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3 authors:

Milton Lopes Filho
Federal University of Rio de Janeiro

Evelyne Miot
MINES ParisTech

Helena Nussenzveig Lopes
Federal University of Rio de Janeiro

EXISTENCE OF A WEAK SOLUTION TO THE VORTEX-WAVE SYSTEM

MILTON C. LOPES FILHO, EVELYNE MIOT, AND HELENA J. NUSSENZVEIG LOPES

Abstract. The vortex-wave system is a coupling of the two-dimensional vorticity equation with the point-vortex system. It is a model for the motion of a finite number of concentrated vortices moving on a distributed vorticity background. In this article, we prove existence of a weak solution to this system with initial background vorticity in L^p, $p > 2$ up to the time of first collision of point vortices.

1. Introduction

The two-dimensional incompressible Euler equations can be written as an active scalar transport equation for vorticity, which takes the form

$$\partial_t \omega + u \cdot \nabla \omega = 0,$$

with the transporting divergence-free velocity $u(\cdot, t)$ determined by $\omega(\cdot, t)$ by means of the Biot-Savart law. We focus on the full-plane case, where the Biot-Savart law takes the form:

$$u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y,t) dy \equiv K * \omega(\cdot, t),$$

with $(a,b)^\perp = (-b,a)$. Existence and uniqueness of a weak solution with initial vorticity $\omega(t=0) \in L^\infty \cap L^1$ is a classical result by Yudovich, see [17] and existence for initial vorticity in the space of measures is known under sign and finite kinetic energy restrictions, see [3, 7].

One special case of solutions with bounded vorticity is the *vortex patch problem*, with initial vorticity the characteristic function of a bounded set with smooth boundary. By Yudovich's Theorem, there exists a unique weak solution of the Euler equations with such initial data, globally defined in time. Moreover, the boundary of the vortex patch stays smooth for all time, see [1]. This special case of the Euler equations is also known as *contour dynamics*.

A second kind of special solution is the *vortex sheet problem*, where the initial vorticity is a Dirac mass supported on a smooth curve. In this case, existence of a weak solution is known under sign restrictions, but very little is known about the solution besides its existence.

A third kind of solution, called *point vortex dynamics*, assumes initial vorticity is a finite sum of Dirac point masses. This type of initial data is not included in the existence results because the associated flow has infinite kinetic energy. Indeed, a single point vortex in the plane at position P induces a velocity of the form $C(x-P)^\perp/|x-P|^2$, which is not square-integrable. However, if one places a finite number of point vortices in the plane, and assumes that each point vortex moves *with the speed induced by the other vortices*, Euler equations appear to reduce to a system of ODEs for the position of the vortices, called *the point vortex system*. This hamiltonian system of ODEs is at the heart of important numerical methods, and represents a good approximation for the motion of sharply concentrated vorticity parcels, see [8].

Although it would be natural to consider point vortex dynamics as a weak solution of the Euler equations, point vortices are too singular to include in the usual weak formulations of the Euler equations, see [15]. In [12], F. Poupaud formulated a treatment of weak solutions

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of the Euler equations which included point vortices, but he could not obtain existence in this context due to the possible appearance of nonlinearity defects. To treat initial vorticities which include point vortices together with continuously distributed vorticity, one alternative is to separate the evolution of the continuous part of the vorticity, evolved using Euler equations, from the evolution of the point vortices, evolved through the point vortex system, coupling these equations by means of the Biot-Savart law. This idea was introduced by Marchioro and Pulvirenti in [9, 10], together with the terminology *vortex-wave system* for the resulting system.

In [9], Marchioro and Pulvirenti proved existence of weak solutions for the vortex-wave system with initial data consisting of a bounded, compactly supported continuous part plus a number of point vortices, and uniqueness when the initial position of the vortices is outside the support of the initial vorticity. They also indicated that uniqueness would still be true if the continuous part of the vorticity was constant near the initial position of the point vortices. Uniqueness in this case was proved by Staravoitov for Lipschitz continuous vorticities in [16] and by Lacave and Miot for bounded vorticities in [2]. The purpose of the present article is to extend the existence result of Marchioro and Pulvirenti to initial vorticities in L^p, $p > 2$.

We add a word regarding the physical meaning of the vortex-wave system. The incompressible 2D Euler equations are a useful simplified model for large-scale geophysical and astrophysical flows and are also used in laboratory modeling of plasma dynamics, among other contexts, see [5] and references therein. The situation where sharply concentrated vorticity appears coupled with continuous background vorticity is common in applications, both with the point vortices embedded in a constant vorticity background, see [5], and in layers of variable vorticity, see [14]. The coupling of point vortices and continuously varying vorticity leads to new phenomena, such as the development of vortex cristals, see [13] and provides an explanation for the generation of Rossby waves, see [11].

The remainder of this article is divided as follows. In the next section we introduce the definitions of Lagrangian and Eulerian weak solutions and give a precise statement of our main result. In the following section we prove our main result in the case where the point vortices are of the same sign. In the last section we present a short-time existence result without the single-sign assumption on the point vortices and present some conclusions.

2. Lagrangian and eulerian solutions

The purpose of this section is to formulate precisely our main result. The unknowns for the vortex-wave system are the background vorticity, which we denote by ω, belonging to $L^1 \cap L^\infty$ and a weighted sum of Dirac masses $\sum d_i \delta_{z_i}$, $i = 1, \dots, N$ called *point vortices*. Let $K = K(x) = x^\perp / 2\pi |x|^2$ be the kernel of the Biot-Savart law in the full plane. Let $v = v(x,t) = K * \omega$ be the part of the flow velocity associated with the background vorticity. We can write the vortex-wave system as follows:

$$\begin{cases} \partial_t \omega + (v + \displaystyle\sum_{j=1}^N d_j K(\cdot - z_j)) \cdot \nabla \omega = 0, \\ v = K * \omega, \\ \dfrac{dz_i}{dt} = v(z_i, t) + \displaystyle\sum_{j \neq i} d_j K(z_i - z_j), \quad i = 1, \dots, N, \\ \omega(x, 0) = \omega_0(x), \quad z_i(0) = z_{i0}. \end{cases} \tag{2.1}$$

Of course, even if ω_0 is smooth, the transport equation for ω has a rather singular velocity, so that the notion of solution for (2.1) should be a weak formulation. There are two weak formulations for (2.1) in the literature, which we describe below.

Definition 2.1 (Lagrangian solutions)**.** Let $\omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$ and let $z_{10}, \dots, z_{N0}$ be N distinct points in $\mathbb{R}^2$. Let $T > 0$. We say that the map $t \mapsto (\omega(\cdot,t), z_1(t), \dots, z_N(t), \phi(\cdot,t))$ is a *Lagrangian solution* to the vortex-wave system (2.1) on $[0,T]$ with initial condition $(\omega_0, z_{10}, \dots, z_{N0})$ if we have

$$\omega \in L^\infty\left([0,T], L^1 \cap L^\infty(\mathbb{R}^2)\right), \qquad v = K * \omega \in C([0,T] \times \mathbb{R}^2)$$

and for all i

$$z_i : [0,T] \to \mathbb{R}^2, \qquad \phi : \mathbb{R}^2 \setminus \{z_{10}, \dots, z_{N0}\} \times [0,T] \to \mathbb{R}^2$$

are such that $z_i \in C^1([0,T], \mathbb{R}^2)$, $\phi(x,\cdot) \in C^1([0,T], \mathbb{R}^2)$ for all $x \neq z_{i0}$ and satisfy

$$\begin{cases} \omega(\phi(x,t),t) = \omega_0(x), \qquad t \in [0,T], \quad v(\cdot,t) = K * \omega(\cdot,t), \\ \dfrac{dz_i}{dt}(t) = v(z_i(t),t) + \displaystyle\sum_{j \neq i} d_j K\left(z_i(t) - z_j(t)\right), \\ z_i(0) = z_{i0}, \\ \dfrac{\partial \phi}{\partial t}(x,t) = v(\phi(x,t),t) + \displaystyle\sum_{j=1}^{N} d_j K\left(\phi(x,t) - z_j(t)\right), \\ \phi_0(x) = x, \ x \neq z_{i0}, \end{cases} \tag{LS}$$

where, for all t, $\phi_t(\cdot) = \phi(\cdot,t)$ is an homeomorphism from $\mathbb{R}^2 \setminus \{z_{10}, \dots, z_{N0}\}$ into $\mathbb{R}^2 \setminus \{z_1(t), \dots, z_N(t)\}$ which preserves Lebesgue's measure.

Marchioro and Pulvirenti [9] proved global existence for (LS) when all the intensities d_i have the same sign. The proof uses the almost-Lipschitz regularity of $v = K * \omega$ and of the explicit form of K. It is shown in particular that characteristics starting in $\mathbb{R}^2 \setminus \cup\{z_{i0}\}$ cannot collide with the point vortices in finite time, and that there is no collision among the vortices in finite time as well. Consequently, all the singular terms involved in (LS) remain well-defined for all time.

This notion of Lagrangian solutions is rather strong, since it requires the existence of a flow ϕ which is continuous in space and time. One can define a weaker notion of solutions: solutions in the sense of distributions of the PDE (without involving the flow ϕ). As we will see, this formulation (called Eulerian formulation) enables us to handle a larger class of solutions. In particular, in contrast with our Lagrangian formulation, it allows the singular fields to become infinite. We define these Eulerian solutions below.

Definition 2.2 (Eulerian solutions)**.** Let $p \in [1, +\infty]$ and $\omega_0 \in L^p(\mathbb{R}^2)$ have compact support. Let $z_{10}, \dots, z_{N0}$ be N distinct points in $\mathbb{R}^2$ and $d_1, \dots, d_N$ be real numbers. We say that $(\omega, z_1, \dots, z_N)$ is an *Eulerian solution* of the vortex-wave equation with initial condition $(\omega_0, z_{10}, \dots, z_{N0})$ on $[0,T]$ if

$$\omega \in L^\infty\left([0,T], L^p(\mathbb{R}^2)\right), \qquad z_i \in L^\infty\left([0,T], \mathbb{R}^2\right)$$

and if we have in the sense of distributions

$$\begin{cases} \partial_t \omega + (v + H) \cdot \nabla \omega = 0, \\ \omega(0) = \omega_0, \\ \dfrac{dz_i}{dt}(t) = v\left(z_i(t),t\right) + \displaystyle\sum_{j \neq i} d_j K\left(z_i(t) - z_j(t)\right), \qquad z_i(0) = z_{i0}, \qquad i = 1, \dots, N, \end{cases} \tag{ES}$$

where v and H are given by

$$v = K * \omega, \qquad H = \sum_{j=1}^{N} d_j K\left(\cdot - z_j\right).$$

In other words, we have for any test function $\varphi \in \mathcal{D}([0,T) \times \mathbb{R}^2)$

$$-\int_{\mathbb{R}^2} \omega_0(x)\varphi(0,x)\,dx = \int_0^T \int_{\mathbb{R}^2} \omega_s(\partial_t \varphi + (v+H)\cdot\nabla\varphi)\,ds\,dx,$$

and for all $t \in [0,T]$, for $i = 1,\ldots,N$,

$$z_i(t) = z_{i0} + \int_0^t \left\{ v(z(s),s) + \sum_{j\neq i} d_j K\left(z_i(s) - z_j(s)\right) \right\} ds.$$

When $p = +\infty$, equivalence between Lagrangian and Eulerian formulations has been proved in [2]. In particular, global existence of Eulerian solutions follows from the existence result for vorticity in L^∞ stated by Marchioro and Pulvirenti [9].

To work with Eulerian solutions it is necessary to give sense to the products ωv and ωH. Assume that $\omega \in L^p$, with $p \geq 1$. Since H belongs to $L^q_{\text{loc}}(\mathbb{R}^2)$ for all $q < 2$, then ωH belongs to $L^1_{\text{loc}}(\mathbb{R}^2)$ provided $p > 2$. On the other hand, as will be recalled in Lemma 3.1, the velocity $v = K * \omega$ is uniformly bounded and continuous in space for all $p > 2$, so that ωv belongs to $L^1_{\text{loc}}(\mathbb{R}^2)$. It is therefore natural to focus on vorticites belonging to $L^p(\mathbb{R}^2)$, for $p > 2$. Our main result will be the following theorem.

Theorem 2.3. *Let $p > 2$ and $\omega_0 \in L^p(\mathbb{R}^2)$ have compact support. Let $\{z_{i0}\}, i = 1,\ldots,N$ be N distinct points in $\mathbb{R}^2$, and let $d_i, i = 1,\ldots,N$ be N positive numbers. Then there exists a global Eulerian solution of the vortex-wave system with this initial data.*

3. Global existence for single signed vortices

The purpose of this section is to prove Theorem 2.3. We start by recalling a useful result.

Lemma 3.1. *Let $p > 2$ and $f \in L^1 \cap L^p(\mathbb{R}^2)$. Let $g = K * f$. Then we have*

$$\|g\|_{L^\infty(\mathbb{R}^2)} \leq C(p)\|f\|_{L^1(\mathbb{R}^2)}^{1-\frac{p'}{2}}\|f\|_{L^p(\mathbb{R}^2)}^{\frac{p'}{2}},$$

where p' denotes the conjugate exponent of p and C depends only on p.

Proof. The proof consists of splitting the integral defining $K * f(x)$ into an integral in a ball of radius ε around x and an integral in the complement. One estimates the integral outside the small ball by $\|f\|_{L^1}/\varepsilon$, then one uses Hölders inequality to estimate the integral inside the small ball. Finally, one chooses the optimal ε.

□

The next lemma concerns the regularity of velocity in our problem.

Lemma 3.2. *Let $p > 2$, $f \in L^1 \cap L^p(\mathbb{R}^2)$ and $g = K * f$. Then g is Hölder continuous with exponent $\alpha(p) = 1 - 2/p$.*

Proof. First we observe that $\nabla g \in L^p(\mathbb{R}^2)$; this follows from Calderón-Zygmund inequality since $f \in L^p(\mathbb{R}^2)$. Next, note that $K \in L^1(\mathbb{R}^2) + L^p(\mathbb{R}^2)$. Hence $g \in L^p(\mathbb{R}^2)$, and the result follows from Morrey's inequality. □

In what follows, we will denote by $\|\omega_0\|$ the quantity $\|\omega_0\|_{L^1(\mathbb{R}^2)} + \|\omega_0\|_{L^p(\mathbb{R}^2)}$.

The proof of Theorem 2.3 relies on Marchioro and Pulvirenti's result [9] which states global existence for initial vorticity belonging to $L^\infty(\mathbb{R}^2)$. For that purpose, we regularize the initial vorticity ω_0 by setting

$$\omega_{0,\delta} = \rho_\delta * \omega_0,$$

where δ is a small parameter and ρ_δ is a standard regularizing kernel. Since all the intensities d_i have the same sign, there exists a global solution $(\omega^\delta, v^\delta, z_i^\delta, \phi_t^\delta)$ of the vortex-wave system in Lagrangian formulation (although nothing can be said about uniqueness).

We first establish some control on the growth of the smooth flow and the vortices.

Proposition 3.3. *Let $(\omega^\delta, v^\delta, z_i^\delta, \phi_t^\delta)$ be defined as above, then there exists a constant C_1 depending only on (ω_0, z_{i0}, d_i) such that for all t and for all δ, we have*

$$|z_i^\delta(t)| \leq C_1(1+t), \qquad i = 1, \dots, N.$$

Moreover, the support of ω^δ grows at most linearly in time. More precisely, there exists C_2 depending only on (ω_0, z_{i0}, d_i) such that for all $x \neq z_{i0}$ belonging to the support of ω_0, we have

$$|\phi_t^\delta(x)| \leq C_2(1+t).$$

Proof. Since ω^δ is transported by a measure preserving flow and ω_0 belongs to $L^1 \cap L^p(\mathbb{R}^2)$, we first infer that ω^δ is uniformly bounded in $L^\infty(\mathbb{R}^+, L^1 \cap L^p(\mathbb{R}^2))$ with respect to δ. In view of Proposition 3.1, there exists a constant C, depending only on $\|\omega_0\|$, such that for all δ

$$\|v^\delta\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)} \leq C. \tag{3.1}$$

Next, we introduce the angular momentum defined by

$$I^\delta(t) = \sum_{i=1}^N d_i |z_i^\delta(t)|^2,$$

which is for the point vortex system constant in time. In the presence of the background part ω^δ, I^δ is no longer conserved, but we can obtain some control of its growth. Indeed, we have, using first symmetry properties of the Kernel K, then the bound (3.1)

$$\frac{dI^\delta}{dt}(t) = 2\sum_{i=1}^N d_i z_i^\delta(t) \cdot v^\delta(z_i^\delta(t), t) \leq C \sum_{i=1}^N |d_i||z_i^\delta(t)|.$$

Since all the d_i are positive, we obtain

$$\frac{dI^\delta}{dt}(t) \leq C\sqrt{I^\delta(t)},$$

which yields for all i

$$|z_i^\delta(t)|^2 \leq C I^\delta(t) \leq C(1+t^2).$$

This is the first part of Proposition 3.3. We now turn to the second part. Given $x \neq z_{i0}$, we observe that the flow $\phi_t^\delta(x)$ has finite velocity except in the neighborhood of the point vortices. Let C_1 be such that, for all t and $\delta > 0$,

$$|z_i^\delta(t)| \leq C_1(1+t). \tag{3.2}$$

Then, we consider a constant $C_2 \geq 2C_1$ so that $\operatorname{supp} \omega_0 \subset B(0, C_2/2)$. Let $x \neq z_{i0} \in \operatorname{supp} \omega_0$. We claim that for all t, we have $|\phi_t^\delta(x)| < 2C_2(1+t)$. Indeed, we have $|\phi_0^\delta(x)| < C_2$. Therefore, if there exists a time t_1 such that $|\phi_{t_1}^\delta(x)| = 2C_2(1+t_1)$, then, since $|\phi_t^\delta(x)|/(1+t)$ is continuous, we can find a time $0 < t_0 < t_1$ such that $|\phi_{t_0}^\delta(x)| = C_2(1+t_0)$ and such that, for $t \in I = [t_0, t_1]$, $|\phi_t^\delta(x)| \geq C_2(1+t)$. In view of (3.1) and (3.2), we have for $t \in I$

$$\left|\frac{d\phi_t^\delta(x)}{dt}\right| \leq \|v^\delta\|_{L^\infty} + \sum_{i=1}^N \frac{d_i}{|\phi_t^\delta(x) - z_i^\delta(t)|} \leq C + \frac{C}{C_1}.$$

Increasing possibly C_2, we therefore obtain for $t \in I$

$$\left|\frac{d\phi_t^\delta(x)}{dt}\right| \leq C_2,$$

thus

$$2C_2(1+t_1) - C_2(1+t_0) = |\phi_{t_1}^\delta(x)| - |\phi_{t_0}^\delta(x)| \leq C_2(t_1 - t_0).$$

This yields a contradiction, and the conclusion follows. □

The next step is to control the minimal distance between the vortices $z_i^\delta(t)$ uniformly with respect to δ.

Proposition 3.4. *There exists a positive and continuous function $t \mapsto d(t)$ depending only on t and (ω_0, z_{i0}, d_i) such that*

$$\inf_{\delta>0} \min_{i\neq j} |z_i^\delta(t) - z_j^\delta(t)| \geq d(t), \qquad \forall t \in \mathbb{R}^+.$$

Proof. We define the generalized Hamiltonian $\mathcal{H}^\delta$ as

$$\mathcal{H}^\delta(t) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} \ln|x-y|\omega^\delta(x,t)\omega^\delta(y,t)\,dx\,dy + \sum_{i\neq j} d_i d_j \ln|z_i^\delta(t) - z_j^\delta(t)|$$
$$+ 2\sum_{i=1}^N d_i \int_{\mathbb{R}^2} \ln|x - z_i^\delta(t)|\omega^\delta(x,t)\,dx.$$

We observe that when $\omega^\delta \equiv 0$, $\mathcal{H}^\delta$ corresponds to the classical Hamiltonian H^δ associated to the point vortex system, which is known to be constant in time. Additionally, in the absence of point vortices ($di = 0$) then $\mathcal{H}^\delta$ corresponds to the *pseudo-energy*, also known to be conserved in time. Conservation of $\mathcal{H}^\delta$ actually still holds in the present situation, as established in Proposition **A**.1 in the appendix. We infer that

$$H^\delta(t) = \mathcal{H}^\delta(0) - \iint_{\mathbb{R}^2\times\mathbb{R}^2} \ln|x-y|\omega^\delta(x,t)\omega^\delta(y,t)\,dx\,dy$$
$$- 2\sum_{i=1}^N d_i \int_{\mathbb{R}^2} \ln|x - z_i^\delta(t)|\omega^\delta(x,t)\,dx,$$

where

$$H^\delta(t) = \sum_{i\neq j} d_i d_j \ln|z_i^\delta(t) - z_j^\delta(t)|.$$

On the one hand, since ω_0^δ is uniformly bounded in $L^p(\mathbb{R}^2)$ and has compact support, we have

$$|\mathcal{H}^\delta(0)| \leq C.$$

On the other hand, we deduce from Proposition 3.3 and from the uniform bounds for ω^δ in $L^\infty(\mathbb{R}^+, L^1 \cap L^p(\mathbb{R}^2))$ that

$$\left|\iint_{\mathbb{R}^2\times\mathbb{R}^2} \ln|x-y|\omega^\delta(x,t)\omega^\delta(y,t)\,dx\,dy\right|$$
$$+2\left|\sum_{i=1}^N d_i \int_{\mathbb{R}^2} \ln|x - z_i^\delta(t)|\omega^\delta(x,t)\,dx\right| \leq C(1+\ln(1+t)),$$

which yields

$$|H^\delta(t)| \leq C\left(1+\ln(1+t)\right).$$

As all the intensities d_i are positive, we obtain, using the first part of Proposition 3.3

$$\ln |z_i^\delta(t) - z_j^\delta(t)| \geq -C\,(1 + \ln(1+t)),$$

which concludes the proof.

□

We are now in position to state some compactness for the trajectories and for the regular vorticity.

Proposition 3.5 (compactness). *There exists $\omega \in L^\infty(\mathbb{R}^+, L^1 \cap L^p(\mathbb{R}^2))$ such that up to a subsequence, ω^δ converges to ω in $C(\mathbb{R}^+, L^1 - w) \cap C(\mathbb{R}^+, L^p - w)$. Let $v = K * \omega$. Then v^δ converges uniformly to v on the compact sets of $\mathbb{R}^+ \times \mathbb{R}^2$. Moreover, there exist N trajectories $z_i \in C(\mathbb{R}^+, \mathbb{R}^2)$ such that for all i, z_i^δ converges to z_i uniformly on compact sets of $\mathbb{R}^+$.*

Remark: We note that we do not expect uniform convergence for the flow ϕ_t^δ. Indeed, this would require to prove that for $x \neq z_{i0}$, the minimal distance between $\phi_t^\delta(x)$ and the vortices $z_i(t)$ is bounded from below uniformly in time. This property is proved in [9] by making use of uniform almost-Lipschitz regularity for v^δ; we do not have this regularity at hand since there is no uniform bound for ω^δ in $L^\infty(\mathbb{R}^+, L^\infty(\mathbb{R}^2))$ available.

Proof. We first deduce from Lemma 3.1 and Proposition 3.3 that the field $v^\delta + H^\delta$ is uniformly bounded in $L^\infty_{\text{loc}}(\mathbb{R}^+, L^q_{\text{loc}}(\mathbb{R}^2))$ for all $q < 2$, where $H^\delta(x,t) = \sum_{i=1}^N d_i K(x - z_i^\delta(t))$. On the other hand, $(\omega^\delta, v^\delta, z_i^\delta)$ is a global Eulerian solution of the vortex-wave system (see [2] for a proof of this fact). Therefore, $\partial_t \omega^\delta = -(v^\delta + H^\delta) \cdot \nabla \omega^\delta$ is uniformly bounded in $L^\infty_{\text{loc}}\left(\mathbb{R}^+, W^{-1,1}_{\text{loc}}(\mathbb{R}^2)\right)$. Moreover, the support of $\omega^\delta(\cdot, t)$ is uniformly bounded for all t. We infer that there exists $\omega \in L^\infty(\mathbb{R}^+, L^1 \cap L^p(\mathbb{R}^2))$ and a subsequence, still denoted by δ, such that ω^δ converges to ω in $C(\mathbb{R}^+, L^1 - w) \cap C(\mathbb{R}^+, L^p - w)$; this is a consequence of the Aubin-Lions Lemma, see [6], Appendix C, and of the Dunford-Pettis Theorem.

Moreover, the support of $\omega(\cdot, t)$ is included in $B\,(0, C(1+t))$. The uniform convergence of v^δ to v on compact sets is a consequence of the convergence of ω^δ to ω in $C(\mathbb{R}^+, L^1 - w)$ combined with the uniform bound of ω^δ in $L^\infty(\mathbb{R}^+, L^p(\mathbb{R}^2))$ and the Hölder bound in Lemma 3.2.

Next, thanks to Lemma 3.1 and Proposition 3.4, we may invoke Ascoli's Theorem and a standard diagonal argument to find continuous $z_i(t)$ such that up to a subsequence, z_i^δ converges to z_i uniformly on the compact sets of $\mathbb{R}^+$. The proposition is proved. □

We can now complete the proof of Theorem 2.3.

Proof of Theorem 2.3

We will prove that the limit $(\omega(\cdot, t), z_1(t), \ldots, z_N(t))$ found above is a global Eulerian solution of (ES) with initial data $(\omega_0, z_{10}, \ldots, z_{N0})$.

For $i \in \{1, \ldots, N\}$, set

$$H_i(x,t) = \sum_{\substack{j=1 \\ j \neq i}}^N d_j K(x - z_j(t)), \qquad u_i = v + H_i.$$

According to Propositions 3.4 and 3.5, we have

$$u_i^\delta\left(z_i^\delta(t), t\right) = \left(v^\delta + H_i^\delta\right)\left(z_i^\delta(t), t\right) \to u_i\left(z_i(t), t\right)$$

uniformly for all i, which means that $\{z_i(t)\}$ satisfies the desired ordinary differential equation.

We now turn to the equation satisfied by ω. To this aim, we define $\chi_0 : \mathbb{R}^2 \to \mathbb{R}$ to be a smooth, *radial* cut-off map such that

$$\chi_0 \equiv 0 \text{ on } B(0, \frac{1}{2}), \quad \chi_0 \equiv 1 \text{ on } B(0,1)^c, \quad 0 \leq \chi_0 \leq 1. \tag{3.3}$$

For a small and positive ε, we set $\chi_\varepsilon(z) = \chi_0(z/\varepsilon)$, so that as ε goes to 0, we have for all $q < 2$

$$\chi_\varepsilon \to 1 \text{ a.e.}, \qquad \|\nabla \chi_\varepsilon\|_{L^q(\mathbb{R}^2)} \to 0. \tag{3.4}$$

We let $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^2)$ be a test function and set

$$\varphi_\varepsilon(x,t) = \varphi(x,t) \prod_{i=1}^N \chi_\varepsilon^i(x,t) = \varphi(x,t)\xi_\varepsilon(x,t),$$

where $\chi_\varepsilon^i(x,t) = \chi_\varepsilon(x - z_i(t))$.

We introduce

$$u = u(x,t) = v(x,t) + H(x,t) = v(x,t) + \sum_{j=1}^N d_j K(x - z_j(t))$$

and

$$u^\delta(x,t) = v^\delta(x,t) + H^\delta(x,t) = v^\delta(x,t) + \sum_{j=1}^N d_j K\left(x - z_j^\delta(t)\right).$$

We claim that for all $t \geq 0$, we have

$$\int_{\mathbb{R}^2} \omega(x,t)\varphi_\varepsilon(x,t) - \int_{\mathbb{R}^2} \omega_0(x)\varphi_\varepsilon(x,0) = \int_0^t \int_{\mathbb{R}^2} \omega\left(\partial_t \varphi_\varepsilon + u \cdot \nabla \varphi_\varepsilon\right) dx\, ds. \tag{3.5}$$

Indeed, since $(\omega^\delta(\cdot,t), v^\delta(t), z_i^\delta(t))$ is a global Eulerian solution for all δ, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \omega^\delta(x,t)\varphi_\varepsilon(x,t)\, dx &- \int_{\mathbb{R}^2} \omega_0^\delta(x)\varphi_\varepsilon(x,0)\, dx \\ &= \int_0^t \int_{\mathbb{R}^2} \omega^\delta\left(\partial_t \varphi_\varepsilon + (v^\delta + H^\delta) \cdot \nabla \varphi_\varepsilon\right) dx\, ds.\end{aligned}$$

For fixed ε, we then let δ go to zero. Since $\varphi_\varepsilon(s)$ vanishes in $\cup_{i=1}^N B(z_i(s), \varepsilon)$ and thanks to the uniform convergence of z_i^δ to z_i, we deduce that $u^\delta \cdot \nabla \varphi_\varepsilon$ converges uniformly to $u \cdot \nabla \varphi_\varepsilon$ on $\mathbb{R}^+ \times \mathbb{R}^2$. The conclusion finally follows from the weak convergence of ω^δ to ω stated in Proposition 3.5.

The last step is to let ε go to zero in (3.5). We compute

$$\partial_t \varphi_\varepsilon + u \cdot \nabla \varphi_\varepsilon = \xi_\varepsilon(\partial_t \varphi + u \cdot \nabla \varphi) + \varphi(\partial_t \xi_\varepsilon + u \cdot \nabla \xi_\varepsilon). \tag{3.6}$$

We have

$$\begin{aligned}(\partial_t \xi_\varepsilon + u \cdot \nabla \xi_\varepsilon)(x,s) &= \sum_{i=1}^N (\prod_{j \neq i} \chi_\varepsilon^j) \left(-\frac{dz_i}{dt} + v + H\right) \cdot \nabla \chi_\varepsilon^i(x,s) \\ &= \sum_{i=1}^N (\prod_{j \neq i} \chi_\varepsilon^j) \left(-\frac{dz_i}{dt} + v + \sum_{k \neq i} d_k K(x - z_k(s))\right) \cdot \nabla \chi_\varepsilon^i(x,s),\end{aligned}$$

where the last equality is due to the fact that $K(x - z_i(s)) \cdot \nabla \chi_\varepsilon^i(x,s)$ vanishes since $\chi_\varepsilon^i(\cdot,s)$ is radial around $z_i(s)$. Now, for $x \in \operatorname{supp}(\nabla \chi_\varepsilon^i)$, we have for all $k \neq i$ and for ε sufficiently small

$$|x - z_k(s)| \geq |z_i(s) - z_k(s)| - \varepsilon \geq \frac{d(s)}{2} \geq \frac{d}{2},$$

where $d(s)$ is the positive function introduced in Proposition 3.4 and d is the minimum of $d(s)$ on $[0,t]$. According to the uniform L^∞ bound for the velocity v, to Proposition 3.4 and in view of the ordinary differential equations satisfied by the point vortices, this yields

$$|\partial_t \xi_\varepsilon + u\cdot\nabla\xi_\varepsilon|(x,s) \leq C\sum_{i=1}^N |\nabla\chi^i_\varepsilon(x,s)|.$$

Thanks to Hölder's inequality, we obtain

$$\Big|\int_0^t\int_{\mathbb{R}^2}\omega\varphi\left(\partial_t\xi_\varepsilon + u\cdot\nabla\xi_\varepsilon\right)\,dx\,ds\Big| \leq Ct\|\varphi\|_{L^\infty}\|\omega\|_{L^\infty(L^p)}\|\nabla\chi_\varepsilon\|_{L^{p'}},$$

where $p' < 2$ denotes the conjugate exponent of p. This last quantity tends to zero when ε goes to zero in view of (3.3). Using (3.6) and the pointwise convergence of ξ_ε to 1, we finally arrive at

$$\int_{\mathbb{R}^2}\omega(x,t)\varphi(x,t)\,dx - \int_{\mathbb{R}^2}\omega_0(x)\varphi(x,0)\,dx = \int_0^t\int_{\mathbb{R}^2}\omega\left(\partial_t\varphi + u\cdot\nabla\varphi\right)dx\,ds.$$

This concludes the proof of Theorem 2.3.

4. Point vortices without sign condition

In this section, we investigate the case where all the intensities d_i do not have necessarily the same sign. In this situation, the boundedness of the linear momentum I_δ and the conservation of the energy $\mathcal{H}_\delta$ do not preclude collision of point vortices; in fact, even in the absence of the background part ω, the point vortex system may evolve towards collisions (see [10] for an explicit example involving three points). Therefore, only local existence is expected. We have the following

Theorem 4.1. *Let $p>2$ and $\omega_0\in L^p(\mathbb{R}^2)$ have compact support. Let $z_{i0}, i=1,\ldots,N$ be N distinct points in $\mathbb{R}^2$, and let $d_i, i=1,\ldots,N$ be N non zero numbers. Then there exists a time T^* and an Eulerian solution of the vortex-wave system in the sense of Definition 2.2 on $[0,T^*]$ with this initial data.*

Proof. The proof of Theorem 4.1 is similar to the one of Theorem 2.3. We first mollify ω_0 to obtain an initial vorticity ω_0^δ belonging to $L^1\cap L^\infty(\mathbb{R}^2)$ and which is uniformly bounded in $L^1\cap L^p(\mathbb{R}^2)$.

We set $K_\delta:\mathbb{R}^2\to\mathbb{R}^2$ to be a smooth, bounded and divergence-free vector field such that $K_\delta(x)=K(x)$ for $|x|\geq\delta$. For instance, if $\varphi=\varphi(r)$ denotes a smooth, compactly supported function with support in $r\in(0,\delta)$, such that $\int_0^\delta r\varphi(r)dr=1$, then take

$$K_\delta = K_\delta(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}\int_0^{|x|} r\varphi(r)\,dr.$$

Next, we consider the following modified vortex-wave system in the Lagrangian formulation

$$\begin{cases} v(\cdot,t) = K * \omega(\cdot,t), \\ \dfrac{dz_i}{dt}(t) = v(z_i(t),t) + \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{N} d_j K_\delta\left(z_i(t)-z_j(t)\right), \\ z_i(0) = z_{i0}, \\ \dfrac{d\phi}{dt}(x,t) = v(t,\phi(x,t)) + \displaystyle\sum_{j=1}^{N} d_j K_\delta\left(\phi(x,t)-z_j(t)\right), \\ \phi_0(x) = x,\ x\neq z_{i0}, \\ \omega(\phi(x,t),t) = \omega_0^\delta(x), \qquad t\in\mathbb{R}^+. \end{cases} \tag{4.1}$$

Since ω_0^δ belongs to $L^\infty(\mathbb{R}^2)$ and in view of the definition of K_δ, all the fields involved in (4.1) are smooth (for each δ) and bounded. It is therefore simple to find a global solution $\left(\omega^\delta,\phi^\delta,z_1^\delta,\dots,z_N^\delta\right)$ of (4.1), and we skip the proof.

We claim that there exists a positive T^* and a positive a which only depend on the initial configuration of vortices and $||\omega_0|| = \|\omega_0\|_{L^1(\mathbb{R}^2)} + \|\omega_0\|_{L^p(\mathbb{R}^2)}$, such that for all $t\leq T^*$

$$\inf_{0<\delta<1}\min_{i\neq j}|z_i^\delta(t)-z_j^\delta(t)|\geq a>0. \tag{4.2}$$

Indeed, we set

$$d^\delta(t) = \min_{i\neq j}|z_i^\delta(t)-z_j^\delta(t)|\geq 0,$$

and, possibly decreasing δ, we may assume that $d^\delta(0) >> \delta$. In fact, since $z_i^\delta(0)$ is independent of δ, it follows that $d^\delta(0)\equiv d>0$. We next introduce a positive time T^δ so that $d^\delta>\delta$ on $[0,T^\delta)$. Setting then

$$r_{ij}^\delta(t) = |z_i^\delta(t)-z_j^\delta(t)|, \qquad f(t) = \sum_{i\neq j}\frac{1}{(r_{ij}^\delta(t))^2},$$

we have for $t\in[0,T^\delta)$

$$\begin{aligned}\frac{d(r_{ij}^\delta)^2}{dt} &= 2\langle z_i^\delta - z_j^\delta, v^\delta(z_i^\delta)-v^\delta(z_j^\delta)\rangle \\ &\qquad + 2\sum_{k\neq i,j} d_k\langle z_i^\delta-z_j^\delta, K(z_k^\delta-z_i^\delta)-K(z_k^\delta-z_j^\delta)\rangle.\end{aligned}$$

According to Lemma 3.1, and since ω^δ is bounded in $L^\infty(L^1\cap L^p)$, we have the uniform bound $\|v^\delta\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^2)}\leq C(||\omega_0||)$. Using the identity

$$|K(a)-K(b)| = C\frac{|a-b|}{|a||b|},$$

(see [4], identity (2.6)), we obtain

$$\left|\frac{d(r_{ij}^\delta)^2}{dt}\right| \leq Cr_{ij}^\delta + C\sum_{k\neq i,j}|d_k|\frac{(r_{ij}^\delta)^2}{r_{ik}^\delta r_{jk}^\delta}\leq Cr_{ij}^\delta + C(r_{ij}^\delta)^2 f.$$

Finally, this yields

$$\frac{df}{dt} = \sum_{i\neq j}\frac{-1}{(r_{ij}^\delta)^4}\frac{d(r_{ij}^\delta)^2}{dt}\leq Cf(1+f).$$

Solving this differential inequality gives

$$f(t) \leq \frac{1}{(1+[f(0)]^{-1})e^{-Ct}-1},$$

with $f(0) \leq N(N-1)d^{-2}$, uniformly in δ. This estimate holds as long as the denominator $(1+[f(0)]^{-1})e^{-Ct}-1>0$, i.e., $0 \leq t < T^* \equiv C\log(1+[f(0)]^{-1})$. Hence, $f \leq C$ for $t < T^*$, and T^* depends only on $f(0)$ and $||\omega_0||$. Since $d^\delta(t) \geq 1/\sqrt{f(t)}$, we conclude that $T^\delta \geq T^*$ and therefore it may be chosen independently on δ. This yields (4.2).

Now, we may apply exactly the same arguments as in the proof of Proposition 3.5 to find $\omega \in L^\infty([0,T^*], L^1 \cap L^p(\mathbb{R}^2))$ so that ω^δ converges weakly to ω in $C([0,T^*], L^1 \cap L^p - w)$ and N trajectories $z_1(t), \ldots, z_N(t)$ defined on $[0,T^*]$ so that for all i, z_i^δ converges uniformly to z_i on $[0,T^*]$. In particular, the lower bound for the minimal distance between the point vortices at every level δ given by (4.2) also holds for the limiting vortices. We may then follow the remainder of the proof of Theorem 2.3 to state that $(\omega, z_1, \ldots, z_N)$ is an Eulerian solution of (ES) on $[0,T^*]$. This concludes the proof of Theorem 4.1. □

We conclude with some final remarks. In summary, we have well-posedness of weak solutions for the vortex-wave system when the initial background vorticity is bounded and constant near the initial position of the vortices, existence of a Lagrangian weak solution when the initial background vorticity is bounded and we just proved existence of an Eulerian weak solution when the initial background vorticity is L^p, $p>2$, without additional restrictions. It would be natural to look for extensions of these results in bounded domains and, more interestingly, in rotating spheres. We do not expect uniqueness for the weak solutions constructed here, because the background velocity field is at best Hölder continuous, which ought to lead to the standard counter-examples for the uniqueness of vortex trajectories. Also existence of weak solutions for less regular background vorticities would be surprising, because we would then expect discontinuous point vortex trajectories. The most interesting open problem in this context is uniqueness when the point vortices move in a continuously varying background, which is an interesting question from the modeling point of view, see [14], and appears very plausible, although technically challenging.

Appendix

We prove here the following

Proposition A.1. Let $\omega_0 \in L^\infty(\mathbb{R}^2)$ be compactly supported and $z_1, \ldots, z_N$ be N distinct points in $\mathbb{R}^2$ with positive masses d_i. Let $(\omega, z_1, \ldots, z_N, \phi)$ be a global Lagrangian solution (in the sense of Definition (2.1)). Then the function

$$\mathcal{H}(t) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} \ln|x-y|\omega(x,t)\omega(y,t)\,dx\,dy \\ + \sum_{i\neq j} d_i d_j \ln|z_i(t)-z_j(t)| + 2\sum_{i=1}^N d_i \int_{\mathbb{R}^2} \ln|x-z_i(t)|\omega(x,t)\,dx.$$

is conserved in time.

Proof. We notice first that according to Proposition 3.3, supp $(\omega(\cdot,t)) \subset B(0, C(1+t))$. Therefore all the integrals involved in the definition of $\mathcal{H}$ are well-defined. In order to simplify the notations, we will sometimes write in the following $f(x)$ or $f_t(x)$ instead of $f(x,t)$ for any function f. We set

$$\mathcal{H}_1(t) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} \ln|x-y|\,\omega_t(x)\omega_t(y)\,dx\,dy,$$

$$\mathcal{H}_2(t) = \sum_{i\neq j} d_i d_j \ln|z_i(t)-z_j(t)|,$$

$$\mathcal{H}_3(t) = 2\sum_{i=1}^N d_i \int_{\mathbb{R}^2} \ln|x-z_i(t)|\omega_t(x)\,dx.$$

Since $\phi(\cdot,t)$ preserves Lebesgue measure, we have

$$\mathcal{H}_1(t) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} \ln|\phi(x,t)-\phi(y,t)|\omega_0(x)\,\omega_0(y)\,dx\,dy.$$

Let us assume for the moment that we may derive inside the integrals. We then have

$$-\frac{1}{2\pi}\frac{d\mathcal{H}_1}{dt}(t) = \iint K^\perp\left(\phi(x,t)-\phi(y,t)\right)\cdot\left(\frac{d\phi}{dt}(x,t)-\frac{d\phi}{dt}(y,t)\right)\omega_0(x)\omega_0(y)\,dx\,dy.$$

In view of the ordinary differential equation solved by $\phi(\cdot,t)$, we obtain, after changing variables

$$-\frac{1}{2\pi}\frac{d\mathcal{H}_1}{dt}(t) = \iint K^\perp(x-y)\cdot(v(x)-v(y))\,\omega_t(x)\omega_t(y)\,dx\,dy$$
$$+\iint K^\perp(x-y)\cdot\sum_{i=1}^N d_i\left(K(x-z_i)-K(y-z_i)\right)\omega_t(x)\omega_t(y)\,dx\,dy.$$

Using the symmetry properties of the Kernel K, we are led to

$$-\frac{1}{2\pi}\frac{d\mathcal{H}_1}{dt}(t) = 2\iint K^\perp(x-y)\cdot v(x)\omega_t(x)\omega_t(y)\,dx\,dy$$
$$+2\iint K^\perp(x-y)\cdot\sum_{i=1}^N d_i K(x-z_i)\,\omega_t(x)\omega_t(y)\,dx\,dy.$$

We observe that

$$\iint K^\perp(x-y)\cdot v(x)\omega_t(x)\omega_t(y)\,dx\,dy = \int v(x)\omega_t(x)\,dx\cdot\int K^\perp(x-y)\omega_t(y)\,dy$$
$$= \int v(x)\cdot v^\perp(x)\omega_t(x)\,dx = 0.$$

Besides, we have for all i

$$\iint K^\perp(x-y)\cdot K(x-z_i)\,\omega_t(x)\omega_t(y)\,dx\,dy = \int K(x-z_i)\,\omega_t(x)\,dx\cdot\int K^\perp(x-y)\omega_t(y)\,dy$$
$$= \int v^\perp(x)\cdot K(x-z_i)\,\omega_t(x)\,dx.$$

This finally yields

$$-\frac{1}{2\pi}\frac{d\mathcal{H}_1}{dt}(t) = 2\sum_{i=1}^N d_i\int v^\perp(x)\cdot K(x-z_i)\,\omega_t(x)\,dx. \tag{a}$$

Similarly, we obtain by change of variables

$$-\frac{1}{2\pi}\frac{d\mathcal{H}_3}{dt}(t) = 2\sum_{i=1}^N d_i\int K^\perp(x-z_i)\cdot\left(v(x)-\frac{dz_i}{dt}\right)\omega_t(x)\,dx,$$

therefore we obtain

$$-\frac{1}{2\pi}\frac{d\mathcal{H}_3}{dt}(t) = -2\sum_{i=1}^{N} d_i \int v^\perp(x)\cdot K(x-z_i)\omega_t(x)\,dx + 2\sum_{i=1}^{N} d_i \frac{dz_i}{dt}\cdot v^\perp(z_i). \tag{b}$$

Finally, we compute

$$-\frac{1}{2\pi}\frac{d\mathcal{H}_2}{dt}(t) = \sum_{i\neq j} d_i d_j K^\perp(z_i - z_j)\cdot\left(\frac{dz_i}{dt} - \frac{dz_j}{dt}\right) = T_1 + T_2,$$

where

$$T_1 = \sum_{i\neq j} d_i d_j K^\perp(z_i - z_j)\cdot \sum_{k\neq i,j} d_k\left(K(z_i - z_k) - K(z_j - z_k)\right)$$

and

$$T_2 = \sum_{i\neq j} d_i d_j K^\perp(z_i - z_j)\cdot\left(v(z_i) - v(z_j)\right).$$

On the one hand, it is well-known that $T_1 = 0$ (see [9]). On the other hand, exchanging i and j in T_2 and using that K is antisymmetric leads to

$$-\frac{1}{2\pi}\frac{d\mathcal{H}_2}{dt}(t) = 2\sum_{i\neq j} d_i d_j K^\perp(z_i - z_j)\cdot v(z_i). \tag{c}$$

Combining (a), (b) and (c) finally yields the conclusion.

We finally give a few indications in order to justify rigorously the previous computations. Let δ be a positive small number. In the definition of $\mathcal{H}(t)$, we replace ln by $\ln_\varepsilon$, where $\ln_\varepsilon|z|$ coincides with $\ln|z|$ on $B(0,\varepsilon)^c$ and is identically equal to $\ln\varepsilon$ in $B(0,\varepsilon)$; this yields a function $\mathcal{H}_\varepsilon(t)$. Since $\omega\in L^\infty(L^1\cap L^\infty)$, $\mathcal{H}_\varepsilon(t)$ converges uniformly to $\mathcal{H}(t)$ when ε goes to zero with respect to t. On the other hand, applying the previous computations to $\mathcal{H}_\varepsilon(t)$, we show that $\frac{d}{dt}\mathcal{H}_\varepsilon(t)$ converges locally uniformly to zero with respect to t. We conclude that $\mathcal{H}(t)$ is conserved in time.

□

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Departamento de Matemática, IMECC, Cx. Postal 6065, Universidade Estadual de Campinas - UNICAMP, Campinas, SP 13083-970, Brazil

E-mail address: `mlopes@ime.unicamp.br`

(E. Miot) Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 4 place Jussieu BC 187, 75252 Paris, France

E-mail address: `miot@ann.jussieu.fr`

Departamento de Matemática, IMECC, Cx. Postal 6065, Universidade Estadual de Campinas - UNICAMP, Campinas SP 13083-970, Brazil

E-mail address: `hlopes@ime.unicamp.br`