Families of Klingen-Eisenstein series and p-adic doubling method

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Geometric constructions of distributions for Klingen-Eisenstein series

will be explained staring from those for Siegel-Eisenstein series at any prime p.

Applications to the standard L-functions of Siegel modular forms are developped.

Generalized Fourier transform of the distributions on the space of symmetric matrices and matrix Gauss sums are used.

Axioms for p-adic doubling method are stated.

Geometric (group theoretic) and algebraic constructions for Eisenstein series

Let p be a prime, $k \geq 4$ then the Eisenstein series is an algebraic q-expansion

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{\substack{n \geq 1 \\ d|n}} d^{k-1} q^n, \; q = e^{2\pi i z}, \; \mathrm{Im}(z) > 0$$

which is a modular form of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$, geometrically described as

$$E_k(z) = \sum_{c,d}{}'(cz+d)^{-k} = \sum_{\gamma \in \{\binom{**}{0*}\} \backslash \mathrm{SL}_2(\mathbb{Z})} (cz+d)^{-k}$$

the summation runs through coprime paires (c, d) with $c \geq 0$, or $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

p-adic meromorphic continuation of Eisenstein series

is described using the p-regular part of the zeta value $\zeta^*(1-k) = \zeta(1-k)(1-p^{k-1})$ as

$$E_k^*(z) = 1 + \frac{2}{\zeta^*(1-k)} \sum_{\substack{n \geq 1 \\ d|n, p \nmid d}} d^{k-1} q^n$$

where the Fourier coefficients are p-adic meromorphic with finite number of poles. These coefficients belong to the quotion field $\mathcal{L} = Quot(\mathbb{Z}_p[[t]])$ of the Iwasawa algebra of the variable $t = (1+p)^k - 1$ (for k mod $p-1$ fixed) such objects are called pseudo-measures due to John Coates; they are uniquely determined by the infinite number of values for $k \geq 4$. Both parts of the algebraic-geometric equality

$$E_k^*(z) = 1 + \frac{2}{\zeta^*(1-k)} \sum_{\substack{n \geq 1 \\ d|n, p \nmid d}} d^{k-1} q^n = (1-p^{k-1})(E_k(z) - p^{k-1} E_k(pz)).$$

p-adic meromorphic continuation for Siegel-Eisenstein series

(see [Yale] (arXiv: 1204.3878)) can be also constructed using the Weierstrass preparation theorem. These series are modular forms for the Siegel modular group $\Gamma_n = \mathrm{Sp}_{2n}(\mathbb{Z})$ of degree n.

Duality and General strategy of p-adic constructions

For any Dirichlet character χ mod p^v consider Shimura's "involuted" Siegel-Eisenstein series assuming their absolute convergence (i.e. $k > m+1$):

$$E_k^*(\chi, z) = \sum_{(c,d)/\sim} \chi(\det(c)) \det(cz+d)^{-k} = \sum_{0<h\in B_m} a_h(k,\chi) q^h.$$

The two sides of the equality produce dual approaches: geometric and algebraic. The Fourier coefficients can be computed by Siegel's method via the singular series

$$a_h(E_k^*(\chi, z)) \\ = \frac{(-2\pi i)^{mk}}{2^{\frac{m(m-1)}{2}} \Gamma_m(k)} \sum_{\mathfrak{R} \bmod 1} \chi(\nu(\mathfrak{R})) \nu(\mathfrak{R})^{-k} \det h^{k-\frac{m+1}{2}} e_m(h\mathfrak{R}) \qquad (1)$$

The orthogonality relations mod p^v produce two families of distributions (notice that terms in the RHS are invariant under sign changes, and (3) is algebraic after multiplying by the factor in (1)):

$$\frac{1}{\varphi(p^v)} \sum_{\chi \bmod p^v} \bar{\chi}(b) \sum_{(c,d)/\sim} \frac{\chi(\det(c))}{\det(cz+d)^k} = \sum_{\substack{(c,d)/\sim \\ \det(c)\equiv b \bmod p^v}} \frac{\mathrm{sgn}(\det(c))^k}{\det(cz+d)^k} \qquad (2)$$

$$\frac{1}{\varphi(p^v)} \sum_{\chi \bmod p^v} \bar{\chi}(b) \sum_{\mathfrak{R} \bmod 1} \frac{\chi(\nu(\mathfrak{R})) e_m(h\mathfrak{R})}{\nu(\mathfrak{R})^k} = \sum_{\substack{\mathfrak{R} \bmod 1 \\ \nu(\mathfrak{R})\equiv b \bmod p^v}} \frac{e_m(h\mathfrak{R}) \mathrm{sgn}\nu(\mathfrak{R})^k}{\nu(\mathfrak{R})^k} \qquad (3)$$

Distributions for Klingen-Eisenstein series and Siegel-Eisenstein series

Fix a Siegel cusp eigenform $f = \sum_T a_T q^T \in S_k^r(\Gamma^r)$, where T runs through half-integral positive symmetric matrices.
For $k > m + r + 1$ and $m \geq r$ then the Klingen – Eisenstein series is defined as the following absolutely convergent series

$$E_k^{m,r}(z, f, \chi) = \sum_{\gamma \in \Delta_{m,r} \cap \Gamma_m \backslash \Gamma_m} \chi(\det(d_\gamma)) f(\omega^{(r)}(\gamma z)) j(\gamma, z)^{-k},$$

with $z \in \mathbb{H}^m$, $\omega(z)^{(r)}$ being the upper left corner of z of size $r \times r$, $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$ and $\Delta_{m,r}$ denotes the set of elements in Γ_m having the form $\begin{pmatrix} * & * \\ 0_{m-r,m+r} & * \end{pmatrix}$ [Kl].
There exist several constructions of Λ-adic Klingen-Eisenstein series for $r = 1$, $m = 2$ (see Section 5 of [PaSE] with Koji Kitagawa (Hokkaido University), [Gue0] for vector-valued case, Section 11 of [MC]. . . .
Modular distributions were constructed coming from the Klingen–Eisenstein series on the symplectic group $G = \mathrm{GSp}_{2m}(m > r)$ These distributions take values in a space of Λ-adic Siegel modular forms and defines a $\mathcal{L}$-adic Siegel modular form where $\mathcal{L}$ denotes the field of fractions of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T]]$.

Boecherer–Garrett–Shimura identity (pull-back formula)

Let $D(s, f, \chi)$ be the standard zeta function of $f \in \mathcal{S}_k^r(\Gamma)$ as above (with local factors of degree $2r+1$) and χ be a Dirichlet character. Then the essential fact for our construction is the Boecherer–Garrett–Shimura identity:

$$\Lambda(k, \chi) D(k-r, f, \eta) E_k^{m,r}(z, f, \chi) = \langle f(w), E_k^{m+r}(\mathrm{diag}[z, w]) \rangle.$$

Here $\Lambda(k, \chi)$ is a product of special values of Dirichlet L-functions and Γ-functions, η is a certain Dirichlet character, E_k^{m+r} a Siegel-Eisenstein series, $(z, w) \in \mathbb{H}_m \times \mathbb{H}_r$ (see [Boe85], [Shi95, (7.4), p.572]).

The p-adic construction is based on the fact the Fourier coefficients of the series $E_k^{m+r}(\mathrm{diag}[z, w])$ produce p-adic measures (the Siegel–Eisenstein measure).

Key ingredients of a general construction ($r \geq 1$, $m \geq r$)

- Siegel-Eisenstein distributions generalizing those in [Yale]

$$E^{m+r}_{k,b,p^v} = \sum_{\substack{(c,d) \\ \det c \equiv b \bmod p^v}} \det(cz+d)^{-k}$$

These are functions on the Siegel upper half plane

$$\mathbb{H}_{m+r} = \{z = {}^t z \in M_{m+r}(\mathbb{C}) | \mathrm{Im}\,(z) > 0\}$$

of degree $m+r$, (c,d) runs over equivalence classes of all coprime symmetric couples,that is $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs over equivalence classes of $\Gamma = \Gamma^{m+r}$ modulo the Siegel parabolic $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

- Higher twists $E^{m+r}_k(\chi_1, \chi_2)$ of Siegel-Eisenstein distributions constructed as in [BP11]. These are certain functions on $(Z, \tau) \in \mathbb{H}_m \times \mathbb{H}_r$

Key ingredients of our construction (continued)

- Pull-back formula

$$\langle E_k^{m+r}(\chi_1,\chi_2)(Z,\tau), f(\tau)\rangle_\tau$$
$$= c_{\chi_1,\chi_2,m,r} D(f,\chi_{1,2},k-r) E_k^{m,r}(f,\chi_1,\chi_2)$$

  representing the Klingen-Eisenstein series $E_k^{m,r}(f,\chi_1,\chi_2)$ on $Z\in\mathbb{H}_{m+r}$, where $D(f,\chi_{1,2},s)$ denotes the standard L-function of f with a certain Dirichlet character $\chi_{1,2}$ attached to f,χ_1,χ_2. In particular, we describe explicitely the series $E_k^{m,r}(f,\chi_1,\chi_2)$.
- Canonical projection of the Klingen-Eisenstein distributions defined by the integrals produces the following family:

$$(\chi_1,\chi_2,f)\mapsto \Phi_{\chi_1,\chi_2,f}(Z) = \langle E_k^{m+r}(\chi_1,\chi_2)(Z,\tau), f_0(\tau)\rangle_\tau$$

  in a fixed-level finite-dimensional subspace using the $U_0(p)$–operator of degree $n=m+r$ and a non-zero eigenvalue $\alpha_0(p)$ of $U_0(p)$, as in [PaTV] and in [BP06]. Here $f_0(\tau)$ denotes an eigenfunction of $U_0(p)$ associated to f.

Axioms for p-adic doubling method in the case $m = r$, $n = 2m$

Next we axiomatize a general p-adic doubling method (see [Boe-Schm], ...) using a family of functions

$$\mathcal{F}_\chi : \mathbb{H}_{2m} \to \mathbb{C}$$

(in place of the Siegel series of degree $m + r$ in the above construction) where χ runs through primitive characters χ mod p^v.
We use the exterior twist

$$\underbrace{\sum_{X \in \mathbb{Z}^{m,n} \bmod p^v} \phi(X)\mathcal{F}_\chi \begin{pmatrix} z_1 & \frac{X}{N} \\ \frac{X^t}{N} & z_4 \end{pmatrix}}_{g_\chi(z_1, z_4)}$$

which will be of Haupttypus of level p^{2v} (with respect to z_1 and z_4). Here $\phi(X)$ is a certain function with a spherical property, represented by a matrix Gauss sum.
As a result we interpolate the following integrals. Let us fix two cusp forms f_1, f_2 and consider

$$\chi \mapsto \int_{\Gamma_0(p^{2v})\backslash\mathbb{H}} \int f_{1,0}, f_{2,0}\overline{g_\chi(z_1, z_4)}dz_1 dz_4 \in \mathbb{C}$$

to get p-adic interpolation and to use U_0^{2v} (in z_1 and z_4) to get finite dimensional space.

Proving congruences using Fourier expansion

$$\mathcal{F}_\chi = \sum_T a(T,\chi)\exp(2\pi i \mathrm{tr}(TZ))$$

then

$$g_\chi(z_1,z_4)|^{z_1}U(p^j)|^{z_4}U(p^j) = \sum_{T_1,T_4} c_j(T_1,T_4,\chi)\exp(2\pi i \mathrm{tr}(T_1 z_1 + T_4 z_4),$$

$$c_j(T_1,T_4,\chi) = \sum_\chi a\left(\begin{pmatrix} p^j T_1 & T_2 \\ T_2 N & p^j T_4 \end{pmatrix},\chi\right)$$

(this is a finite sum because of the semi positivity of T).
If $r \geq 1$

$$c_j(T_1,T_4,\chi) = \frac{1}{p^r}\sum_{\substack{\chi \\ conductor(\chi)\leq p^r}} a\left(\begin{pmatrix} p^{n-r} T_1 & T_2 \\ T_2 N & p^{n-r} T_4 \end{pmatrix},\chi\right)$$

$$\sum_{\substack{X\in\mathbb{Z}^{(n,r)} \\ \bmod p^v}} \phi(X)\exp(\mathrm{tr}2\pi i\frac{1}{p^v}T_2 X)$$

must satisfy certain congruences
Result: p-adic interpolation of the integrals

$$\alpha_{f_1}^{-2v}\alpha_{f_2}^{-2v}\int_{\Gamma_0(p^{2v})\backslash\mathbb{H}}\int f_{1,0}, f_{2,0}\overline{g_\chi(z_1,z_4)}dz_1 dz_4 \in \mathbb{C}$$

Special cases and problems

- To study the case of the theta series

$$\Theta_S \mapsto \Theta_{S,\chi} = \underbrace{\sum_X \chi(\det X) \exp(2\pi i \mathrm{tr}(S[X]Z))}_{a\ \ possibility}$$

- To state axioms for p-adic pull back formula for starting with a family of functions

$$\mathcal{F}_\chi : \mathbb{H}_{m+r} \to \mathbb{C}$$

where χ runs through primitive characters χ mod p^v

- To define a general exterior twist

$$\underbrace{\sum_{X \in \mathbb{Z}^{m,n} \bmod p^v} (**) \mathcal{F}_\chi \begin{pmatrix} z_1 & \frac{X}{N} \\ \frac{X^t}{N} & z_4 \end{pmatrix}}_{g_\chi(z_1, z_4)}$$

which will be of Haupttypus of level p^{2v} of degree $r + m$.

Why study L-values attached to modular forms?

A popular proceedure in Number Theory is the following:

Construct a generating function $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[[q]]$ of an arithmetical function $n \mapsto a_n$, for example $a_n = p(n)$ $\rightsquigarrow$ Compute f via modular forms, for example $\sum_{n=0}^{\infty} p(n) q^n = (\Delta/q)^{-1/24}$ $\rightsquigarrow$ A number (solution)

↑ Good bases, finite dimensions, many relations and identities …

↑ Values of L-functions, (complex and p-adic), periods, congruences, …

Example (Hardy-Ramanujan)

$$p(n) = \frac{e^{\pi\sqrt{2/3(n-1/24)}}}{4\sqrt{3}\lambda_n^2} + O(e^{\pi\sqrt{2/3(n-1/24)}}/\lambda_n^3),$$
$$\lambda_n = \sqrt{n - 1/24},$$

Other examples: Birch and Swinnerton-Dyer conjecture, L-values attached to modular forms, Wiles's proof of Fermat's Last Theorem,…(see [Ma-Pa05])

Generalities about p-adic L-functions

There exist two kinds of L-functions

- Complex-analytic L-functions (Euler products)
- p-adic L-functions (Mellin transforms L_μ of p-adic measures)

Both are used in order to obtain a number (L-value) from an automorphic form. Usually such a number is algebraic (after normalization) via the embeddings

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \quad \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p.$$

How to define and to compute p-adic L-functions? We use Mellin transform of a $\mathbb{Z}_p$-valued distribution μ on a profinite group

$$Y = \varprojlim_i Y_i, \ \mu \in Distr(Y, \mathbb{Z}_p) = \mathbb{Z}_p[[Y]] = \varprojlim_i \mathbb{Z}_p[Y_i] =: \Lambda_Y$$

(the Iwasawa algebra of Y).

$$L_\mu(x) = \int_Y x(y) d\mu, \quad x \in X_Y = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$$

(the Mellin transform of μ on Y).

A general idea (J.-P. Serre [Se73]) is to construct p-adic L-functions directly from Fourier coefficients of modular forms.

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