Exercise list 3

Exercice 1. Study the pointwise, normal, and uniform convergence of the series of functions $\sum_{n\geq 0} u_n$ where:

1. $u_n(x) = e^{-nx}$, $x \in \mathbb{R}_+$.
2. $u_n(x) = x^n$, $x \in [0, 1]$.
3. $u_n(x) = \dfrac{1}{2^n} \sin(3^n x)$, $x \in \mathbb{R}$.
4. $u_n(x) = \dfrac{1}{1 + (n-x)^2}$, $x \in \mathbb{R}$.

Exercice 2. Study the pointwise, normal, and uniform convergence of the series of functions $\sum_{n\geq 0} u_n$ where:

1. $u_n(x) = n^x$, $x \in \mathbb{R}$.
2. $u_n(x) = (-1)^n n^x$, $x \in \mathbb{R}$.
3. $u_n(x) = e^{-n(x^2+1)}$, $x \in \mathbb{R}$.
4. $u_n(x) = \frac{1}{n} \arctan(\frac{x}{n})$, $x \in \mathbb{R}$.

Exercice 3. Study the pointwise, normal, and uniform convergence of the series of functions $\sum_{n\geq 0} u_n$ where:

1. $u_n(x) = ne^{-nx}$, $x \in \mathbb{R}_+^*$.
2. $u_n(x) = \begin{cases} n^2 x(1 - nx) & \text{if } x \in [0, \frac{1}{n}], \\ 0 & \text{if } x \in [\frac{1}{n}, 1]. \end{cases}$
3. $u_n(x) = e^{-nx} \sin x$, $x \in \mathbb{R}_+$.
4. $u_n(x) = \dfrac{\sin(nx)}{1 + n^2 x^2}$, $x \in \mathbb{R}$.

Exercice 4. Let $(u_n)_{n\geq 1}$ be the sequence of functions where, for all $n \geq 1$, $u_n : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$u_n(x) = \frac{1}{n + xn^2} \quad (n \geq 1, x \in \mathbb{R}).$$

1. Find the pointwise convergence domain $D \subset \mathbb{R}$ of the series of functions $\sum_{n\geq 1} u_n$.
2. Study normal convergence of the series $\sum_{n\geq 1} u_n$ on D, and then on $[a, +\infty[$ for all $a > 0$.
3. Does the series $\sum_{n\geq 1} u_n$ converge uniformly on D?
4. Is the function $f = \sum_{n=1}^{+\infty} u_n$ differentiable on $\mathbb{R}_+^*$?
5. Show that f is integrable on $[1, 2]$ and express $\int_1^2 f(t)\, dt$ as the sum of a series of real numbers.

Exercice 5. Study the convergence of the series $\sum u_n$ where, for all $n \geq 1$, u_n is given by:

$$u_n(x) = \frac{1}{n^3 + n^4 x^2}, \quad x \in \mathbb{R}.$$

Is the sum continuous on $\mathbb{R}$? differentiable on $\mathbb{R}$?

Exercice 6. Same questions with $\sum v_n$ where, for all $n \geq 1$, v_n is given by:

$$v_n(x) = \frac{1}{n^2 + n^4 x^2}, \quad x \in \mathbb{R}.$$

Exercice 7. Show that the series $\sum u_n$ where, for all $n \geq 1$, u_n is given by:

$$u_n(x) = \frac{(-1)^n}{2\sqrt{n} + \cos x}, \quad x \in \mathbb{R},$$

converges uniformly on $\mathbb{R}$. Does the series converges normally.

Exercice 8. We want to show that the series $\sum u_n$ where, for all $n \geq 1$, u_n is given by:

$$v_n(x) = \frac{\cos(nx)}{\sqrt{n + x}}, \quad x \in \mathbb{R},$$

converges uniformly on the interval $I = [\alpha, 2\pi - \alpha]$, where $0 < \alpha < \pi$.

Let $x \in \mathbb{R}$ and $n, p, q \in \mathbb{N}^*$, such that $n \geq p$. We set $S_{p,n}(x) = \sum_{k=p}^{n} \cos(kx)$.

1. Show, using $\cos a = \frac{1}{2}(e^{ia} + e^{-ia})$ that if $x \neq 2k\pi$, then $|S_{p,n}(x)| \leq \frac{1}{|\sin(x/2)|}$.
2. Check that $\sum_{n=p}^{p+q} v_n(x) = \sum_{n=p}^{p+q-1} S_{p,n}(x) \Big[\frac{1}{\sqrt{n+x}} - \frac{1}{\sqrt{n+1+x}} \Big] + \frac{S_{p,p+q}(x)}{\sqrt{p+q+x}}$.
3. Deduce that for all $x \in I$, $\left| \sum_{n=p}^{p+q} v_n(x) \right| \leq \frac{1}{\sqrt{p+x}} \frac{1}{|\sin(\frac{x}{2})|}$. and deduce the result.

Exercice 9. For $x > 0$, we set

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{x(x+1)\dots(x+n)}.$$

1. Show that $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is well defined and continuous.
2. Express $f(x+1)$ with $f(x)$ for all $x > 0$.
3. Study the differentiability of f on $\mathbb{R}_+^*$.
4. Show that the function f is monotonous, and give an equivalent of $f(x)$ when $x \to 0$, and when $x \to +\infty$.
5. Draw the graph of f.

Exercice 10. We consider the sequence of functions $(u_n)_{n\in\mathbb{N}}$ where $u_n : [0,1] \to \mathbb{R}$ is defined by $u_0(x) = 1$ and, if $n \geq 1$,

$$u_n(x) = \begin{cases} \dfrac{(-1)^n}{n!}(x\ln(x))^n & \text{if } x \in \,]0,1]\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

1. For all $n \in \mathbb{N}$, check that u_n is continuous on $[0,1]$.
2. Calculate, using integration by part, $\int_0^1 u_n(x)\,\mathrm{d}x$ for all $n \in \mathbb{N}$.
3. Show that the series of function $\left(\sum_{n\in\mathbb{N}} u_n\right)$ converges normally on $[0,1]$, and calculate the sum.
4. Deduce that
$$\int_0^1 \frac{1}{x^x}\,\mathrm{d}x = \sum_{n=1}^{\infty} \frac{1}{n^n}\,.$$

Exercice 11. We want to study the sum S of the series of function $\sum v_n$ where, for all $n \geq 1$, we have

$$v_n(X) = \frac{1}{n^2 + n^4x^2}\,, \qquad x \in \mathbb{R}.$$

1. Show that $\sum_{n\geq 1} v_n$ converges pointwise to $\mathbb{R}$. Its sum S is a function from $\mathbb{R}$ to $\mathbb{R}$.
2. Show that S is even, and (strictly) decreasing on $\mathbb{R}_+$.
3. Show that S is continuous on $\mathbb{R}$, and that it converges to 0 when x goes to $+\infty$.
4. Study the normal convergence of $\left(\sum_{n\geq 1} v_n'\right)$: show that it is not happening on $\mathbb{R}$, but it is on $\mathbb{R} \setminus [-a,a]$, for all $a > 0$. Deduce that S is differentiable on $\mathbb{R}_+^*$.
5. To study the differentiability of the function S at 0, we look at $\tau_h S(0) = \dfrac{1}{h}(S(h) - S(0))$, firstly when $h > 0$.

   Write it down as $h\sum_{n=1}^{+\infty} w_h(n)$, and compare the partial sums of the series $\sum_{n\geq 1} w_h(n)$ with integrals (of the function $w_h : t \mapsto (1 + h^2t^2)^{-1}$). Deduce that $\tau_h S(0)$ tends to $-\pi/2$ when h goes to 0^+.
6. Give the shape of the graph of S.
7. Is the function S differentiable at 0?

Exercice 12. The Cantor function or Devil staircase("I have so many names...")
Let $(f_n)_{n\in\mathbb{N}}$ be the function from $[0,1]$ to $\mathbb{R}$ defined by induction by: $f_0(x) = x$ for all $x \in [0,1]$ and, for $n \geq 0$, $f_{n+1}(x) = \begin{cases} \frac{1}{2}f_n(3x) & \text{si } x \in [0,\frac{1}{3}], \\ \frac{1}{2} & \text{si } x \in [\frac{1}{3},\frac{2}{3}], \\ \frac{1}{2} + \frac{1}{2}f_n(3x-2) & \text{si } x \in [\frac{2}{3},1]. \end{cases}$

1. Draw, on th same picture, the graphs of f_0, f_1, f_2.
2. Show that for all $n \geq 0$, f_n is continuous and increasing.
3. We consider the series $\sum u_n$ where $u_n = f_n - f_{n-1}$ $(n > 0)$. Show that $\left(\sum u_n\right)$ converges normally.

4. Deduce that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function f which is continuous and increasing.

Exercice 13. Let $a \in]0,1[$ and $b > 0$. For all $n \in \mathbb{N}$, we define u_n from $\mathbb{R}$ to $\mathbb{R}$ by

$$u_n(x) = a^n \sin(b^n x) \text{ for all } x \in \mathbb{R}.$$

1. Show that $\left(\sum_{n\in\mathbb{N}} u_n\right)$ converges normally. We denote by f its sum.
2. Show that f is continuous on $\mathbb{R}$.
3. Show that if $ab < 1$, f is a function of class C^1.
4. Show that for all $x \in \mathbb{R}$, $f(x) = af(bx) + \sin(x)$.
5. We assume that $ab = 1$ ans that b is an integer ≥ 2. Show that f is not differentiable at the points
$$x = 2kb^n\pi,\ (k,n) \in \mathbb{Z}^2.$$
We can start with $x = 0$ and the show that f is 2π-periodic. What can we say about this family of points (look at $b = 10$ for example)?