MAT402 **Year 2018-2019**

Exercise list 1 : reminders and warming up

About supremum (and infimum)

As in the lecture notes, if f is a function from a set A to $\mathbb{R}$, we will denote by $\sup_A f$ the supremum of $f(A) = \{f(x),\, x \in A\}$.

Exercise 1. Find, when it exists the supremum and infimum of the following sets

$$D_n = \left\{ \frac{x^2 - n^2}{x^2 + n^2},\, x \in \mathbb{R}_+ \right\}.$$

It may depends on the value of the parameter n.
I there a maximum and/or a minimum ?

Exercise 2. Let f and g be two function from $\mathbb{R}$ to $\mathbb{R}$. show that

$$\sup_{\mathbb{R}}(f+g) \leqslant \sup_{\mathbb{R}} f + \sup_{\mathbb{R}} g.$$

Exercise 3. Let $A = \{x \in \mathbb{Q} | x < \sqrt{2}\}$ and $f : \mathbb{R} \to \mathbb{R}$ an increasing and continuous function.
Find $\sup_A f$.

Exercise 4. Let $B = \mathbb{Q} \cap]0,1[$. We consider the function g from $\mathbb{R}$ to $\mathbb{R}$ given by $g(x) = x - x^3$. Determine $\sup_B g$ and $\inf_B g$.

Exercise 5. Let $(u_{i,j})_{i,j\in\mathbb{N}}$ be a sequence depending on two parameters.
show that

$$\sup_{i\in\mathbb{N}}\big(\sup_{j\in\mathbb{N}}(u_{i,j})\big) = \sup_{j\in\mathbb{N}}\big(\sup_{i\in\mathbb{N}}(u_{i,j})\big).$$

Exercise 6. let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers.
for $n \in \mathbb{N}$ we set $X_n = \{u_k, k \geqslant n\}$, et $s_n = \sup X_n$.

1. Show that $(s_n)_{n\in\mathbb{N}}$ is decreasing.
2. Show that $(s_n)_{n\in\mathbb{N}}$ is convergent. we denote its limit by $\limsup u_n$.
3. Define, in the same way, $\liminf u_n$.
4. Show that, if u_n converges to ℓ, then $\liminf u_n = \limsup u_n = \ell$.
5. Show that, if $\liminf u_n = \limsup u_n$, then u_n converges to this value.
6. Determine $\limsup u_n$ and $\liminf u_n$ for the sequence defined for all $n \in \mathbb{N}$ by $u_n = \cos(2\pi n/3)$.

Exercise 7. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. We set $L = \limsup u_n$.

1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence which converges to a. Express $\limsup(a_n + u_n)$ with a and L.
2. If $(a_n)_{n\in\mathbb{N}}$ is only bounded, do we have $\limsup(a_n + u_n) = \limsup a_n + \limsup u_n$?
3. Express $\limsup e^{u_n}$ with L.

Exercise 8. Let f be an increasing function from $[0,1]$ to $[0,1]$. We want to show that f as a fixed point, i.e. a point $x \in [0,1]$ such that $f(x) = x$. (Note : f we do not assume that f is continuous. This would make the exercise easier, you should try)

For that purpose, we set A to be the set of all $x \in [0,1]$ such that $f(x) \leqslant x$.

a) Show that A is non empty and has an infimum, that we may denote α, and that $\alpha \in [0,1]$. The rest of the exercise is about proving that α is a fixed point of f.

b) Use that f is increasing to show that

i) If $x \in [0,1]$ is a lower bound of A, then $f(x)$ is also a lower bound of A.

ii) If $x \in [0,1]$ is an element of A, then $f(x)$ is also an element of A.

c) Applying i) to $x = \alpha$, show that $f(\alpha) \leqslant \alpha$, i.e. $\alpha \in A$.

Applying ii) to $x = \alpha$, show that $f(\alpha) \geqslant \alpha$, and then conclude.

On sequences and series

Exercise 9. Soit $(a_n)_{n\in\mathbb{N}}$ a sequence of real numbers and l a real number. For each of the following state if they are true or false and justify your answer.

1. $(a_n)_{n\in\mathbb{N}}$ converges to 0 if and only if $(|a_n|)_{n\in\mathbb{N}}$ converges to 0.
2. If the sequence $(|a_n|)_{n\in\mathbb{N}}$ converges to l, then the sequence $(a_n)_{n\in\mathbb{N}}$ converges to l or $-l$.
3. If the sequence $(a_n)_{n\in\mathbb{N}}$ converges to l, then the sequence $(|a_n|)_{n\in\mathbb{N}}$ converges $|l|$.

Exercise 10. Determine the nature of the series with terms u_n given as follow (this may depends on the parameters $x \in \mathbb{R}$ or $z \in \mathbb{C}$).

1) $u_n = \dfrac{1}{(n+1)!}$ 2) $u_n = \dfrac{4^n}{n!}$ 3) $u_n = \dfrac{n^n}{3^{1+2n}}$ 4) $u_n = e^{-n^3-n}$

5) $u_n = \dfrac{n^{n+1}}{n!}\ln\left(1+\dfrac{1}{n}\right)$ 6) $u_n = \left(1+\dfrac{1}{n}\right)^n - e$ 7) $u_n = (-1)^n \sin\dfrac{1}{n}$ 8) $u_n = \dfrac{2^n}{(n+2)3^{n+1}}$

9) $u_n = \dfrac{x^2}{x^2+n}$ 10) $u_n = \dfrac{x^2}{x^2+n^2}$ 11) $u_n = e^{-nx}$ 12) $u_n = z^n$

Exercise 11. (Cesaro) Let $(a_n)_{n\in\mathbb{N}^*}$ be a sequence of natural numbers. we want to show that if the sequence $(a_n)_{n\in\mathbb{N}^*}$ converges to a limit l, then the sequence of arithmetical means $(b_n)_{n\in\mathbb{N}^*}$ defined by $b_n = \dfrac{1}{n}(a_1 + ... + a_n)$ converges to l.

1. For all $n_0 \in \mathbb{N}^*$, what can we say about $\left(\dfrac{1}{n}(a_1 + ... + a_{n_0})\right)_{n\in\mathbb{N}^*}$?
2. For all $n \in \mathbb{N}^*$, express $b_n - l$ with $a_i - l$ for $i = 1, ..., n$.
3. Let $\epsilon > 0$ and real numbersx_1,..., x_k such that $x_1, ..., x_k \in]-\epsilon, +\epsilon[$. Show that for all $m \geqslant k$, we have $\dfrac{1}{m}(x_1 + ... + x_k) \in]-\epsilon, +\epsilon[$.
4. Show that if the sequence $(a_n)_{n\in\mathbb{N}}$ converge to a finite limit l, then the sequence $(b_n)_{n\in\mathbb{N}}$ converges also to l.
5. Is the converse true ?
6. What can we say if $(a_n)_{n\in\mathbb{N}^*}$ tends to $+\infty$?

Exercise 12. Let $(u_n)_{n\in\mathbb{N}}$ a sequence with value in $D = \{0, 1, \ldots, 9\}$.

1. Show that the series with terms given by $u_n \cdot 10^{-n}$ converges and that its limit x lies in $[0, 10]$.
2. Show that if the $(u_n)_{n\in\mathbb{N}}$ converges, there is $n_0 \in \mathbb{N}$ such that $u_n = u_{n_0}$ for all $n \geqslant n_0$.
3. Show that if the $(u_n)_{n\in\mathbb{N}}$ converges to 9, then there exists a sequence $(v_n)_{n\in\mathbb{N}} \in D^{\mathbb{N}}$ which converges to 0 so that $\sum_{n=0}^{+\infty} u_n 10^{-n} = \sum_{n=0}^{+\infty} v_n 10^{-n}$.
4. For all $x \in [0, 10[$, show that there is a unique sequence $(u_n)_{n\in\mathbb{N}}$ which do not converges to 9 and such that $\sum_{n=0}^{+\infty} u_n 10^{-n} = x$.

Exercise 13. We consider the series $\displaystyle\sum_{n\in A} \frac{1}{n}$ where A is the set of all the integers with no 2 in their decomposition in base 10. Evaluating, for all $n \geqslant 0$ the nimber of terms in $A \cap [10^n, 10^{n+1}[$, study the nature of this series.

Exercise 14. Let $a > 0$ be a fixed positive number. We define the sequence $(P_n(a))_{n\in\mathbb{N}}$ by

$$P_0(a) = 1, \quad \text{and} \quad P_{n+1}(a) = (n + a)P_n(a) \quad \text{for all } n \in \mathbb{N}.$$

We want to prove that

$$L(a) = \lim_{n\to\infty} \frac{P_n(a)}{n!\, n^{a-1}}$$

exists and is strictly positive. for that purpose, we consider the series with terms u_n, where

$$u_n = \ln(n + a) - a\ln(n + 1) + (a - 1)\ln n.$$

1. Compare the partial sum of order $n - 1$ of $\sum u_n$ with $\ln \dfrac{P_n(a)}{n!\, n^{a-1}}$.
2. using Taylor expansion, show that $\sum_{n\in\mathbb{N}^*} u_n$ converges.
3. Conclude.