Actions of finite group schemes on curves

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Finite groups of automorphisms of curves

The ground field k is algebraically closed of characteristic $p \geq 0$.

Let X be a smooth projective curve, and $G \subset \mathrm{Aut}(X)$ a finite subgroup. Then $Y = X/G$ is a smooth projective curve and the quotient morphism $\pi : X \to Y$ is a ramified Galois cover with group G.

More specifically, the subset $X_{\mathrm{fr}} \subset X$ of **free points** (that is, points with trivial stabilizer) is open, dense and G-stable, and π restricts to a G-**torsor** $X_{\mathrm{fr}} \to X_{\mathrm{fr}}/G = Y_{\mathrm{fr}}$.

The complement $X \setminus X_{\mathrm{fr}}$ is the finite set of **ramification points**, with image in Y the set of **branch points**. The morphism π is finite and flat of degree $n = |G|$.

The field of rational functions $k(X)$ is a **function field of one variable**, i.e., a finitely generated extension of k with transcendence degree 1. Moreover, $k(Y) = k(X)^G$ and hence $k(X)/k(Y)$ is a Galois extension with group G.

The Hurwitz formula

Assume either that $p = 0$ or p does not divide n. Then the stabilizer G_x is cyclic for any $x \in X$. Moreover, we have the equality of divisor classes on X

$$K_X = \pi^*(K_Y) + \sum_{x \in X} (e_x - 1)x,$$

where K_X denotes the canonical class, i.e., $\mathcal{O}_X(K_X) = \Omega^1_X$ (the line bundle of differential forms on X), and $e_x = |G_x|$ is the **ramification index** at x.

We have $e_x \geq 1$ with equality if and only if $x \in X_{\mathrm{fr}}$. Thus, only the ramification points contribute to the above sum. Moreover, the ramification index is constant on G-orbits.

Taking degrees, we obtain the Hurwitz formula (1893)

$$2g(X) - 2 = n(2g(Y) - 2 + \sum_{y \in Y} (1 - \frac{1}{e_y})),$$

where g denotes the genus, and $e_y = e_x$ for any point $x \in X$ such that $\pi(x) = y$.

The Chevalley–Weil formula

We still consider a finite group G of automorphisms of a smooth projective curve X.

Then G acts on the line bundle Ω^1_X and hence on its tensor powers $(\Omega^1_X)^{\otimes m}$, where $m \in \mathbb{Z}$. This yields representations of G on the cohomology groups $H^0(X, (\Omega^1_X)^{\otimes m})$ and $H^1(X, (\Omega^1_X)^{\otimes m})$.

If $g(X) \geq 2$, then $H^1(X, \Omega^1_X) = k$ and $H^1(X, (\Omega^1_X)^{\otimes m}) = 0$ for $m \geq 2$, by Serre duality. Moreover, $\mathrm{Aut}(X)$ is finite.

If in addition k is the field of complex numbers, the Chevalley–Weil formula (1934) determines the character of the representations of $\mathrm{Aut}(X)$ in the spaces $H^0(X, (\Omega^1_X)^{\otimes m})$, where $m \geq 1$.

This formula has been generalized by Ellingsrud–Lønsted (1980) to our original setting of an algebraically closed field k of characteristic $p \geq 0$, a smooth projective curve X, a finite subgroup $G \subset \mathrm{Aut}(X)$ of order prime to p, and a G-equivariant line bundle L on X.

The Ellingsrud–Lønsted formula

This formula determines the character $\chi_G(X, L)$ of the virtual G-module $H^0(X, L) - H^1(X, L)$. It gives back the Chevalley–Weil formula when L is a positive tensor power $(\Omega^1_X)^{\otimes m}$.

For simplicity, we state the Ellingsrud–Lønsted formula in the unramified case, where $\pi : X \to Y$ is a G-torsor. Then we have $L = \pi^*(M)$ for a unique line bundle M on Y.

Theorem

With the above notation, we have

$$\chi_G(X, L) = \chi(Y, M)\, \mathrm{ch}\, \mathcal{O}(G),$$

where $\chi(Y, M) = \dim H^0(Y, M) - \dim H^1(Y, M) = \deg(M) + 1 - g(Y)$, *and* $\mathrm{ch}\, \mathcal{O}(G)$ *denotes the character of the regular representation.*

In the ramified case, the formula has additional terms associated to local data at the ramification points.

Finite group schemes

Our aim is to generalize the Hurwitz and Ellingsrud–Lønsted formulas to finite group schemes of automorphisms of curves. We begin with some general background on these objects.

Definition

- A **group functor** is a (covariant) functor from the category of algebras to that of groups.
- An **affine group scheme** is a representable group functor.

(By an algebra, we mean a commutative, associative k-algebra with unit).

Equivalently, a functor G from algebras to groups is an affine group scheme if there exists an algebra A such that $G(R) = \mathrm{Hom}_{\mathrm{alg}}(A, R)$ for any algebra R. Then A is unique; we write $G = \mathrm{Spec}(A)$ and $A = \mathcal{O}(G)$.

The functorial group structure on each $G(R)$ corresponds to a **Hopf algebra** structure on $\mathcal{O}(G)$.

Finite group schemes (continued)

Examples

- The assignement $R \mapsto (R, +)$ yields a group functor $\mathbb{G}_a$: the **additive group**. It is represented by the algebra $k[T]$.
- Likewise, $R \mapsto (R^\times, \times)$ yields a group functor $\mathbb{G}_m$: the **multiplicative group**, represented by the algebra $k[T, T^{-1}]$.

Definition

An affine group scheme G is **finite** if the algebra $\mathcal{O}(G)$ is finite-dimensional as a k-vector space. The dimension of $\mathcal{O}(G)$ is then called the **order** of G, and denoted by $|G|$.

Example

Let G be a finite group of order n, and $\mathcal{O}(G)$ the algebra of k-valued functions on G equipped with pointwise multiplication. (This is the dual of the group algebra $k[G]$). Then G is a constant finite group scheme with Hopf algebra $\mathcal{O}(G)$, of dimension n. Moreover, $G \simeq \operatorname{Spec} \mathcal{O}(G)$ via evaluation of functions.

Finite group schemes (continued)

The following is a special case of a theorem of Cartier :

Proposition

If $p = 0$ then every finite group scheme is a (genuine) finite group.

In characteristic $p > 0$, there are additional examples :

- The group functor $R \mapsto \{t \in R \mid t^p = 0\}$ is represented by a finite group scheme α_p such that $\mathcal{O}(\alpha_p) = k[t]/(t^p)$. Moreover, α_p is the (schematic) kernel of the Frobenius endomorphism

$$F : \mathbb{G}_a \longrightarrow \mathbb{G}_a, \quad t \longmapsto t^p.$$

- The group functor $R \mapsto \{t \in R^\times \mid t^p = 1\}$ is represented by a finite group scheme μ_p such that $\mathcal{O}(\mu_p) = k[t, t^{-1}]/(t^p - 1)$. Moreover, μ_p is the kernel of the Frobenius endomorphism of $\mathbb{G}_m$.

Both α_p and μ_p have order p. One can show that every finite group scheme of order p is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, α_p or μ_p, and these are pairwise non-isomorphic.

Actions of affine group schemes

Let G be an affine group scheme, and X a scheme. An **action** of G on X is an action of the group $G(R)$ on the set $X(R)$ for any algebra R, which is functorial in R. We then say that X is a G-**scheme**.

The action is **faithful** (resp. **free**) if $G(R)$ acts faithfully (resp. freely) on $X(R)$ for all R.

If $X = \mathrm{Spec}(A)$, then the G-actions on X correspond bijectively to the G-actions on A by algebra automorphisms (given for any algebra R by an action of $G(R)$ on the R-algebra $A \otimes_k R$, functorial in R.

Proposition

- *The actions of α_p on an affine scheme $X = \mathrm{Spec}(A)$ correspond bijectively to the k-derivations D of the algebra A such that $D^p = 0$.*
- *The actions of μ_p on $\mathrm{Spec}(A)$ correspond bijectively to the k-derivations D of A such that $D^p = D$.*

Thus, the μ_p-actions on $\mathrm{Spec}(A)$ correspond bijectively to the $\mathbb{Z}/p\mathbb{Z}$-gradings of the algebra A.

Quotients by finite group schemes

Let G be a finite group scheme acting on an affine scheme $X = \mathrm{Spec}(A)$. Consider the subalgebra of invariants

$$A^G = \{f \in A \mid f(g \cdot x) = f(x) \text{ for all } R,\ g \in G(R),\ x \in X(R)\}.$$

The inclusion $A^G \subset A$ corresponds to a morphism of affine schemes

$$\pi : X \longrightarrow Y = X/G,$$

the **(categorical) quotient** by G. If A is finitely generated, then A^G is finitely generated as well.

More generally, let G act on a scheme X. Assume that X is covered by open affine G-stable subsets. Then the corresponding quotient morphisms glue to a morphism $\pi : X \to Y$, the quotient by G. It restricts to a G-torsor $X_{\mathrm{fr}} \to Y_{\mathrm{fr}}$ on the largest G-stable open subset for which the action is free.

The above assumption is satisfied if X is quasi-projective, or if G is **infinitesimal**, that is, $G(k) = \{e\}$ (then every open subset is G-stable). Examples of infinitesimal group schemes include α_p and μ_p.

Actions of finite group schemes : some issues

Let G be a finite group, and X a variety equipped with a faithful action of G. Then the following hold :

(i) The subset $X_{\mathrm{fr}} \subset X$ is non-empty.

(ii) If G acts freely and Y is smooth, then X is smooth.

(iii) The G-action on X lifts uniquely to a G-action on the normalization $\nu : \tilde{X} \to X$.

None of these assertions extend to faithful actions of finite group schemes :

(i) fails for example when $G = \alpha_p \rtimes \mu_p$ acts on the projective line $\mathbb{P}^1$ by affine transformations.

(ii) and (iii) fail for 'most' α_p-torsors $X \to \mathbb{A}^1$, see the next slide.

For any such torsor, X admits no α_p-equivariant resolution of singularities (since the normalization of a curve is its unique desingularization).

α_p-torsors over the affine line

Let $f(x)$ be a polynomial in one variable, and X the zero subscheme of $y^p - f(x)$ in the affine plane $\mathbb{A}^2$ with coordinates x, y.

The group scheme α_p acts on $\mathbb{A}^2$ via

$$g \cdot (x, y) = (x, g + y).$$

(The corresponding derivation of $\mathcal{O}(\mathbb{A}^2) = k[x, y]$ is $\partial/\partial y$). This action is free, with quotient morphism

$$(x, y^p) : \mathbb{A}^2 \longrightarrow \mathbb{A}^2.$$

Moreover, X is α_p-stable with quotient morphism $x : X \to \mathbb{A}^1$, an α_p-torsor.

One may show that every α_p-torsor over $\mathbb{A}^1$ is obtained in this way. Also, X is a variety if and only if f is not a pth power of a polynomial ; then X is a rational curve, with normalization isomorphic to $\mathbb{A}^1$.

α_p-torsors over the affine line (continued)

Now assume that the derivative f' is non-constant (this condition holds for a general polynomial f). Then X is singular by the jacobian criterion.

We claim that the α_p-action on X does not lift to an action on its normalization. Otherwise, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^1 & \xrightarrow{\nu} & X \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \mathbb{A}^1/\alpha_p & \xrightarrow{f} & \mathbb{A}^1, \end{array}$$

where the vertical arrows are the quotient morphisms, the normalization ν is equivariant, and f is obtained from ν by the universal property of quotients.

Since ν is finite and birational, f is finite and birational as well. Also, $\mathbb{A}^1/\alpha_p$ is normal, since so is $\mathbb{A}^1$. Thus, f is an isomorphism. So $\tilde{\pi}$ is an α_p-torsor and ν is a morphism of such torsors. But then ν is an isomorphism, a contradiction.

Linearly reductive group schemes

A finite group scheme G is **linearly reductive** if every G-module is semi-simple.

To describe the structure of these group schemes, recall that every finite group scheme G has a largest infinitesimal normal subgroup scheme G^0. Moreover, we have a canonical isomorphism of group schemes $G \simeq G^0 \rtimes G(k)$, where the group of k-points $G(k)$ is viewed as a constant group scheme.

We may now state the following result, due to Nagata and Demazure–Gabriel :

Theorem

G is linearly reductive if and only if it satisfies the following conditions : G^0 is diagonalizable and the order of $G(k)$ is prime to p.

As an easy consequence, one obtains :

Proposition

Let X be a variety equipped with a faithful action of a linearly reductive group scheme G. Then X_{fr} is non-empty.

Equivariantly normal varieties

Let G be a finite group scheme, and X a G-variety.

Definition

We say that X is G-**normal** if it satisfies the following condition :
For any G-variety Y, every G-equivariant finite birational morphism $Y \to X$ is an isomorphism.

This is equivalent to the quotient stack $[X/G]$ being normal.

When X is affine, this is also equivalent to the condition that $\mathcal{O}(X)$ is the largest G-stable subalgebra of its integral closure in the function field $k(X)$.

Proposition

Let X be a G-variety.

- *Then X admits a G-normalization $X' \to X$, unique up to unique G-equivariant isomorphism.*
- *If X is G-normal, then the quotient variety X/G is normal.*
- *Conversely, if X/G is normal and G acts freely on X, then X is G-normal.*

A G-normality criterion

Let again G be a finite group scheme, and X a G-variety that admits a covering by open affine G-stable subsets.

We may realize G as a subgroup scheme of a smooth connected algebraic group $G^{\#}$ (e.g., $G^{\#} = \mathrm{GL}_n$ in which G embeds via the regular representation). We may then form the "induced" scheme

$$X^{\#} = (G^{\#} \times X)/G,$$

where G acts on $G^{\#} \times X$ via $g \cdot (h, x) = (hg^{-1}, g \cdot x)$.

Then $X^{\#}$ is a variety equipped with an action of $G^{\#}$ (via left multiplication on itself) and with a faithfully flat, $G^{\#}$-equivariant morphism to $G^{\#}/G$ with fiber X at the base point.

Proposition

- *A G-variety X is G-normal if and only if $X^{\#}$ is normal.*
- *A G-curve X is G-normal if and only if $X^{\#}$ is smooth. Then X is a local complete intersection.*

More on G-normal curves

Classically, the assignment $X \mapsto k(X)$ yields an anti-equivalence of categories between smooth (or equivalently, normal) projective curves and function fields of one variable. This can be generalized as follows :

Theorem

Let G be a finite group scheme. Then the above assignment yields an anti-equivalence of categories between G-normal projective curves and function fields of one variable equipped with a (functorial) G-action.

We also have the following criterion for equivariant normality, which follows from Serre's normality criterion together with the proposition on the previous slide :

Theorem

Let G be a finite group scheme, and X a G-curve. Then X is G-normal if and only if the orbit $G \cdot x$ is a Cartier divisor for any $x \in X(k)$.

Here $G \cdot x$ is the finite subscheme of X obtained as the schematic image of the morphism $G \to X$, $g \mapsto g \cdot x$.

G-normal curves (continued)

Corollary

Let $G = \alpha_p$ or μ_p, and X a G-normal curve with quotient Y.
Then X is G-normal if and only if the following conditions hold :
X is smooth at each G-fixed point and Y is smooth.

Example

Let $f(x, y)$ be a homogeneous polynomial of degree p in x, y, which is not a pth power. Let $X \subset \mathbb{P}^2$ the zero subscheme of $z^p - f(x, y)$; then X is a projective curve.

The group scheme μ_p acts on $\mathbb{P}^2$ via

$$t \cdot (x, y, z) = (x, y, tz),$$

and this action stabilizes X. The projection $(x, y) : X \to \mathbb{P}^1$ is the quotient morphism. The fixed points are exactly the $[x : y : 0]$, where $f(x, y) = 0$.

It follows that X is μ_p-normal if and only if f has simple roots.

Further examples

Let K be a function field in one variable, and $K^p \subset K$ the subfield of pth powers. Then there exists $t \in K$ such that

$$K = \bigoplus_{m=0}^{p-1} K^p \, t^m$$

(this is equivalent to K being separable algebraic over $k(t)$).

The above decomposition is a $\mathbb{Z}/p\mathbb{Z}$-grading of K, and hence corresponds to a μ_p-action. The corresponding derivation D is uniquely determined by the condition that $D(t) = t$, i.e., $D = t \, \frac{\mathrm{d}}{\mathrm{d}t}$.

The data of K and t yield a unique μ_p-normal projective curve X such that $k(X) = K$. If the geometric genus of X is at least 2, then X is non-normal (since a smooth projective curve of genus $g \geq 2$ has no nonzero global vector fields).

One can show that every projective μ_p-normal curve is obtained via this construction. Likewise, every projective α_p-normal curve arises from a derivation of the form $\frac{\mathrm{d}}{\mathrm{d}t}$ for t as above.

A Hurwitz formula for G-normal curves

Theorem

Let G be a finite linearly reductive group scheme, and X a G-normal curve with quotient $\pi : X \to Y = X/G$. Then

$$\omega_X = \pi^*(\omega_Y)(\sum(e_x - 1)G \cdot x),$$

where the sum runs over the G-orbits $G \cdot x$, $x \in X(k)$, and e_x denotes the order of the stabilizer G_x.

Here ω_X denotes the dualizing sheaf of X (it is invertible, since X is a local complete intersection). If X is projective, then we obtain by taking degrees

$$2p_a(X) - 2 = n(2p_a(Y) - 2 + \sum_{y \in Y}(1 - \frac{1}{e_y})),$$

where $n = |G|$, and p_a denotes the arithmetic genus.

If G is a genuine finite group, then X is smooth and hence $\omega_X = \Omega^1_X$.
We then recover the Hurwitz formula for ramified Galois covers of smooth projective curves of degree prime to p.

A version of the Ellingsrud-Lønsted formula

Let G be a finite **diagonalizable** group scheme, and X a G-curve with quotient morphism $\pi : X \to Y$. We assume that the action is faithful, and X is G-normal.

Proposition

- *For any $x \in X(k)$, the stabilizer G_x acts faithfully on the fiber at x of the line bundle $\mathcal{O}_X(-G \cdot x)$ via a character ν_x.*
- *Every G-equivariant line bundle L on X can be written uniquely as $\pi^*(M)(-\sum n_x G \cdot x)$, where the sum runs over the G-orbits in $X(k)$, and the n_x are integers such that $2 \leq n_x \leq e_x$.*

Theorem

With the above notation, we have

$$\chi_G(X, L) = \chi(Y, M)\, \mathrm{ch}\, \mathcal{O}(G) + \sum \gamma(e_x, \nu_x, n_x)\, \mathrm{ch}\, \mathcal{O}(G/G_x),$$

where the sum runs again over the G-orbits in $X(k)$, and γ is an explicit function with values in the character ring of G.

G-normal curves and smooth projective surfaces

Let E be an elliptic curve, and $G \subset E$ a finite subgroup scheme. Given a projective G-normal curve X, the quotient

$$S = (E \times X)/G$$

is a smooth projective surface equipped with an action of E.

One can show that every smooth projective surface whose automorphism group contains an elliptic curve is obtained by this construction.

The quotient $\pi : X \to Y = X/G$ defines a morphism

$$f : S \longrightarrow Y = S/E,$$

which is an elliptic fibration with general fiber E.

If G is linearly reductive, then the Hurwitz formula for ω_X yields a version of the canonical bundle formula for an elliptic fibration due to Bombieri and Mumford.

The geometry of these elliptic surfaces S has been recently investigated by Fong and Maccan. The canonical ring $R(S)$ is the ring of G-invariants in $R(X) = \bigoplus_{m=0}^{\infty} H^0(X, \omega_X^{\otimes m})$.

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