(\deg(P_i)) = d$ then W_d is a $\mathbf{G}_m$ torsor over the space $\mathcal{H}om^d(\mathbf{P}^1_k, \mathbf{P}^n_k)$ which is locally trivial for Zariski topology. Hence

$$(\boldsymbol{L}-1)\left[\mathcal{H}om^d(\mathbf{P}^1_k, \mathbf{P}^n_k)\right] = [W_d]. \tag{10}$$

But if we consider the space of $(n+1)$-tuples of polynomials $(P_0, \dots, P_n)$ such that $\max_{0 \leqslant i \leqslant n}(\deg(P_i)) = d$, then it is naturally isomorphic to $\mathbf{A}^{(n+1)(d+1)} - \mathbf{A}^{(n+1)d}$ and we may decompose it as a disjoint union according to the degree of the gcd of the polynomials. The piece corresponding to the families with $\deg(\gcd_{0 \leqslant i \leqslant n}(P_i)) = k$ is isomorphic to $[W_{d-k}] \times \mathbf{A}^k$ where $\mathbf{A}^k$ parametrizes the gcd which is a unitary polynomial of degree k. We get the formula

$$\boldsymbol{L}^{(n+1)(d+1)} - \boldsymbol{L}^{(n+1)d} = \sum_{k=0}^{d} \boldsymbol{L}^k [W_{d-k}].$$

We may introduce formal series in $\widehat{\mathscr{M}_k}[[T]]$ to get the formula

$$\sum_{d \geqslant 0} (\boldsymbol{L}^{n+1}-1)\boldsymbol{L}^{(n+1)d} T^d = \Big(\sum_{k \geqslant 0} \boldsymbol{L}^k T^k\Big)\Big(\sum_{d \geqslant 0} [W_d] T^d\Big).$$

From which we deduce

$$\sum_{d \geqslant 0} [W_d] T^d = (1-\boldsymbol{L}T)(\boldsymbol{L}^{n+1}-1)\sum_{d \geqslant 0} \boldsymbol{L}^{(n+1)d} T^d.$$

Therefore, if $d \geqslant 1$, we get

$$\begin{aligned}
[W_d] &= (\boldsymbol{L}^{n+1}-1)(\boldsymbol{L}^{(n+1)d} - \boldsymbol{L}\boldsymbol{L}^{(n+1)(d-1)}) \\
&= (\boldsymbol{L}^{n+1}-1)\boldsymbol{L}^{(n+1)d}(1-\boldsymbol{L}^{-n}).
\end{aligned}$$

Combining with formula (10) gives the formula of the proposition. □

Remarks 5.6. — a) Let us quickly explain how the constant obtained might be interpreted as an adelic volume. First, for the projective space the L function associated to the Picard group coincide with the usual zeta function. This has a motivic analogue decribed by M. Kapranov in [**Ka**]:

$$Z_{\mathbf{C}(T)}(U) = \sum_{d\geqslant 0} [(\mathbf{P}^1_k)^{(d)}] U^d$$

where $(\mathbf{P}^1_k)^{(d)}$ is the symmetric product $(\mathbf{P}^1_k)^d/\mathfrak{S}_d$ and is isomorphic to $\mathbf{P}^d_k$. The parameter U should be understood as $\boldsymbol{L}^{-s}$. The residue of the zeta function at $s = 1$ corresponds to

$$\begin{aligned}
&\Big((1-\boldsymbol{L}U)Z_{\mathbf{C}(T)}(U)\Big)(\boldsymbol{L}^{-1})\\
&= \left((1-\boldsymbol{L}U)\sum_{d\geqslant 0}\frac{\boldsymbol{L}^{d+1}-1}{\boldsymbol{L}-1}U^d\right)(\boldsymbol{L}^{-1})\\
&= \frac{1}{\boldsymbol{L}-1}\left((1-\boldsymbol{L}U)\left(\frac{\boldsymbol{L}}{1-\boldsymbol{L}U}-\frac{1}{1-U}\right)\right)(\boldsymbol{L}^{-1})\\
&= \frac{1}{\boldsymbol{L}-1}\left(\frac{\boldsymbol{L}-1}{1-U}\right)(\boldsymbol{L}^{-1})\\
&= \frac{1}{1-\boldsymbol{L}^{-1}}.
\end{aligned}$$

By translating the formula (3), the expected constant should formally have the form

$$C = \frac{\boldsymbol{L}^n}{1-\boldsymbol{L}^{-1}}\prod_{P\in\mathbf{P}^1_k}(1-\boldsymbol{L}^{-\deg(P)})[\mathbf{P}^n_{\kappa(P)}]\boldsymbol{L}^{-n\deg(P)},$$

where $\mathbf{L}^{-1}$ plays the rôle of the square root of the discriminant. The term appearing in the product may be simplified as $1-\boldsymbol{L}^{-(n+1)\deg(P)}$. However this formal constant involves a product over a possibly uncountable set $\mathbf{P}^1_k$. Nevertheless, in this very particular case, we may consider the *inverse* of this product. Then, we get

$$\prod_{P\in\mathbf{P}^1_k}\sum_{m\geqslant 0}\boldsymbol{L}^{-(n+1)m\deg(P)},$$

where the product is taken over closed points of $\mathbf{P}^1_k$. If we admit that it makes sense to develop this product, we get, noting that we get a sum over all divisors of $\mathbf{P}^1_k$,

$$\sum_{m\geqslant 0}\sum_{P\in(\mathbf{P}^1_k)^{(m)}(k)} \boldsymbol{L}^{-(n+1)m}.$$

But we may now interpret each interior sum as a motivic integral and get, using the fact that $(\mathbf{P}^1_k)^{(m)}$ is isomorphic to the projective space $\mathbf{P}^m_k$,

$$\begin{aligned}\sum_{m\geqslant 0}[\mathbf{P}^m_k]\boldsymbol{L}^{-(n+1)m} &= \sum_{m\geqslant 0}\frac{1-\boldsymbol{L}^{m+1}}{1-\boldsymbol{L}}\boldsymbol{L}^{-(n+1)m}\\ &= \frac{1}{1-\boldsymbol{L}}\left(\frac{1}{1-\boldsymbol{L}^{-n-1}}-\frac{\boldsymbol{L}}{1-\boldsymbol{L}^{-n}}\right)\\ &= \frac{1}{1-\boldsymbol{L}}\times\frac{1-\boldsymbol{L}}{(1-\boldsymbol{L}^{-n})(1-\boldsymbol{L}^{-n-1})}\end{aligned}$$

Finally we get

$$C=\frac{\boldsymbol{L}^{n+1}-1}{\boldsymbol{L}-1}(1-\boldsymbol{L}^{-n})$$

as wanted.

b) This type of result is compatible with products and we get a result for products of projective spaces for free. D. Bourqui has more general results for toric varieties [**Bou**].

c) M. Bilu in [**Bil**] has defined an Euler product giving a precise meaning for the expected constant in this setting [1].

5.3. Equidistribution in the geometric setting. — In the geometric setting equidistribution may be described as follows.

Construction 5.7. — Let $\mathscr{S}$ be a subscheme of dimension 0 of $\mathscr{C}$, then we may consider the moduli space $\mathfrak{Hom}(\mathscr{S},V)$ which parametrizes the morphisms from $\mathscr{S}$ to V. For any subvariety W of $\mathfrak{Hom}(\mathscr{S},V)$, we may then consider the set of morphisms $f:\mathbf{P}^1_k\to V$ of multidegree d such that the restriction $f_{|\mathscr{S}}$ belongs to W. This is parametrized by a variety $\mathfrak{Hom}^d_W(\mathscr{C},V)$ contained in $\mathfrak{Hom}^d(\mathscr{C},V)$.

1. The construction of M. Bilu and the work of D. Bourqui suggest that the filtration described here is not the correct one to get the expected limit. In fact, one may need a filtration such that if X and Y are geometrically irreducible varieties then $[\mathbf{L}^{-\dim(X)}X]-[\mathbf{L}^{-\dim(Y)}Y]$ belongs to $F^1\mathscr{M}_{k,\mathrm{loc}}$.

Naïve geometric equidistribution 5.8. *We shall say that* naïve equidistribution *holds for V if for any subscheme $\mathscr{S}$ of dimension* 0 *in $\mathscr{C}$ and any subvariety W of $\mathscr{H}om(\mathscr{S},V)$, the symbol*

$$\left(\left[\mathscr{H}om^d_W(\mathscr{C},V)\right]\left[\mathscr{H}om(\mathscr{S},V)\right]-\left[\mathscr{H}om^d(\mathscr{C},V)\right][W]\right)\mathbf{L}^{-\langle\omega_V^{-1},d\rangle}$$

converges to 0 *in $\widehat{\mathscr{M}_k}$ for $d\in\mathrm{Pic}(V)^\vee\cap C^\circ_{\mathrm{eff}}(V)^\vee$ as the distance from d to $\partial C_{\mathrm{eff}}(V)^\vee$ goes to infinity.*

Remark 5.9. — This statement gives a precise meaning to the idea of a convergence

$$\frac{\left[\mathscr{H}om^d_W(\mathscr{C},V)\right]}{\left[\mathscr{H}om^d(\mathscr{C},V)\right]}\longrightarrow\frac{[W]}{\left[\mathscr{H}om(\mathscr{S},V)\right]}.$$

5.4. Crash course about obstruction theory. — Obstruction theory gives a sufficient condition for the moduli spaces to have the expected dimension. Let us give a very short introduction to these tools, the interested reader may turn to the book of O. Debarre [**De**] for a more serious introduction to this subject.

let $f:\mathbf{P}^1_k\to V$ be a morphism of multidegree d then we may consider the tangent space at f and the dimension at f. There is a natural isomorphism

$$T_f\mathscr{H}om^d(\mathbf{P}^1_k,V)\xrightarrow{\sim}H^0(\mathbf{P}^1_k,f^*(T_V))$$

and

$$\dim_f\left(\mathscr{H}om^d(\mathbf{P}^1_k,V)\right)\geqslant h^0(\mathbf{P}^1_k,f^*(T_V))-h^1(\mathbf{P}^1_k,f^*(T_V)).$$

On the other hand, on $\mathbf{P}^1_k$, any vector bundle splits into a direct sum of line bundles. In other words, there exists an isomorphism

$$f^*(TV)\xrightarrow{\sim}\bigoplus_{i=1}^n\mathscr{O}_{\mathbf{P}^1_k}(a_i)$$

with $a_1\geqslant a_2\geqslant\cdots\geqslant a_n$ and $(a_1,\dots,a_n)$ is uniquely determined. If $a_n\geqslant 0$, then we get that $h^1(\mathbf{P}^1_k,f^*(TV))=0$ and

$$\begin{aligned}\dim_f\left(\mathscr{H}om(\mathbf{P}^1_k,V)\right)&=h^0(\mathbf{P}^1_k,f^*(T_V))=\sum_{i=1}^n h^0(\mathscr{O}_{\mathbf{P}^1_k}(a_i))\\&=\sum_{i=1}^n a_i+1=n+\langle d,\omega_V^{-1}\rangle,\end{aligned}$$

which is the expected dimension. Thus a sufficient condition to get the expected dimension is $a_n \geqslant 0$.

But let us now add some conditions related to equidistribution. Let $\mathscr{S}$ be a subscheme of $\mathbf{P}^1_k$ of dimension 0. Then $\mathscr{S}$ corresponds to a divisor $D = \sum_{P \in I} n_P P$ on $\mathbf{P}^1_k$ and may described as $\mathrm{Spec}(\times_{P\in I}\mathscr{O}_{\mathbf{P}^1_k,P}/\mathfrak{m}_P^{n_P})$, where $\mathfrak{m}_P$ is the maximal ideal of the local ring $\mathscr{O}_{\mathbf{P}^1_k,P}$. Let s be the degree of D, that is $\sum_{P\in I} n_P[\kappa(P):k]$. Then $\mathscr{H}om(\mathscr{S}, V)$ has dimension ns; therefore if we fix $\varphi : \mathscr{S} \to V$, the expected dimension of $\mathscr{H}om^d_{\{\varphi\}}(\mathbf{P}^1_k, V)$ ought to be $n(1-s) + \langle d, \omega_V^{-1}\rangle$. But obstruction theory in that setting relates the deformation at f to the vector bundle $f^*(TV)\otimes\mathscr{O}(-D)$ therefore a sufficient condition for the dimension of the moduli space $\mathscr{H}om^d_{\{\varphi\}}(\mathbf{P}^1_k, V)$ at f to be the correct one is $a_n - s \geqslant 0$. In particular a curve is said to be *very free* if $a_n > 0$. Therefore if one wishes to have geometric equidistribution, then one ought to look at the limit as a_n goes to $+\infty$.

One should note that the counter-examples introduced in section 4.4, like the intersection of two quadrics, also show the necessity to go beyond degrees in the geometric setting.

6. Slopes à la Bost

Following the geometric analogue, we need a notion which is the arithmetic translation of the notion of very free curves. This analogue, introduced in [**Pe4**], is given by Arakelov geometry and is based upon the slopes as they are considered by J.-B. Bost.

6.1. Definition. — In this section, we again consider a nice variety V over a number field $\mathbf{K}$.

6.1.1. *Slopes of an adelic vector bundle over* $\mathrm{Spec}(\mathbf{K})$. — The following definition is a variant of the definition described in chapter II, §3.2 of this volume.

Definition 6.1. — Let E be a $\mathbf{K}$-vector space of finite dimension n equipped with

— A projective $\mathscr{O}_{\mathbf{K}}$-submodule Λ_E of rank n;

— For any complex place $w \in \mathrm{Val}(\mathbf{K})$, a map

$$||\cdot||_w : E_w = E \otimes_{\mathbf{K}} \mathbf{K}_w \longrightarrow \mathbf{R}_{\geqslant 0}$$

such that there exists a positive definite hermitian form ϕ on E_w so that $||y||_w = \phi(y, y)$;

— For any real place $w \in \mathrm{Val}(\mathbf{K})$ a euclidean norm

$$||\cdot||_w : E_w \longrightarrow \mathbf{R}_{\geqslant 0}.$$

Let F be a vector subspace of E. We equip it with $\Lambda_F = \Lambda \cap F$ and the restrictions of the norms. The *Newton polygon*, which we denote by $\mathscr{P}(E)$ is defined as the convex hull of the set of pairs $(\dim(F), \widehat{\deg}(F))$ where F describes the set of vector subspaces of E.

Remark 6.2. — Let us assume that $\mathbf{K} = \mathbf{Q}$. If we consider the subspaces F of dimension 1, then $\widehat{\deg}(F)$ is given as $-\log(||y_0||_\infty)$ where y_0 is a generator of $\Lambda \cap F$. Thus we get the points $(1, -\log(||y||_\infty))$ where y goes over the primitive elements of the lattice Λ. In particular, there is an upper bound for the possible values of the second coordinate. More generally $\mathscr{P}(E)$ is bounded from above. In the figure 5, we represented how the points $(\dim(F), \widehat{\deg}(F))$ and the upper part of the convex hull may look like.

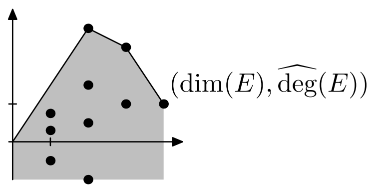

FIGURE 5. Convex hull

Construction 6.3. — Since the set $\mathscr{P}(E)$ is bounded from above, we may define the function $m_E : [0, n] \to \mathbf{R}$ by

$$m_E(x) = \max\{ y \in \mathbf{R} \mid (x, y) \in \mathscr{P}(E) \}.$$

This function is concave and affine in each interval $[i-1, i]$ for $i \in \{1, \dots, \dim(E)\}$. The slopes of E are then given as

$$\mu_i(E) = m_E(i) - m_E(i-1)$$

for $i \in \{1, \dots, \dim(E)\}$.

Remarks 6.4. — a) By construction, we have the inequalities

$$\mu_1(E) \geqslant \mu_2(E) \geqslant \cdots \geqslant \mu_{\dim(E)}(E).$$

These inequalities might not be strict. Moreover

$$\widehat{\deg}(E) = \sum_{i=1}^{\dim(E)} \mu_i(E).$$

Therefore the *slope* of E, which is defined as $\mu(E) = \frac{\widehat{\deg}(E)}{\dim(E)}$ is the mean of the slopes:

$$\mu(E) = \frac{1}{\dim(E)} \sum_{i=1}^{\dim(E)} \mu_i(E).$$

b) The value of $m_E(i)$ may differ from $\max_{\dim(F)=i}(\widehat{\deg}(F))$. However, following E. Gaudron [**Ga**, definition 5.18], we may define the successive minima of the arithmetic lattice E as follows: for $i \in \{1,\dots,\dim(E)\}$, the i-th minima $\lambda_i(E)$ is the infimum of the numbers $\theta \in \mathbf{R}_{>0}$ such that there exists a family of strictly positive real numbers $(\theta_w)_{w\in\mathrm{Val}(\mathbf{K})}$ and a free family $(x_1,\dots,x_i)$ in E such that

(i) The set $\{\, w \in \mathrm{Val}(\mathbf{K}) \mid \theta_w \neq 1 \,\}$ is finite;

(ii) The product $\prod_{w\in\mathrm{Val}(\mathbf{K})} \theta_w$ is equal to θ;

(iii) We have the inequalities

$$||x_j||_w \leqslant \theta_w$$

for $j \in \{1,\dots,i\}$ and $w \in \mathrm{Val}(\mathbf{K})$.

Then Minkowski's theorem gives an explicit constant $C_{\mathbf{K}}$ such that

$$0 \leqslant \log(\lambda_i(E)) + \mu_i(E) \leqslant C_{\mathbf{K}}$$

for $i \in \{\,1,\dots,\dim(E)\,\}$. Other definitions of successive minima are given in chapter II, §3.1 and are similarly related to slopes.

c) In this chapter, the slopes are not invariant under field extensions since we did not normalise them by $\frac{1}{[\mathbf{K}:\mathbf{Q}]}$. This conforms to the usual convention for heights in Manin's program, which has been chosen to get a formulation of the expected estimate which does not depend on the degree of the field.

6.1.2. *Slopes on varieties, freeness.* — We now apply the constructions of last paragraph to vector bundles on varieties.

Definition 6.5. — Let E be a vector bundle on the nice variety V of dimension n. We assume that E is equipped with an adelic norm $(||\cdot||_w)_{w\in\mathrm{Val}(\mathbf{K})}$ then for any rational point $P \in V(\mathbf{K})$, the fibre E_P is an adelic vector bundle over $\operatorname{Spec}(\mathbf{K})$ and we may define

$$\mu_i^E(P) = \mu_i(E_P).$$

In particular, if V is equipped with an adelic metric (see definition 2.5), we may define the *slopes* of a rational point $P \in V(\mathbf{K})$ as

$$\mu_i(P) = \mu_i(T_P V)$$

for $i \in \{1, \dots, n\}$.

Remarks 6.6. — a) From remark 6.4 (i), we deduce that for any rational point $P \in V(\mathbf{K})$, we have

$$\mu_n(P) \leqslant \mu_{n-1}(P) \leqslant \dots \leqslant \mu_1(P)$$

and $\widehat{\deg}(T_P V) = \sum_{k=1}^n \mu_i(P)$. But we may interpret this degree $\widehat{\deg}(T_P V) = \widehat{\deg}((\omega_V^{-1})_P)$ as the logarithmic height of P, that is $h(P) = \log(H(P))$, where the height H is defined by the induced metric on the anticanonical line bundle.

b) From the previous remark we deduce the inequalities

$$\mu_n(P) \leqslant \frac{h(P)}{n} \leqslant \mu_1(P)$$

for any rational point $P \in V(\mathbf{K})$.

Definition 6.7. — The *freeness* of a rational $P \in V(\mathbf{K})$ is defined by

$$l(P) = \begin{cases} n\frac{\mu_n(P)}{h(P)} \text{ if } \mu_n(P) > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Remarks 6.8. — a) By definition the freeness of a point $l(P)$ belongs to the interval $[0, 1]$.

b) We have the equality $l(P) = 0$ if and only if the minimal slope $\mu_n(P) \leqslant 0$.

c) The equality $l(P) = 1$ occurs if and only if the lattice $T_P V$ is semi-stable, that is $\mu_1(P) = \dots = \mu_n(P)$ and $h(P) > 0$. In other words this means that $\mu(F) \leqslant \mu(T_P V)$ for any subspace F of $T_P V$. This is, for example, the case if the lattice is the usual lattice $\mathbf{Z}^n$ in $\mathbf{R}^n$ equipped with its standard euclidean structure. Up to scaling, this occurs for a point $(P, \dots, P)$ on the diagonal of $(\mathbf{P}^1_{\mathbf{K}})^n$. Another example of a semi-stable lattice in dimension 2 is the classical hexagonal lattice $\mathbf{Z}[j]$ generated by a primitive third root of 1, as shown in figure 6. More generally for two dimensional lattices we may consider that Λ is isomorphic to the lattice $a(\mathbf{Z}+\mathbf{Z}\tau) \subset \mathbf{C}$, where a is some positive real number, $\Re(\tau) \in [-1/2, 1/2]$, $|\tau| \geqslant 1$

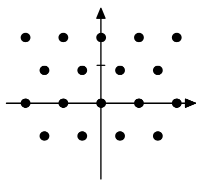

FIGURE 6. Hexagonal lattice

and $\Im(\tau) > 0$. Then a lattice is semistable if and only if $\Im(\tau) \leqslant 1$, which is drawn in grey on figure 7.

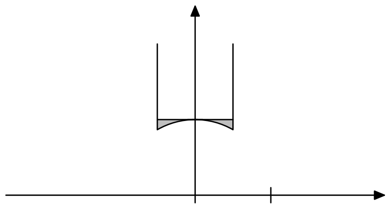

FIGURE 7. Semi-stable lattices

d) For any rational point on a curve, we have $l(P) = 1$.

e) For a surface S over $\mathbf{Q}$, an adelic metric define two invariants, namely the height H and a map $S(\mathbf{Q}) \to \mathbf{H}/\mathrm{PSL}_2(\mathbf{Z})$, where $\mathbf{H}$ denotes the Poincaré half-plane $\{z \in \mathbf{C} \mid \Im(z) > 0\}$ which sends a point P to the class of τ_P such that the lattice in $T_P S$ is isomorphic to $a_P(\mathbf{Z} + \mathbf{Z}\tau_P)$ with $a_P \in \mathbf{R}_{>0}$. Then, taking τ_P in the usual fundamental domain, the freeness of P is given by

$$l(P) = \begin{cases} 1 \text{ if } \Im(\tau_P) \leqslant 1 \text{ and } h(P) > 0, \\ 1 - \frac{\log(\Im(\tau_P))}{h(P)} \text{ if } 1 < \Im(\tau_P) < H(P), \\ 0 \text{ otherwise.} \end{cases}$$

Indeed, in that case, we have $h(P) = -2\log(a_p) - \log(\Im(\tau_P))$ and, since $|\tau_P| \geqslant 1$, the first slope is given by

$$\mu_1(P) = \max\left(-\log(a_P), \frac{h(P)}{2}\right)$$

We get that $\mu_1(P) = \mu_2(P)$ if and only if $\Im(\tau_P) \leqslant 1$ and

$$\mu_2(P) = -\log(a_p) - \log(\Im(\tau_P)) = \frac{h(P) - \log(\Im(\tau_P))}{2}$$

otherwise.

f) By definition, the freeness $l(P)$ is invariant under field extensions. Thus a condition of the form $l(P) > \varepsilon$ does not depend on the field of definition and makes sense for algebraic points in $V(\overline{\mathbf{K}})$. On the other hand the defining condition for a thin subset, namely $P \in \varphi(X(\mathbf{K}))$ for a morphism φ as in definition 3.34 does not make sense for algebraic points.

6.2. Properties. — Let us first describe how the freeness depends on the choice of the metric.

Proposition 6.9. — *Let $\varphi : E \to F$ be a morphism of vector bundles and let $(\|\cdot\|_w)_{w\in\mathrm{Val}(\mathbf{K})}$ (resp. $(\|\cdot\|'_w)_{w\in\mathrm{Val}(\mathbf{K})}$) be an adelic norm on E (resp. F) then there exists a family $(\lambda_w)_{w\in\mathrm{Val}(\mathbf{K})}$ such that*

(i) *For any $w \in \mathrm{Val}(\mathbf{K})$, any $P \in V(\mathbf{K}_w)$, and any $y \in E_P$, we have*
$$\|\varphi(y)\|'_w \leqslant \lambda_w \|y\|_w;$$
(ii) *The set $\{\, w \in \mathrm{Val}(\mathbf{K}) \mid \lambda_w \neq 1 \,\}$ is finite.*

Proof. — Let $\mathbf{P}(E)$ be the projective bundle of the lines in E and $E^\times$ be the complement of the zero section in E. Then for any place w of $\mathbf{K}$, we may define a map $f_w : E^\times(\mathbf{K}_w) \to \mathbf{R}_{\geqslant 0}$ by $f_w(y) = \frac{\|\varphi(y)\|'_w}{\|y\|_w}$. This map is constant on the lines and induces a continuous map $\mathbf{P}(E)(\mathbf{K}_w) \to \mathbf{R}_{\geqslant 0}$. Since the space $\mathbf{P}(E)(\mathbf{K}_w)$ is compact, this function is bounded from above by a constant λ_w. Moreover for almost all $w \in \mathrm{Val}(\mathbf{K})$ the norms on E and F are defined by model and the morphism φ is defined over $\mathscr{O}_w$. For such a place w, for any $P \in V(\mathbf{K}_w)$, we get that
$$\varphi(\{\, y \in E_P \mid \|y\|_w \leqslant 1 \,\}) \subset \{\, y \in F_P \mid \|y\|'_w \leqslant 1 \,\},$$
therefore we may take $\lambda_w \leqslant 1$. □

Remark 6.10. — From this proposition, it follows that, if we consider norms $(\|\cdot\|_w)_{w\in\mathrm{Val}(\mathbf{K})}$ and $(\|\cdot\|'_w)_{w\in\mathrm{Val}(\mathbf{K})}$ on a vector bundle, then the quotient $\frac{\|\cdot\|'_w}{\|\cdot\|_w}$ is bounded from above and from below by a strictly positive constant. Moreover, by definition the norms are equal for almost all places. This implies the existence of a constant C such that, for any rational point $P \in V(\mathbf{K})$ and any subspace F of $T_P V$,
$$|\,\widehat{\deg}(F) - \widehat{\deg}'(F)| \leqslant C.$$
where $\widehat{\deg}'$ is the degree corresponding to the second norm.

Corollary 6.11. — *Let μ_i and μ'_i be the slopes defined by two different metrics on V and let l and l' be the corresponding freeness, then*

(i) *The difference $|\mu_i - \mu'_i|$ is bounded on $V(\mathbf{K})$;*

(ii) *There exists $C \in \mathbf{R}_{>0}$ such that*

$$|l(P) - l'(P)| < \frac{C}{h(P)}$$

for any $P \in V(\mathbf{K})$ such that $h(P) > 0$.

We now wish to describe a strong link between the geometric and arithmetic settings. Let us first define the freeness in the geometric setting.

Definition 6.12. — Let $\varphi : \mathbf{P}^1_{\mathbf{K}} \to V$ be a morphism of varieties. The pull-back of the tangent bundle $\varphi^*(TV)$ is isomorphic to a direct sum $\bigoplus_{i=1}^n \mathscr{O}_{\mathbf{P}^1_{\mathbf{K}}}(a_i)$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$. The slopes of φ are the integers $\mu_i(\varphi) = a_i$. We may consider $\deg_{\omega_V^{-1}}(\varphi) = \sum_{i=1}^n \mu_i(\varphi)$ and the *freeness of φ* is defined by

$$l(\varphi) = \begin{cases} \frac{na_n}{\deg_{\omega_V^{-1}}(\varphi)} \text{ if } a_n > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Remark 6.13. — By construction $l(\varphi) \in [0,1] \cap \mathbf{Q}$ and $l(\varphi) > 0$ if and only if φ is very free, that is $a_n > 0$.

Proposition 6.14. — *Let $\varphi : \mathbf{P}^1_{\mathbf{K}} \to V$ be a non constant morphism of varieties and assume that V is equipped with an adelic metric. Then*

$$l(\varphi(P)) \longrightarrow l(\varphi)$$

as $h_{\mathscr{O}(1)}(P) \to +\infty$.

Proof. — Let us fix an isomorphism from $\varphi^*(TV)$ to a direct sum $\bigoplus_{i=1}^n \mathscr{O}_{\mathbf{P}^1_{\mathbf{K}}}(a_i)$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$. On $\varphi^*(TV)$ we consider the pull-back of the adelic metric on V and we equip the sum $\bigoplus_{i=1}^n \mathscr{O}_{\mathbf{P}^1_{\mathbf{K}}}(a_i)$ with the direct sums of the norms induced by a norm on $\mathscr{O}_{\mathbf{P}^1_{\mathbf{K}}}(1)$. Using the corollary 6.11, we get that the differences $|\mu_i(\varphi(P)) - a_i h_{\mathscr{O}(1)}(P)|$ is bounded, as well as $|h(\varphi(P)) -$

$\sum_{i=1}^{n} a_i h_{\mathscr{O}(1)}(P)|$. If $a_n \geqslant 0$, then the sum $\sum_{i=1}^{n} a_i$ is strictly positive since the morphism is not constant and we get

$$\left| l(\varphi(P)) - \frac{a_n n}{\sum_{i=1}^{n} a_i} \right| < \frac{C}{h_{\mathscr{O}(1)}(P)}.$$

If $a_n < 0$, then we get that $l(\varphi(P)) = 0$ except for a finite number of $P \in \mathbf{P}^1_{\mathbf{K}}$. $\square$

6.3. Explicit computations

6.3.1. *In the projective space.* — Let us compute the freeness for points of the projective space. We denote by H the usual height on $\mathbf{P}^N_{\mathbf{K}}$ relative to $\omega^{-1}_{\mathbf{P}^N_{\mathbf{K}}} = \mathscr{O}_{\mathbf{P}^N_{\mathbf{K}}}(N+1)$ and write $h = \log \circ H$.

Proposition 6.15. — *Let $P \in \mathbf{P}^n(\mathbf{K})$, then*

$$l(P) = \frac{n}{n+1} + \min_F \left(\frac{-n \widehat{\deg}(F)}{\operatorname{codim}(F) h(P)} \right)$$

where F goes over the subspaces $F \subsetneq \mathbf{K}^{n+1}$ such that $P \in \mathbf{P}(F)$.

Proof. — Let $D \subset E$ be the line in E corresponding to the projective point P. There is a canonical isomorphism from the tangent space $T_P \mathbf{P}^n_{\mathbf{K}}$ to the quotient $D^\vee \otimes E/D^\vee \otimes D$ where $D^\vee$ is the dual of D. This gives a bijection from the set of subspaces F of E such that $D \subset F \subsetneq E$ to the strict subspaces of $T_P \mathbf{P}^n_{\mathbf{K}}$ which maps the subspace F to the quotient $D^\vee \otimes F/D^\vee \otimes D$. Since $D^\vee \otimes D$ is canonically isomorphic to $\mathbf{K}$, the arithmetic degree of the subspace of $T_P \mathbf{P}^n_{\mathbf{K}}$ is given by

$$\begin{aligned}\widehat{\deg}(D^\vee \otimes F/D^\vee \otimes D) = \widehat{\deg}(D^\vee \otimes F) - \widehat{\deg}(\mathbf{K}) \\ = \widehat{\deg}(F) - \dim(F)\,\widehat{\deg}(D).\end{aligned}$$

On the other hand, by the description of the tangent space,

$$h(P) = -(n+1)\,\widehat{\deg}(D).$$

We get that the smallest slope is given by

$$\mu_n(P) = -\widehat{\deg}(D) + \min_F \left(\frac{-\widehat{\deg} F}{\operatorname{codim}_E(F)} \right)$$

and the freeness by

$$l(P) = \frac{n}{n+1} + \min_F \left(\frac{-n\,\widehat{\deg}(F)}{\operatorname{codim}_E(F) h(P)} \right). \quad \square$$

Corollary 6.16. — *For any point $P \in \mathbf{P}^n(\mathbf{K})$, we have*

$$l(P) \geqslant \frac{n}{n+1}.$$

Remarks 6.17. — a) If we take a fixed projective subspace F in E, then $l(P)$ converges to $\frac{n}{n+1}$ as $h(P)$ goes to $+\infty$ with $P \in F$.

b) One can show that for any $\eta > 0$, there exists a constant $C > 0$ such that, for $B > 1$,

$$\sharp\{ P \in \mathbf{P}^n(\mathbf{K}) \mid H(P) \leqslant B \text{ and } l(P) < 1-\eta \} < CB^{1-\eta}.$$

Since we have an equivalence

$$\sharp\{ P \in \mathbf{P}^n(\mathbf{K}) \mid H(P) \leqslant B \} \sim C(\mathbf{P}^n_{\mathbf{K}})B$$

as B goes to infinity, this means that the number of points P with a freeness $l(P) < 1-\eta$ is in fact asymptotically negligible.

6.3.2. *Products of lines.* — Despite the previous example, the freeness of points can be very small even on a homogeneous variety. Let us prove that for $(\mathbf{P}^1_{\mathbf{K}})^n$. We equip $(\mathbf{P}^1_{\mathbf{K}})^n$ with the product of the adelic metrics. We denote by H the usual height on $\mathbf{P}^1_{\mathbf{K}}$ relative to $\omega^{-1}_{\mathbf{P}^1_{\mathbf{K}}} = \mathscr{O}_{\mathbf{P}^1_{\mathbf{K}}}(2)$ and write $h = \log \circ H$. We shall also use h (resp.H) to denote the logarithmic (resp exponential) height on $(\mathbf{P}^1_{\mathbf{K}})^n$.

Proposition 6.18. — *For any $\boldsymbol{P} = (P_1, \dots, P_n) \in \mathbf{P}^1(\mathbf{K})^n$, one gets*

$$l(\boldsymbol{P}) = \frac{n \min_{1\leqslant i\leqslant n}(h(P_i))}{\sum_{i=1}^n h(P_i)}.$$

Proof. — The tangent space $T_{\boldsymbol{P}}(\mathbf{P}^1_{\mathbf{K}})^n$ is canonically isomorphic to $\bigoplus T_{P_i}\mathbf{P}^1_{\mathbf{K}}$ and, by construction, this isomorphism is compatible with the norms. Let us choose a permutation $\sigma \in \mathfrak{S}_n$ such that

$$h(P_{\sigma(1)}) \geqslant h(P_{\sigma(2)}) \geqslant \cdots \geqslant h(P_{\sigma(n)}).$$

Then we get that $\mu_i(\boldsymbol{P}) = h(P_{\sigma(i)})$, since the the subspace of dimension i with the biggest arithmetic degree is given by $\bigoplus_{j=1}^i T_{P_{\sigma(j)}}\mathbf{P}^1_{\mathbf{K}}$. $\square$

Corollary 6.19. — *For any* $\varepsilon > 0$, *there exist a constant* C_ε *such that*

$$\frac{\sharp\{P \in \mathbf{P}^1(\mathbf{K})^n \mid H(P) \leqslant B \text{ and } l(P) > \varepsilon\}}{\sharp\{P \in \mathbf{P}^1(\mathbf{K})^n \mid H(P) \leqslant B\}} \longrightarrow C_\varepsilon$$

as $B \to +\infty$. *Moreover* $1 - C_\varepsilon = O(\varepsilon)$.

Proof. — Let us consider the map $\boldsymbol{h} : \mathbf{P}^1(\mathbf{K})^n \to \mathbf{R}^n_{\geqslant 0}$ given by $(P_i)_{1\leqslant i\leqslant n} \mapsto (h(P_i))_{1\leqslant i\leqslant n}$ and, for $\boldsymbol{t} = (t_i)_{1\leqslant i\leqslant n}$, write $|\boldsymbol{t}| = \sum_{i=1}^n t_i$. The height of point $\boldsymbol{P}$ in $\mathbf{P}^1(\mathbf{K})^n$ is given by $h(\boldsymbol{P}) = |\boldsymbol{h}(\boldsymbol{P})|$. By proposition 6.18, we only have to estimate the cardinal of the set

$$\left\{ (P_i)_{1\leqslant i\leqslant n} \in \mathbf{P}^1(\mathbf{K})^n \,\middle|\, \sum_{i=1}^n h(P_i) \leqslant \min\left(\log(B), \frac{n}{\varepsilon} \min_{1\leqslant i\leqslant n}(h(P_i))\right) \right\}.$$

Let us introduce the compact simplex $\Delta_\varepsilon(B)$ in $\mathbf{R}^n_{\geqslant 0}$ defined by

$$|\boldsymbol{t}| \leqslant \min\left(\log(B), \frac{n}{\varepsilon} \min_{1\leqslant i\leqslant n}(t_i)\right).$$

Then we may write the above set as

$$\{\boldsymbol{P} \in \mathbf{P}^1(\mathbf{K})^n \mid \boldsymbol{h}(\boldsymbol{P}) \in \Delta_\varepsilon(B)\}.$$

Using the estimate of S. H. Schanuel [**Sc**, theorem 1], we get

$$\sharp\{P \in \mathbf{P}^1(\mathbf{K}) \mid H(P) \leqslant B\} = C(\mathbf{P}^1_{\mathbf{K}})B + O(B^{1/2}\log(B)),$$

we get that, for real numbers η, δ with $0 < \eta < 1$ and $0 < \delta < 1/2$ and any $\boldsymbol{t} = (t_1, \dots, t_n) \in \mathbf{R}^n_{\geqslant 0}$, we have

$$\begin{aligned}
&\sharp\left\{ \boldsymbol{P} \in \mathbf{P}^1(\mathbf{K})^n \,\middle|\, \boldsymbol{h}(\boldsymbol{P}) \in \prod_{i=1}^n [t_i, t_i + \eta] \right\} \\
&= C(\mathbf{P}^1_{\mathbf{K}})^n e^{|\boldsymbol{t}|}(e^\eta - 1)^n + O(e^{|\boldsymbol{t}| - \delta \min_{1\leqslant i\leqslant n}(t_i)}) \\
&= C(\mathbf{P}^1_{\mathbf{K}})^n e^{|\boldsymbol{t}|}\eta^n + O(e^{|\boldsymbol{t}|}\eta^{n+1}) + O(e^{|\boldsymbol{t}| - \delta \min_{1\leqslant i\leqslant n}(t_i)}).
\end{aligned} \tag{11}$$

Covering $\Delta_\varepsilon(B)$ with cubes with edges of length η, the number of such cubes meeting the boundary of the simplex is bounded by $O((\log(B)/\eta)^{n-1})$. Therefore comparing sum and integral, we get the following estimate for the cardinal of our set:

$$C(\mathbf{P}^1_{\mathbf{K}})^n \int_{\Delta_\varepsilon(B)} e^{|\boldsymbol{t}|}\mathrm{d}\boldsymbol{t} + O(B(\log(B))^n\eta) + O\left(\left(\frac{\log(B)}{\eta}\right)^n B^{1-\delta\varepsilon/n}\right).$$

We may take $\eta = B^{-\delta\varepsilon/(2n^2)}$ to have a sufficiently small error term. The computation of the integral gives $BP_\varepsilon(\log(B))$ where P_ε is a polynomial of degree $n-1$ and leading coefficient $\frac{1}{(n-1)!} + O(\varepsilon)$. To conclude, we note that $C((\mathbf{P}^1_{\mathbf{K}})^n) = \frac{1}{(n-1)!}C(\mathbf{P}^1_{\mathbf{K}})^n$. $\square$

Remarks 6.20. — a) The proof shows that the number of points with freeness $< \varepsilon$ is not negligible in this case!

b) If we consider as in section 4 the points P in $\mathbf{P}^1(\mathbf{K})^n_{h\in\mathscr{D}_B}$ where $\mathscr{D}_B = \mathscr{D}_1 + \log(B)u$, with $u = (u_i)_{1\leqslant i\leqslant n}$, then

$$l(P) \longrightarrow \frac{n\min_{1\leqslant i\leqslant n}(u_i)}{\sum_{i=1}^n u_i}$$

as B goes to infinity. Thus, in this case, the set

$$\{\, P \in V(\mathbf{K}) \mid h(P) \in \mathscr{D}_B,\ l(P) < \varepsilon \,\}$$

is empty for B big enough.

6.4. Accumulating subsets and freeness. — We are now going to show that the freeness gives valuable information about points related to accumulating phenomena.

6.4.1. *Rational curves of low degree.* — Conjecturally the accumulating subsets on projective surfaces are rational curves of low degree. More precisely, the number of points on a rational curve L in a nice variety V for a height given by an adelic metric is equivalent to $C(L)B^{2/\langle L,\omega_V^{-1}\rangle}$. Therefore such a curve would be accumulating if $\langle L,\omega_V^{-1}\rangle < 2$ and could be weakly accumulating if $\langle L,\omega_V^{-1}\rangle = 2$ and the rank of the Picard group of the variety is 1. On a surface S, by the adjunction formula,

$$-2 = \deg(\omega_L) = \langle L, L\rangle + \langle L, \omega_S\rangle.$$

If the rank of the Picard group $\mathrm{Pic}(V)$ is one, any effective divisor is ample since S is projective, in that case $\langle L, L\rangle > 0$, hence $\langle L,\omega_S^{-1}\rangle > 2$ which excludes the last case for a surface. The remaining cases are covered by the following proposition.

Proposition 6.21. — *Let V be a nice variety on the number field $\mathbf{K}$, and let L be a rational curve in V such that $\langle L,\omega_V^{-1}\rangle < 2$. Then the set*

$$\{\, P \in L(\mathbf{K}) \mid l(P) > 0 \,\}$$

is finite.

Proof. — Choose a morphism $\varphi : \mathbf{P}^1_{\mathbf{K}} \to L$ which is birational and an isomorphism $\varphi^*(TS) \xrightarrow{\sim} \bigoplus_{i=1}^n \mathscr{O}_{\mathbf{P}^1_{\mathbf{K}}}(a_i)$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$. Then $\mu_i(\varphi) = a_i$ and $\sum_{i=1}^n \mu_i(\varphi) = \langle L, \omega_V^{-1} \rangle < 2$. We have a natural morphism $T\mathbf{P}^1_{\mathbf{K}} \to \varphi^*(TV)$ which implies that $a_1 \geqslant 2$. Therefore $a_2 < 0$ and we may apply proposition 6.14. □

Remarks 6.22. — a) If we consider only the rational points which satisfy the condition $l(P) > \varepsilon(B)$ for some decreasing function ε with values in $\mathbf{R}_{>0}$, then we exclude all points of L outside a finite set.

b) In dimension $\geqslant 3$, if $\langle L, \omega_V^{-1} \rangle = 2$, then we get that the freeness $l(P)$ goes to 0 on L. This applies to the projective lines in cubic volumes or complete intersections of two quadrics in $\mathbf{P}^6$.

6.4.2. *Fibrations.* — We remind the reader that, in the counter-example of Batyrev and Tschinkel [**BT1**], the accumulating subset is the reunion of fibers of a fibration. We are now going to explain that the freeness also detects such abnormality.

Proposition 6.23. — *Let $\varphi : X \to Y$ be a dominant morphism of nice varieties. Then there exists a constant C such that for any $P \in X(\mathbf{K})$ such that the linear map $T_P\varphi$ is onto,*

$$\mu_{\dim(X)}(P) \leqslant \mu_{\dim(Y)}(\varphi(P)) + C.$$

If, moreover, the logarithmic height of P is strictly positive, we get the inequality:

$$l(P) \leqslant \frac{mh(\varphi(P))}{nh(P)} l(\varphi(P)) + \frac{mC}{h(P)}$$

with $m = \dim(X)$ and $n = \dim(Y)$.

Proof. — The linear map $T_P\varphi$ induces a dual map $T_P\varphi^\vee : T_{\varphi(P)}Y^\vee \to T_PX^\vee$ which is injective. Using $|||\cdot|||$ to denote the usual operator norm, we get an inequality

$$\begin{aligned}\mu_1(T_{\varphi(P)}Y^\vee) &\leqslant \mu_1(T_PX^\vee) + \max_{1 \leqslant k \leqslant \dim(Y)} \left(\frac{\log\left(|||\textstyle\bigwedge^k T_P\varphi^\vee|||\right)}{k} \right) \\ &\leqslant \mu_1(T_PX^\vee) + C.\end{aligned}$$

We conclude with the duality formula for slopes. □

Corollary 6.24. — *Let $Q \in Y(\mathbf{K})$ be a non critical value of φ, then $l(P)$ converges to 0 as $h(P)$ goes to $+\infty$ with P in the fibre $X_Q(\mathbf{K})$.*

Remark 6.25. — In particular, this detects bad points in the counter-example of Batyrev and Tschinkel. Of course this result applies to $(\mathbf{P}^1_{\mathbf{K}})^2$ as well. In fact it is the very property which makes freeness efficient to detect bad points in the counter-example of Batyrev and Tschinkel which implies that the proportion of rational points in $(\mathbf{P}^1_{\mathbf{K}})^2$ with small freeness is not negligible. Section 7 will show how the freeness reveals subvarieties which are locally accumulating even if they are not globally accumulating.

6.5. Combining freeness and heights. — To conclude this part, let us suggest a formula which takes into account both the freeness and all the heights.

Definition 6.26. — Let $\mathscr{D}_1$ be a compact polyhedron in $\operatorname{Pic}(V)^\vee_{\mathbf{R}}$ and let $u \in C^\circ_{\mathrm{eff}}(V)^\vee$. For any $B > 1$ we define $\mathscr{D}_B = \mathscr{D}_1 + \log(B)u$. Let $\varepsilon \in \mathbf{R}_{>0}$ be small enough, relatively to the distance from u to the boundary of $C_{\mathrm{eff}}(V)^\vee$. Then we define

$$V(\mathbf{K})^{l>\varepsilon}_{\boldsymbol{h}\in\mathscr{D}_B} = \{\, P \in V(\mathbf{K}) \mid \boldsymbol{h}(P) \in \mathscr{D}_B, l(P) > \varepsilon \,\}.$$

Instead of using a constant ε, we could also consider a slowly decreasing function in B as in [**Pe4**]. With these notations, we can present our final problematic:

Question 6.27. — *We assume that our nice variety V satisfies the conditions of the hypothesis 3.27. Do we have an equivalence*

$$\sharp V(\mathbf{K})^{l>\varepsilon}_{\boldsymbol{h}\in\mathscr{D}_B} \sim \beta(V)\nu(\mathscr{D}_1)\boldsymbol{\omega}_V(V(\boldsymbol{A}_{\mathbf{K}})^{\mathrm{Br}})B^{\langle\omega_V^{-1},u\rangle} \tag{12}$$

as B goes to infinity?

Equidistribution 6.28. *We shall say that* free points are equidistributed for $\boldsymbol{h}$ *if the measure $\boldsymbol{\delta}_{V(\mathbf{K})^{l>\varepsilon}_{\boldsymbol{h}\in\mathscr{D}_B}}$ converges weakly to $\boldsymbol{\mu}^{\mathrm{Br}}_V$ as B goes to infinity.*

7. Local accumulation

The rational points on $\mathbf{P}^2_{\mathbf{K}}$ and $(\mathbf{P}^1_{\mathbf{K}})^2$ are equidistributed in the sense of naïve equidistribution 3.9. But if one looks at figures 1 and 2, we see lines, which are all projective lines for the projective plane and the fibres of the two projections for the product of two projective lines. To interpret these lines, we need to go beyond the global distribution.

7.1. Local distribution. — Let us assume that $\mathbf{K} = \mathbf{Q}$ to simplify the discussion. Instead of looking at the proportion of points in a fixed open subset U in the adelic space, we may look at the rational points of bounded height in a open subset U_B depending on B and ask the very broad question

Question 7.1. — *For which families $(U_B)_{B>1}$ of open subsets in $V(\boldsymbol{A}_{\mathbf{Q}})$ can we hope to have*

$$\frac{\sharp U_B \cap V(\mathbf{Q})_{\boldsymbol{h}\in\mathscr{D}_B}}{\sharp V(\mathbf{Q})_{\boldsymbol{h}\in\mathscr{D}_B}} \sim \boldsymbol{\mu}_V^{\mathrm{Br}}(U_B)$$

as B goes to infinity?

A particularly interesting case is the distribution around a rational point. Fix $P_0 \in V(\mathbf{Q})$ and choose a local diffeomorphism $\rho : W \to W'$, where W is an open subset in $V(\mathbf{R})$ and W' is an open subset of $T_{P_0}V_{\mathbf{R}}$, which maps P_0 to 0 and such that the differential at P_0 is the identity map. Then we may try to zoom in on the point P_0 with some power of B. More precisely, let us consider the ball

$$\mathscr{B}(0,R) = \{y \in T_{P_0}V_{\mathbf{R}} \mid \|y\|_\infty \leqslant R\}.$$

We may then introduce the probability measure on $\mathscr{B}(0,R)$ defined by

$$\delta_{R,B}^{\alpha} = \frac{1}{\sharp(V(\mathbf{Q})_{H\leqslant B} \cap \rho^{-1}(\mathscr{B}(O,RB^{-\alpha})))} \sum_{\substack{P\in V(\mathbf{Q})_{H\leqslant B}\\ \rho(P)\in\mathscr{B}(0,RB^{-\alpha})}} \delta_{B^{\alpha}\rho(P)}.$$

Remarks 7.2. — a) Let us assume that P_0 belongs to a Zariski open subset of V on which the rational points of bounded height are equidistributed in the sense of 3.31. For $\alpha = 0$, we get the measure induced on $B(0,R)$ by $\rho_*(\mu_\infty)$.

b) Under the same hypothesis, if α is small, corresponding to a small zoom, we may expect that the points are evenly distributed: the measure converges to the probability measure induced by the Lebesgue measure.

c) If α is big enough, diophantine approximation tells us that there is no rational point that near to the rational point P_0. In other words, for α big enough the above measure is the Dirac measure at P_0.

We are interested in the critical values of α, that is those for which the asymptotic behaviour of the measure $\delta_{R,B}^{\alpha}$ changes. In particular, we can consider the smallest value of α for which the measure is not the Dirac measure at P_0, which is the biggest of the critical values. This is directly related to the generalisation of the measures of irrationality introduced by D. McKinnon and M. Roth in [**MR**].

In our context, with a height defined by an adelic metric on V, the archimedean metric defines a distance d_∞ on $V(\mathbf{R})$. Then if W is a constructible subset of V containing P_0, we define in this text $\alpha_W(P_0)$ as the infimum of the set of $\alpha \in \mathbf{R}_{>0}$ such that for any $C \in \mathbf{R}$ the set

$$\left\{ Q \in W(\mathbf{Q}) \,\middle|\, d_\infty(Q, P_0) < \frac{C}{H(Q)^\alpha} \right\}$$

is finite. Since ρ is a diffeomorphism, $\alpha_V(P_0)$ corresponds to the biggest critical value.

Remark 7.3. — In this text, we take the inverse of the constant defined by D. McKinnon and M. Roth in their paper (*loc. cit.*), since it better expresses the power appearing in the zoom factor.

In [**Mc**], D. McKinnon suggests that there should exist rational curves L in V such that $\alpha_V(P_0) = \alpha_L(P_0)$. In other words the best approximations should come from rational curves. On the other hand D. McKinnon and M. Roth [**MR**, theorem 2.16] give the following formula for $\alpha_L(P_0)$: let $\varphi : \mathbf{P}^1_{\mathbf{K}} \to L$ be a normalisation of the curve L

$$\alpha_L(P_0) = \max_{Q \in \varphi^{-1}(P_0)} \frac{r_Q m_Q}{d}$$

where $d = \deg(\varphi^*(\omega_V^{-1}))$, m_Q is the multiplicity of the branch of L through x corresponding to Q and r_Q corresponds to the approximation of Q by rational points in $\mathbf{P}^1_{\mathbf{Q}}$ and is given by Roth theorem [**Ro**]:

$$r_Q = \begin{cases} 0 \text{ if } \kappa(Q) \not\subset \mathbf{R}, \\ 1 \text{ if } \kappa(Q) = \mathbf{Q}, \\ 2 \text{ otherwise.} \end{cases}$$

On the other hand, if we take a sequence of rational points $(Q_n)_{n \in \mathbf{N}}$ on $L(\mathbf{Q})$ which converges to P_0 then $(H(Q_n))_{n \in \mathbf{N}}$ goes to $+\infty$ and therefore, by proposition 6.14, we have that $(l(Q_n))_{n \in \mathbf{N}}$ converges to $l(\varphi)$. In the case where there exists a branch of degree 1 through P_0, if the deformations of the morphism φ are contained in a strict subvariety, this means that all the tangent vectors in $T_{P_0}V$ can not be obtained by a deformation of φ and thus φ can not be very free. Under these assumptions, we get that $l(\varphi) \leqslant 0$ and therefore $(l(Q_n))_{n \in \mathbf{N}}$ converges to 0. Therefore, if the locally accumulating subvarieties are dominantly covered by rational curves, we may expect that the freeness of the points on these locally accumulating subvarieties tends to 0.

In [**Hu1**], [**Hu2**], and [**Hu3**], 黄治中 studies the local distribution of points on various toric surfaces, exhibiting phenomena like local accumulating subvarieties, and locally accumulating thin subsets.

8. Another description of the slopes

Construction 8.1. — For any vector bundle E of rank r on V, we may define the *frame bundle of* E, denoted by $F(E)$, as the GL_r-torsor of the basis in E: for any extension $\mathbf{L}$ of $\mathbf{K}$ and any point $P \in V(\mathbf{L})$, the fibre of $F(E)$ at P is the set of basis of the fibre E_P. For a line bundle L, the frame bundle $F(L)$ is equal to $L^{\times}$.

Let us now assume that the vector bundle E is equipped with an adelic norm $(\|\cdot\|_w)_{w\in\mathrm{Val}(\mathbf{K})}$. Then for any place w, any point $P \in V(\mathbf{K}_w)$ and any basis $\boldsymbol{e} = (e_1, \dots, e_r) \in F(E)_P$ we get an element M_w in $\mathrm{GL}_r(\mathbf{K}_w)/K_w$ where

$$K_w = \begin{cases} \mathrm{GL}_r(\mathscr{O}_w) \text{ if } w \text{ is ultrametric,} \\ O_r(\mathbf{R}) \text{ if } w \text{ is real,} \\ U_r(\mathbf{R}) \text{ if } w \text{ is complex.} \end{cases}$$

which is the class of the matrix of the coordinates of $(e_1, \dots, e_r)$ in a basis of the $\mathscr{O}_w$ lattice (resp. orthonornal basis) defined by $\|\cdot\|_w$ if w is ultrametric (resp. non-archimedean). We get a map

$$F(E)(\boldsymbol{A}_\mathbf{K}) \longrightarrow \mathrm{GL}_r(\boldsymbol{A}_\mathbf{K})/K,$$

where K is the compact subgroup $\prod_{w\in\mathrm{Val}(\mathbf{K})} K_w$. Taking the quotient by $\mathrm{GL}_r(\mathbf{K})$ for the rational points we get a map

$$V(\mathbf{K}) \longrightarrow \mathrm{GL}_r(\mathbf{K})\backslash \mathrm{GL}_r(\boldsymbol{A}_\mathbf{K})/K.$$

Let us denote by Q_r the biquotient on the right, we get a map

$$\tau_E : V(\mathbf{K}) \longrightarrow Q_r.$$

The determinant composed with product of the norms gives a morphism of groups from the adelic group $\mathrm{GL}_n(\boldsymbol{A}_\mathbf{K})$ to $\mathbf{R}_{>0}$ which is invariant under the action of K on the right and the action of $\mathrm{GL}_n(\mathbf{K})$ on the left, this gives a map $|\det| : Q_r \to \mathbf{R}_{>0}$. The composition $|\det| \circ \tau_E$ coincides with the exponential height H_E defined by E with its adelic norm.

Similarly, since the slopes μ_i^E are defined in terms of the $\mathscr{O}_\mathbf{K}$-module defined by the norms at the ultrametric places equipped with the non-archimedean norms, we may factorise the slopes through $\mathcal{Q}_r$, and the freeness of a rational point P may also be computed in terms of $\tau_{TV}(P)$.

Remarks 8.2. — a) In $\mathcal{Q}_r$, we may consider the subset $\mathcal{Q}_r^1$ of points P such that $|\det|(P) = 1$. The determinant map then defines a map $\mathcal{Q}_r^1 \to \mathbf{K}^*\backslash\mathbf{G}_m(\boldsymbol{A}_\mathbf{K})^1/K_{\mathbf{G}_m}$ where $K_{\mathbf{G}_m}$ is the product over the places w of the maximal compact subgroup in $\mathbf{G}_m(\mathbf{K}_w)$. We get a map $c : \mathcal{Q}_r^1 \to \mathrm{Pic}(\mathscr{O}_\mathbf{K})$; the composition map $c \circ \tau_E$ maps a rational point P onto the class of the projective $\mathscr{O}_\mathbf{K}$-module defined by the ultrametric norms in E_P. As an example, for the projective space $\mathbf{P}^n_\mathbf{K}$, with $E = TV$, this maps a point $P = [y_0 : \dots : y_n]$ with integral homogeneous coordinates to $(n+1)$ times the class of the ideal $(y_0, \dots, y_n)$.

b) For surfaces, as described in remark 6.8 e), the slopes, and thus the freeness, measures the deformation of the lattice or the proximity to the cusp in the modular curve $X(1)$. The above construction generalises this description in higher dimension.

c) The frame bundle would enable GL_n descent on varieties for which the lifting to versal torsors is not sufficient. In fact we may extend this and consider bundles giving geometric elements in the Brauer group. This may provide a method to generalise the construction of Salberger [**Sal**] in the case the geometric Brauer group is not trivial.

9. Conclusion and perspectives

In these notes we made a quick survey of the various directions to upgrade the principle of Batyrev and Manin to include the cases of Zariski dense accumulating subsets. Let me summarize these options:

1. Remove accumulating thin subsets. This method has been successful in several cases. However, this notion depends on the ground field and we could imagine situations in which there are infinitely many thin subsets to remove, similar to the situation of K3-surfaces containing infinitely many rational lines which are all accumulating.
2. Consider all heights. This method may apply to fibrations and other cases in which the accumulating subsets come from line bundles. However, as shown by examples of Picard rank one, this is not enough to detect accumulating subsets of higher codimension.
3. As in [**Pe4**], we could use a height defined by an adelic metric and the freeness. But the freeness condition tends to remove too many points as

shown by the product of projective lines. A recent example by W. Sawin [**Sa**] indicates this is not enough.

4. Combine all heights and freeness. This combination is inspired by the geometric analogue.

This list is far from exhaustive. In fact, we could consider the slopes given by norms on any vector bundle on our variety which gives a profusion of probably redundant invariants. Arakelov geometry is a very natural tool to attack this question of redundancy and look for a minimal set of slopes controlling the distribution of points.

The freeness, which is in part suggested by the analogy with the geometric setting, is very efficient to detect local adelic deformations which correspond to local or global accumulation. However this invariant is particularly difficult to compute efficiently. Indeed its explicit computation is related to the finding of a non-zero vector of minimal length in a lattice which is known to be computationally difficult. At the time of writing, the following question is still open [(2)]:

Question 9.1. — *Let V be a smooth hypersurface of degree d in $\mathbf{P}^N_{\mathbf{Q}}$, with $d \geqslant 3$ and $N > (d-1)2^d$. Is the cardinal of points $x \in V(\mathbf{Q})$ with $l(x) < \varepsilon$ and $H(x) < B$ negligible as B goes to infinity?*

In other words, the author is still lacking methods giving lower bounds for the smallest slope, but again we may hope that the techniques of Arakelov geometry may provide the necessary tools.

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