MAT402 **Year 2018-2019**

Exercise list 2

Exercise 1. Study pointwise convergence and uniform convergence of the following sequence of functions $(f_n)_{n\geq 1}$, $(g_n)_{n\geq 1}$, $(h_n)_{n\geq 1}$ and $(k_n)_{n\geq 1}$ defined on the specified intervals I. If there is no uniform convergence find intervals on which there is.

$$f_n(x) = \frac{x}{x+n} \text{ sur } \mathbb{R}_+, \quad g_n(x) = xne^{-xn} \text{ sur } \mathbb{R}_+, \quad h_n(x) = (\sin x)^n \text{ sur } \mathbb{R};$$

The function $k_n : \mathbb{R} \to \mathbb{R}$ is continuous, defined for all $n \geq 1$ by $k_n(x) = 0$ if $x \leq -1/n$, $k_n(x) = 1$ if $x \geq 1/n$, and such that k_n is affine on the interval $[-1/n, 1/n]$.

Exercise 2.

for $n \in \mathbb{N}$, we define the functions c_n and s_n, from $\mathbb{R}$ to $\mathbb{R}$, by $c_n(x) = \cos(nx)$ and $s_n(x) = \sin(nx)$. What are the intervals on which the functions $(c_n)_{n\geq 0}$ and $(s_n)_{n\geq 0}$ converges pointwise? (indication: it might be useful to use formulas looking like $\cos(a+b) = \cdots$ and $\sin(a-b) = \cdots$)

Exercise 3. Let $n \in \mathbb{N}^*$. We set $f_n(x) = 0$ and $|x - \frac{1}{2}| \geq \frac{1}{n}$, $f_n(\frac{1}{2}) = 1$, extend f_n affinely on $[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}]$ and $[\frac{1}{2}, \frac{1}{2} + \frac{1}{n}]$ in order to be continuous.

1. Draw the graph of f_n and give a formula for $f_n(x)$ with respect to x.
2. Study the convergence of $(f_n)_{n\geq 1}$ on $[0, 1]$, then the convergence of the sequence $\left(\int_0^1 f_n(x)\,\mathrm{d}x\right)_{n\geq 1}$.
3. Same questions when the functions f_n are defined on $[0, 1]$ by:
   a) $f_n(x) = 0$ if $x \in [0, \frac{1}{n}[$, $f_n(x) = \frac{1}{x}$ if $x \in [\frac{1}{n}, 1]$.
   b) $f_n(x) = n$ if $x \in]0, \frac{1}{n}]$, $f_n(x) = 0$ if $x \in \{0\} \cup]\frac{1}{n}, 1]$.

Exercise 4. For $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$, we set $f_n(x) = \frac{1}{n}\sin(nx)$.

1. Study the convergence of the sequence $(f_n)_{n\geq 1}$.
2. Study the convergence of the sequence of derivatives $(f_n')_{n\geq 1}$. What can we observe ?

Exercise 5. For all $n \in \mathbb{N}^*$, we define the function $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(t) = \sqrt{t^2 + \frac{1}{n}}$.

1. Study the pointwise convergence of the sequence $(f_n)_{n\geq 1}$ on $\mathbb{R}$.
2. Study the pointwise and uniform convergence of the sequence $(f_n')_{n\geq 1}$ on $\mathbb{R}$.

Exercise 6. Pólya's Lemma

Let $(f_n)_{n\geq 0}$ be a sequence of continuous functions on $[a, b]$ which converges uniformly on $[a, b]$ to a function f.

let $(x_n)_{n\geq 0}$ be a sequence of real numbers in $[a, b]$ which converges to a limit denoted l.

Show that the sequence $(f_n(x_n))_{n\geq 0}$ converges to $f(l)$.

Can we avoid the uniform convergence hypothesis ?

Exercise 7. Let $(f_n)_{n\geq 0}$ be the sequence of function defined on $\mathbb{R}$ by $f_n(x) = \dfrac{2^n x}{1 + n2^n x^2}$.

Study the pointwise convergence of the sequence.

Show, with different methods that there is no uniform convergence on $\mathbb{R}$.

Show that there is uniform convergence on an interval of the form $I_a =]-\infty, -a] \cup [a, +\infty[$ where $a > 0$.

Exercise 8. Find a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ such that:

(i) for all n, the improper integral $\int_{-\infty}^{+\infty} f_n$ converges;

(ii) the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $\mathbb{R}$ to a function f ;

(iii) the improper integral $\int_{-\infty}^{+\infty} f$ converges ;

(iv) $\int_{-\infty}^{+\infty} f_n$ doesn't converges to $\int_{-\infty}^{+\infty} f$ when n goes to $+\infty$.

Exercise 9. Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of functions defined on $[0,1]$ by $f_n(x) = \sin(x^n(1-x))$.
1) Study the pointwise convergence of the sequence $(f_n)_{n\in\mathbb{N}}$.
2) Study the uniform convergence of the sequence $(f_n)_{n\in\mathbb{N}}$.
3) What can we conclude about the sequence $(I_n)_{n\in\mathbb{N}}$, where $I_n = \displaystyle\int_0^1 f_n(t)\,\mathrm{d}t$?
4) Does the sequence $(f_n')_{n\in\mathbb{N}}$ converges pointwise ?
5) Does the sequence $(f_n')_{n\in\mathbb{N}}$ converges uniformly on $[0,1]$?

Exercise 10.
We consider the sequence of functions $(f_n)_{n\in\mathbb{N}^*}$ defined on $\mathbb{R}$ by $f_n(x) = \arctan(x/n)$.
1) Show that the sequence of functions $(f_n')_{n\in\mathbb{N}^*}$ converges uniformly on $\mathbb{R}$ but that the sequence $(f_n)_{n\in\mathbb{N}^*}$ doesn't converges uniformly on $\mathbb{R}$.
2) on which intervals does $(f_n)_{n\in\mathbb{N}^*}$ converge uniformly?