A few reminders: Alternated series, Abel summation process and integral test for convergence

Alternated series

A typical example: $\sum_{n\geqslant 1} \frac{(-1)^n}{\sqrt{n}}$.

show that, if $(a_n)_{n\in\mathbb{N}}$ is a decreasing sequence of positive numbers which converges to 0, then the series $\sum_{n\in\mathbb{N}}(-1)^n a_n$ converges.

For that, we can show that $(S_k)_{k\in\mathbb{N}}$ and $(T_k)_{k\in\mathbb{N}}$, defined for all $k \in \mathbb{N}$ by

$$S_k = \sum_{n=0}^{2k}(-1)^n a_n \quad \text{and} \quad T_k = \sum_{n=0}^{2k+1}(-1)^n a_n,$$

are adjacent.

Partial sum error bound: under the previous hypotheses, show that for all $N \in \mathbb{N}$ the partial sum error

$$r_N = \sum_{n=N}^{\infty}(-1)^n a_n$$

has sign $(-1)^N$, and that

$$|r_N| \leqslant a_N$$

(we can try to regroup the terms in the sum by pairs as follow $(-1)^N r_N = (a_N - a_{N+1}) + (a_{N+2} - a_{N+3}) + \cdots = a_N - (a_{N+1} - a_{N+2}) - (a_{N+3} - a_{N+4}) - \dots$).

can we apply these results to the following sequence $\sum_{n\geqslant 2} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$?

Abel summation process (summation by part).

A typical example: $\sum_{n\geqslant 1} \frac{\cos(n\theta)}{\sqrt{n}}$, where θ is a fixed real number which is not a multiple of 2π.

show that, if

- $(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ is a decreasing sequence of positive numbers which converges to 0,
- $(b_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ is such that the sequence $(B_n)_{n\in\mathbb{N}}$ of partial sums $B_n = \sum_{k=0}^{n} b_k$ is bounded,

then the series $\sum_{n\in\mathbb{N}} a_n b_n$ converges.

For that, we can show that, if $1 \leqslant p < q$, then we have

$$\sum_{n=p}^{q} a_n b_n = -a_p B_{p-1} + \sum_{n=p}^{q-1}(a_n - a_{n+1})B_n + a_q B_q.$$

Partial sum error bound: under the previous hypotheses, show that

$$\forall N \in \mathbb{N}, \quad |\sum_{n=N}^{\infty} a_n b_n| \leqslant 2a_N B,$$

where B is an upper bound of $\{|B_n|, n \in \mathbb{N}\}$.

Remark: we can replace the hypotheses on $(a_n)_{n\in\mathbb{N}}$ by "$(a_n)_{n\in\mathbb{N}}$ is a sequence of complexes number which converges to 0 and such that the series $\Sigma_{n\in\mathbb{N}}(a_n - a_{n+1})$ converges absolutely."

Integral test for convergence.

let $f : \mathbb{R}_+ \to \mathbb{R}$ be a decreasing continuous function such that $f(x)$ converges to 0 when x tends to $+\infty$. Show that, if $1 \leqslant p \leqslant q$ then

$$\int_p^{q+1} f(t)\,\mathrm{d}t \leqslant \sum_{n=p}^{q} f(n) \leqslant \int_{p-1}^{q} f(t)\,\mathrm{d}t.$$

Deduce from the previous inequalities that

- the series $\sum_{n\in\mathbb{N}} f(n)$ converges if and only if the integral $\int_0^{\infty} f(t)\,\mathrm{d}t$ converges;
- If there is convergence, then we have:

$$\forall p \in \mathbb{N}, \quad \int_p^{\infty} f(t)\,\mathrm{d}t \leqslant \sum_{n=p}^{\infty} f(n) \leqslant \int_{p-1}^{\infty} f(t)\,\mathrm{d}t;$$

- if there is no convergence, then we have:

$$\forall q \in \mathbb{N}, \quad \int_1^{q+1} f(t)\,\mathrm{d}t \leqslant \sum_{n=1}^{q} f(n) \leqslant \int_0^{q} f(t)\,\mathrm{d}t.$$

This can for example be applied to the Riemann series $\sum_{n\geqslant 1} \frac{1}{n^{\alpha}}$ and Bertrand series $\sum_{n\geqslant 1} \frac{1}{n^{\alpha}(\ln(n))^{\beta}}$.